Boundary Criticality of Complex Conformal Field Theory: A Case Study in the Non-Hermitian 5-State Potts Model

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Abstract

Conformal fields with boundaries give rise to rich critical phenomena that can reveal information about the underlying conformality. While the existing studies focus on Hermitian systems, here we explore boundary critical phenomena in a non-Hermitian quantum 5-state Potts model which exhibits complex conformality in the bulk. We identify free, fixed and mixed conformal boundary conditions and observe the conformal tower structure of energy spectra, supporting the emergence of conformal boundary criticality. We also studied the duality relation between different conformal boundary conditions under the Kramers-Wannier transformation. These findings should facilitate a comprehensive understanding for exotic irrational CFTs and stimulate further exploration on the boundary critical phenomena within non-Hermitian strongly-correlated systems.

Contents

1 Introduction

The emergence of conformal invariance offers strong advances in the study of quantum critical phenomena [1–3]. It has been well established that the long-wavelength limit of continuous phase transitions of many lattice models is consistently described by the corresponding conformal field theories (CFTs). Given that the bulk conformality is already very rich and important, the associated boundary critical behavior by imposing boundary conditions to critical systems is even more abundant and holds the potential to understand wide ranges of physical problems, ranging from surface critical phenomenon in statistical models [4] to quantum transport through impurities [5]. Just like the bulk CFTs, these boundary critical behaviors are believed to fall into corresponding universality classes that are described by boundary CFTs (BCFTs) [6,7].

For a given CFT, there are many possible ways to impose boundary conditions (b.c.s) onto it. In general, all physical boundary conditions are believed to flow to certain boundary fixed points, known as *conformal boundary conditions*. So far, for rational CFTs, the classification of conformal boundary states has achieved great success by using the Verlinde formula, which establishes the correspondence between conformal boundary states and boundary CFT operators [8–11]. In contrast, the investigation on irrational cases is difficult from analytical and numerical point of view, with few existing result that limited to numerical calculations [12]. Meanwhile, even when the conformal b.c.s are completely identified within some simplest scenarios (such as minimal models), it is still highly non-trivial to align specific boundary states with concrete lattice models, since there is no any general principle to ensure the emergency of specific boundary criticality at low-energy limit for a given boundary interaction term. Notably, the presence of boundaries not only causes local modifications of the correlations, but also leads to a global change in the Hilbert space of CFTs, selecting allowed operators through appropriate boundary conditions and modular invariance [8, 13, 14]. This provides an approach to verify realized conformal boundary states in lattice models by studying the operator content.

This work will address a new class of irrational CFT, dubbed as the *complex CFT*, and their conformal b.c.s. Complex CFT is a branch of conformal invariant theory that greatly violates unitarity, while intrinsically different from conventional non-unitary CFTs with real conformal data and real central charge (see detailed discussion in [15]). This kind of CFT leads to many interesting physical phenomena. For example, when the unitarity-breaking term is not too strong, the physical system defined in the real parameter space could exhibit approximate conformal symmetry, which is conjectured to be the origin of walking renormalization group (RG) flows in gauge theories [16–20] and certain weakly first-order phase transitions in statistical physics [15,21–24]. Very recently, the existence of complex CFTs has been numerically verified in 2 dimensional classical and $1+1$ dimensional quantum 5 state Potts model, where the emergent complex Virasoro symmetry and underlying conformal data are revealed explicitly [25,26].

Moreover, studying complex CFT presents several formidable difficulties and remains a less explored land, leaving many open questions waiting to be mined. A question of particular interest is the generalization of (non-)unitary BCFT to its complex counterpart. On the one hand, as the complex CFT is irrational, the well-established Verlinde formula [27] does not provide a systematic approch to construct the BCFT from fusion rule of bulk operators. On the other hand, the boundary conditions of complex CFT are generally complex, for which the stability under RG flows is not aware. Under these grounds, we address the following questions

1. Are there conformal b.c.s, i.e. conformal fixed points under boundary RG flows, for the complex CFT?
2. If exist, what is the typical feature for the boundary conformal invariance of the complex CFT?

These questions are not only important for studying complex CFTs, but also potentially helpful for establishing general boundary conditions for such kind of irrational CFT.

Here, in this paper we will explore these questions and answer them in the affirmative. We initiate the numerical study on boundary complex CFTs based on the $1+1$ dimensional non-Hermitian quantum 5-state Potts model with free, fixed and mixed boundaries. The emergent boundary complex fixed point is evidenced through the conformal spectrum extracted through an extrapolation of rescaled energy gaps, which is a typical feature for (non-)unitary BCFTs. For the case of free b.c. and fixed b.c., we numerically verify the correspondence between lowest eigenspectrum with Virasoro characters and confirm the duality between free and fixed boundary fixed points. For mixed b.c., we observe emergent complex conformal tower structure at the thermodynamic limit, while a complete determination of all of these Virasoro characters requires further theoretical insight. None of the boundary conformal data shown in this work has been reported before. These observations indicate that the existing knowledge of ordinary BCFT can be (partially) extended to the boundary critical behaviors of the complex CFT, but a complete characterization of them requires new tools for complex conformal b.c.s.

This paper is organized as follows. In Sec. 2, we review the known results of the Potts complex CFT and the well-established BCFT formalism for unitary CFTs. Then we introduce the non-Hermitian 5-state Potts lattice model and its possible conformal b.c.s in Sec. 3. The main results are presented in Sec. 4, including a detailed analysis of boundary fixed point and related conformal operator content. The duality condition among different b.c.s is also discussed. In Sec. 5, we summarize this work and discuss some open questions for future studies.

2 Review of background

2.1 Complex CFT and the Potts realization

A generic 2D CFT is characterized by its conformal data, which includes the central charge, scaling dimensions of local primary operators and operator product coefficients between them [1,3]. The central charge, denoted by a single number c, plays a crucial role in the conformal algebra, which can be separated into a direct product of two independent copies of the Virasoro algebra $\mathcal{V}\otimes\bar{\mathcal{V}}$ given by their generators $\{L_n\}$, $\{\bar{L}_n\}$ satisfying

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \\
[L_m, \bar{L}_n] &= 0
\end{aligned} \tag{1}$$

Moreover, each conformal multiplet, including a primary field with all of its descendants, constitutes an irreducible representation of the underlying conformal algebra [1] that is specified by

[1]This statement does not hold for some special cases, such as logarithmic CFTs.

the central charge c [28]. Since unitarity is believed to be an indispensable feature for quantum theories, the majority of previous exploration concerned with real c [28]. Nonetheless, the representation theory of Virasoro algebra with complex c is still self-consistent from the perspective of mathematics [29].

By extending the definition of conformal data from real domain to the complex domain, one can define the so-called complex CFT, different from usual unitary CFT or non-unitary CFT. [2] Formally, a CFT is said to be unitary, if it contains no negative norm states [3]. If the norm of some CFT states is negative, the corresponding CFT is non-unitary. Non-unitary CFT could have negative central charge, or negative scaling dimensions or some other possibilities. The most famous example is the Lee-Yang edge singularity, with central charge $c = -22/5$ and a single nontrivial primary field with $\Delta = -2/5$ besides the identity operator. Compared with the ordinary (non-)unitary CFT, complex CFT has several notable features. First of all, the conformal data (scaling dimensions, OPE coefficients) and central charge of complex CFT are all complex numbers, in sharp contrast to the (non-)unitary CFT. Second, in complex CFT the left CFT state is not the complex conjugate of right CFT state ${}_L\langle\phi| \neq (|\phi\rangle_R)^\dagger$. We use subscripts L and R to distinguish left and right eigenstates satisfying biorthogonal relations under radial quantization:

$$|\phi\rangle_R = \lim_{r\to 0} \hat{\phi}(r,0)|0\rangle_R, \quad {}_L\langle\phi| = \lim_{r\to\infty} r^{2\Delta_\phi}\ {}_L\langle 0|\hat{\phi}(r,0) \tag{2}$$

Thirdly, as the central charge is associated with the anomaly of the energy-stress tensor, it appears in the Hamiltonian and controls the finite-size scaling of ground-state energy of CFTs and therefore observable [30,31]. In particular, the CFT Hamiltonian (generator of time translation) defined on the cylinder with circumference L (which corresponds to a finite-size system with length L) is the dilation operator $D = L_0 + \overline{L}_0$ on the complex plane that connected by a conformal mapping $z \to (\frac{L}{2\pi})\ln z$, which gives

$$H = \frac{2\pi}{L}(L_0 + \bar{L}_0 - \frac{c}{12}) \tag{3}$$

As c belongs to complex for complex CFT, the above expression of the Hamiltonian indicates that one has to search for the complex fixed points in potential non-Hermitian models. Previously, the Potts model was proposed to be a candidate platform implementing such possibilities, and this conjecture has been validated in a non-Hermitian 5-state Potts model in recent works [25,26].

Here we briefly recall the two dimensional critical Potts CFT, which can be equivalently formulated through Coulomb gas construction [34] that connects the Potts component Q and the Kac parameter m as

$$Q = 4\cos^2(\frac{\pi}{m+1}) \tag{4}$$

The central charge and scaling dimension for each representation could be expressed through the Kac formula

$$\begin{aligned} c &= 1 - \frac{6}{m(m+1)} \\ h_{r,s} &= \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \end{aligned} \tag{5}$$

This formalism was originally proposed for integer $Q \leq 4$, and recently has been extended to

[2]Here we distinguish non-unitary CFT from complex CFT. We denote theories containing real while negative norm states as the non-unitary CFT.

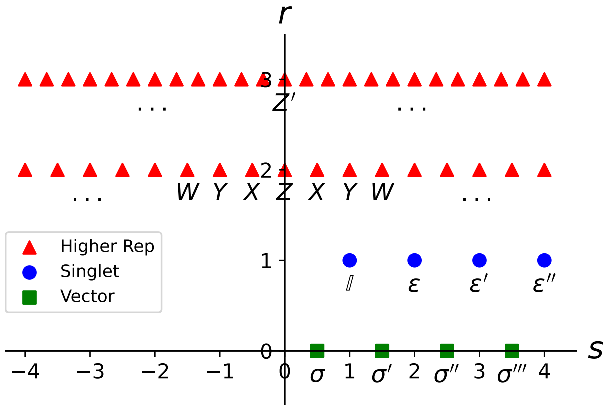

Figure 1: Kac index within $0 \leq r \leq 3$ and $-4 \leq s \leq 4$ for bulk Potts CFT with generic Q [32, 33]. The representations for S_Q singlet operators correspond to the direct product of two irreducible Verma module $V_{1,n} \otimes \bar{V}_{1,n}$ ($n \in \mathbb{N}^*$), marked by blue circles. The S_Q vector fields are also diagonal with Kac index $(0, m+1/2)$ ($m \in \mathbb{N}$), marked by green squares, while each module is non-degenerate in this situation since $r = 0$. The primary fields with S_Q higher representation correspond to logarithmic operator pairs combining Verma modules with opposite s: $V_{r,n/r} \otimes \bar{V}_{r,-n/r}$ or $V_{r,-n/r} \otimes \bar{V}_{r,n/r}$ ($r, n \in \mathbb{N}$ and $r \geq 2$).

generic Q [24, 32, 33], giving the representation of torus partition function as [3] [33]

$$\mathcal{S}_Q = \bigoplus_{s \in \mathbb{N}^*} \mathcal{R}_{(1,s)} \otimes [\,] \oplus \bigoplus_{s \in \mathbb{N}+\frac{1}{2}} \mathcal{W}_{(0,s)} \otimes [1] \oplus \bigoplus_{r \in \mathbb{N}+2} \bigoplus_{s \in \frac{1}{r}\mathbb{Z}} \mathcal{W}_{(r,s)} \otimes \Xi_{(r,s)} \tag{6}$$

Where $[\,]$, $[1]$ and $\Xi_{(r,s)}$ denotes S_Q singlet, vector and higher representations, and $\mathcal{R}_{(1,s)}$ and $\mathcal{W}_{(r,s)}$ labels Virasoro representations. See an illustration for the diagram of Kac index in Fig. 1. The partition function for the former two kinds of field is always diagonal, while the last field combines $(h_{r,s}, \bar{h}_{r,-s})$ and $(h_{r,-s}, \bar{h}_{r,s})$ forming indecomposable while not fully reducible representation, leading to rank-two Jordan cells under L_0 and $\bar{L}_0$. This establishes the fact that the Potts CFT is logarithmic for generic Q [35–37]. Meanwhile, the existence of exact null states for S_Q singlet fields $(h_{1,s}, \bar{h}_{1,s})$ with $s \in \mathbb{N}^*$ imposes strong constraint on the structure of operator algebra, and the associated three-point OPE coefficient and four-point conformal block could be analytically bootstrapped with numerical confirmation [26, 35, 38–43].

Interestingly, when $Q > 4$, from Eq. (4) we have $\cos^2\left(\frac{\pi}{m+1}\right) > 1$, which has no real solution for the parameter m. However, it was argued that the Coulomb Gas partition function remains valid after an analytical continuation with complexified m [24]. Consequently, all conformal data, including the central charge, scaling dimensions, and OPE coefficients, are analytically continued into the complex plane. The norm of most CFT states is no longer real simultaneously and the reflection positivity of correlators is not applicable, which strongly conflicts with unitarity. These theories are classified as a novel branch of theories called complex CFTs.

For the Potts case, this means that the existence of real fixed points persists until $Q = 4$, after which real critical points disappear while turning into complex critical points [24]. Since these complex CFTs could only emerge in non-Hermitian models, the phase transition point

[3] Note that $Q = 2, 3$ lead to unitary minimal models with finite primary fields, outside of the scope of this partition function.

within the original Potts model could only approach the true RG fixed points, while never reaching it. Thus the nature of phase transitions of 2D Potts model change from continuous to discontinuous at $Q = 4$. As the complex fixed points moving away from the real axis with increasing Q, the correlation length at the phase transition points gradually decreases. Importantly, when Q is slightly above 4, a pair of complex fixed points reside extremely close to the real axis, therefore the RG flows between these fixed points greatly slow down and the phase transitions become weakly first-order exhibiting pseudo universal scaling behavior across a large length scale.

2.2 Two-dimensional BCFT

In this section, we revisit the basic ingredient for 2D BCFT. Soon after the birth of two dimensional CFT, Cardy formulated the essential criterion for the conformal b.c.s, stating that the off-diagonal component of the stress tensor, either parallel or perpendicular to the boundary, should vanish [4,14]. When the boundary aligns with the time axis, this implies the absence of momentum flow across the boundary [7]. Under the RG, any uniform b.c. is expected to flow into a conformally invariant fixed point. Additionally, given a bulk CFT, there may be multiple conformal b.c.s, that are distinctively characterized by their boundary operator content.

For 2D CFT defined in the upper half-plane, the boundary corresponds to the real axis. The conformal b.c. suggests that the energy-momentum tensors fulfill $T(z) = \overline{T}(\overline{z})$ when z lies on the real axis. Consequently, the correlators of $\overline{T}$ are analytically continued from those of T into the lower half-plane. The conformal Ward identity takes the form:

$$\langle T(z)\Pi_j\varphi_j(z_j,\overline{z}_j)\rangle = \sum_j\Big(\frac{h_j}{(z-z_j)^2}+\frac{1}{z-z_j}\partial_{z_j}+\frac{\overline{h}_j}{(\overline{z}-\overline{z}_j)^2}+\frac{1}{\overline{z}-\overline{z}_j}\partial_{\overline{z}_j}\Big)\langle\Pi_j\varphi_j(z_j,\overline{z}_j)\rangle \tag{7}$$

In radial quantization, to ensure equivalence among Hilbert spaces defined on different manifolds, one selects semicircles centered at a boundary point, conventionally the origin. Leveraging the conformal b.c., the dilation operator can be expressed as:

$$D = L_0 = \frac{1}{2\pi i}\oint_C zT(z)dz \tag{8}$$

where C denotes a full circle around the origin. Notably, there is now only a single copy of the Virasoro algebra, due to the fact that conformal mappings preserving the real axis correspond to real analytic functions.

Consideration of the partition function on the torus constrains the bulk operator content through modular invariance. Similarly, consistency on an annulus helps in classifying both permissible b.c.s and boundary operator content. Imagine a CFT on an annulus formed by a rectangle of unit width and height δ, with b.c.s a and b on either edges. The partition function with these b.c.s is denoted as $Z_{ab}(\delta)$.

One approach to compute $Z_{ab}(\delta)$ involves considering the CFT on an infinitely long strip of unit width, conformally related to the upper half-plane by the conformal mapping $z \to (\frac{1}{\pi})\ln z$. The Hamiltonian equals the generator of infinitesimal translations along the strip, given by:

$$H_{ab} = \pi D - \frac{\pi c}{24} = \pi L_0 - \frac{\pi c}{24} \tag{9}$$

For the annulus, we have:

$$Z_{ab}(\delta) = \mathrm{Tr} e^{-\delta H_{ab}} = \mathrm{Tr} q^{L_0 - \frac{c}{24}} \tag{10}$$

where $q \equiv e^{-\pi\delta}$. This can be decomposed into characters:

$$Z_{ab}(\delta) = \sum_h n^h_{ab}\chi_h(q) \tag{11}$$

with the non-negative integers n^h_{ab}. Evaluating the coefficients n^h_{ab} fully specifies the operator content for corresponding boundary conformal fixed point, which has been accomplished only in rational CFTs.

Alternatively, the annulus partition function can be interpreted, up to an overall rescaling, as the path integral for a CFT on a circle of unit circumference, being propagated for imaginary time δ^{-1}. From this perspective:

$$Z_{ab}(\delta) = \langle a|e^{-H/\delta}|b\rangle \tag{12}$$

Since the conformal b.c. requires $L_n = \overline{L}_{-n}$, the solution space for boundary states $|B\rangle$ must reside in the subspace satisfying $L_n|B\rangle = \overline{L}_{-n}|B\rangle$. For diagonal CFTs, a special series of boundary states could be constructed systematically through

$$|h\rangle\rangle = \sum_{N=0}^{\infty} \sum_{j=1}^{d_h(N)} |h,N;j\rangle \otimes \overline{|h,N;j\rangle} \tag{13}$$

called Ishibashi states. Then, the overlap between different Ishibashi states is given by

$$\langle\langle h|e^{-\frac{2\pi}{\delta}(L_0+\bar{L}_0-\frac{c}{12})}|h'\rangle\rangle = \delta_{hh'}\chi_h(e^{-\frac{4\pi}{\delta}}) \tag{14}$$

The Cardy states correspond to physical boundary states fulfilling Eq.(11), which are linear combinations of the Ishibashi states:

$$|a\rangle = \sum_h \langle\langle h|a\rangle |h\rangle\rangle \tag{15}$$

where $|a\rangle$ and $|h\rangle\rangle$ represent conformal boundary states associated with this b.c. and Ishibashi states linked to boundary primary operator h respectively. Equating the two expression for the annlus partition function, the Cardy conditions are derived as

$$\begin{aligned} n^h_{ab} &= \sum_{h'} S_{hh'}\langle a|h'\rangle\rangle\langle\langle h'|b\rangle \\ \langle a|h'\rangle\rangle\langle\langle h'|b\rangle &= \sum_h S_{h'h} n^h_{ab} \end{aligned} \tag{16}$$

where S is the modular matrix, which characterizes the transformation among different characters under modular transformation $\delta \to -1/\delta$: $\chi_i(q) = \sum_j S_{ij}\chi(\tilde{q})$, with ($\tilde{q} = e^{-\pi/\delta}$) and $\chi_i(q)$ being the conformal character of the corresponding irreducible module labeled by i. These requirements impose strong restriction on the structure of permissible boundary states.

It can be demonstrated that the elements $S_{hh'}$ of the modular transformation matrix S also appear in the Verlinde formula [27], derived from considering the consistency of the CFT on a torus. This states that the right-hand side of:

$$n^h_{h'h''} = \sum_\ell \frac{S_{h\ell}S_{\ell h'}S_{\ell h''}}{S_{0\ell}} \tag{17}$$

is equal to the fusion algebra coefficient $n^h_{h'h''}$. Since these are non-negative integers, the consistency of the above boundary states ansatz is confirmed. At least for diagonal models, there is a bijection between the allowed primary fields in the bulk CFT and the permissible conformally invariant b.c.s.

The construction of Cardy state is helpful, since it shows how boundary primary fields are connected with the modular invariance in the bulk CFT. The aforementioned formalism and the conformal b.c.s have been intensively studied within rational BCFTs, where the Verlinde formula works well. In comparison, there is little known results for irrational BCFTs. An obvious obstruction is that the number of primaries is infinite in irrational CFTs and evaluating the modular matrix S is impossible for a generic case. For this reason, we explore properties of boundary ingredients: the boundary conformal spectrum, the correspondence between boundary fields and the representation of Virasoro algebra, in the context of Potts complex CFT in this work. For this purpose, the numerical investigation provides a promising avenue, advocating a first step in the understanding of BCFTs in the context of irrational CFTs.

3 Lattice Model

3.1 Original Potts Model

The original Q-state quantum Potts model is defined through [44]

$$\begin{aligned} H_{\texttt{Potts}} &= H_0(J,h) - h_L\eta_L - h_R\eta_R \\ H_0(J,h) &= -J\sum_{i=1}^{L-1}\sum_{k=1}^{Q-1}(\sigma_i^\dagger\sigma_{i+1})^k - h\sum_{i=1}^{L}\sum_{k=1}^{Q-1}\tau_i^k \end{aligned} \tag{18}$$

The Hilbert space is a tensor-product of Q dimensional local spaces spanned by $|0,1,\cdots,(Q-1)\rangle$. H_0 is the bulk hamiltonian and $\eta_{L(R)}$ is the boundary term.

We firstly focus on the bulk hamiltonian H_0. The Potts spin phase operator is defined as

$$\hat{\sigma}|n\rangle = \omega^n|n\rangle, \quad \omega = e^{2\pi i/Q} \tag{19}$$

and the spin flip operator is

$$\hat{\tau}|n\rangle = |(n+1) \bmod Q\rangle \tag{20}$$

Importantly, they satisfy the relations

$$\sigma_i^Q = \tau_i^Q = 1, \quad \sigma_i\tau_i = \tau_i\sigma_i\omega. \tag{21}$$

$H_0(J,h)$ has two essential properties:

(1) The Hamiltonian is invariant under the S_Q spin permutation symmetry. For example, for $Q=3$, this symmetry is generated by Z_3 symmetry generator $\mathcal{S}$ and charge conjugation $\mathcal{C}$. Their actions do not commute, and the fact that S_Q is non-Abelian has interesting consequences. Charge conjugation obeys $\mathcal{C}^2=1$. It acts on the operators as

$$\mathcal{C}\sigma_j\mathcal{C} = \sigma_j^\dagger, \quad \mathcal{C}\tau_j\mathcal{C} = \tau_j^\dagger. \tag{22}$$

Cyclic permutations are generated by $\mathcal{S} = \Pi_j\tau_j$, and they satisfy

$$\mathcal{S}^\dagger\sigma_j\mathcal{S} = \omega\sigma_j, \quad \mathcal{S}^\dagger\tau_j\mathcal{S} = \tau_j \tag{23}$$

(2) For general Q and periodic b.c., the Potts model is self-dual at the point $J=h$, under Kramers-Wannier transformation

$$\tau_i \to \sigma_i^\dagger\sigma_{i+1}, \quad \sigma_i^\dagger\sigma_{i+1} \to \tau_{i+1}. \tag{24}$$

By tuning the parameter J/h, $H_0(J,h)$ has two phases: an ordered phase at $J>h$ that spontaneously breaks S_Q symmetry, and an disordered phase at $J<h$ that respects S_Q symmetry. The order-disorder transition occurs at $J=h$, with its precise position determined by

the Kramers-Wannier duality. It is well established that the phase transition in this model is continuous for $Q \leq 4$ but turns into first-order for $Q > 4$ [45–48]. Notably, for Q just above 4, such as $Q = 5$, the first-order transition is very weak characterized by a large correlation length and a small energy gap [49].

Next we turn to the boundary terms. From the perspective of Q state quantum Potts model, different known boundary states are achieved through adding a large boundary field that project the boundary spin to a specific configuration. For example, in the 3-state Potts CFT, a D series of minimal model $\mathcal{M}_{5,6}$ with central charge $c = \frac{4}{5}$ [50], the free, fixed (A,B or C) and mixed (AB,BC,AC) b.c.s [51–53] have been established as conformal b.c.s that renormalize into corresponding boundary fixed points of 3-state Potts BCFT [8, 54], where A, B and C correspond to the three directions of spin polarization at each local site related via the Z_3 cyclic permutation. The complete classification of Cardy states includes another conformal boundary state $|\phi_{2,2}\rangle$ [54, 55], realized as the "new" boundary condition [4], preserving the S_3 permutation symmetry and could be reached by changing the sign of boundary transverse fields and polarizing the boundary spins into the direction along this transverse field $\frac{|A\rangle+|B\rangle+|C\rangle}{\sqrt{3}}$ [53].

3.2 Non-Hermitian Potts Model

The original Potts model $H_{\texttt{Potts}}$ in Eq.(18) exhibits second-order ferromagnetic to paramagnetic phase transitions for $Q \leq 4$. For $Q > 4$ the phase transition becomes first-order type, so in the traditional viewpoint there is no critical phenomena that should be concerned. On the other hand, it has been proposed that the $Q > 4$ Potts model could host the conformal fixed points living in the complex parameter space [24]. This idea has been realized and validated in an extended quantum lattice model [26] and a classical model [25] very recently.

In this work, we mainly follow Ref. [26] and consider a 1 + 1D non-Hermitian 5-state Potts subjected to different b.c.s:

$$\begin{aligned}
H_{\text{NH-Potts}} =& H_0(J,h) + H_1(\lambda) - h_L\eta_L - h_R\eta_R \\
H_1(\lambda) =& \lambda \sum_{i=1}^{L-1} \sum_{k_1,k_2=1}^{Q-1} [(\tau_i^{k_1} + \tau_{i+1}^{k_1})(\sigma_i^\dagger \sigma_{i+1})^{k_2} + (\sigma_i^\dagger \sigma_{i+1})^{k_1}(\tau_i^{k_2} + \tau_{i+1}^{k_2})]
\end{aligned} \tag{25}$$

where $H_1(\lambda)$ is a nearest-neighbor interaction term which ensures the Kramers-Wannier duality and S_Q spin permutation symmetry. By choosing λ as a complex number, $H_1(\lambda)$ breaks the hermiticity of the original Potts model. For the bulk Hamiltonian ($h_L = h_R = 0$) with periodic b.c., $H_{\text{NH-Potts}}$ realizes two fixed points in the complex parameter space and these two complex-onjugated-partner fixed points are described by the complex conformal field theory [26]. The critical parameters are determined to be $J_c = h_c = 1$ and $\lambda_c \approx 0.079 + 0.060i$ for $Q = 5$.

The boundary terms in Eq.(25) introduces different b.c.s through different choices of η applied to the ends of the spin chain ('L(R)' labels left (right) boundary site). Inspired by previous conformal b.c.s in two dimensional 2, 3 and 4-state Potts model [8, 52, 53, 56], we consider free, fixed and mixed b.c.s in this work. These are also known as blob b.c.s, where the boundary spin could only take $Q_b \leq Q$ values. For $Q_b = Q$, η are just trivial null matrix and the boundary is free. For $Q_b = 1$, the boundary spin could only take a fixed value. This correspond to fixed b.c., which could be realized through adding a strong magnetic field that project the boundary spin to a certain direction. Similarly, for mixed b.c. $1 < Q_b < Q$, η forbids several boundary spins while taking the rest spins with equal possibilities. In all cases, we take h_L and h_R to be a large number to ensure the long-wavelength limit of this model could flow to corresponding boundary fixed points. The correspondence between b.c.s and different choices of η is summarized in Table.1.

[4] In particular, this new boundary condition was found by implementing a duality between different boundary

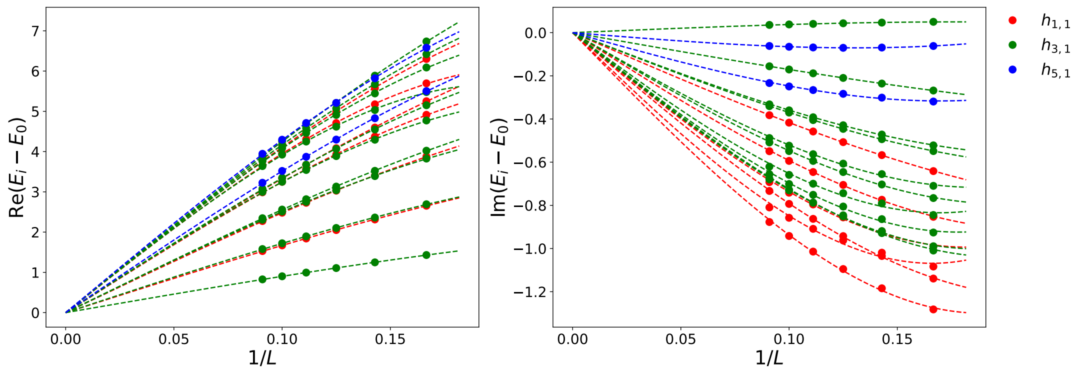

Figure 2: Finite-size scaling of raw energy gaps with free-free boundary condition. We numerically compute several low-lying energy gaps with total system size $L = 6, 7, \cdots, 11$ and fit them through $(E_n - E_0) \propto A/L + B/L^3$. Vanishing energy gaps in the thermodynamic limit signals the boundary criticality. Different colors label states belonging to different conformal multiplets, which will be elucidated in detail below.

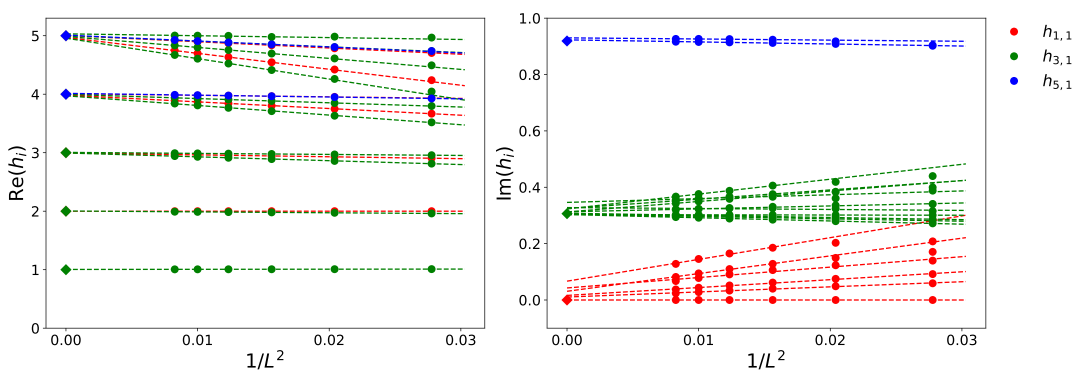

Figure 3: Finite-size scaling of rescaled energy gaps with free-free boundary condition. At each size, we rescale the whole spectrum by setting the first descendant of the identity operator to be $h_{L_{-2}\mathbb{I}} = 2$. Then an extrapolation is performed through $h(L) = h(\infty) + C/L^2$. Up to Re$(h) \leq 5$, we classified three conformal multiplets according to their degeneracy and conformal tower structure. The $h_{1,1}$ multiplet labels the conformal family of identity operators, whose lowest field corresponds to the ground state in this case. The $h_{3,1}$ multiplet denotes lowest S_5 vector operators with 4-fold degeneracy, corresponding to the scaling dimension of boundary magnetization. Lastly, $h_{5,1}$ is the lowest higher-representation field.

Table 1: Conformal boundary conditions for non-Hermitian 5-state Potts model. We use {A,B,C,D,E} to label allowable values of boundary spins.

Boundary condition	η
free	Null matrix
fixed (A)	$\|0\rangle\langle 0\| - \sum_{i=1}^{4} \|i\rangle\langle i\|$
mixed (AB)	$\sum_{i=0}^{1} \|i\rangle\langle i\| - \sum_{j=2}^{4} \|j\rangle\langle j\|$
mixed (ABC)	$\sum_{i=0}^{2} \|i\rangle\langle i\| - \sum_{j=3}^{4} \|j\rangle\langle j\|$
mixed (ABCD)	$\sum_{i=0}^{3} \|i\rangle\langle i\| - \|4\rangle\langle 4\|$

4 Numerical Results

4.1 Gaplessness and finite-size scaling

We numerically study the non-Hermitian hamiltonian Eq.(25) through the exact diagonalization (ED) and extract its low-lying eigenvalue spectra at different total system size L. Near the boundary critical point, the energy spectrum should follow the scaling form of BCFT as [31,57]

$$(E_n - E_0) \propto \frac{\pi}{L}(h_n - \frac{c}{24}) + \delta E_n^{\text{bulk}} + \delta E_n^{\text{boundary}}, \tag{26}$$

where E_n and E_0 are the eigenenergies for n-th excited state and ground state, h_n is the scaling dimension of corresponding boundary operator, c is the central charge. δE_n^{bulk} and $\delta E_n^{\text{boundary}}$ are finite-size correction due to bulk and boundary irrelevant operators [13, 58–60], whose contributions should vanish at thermodynamic limit. Eq.(26) looks exactly the same as the usual BCFT, because it comes from the global conformal symmetry. The only difference is that, here in the complex CFT conformal data such as central charge c and scaling dimensions h_n are not real numbers. Instead, these (boundary) conformal data take complex values. As a result, eigenenergies E_n have both real and imaginary part, which are separately shown in Fig. 2. We illustrate the procedure for free-free b.c.. Both the real and imaginary parts of the raw energy gaps scale to nearly zero in the thermodynamic limit, signaling the boundary criticality.

The bulk correction δE_n^{bulk} takes the same form as the periodic boundary condition case regardless of the b.c., as analyzed in [13,26], where the leading order correction $\delta E_n^{\text{bulk}} \propto (1/L)^3 + \cdots$ comes from $T^2 + \bar{T}^2$ and $T\bar{T}$. The boundary correction $\delta E_n^{\text{boundary}}$ depends on the operator content for corresponding boundary fixed points and need to be treated case by case [60]. For example, for free-free b.c., the leading order term of $\delta E_n^{\text{boundary}}$ is higher than $(1/L)^4$. [5] So in Fig. 2 we use the finite-size scaling function as $(E_n - E_0) \propto A/L + B/L^3$.

4.2 Conformal Tower

After determining the existence of boundary criticality, we turn to extract the scaling dimensions for boundary operators through the rescaled gap:

$$\frac{2(E_n - E_0)}{E_{L_{-2}\mathbb{I}} - E_0} \approx h_n + \frac{\alpha}{L^2} + \cdots \tag{27}$$

conditions, which we will discuss later.

[5] The boundary perturbation is able to be inspected from its boundary conformal spectrum, which will be presented within the next subsection. Currently, we may assume this is the case and recheck this condition when the boundary operator content is extracted.

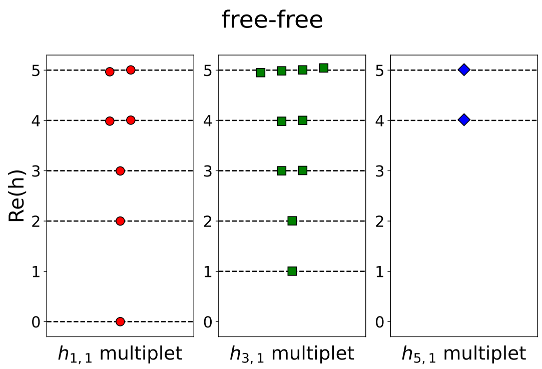

Figure 4: Real parts of boundary conformal spectrum of free-free boundary condition. The solid symbols label numerical results after an extrapolation (see main text), while the dashed lines marked theoretical values from each identified Verma module. Different multiplets are distinguished through their ground states degenercay, originating from different S_Q representations. The imaginary parts belonging to the same conformal multiplet are almost the same, see Tab.2.

Table 2: Boundary conformal multiplet for free-free boundary condition. The first column presented the representation of different boundary operators $\phi_{r,s}$ specified by Kac index r,s. Then the theoretical prediction and numerical results after extrapolation are presented within the second and third column.

Operators	theoretical	numerical
$\phi_{1,1}$	0	0
$L_{-2}\phi_{1,1}$	2	2
$L_{-3}\phi_{1,1}$	3	$2.9961+0.0069i$
$L_{-2}^2\phi_{1,1}$	4	$3.9868+0.0287i$
$L_{-4}\phi_{1,1}$	4	$4.0060+0.0115i$
$L_{-2}L_{-3}\phi_{1,1}$	5	$4.9681+0.0764i$
$L_{-5}\phi_{1,1}$	5	$5.0044+0.0351i$

Operators	theoretical	numerical
$\phi_{3,1}$	$1+0.3063i$	$1.0037+0.3060i$
$L_{-1}\phi_{3,1}$	$2+0.3063i$	$2.0042+0.3045i$
$L_{-1}^2\phi_{3,1}$	$3+0.3063i$	$2.9990+0.3065i$
$L_{-2}\phi_{3,1}$	$3+0.3063i$	$3.0055+0.3033i$
$L_{-1}^3\phi_{3,1}$	$4+0.3063i$	$3.9836+0.3200i$
$L_{-1}L_{-2}\phi_{3,1}$	$4+0.3063i$	$4.0010+0.3090i$
$L_{-1}^4\phi_{3,1}$	$5+0.3063i$	$4.9512+0.3524i$
$L_{-1}^2L_{-2}\phi_{3,1}$	$5+0.3063i$	$4.9857+0.3281i$
$L_{-2}^2\phi_{3,1}$	$5+0.3063i$	$5.0047+0.3142i$
$L_{-4}\phi_{3,1}$	$5+0.3063i$	$5.0435+0.3223i$

Operators	theoretical	numerical
$\phi_{5,1}$	$4+0.9190i$	$4.0148+0.9216i$
$L_{-1}\phi_{5,1}$	$5+0.9190i$	$5.0097+0.9319i$

The extrapolation of rescaled energy gaps is shown in Fig. 3. Here we found three boundary primaries up to $\mathrm{Re}(h_n) \leq 5$ for the free-free b.c., labeled by $\phi_{1,1}, \phi_{3,1}$ and $\phi_{5,1}$ (see Kac table in Fig. 1), corresponding to lowest S_5 singlet, vector and higher representation boundary operators. Since this b.c. preserves the full permutation symmetry, only perturbations from S_5 singlet boundary operators contribute to the energy spectrum. From our numerical results, the only singlet primary operator with (the real part of) dimension lower than 5 is the identity operator, indicating that the lowest boundary contribution to $\delta E_n^{\text{boundary}}$ is higher than $1/L^4$. Accordingly, the boundary conformal spectrum could be extracted through an extrapolation of rescaled energy gaps in Eq. (27), and the results are shown in Fig. 3. In the thermodynamic limit, it is expected that all states belonging to the same conformal family should have the same imaginary part, but the real parts are different by integer numbers.

Next we will separately discuss the conformal tower structures for different b.c.s.

4.2.1 Free-Free b.c.

Now we are ready to discuss the boundary conformal towers within this model. The lowest scaling dimensions of boundary operators for free-free b.c. are summarized in Tab. 2 and Fig. 4. The boundary conformal spectrum exhibits many universal features and unveils underlying conformal information. Firstly, within each conformal family, the scaling dimension with lowest real part is identified as a primary boundary operators $\hat{\phi}$, with its descendants deviating from it with integer values. This is attributed to that the descendant fields are produced through applying Virasoro generators to the primary $L_{-\nu_1}L_{-\nu_2}\cdots L_{-\nu_m}\hat{\phi}$ with positive integer ν. For the BCFT case, there is only one copy of Virasoro algebra. This is in contrast with the bulk criticality, where both of the holomorphic and anti-holomorphic parts should be taken into account. Additionally, most scaling dimensions for boundary operators take complex values, sharply different from BCFTs of real fixed points, indicating the non-unitary nature of the hidden fixed points. Comparing the scaling dimensions with the highest weight of existing Verma Module, we could align these lowest boundary operators with their representation and Kac index for the underlying Virasoro algebra. In addition, the rs-th level descendants contains a null state for the Verma module with integer r and s, which is explicitly observed in our numerics. For example, the third order descendants for the Kac operators $\phi_{3,1}$ only have two fold degeneracy, due to the extra constraint: $f_1 L_{-1}^3\phi_{3,1} + f_2 L_{-1}L_{-2}\phi_{3,1} + f_3 L_{-3}\phi_{3,1} = 0$ (f_i are some constants related to the central charge c). Both the theoretical value and degeneracy agree with numerical results.

Based on the numerical results, the lowest parts of the cylinder partition function of S_5 complex fixed points with free-free b.c. can be determined as

$$Z_{\text{free,free}} = \chi_{1,1} + 4\chi_{3,1} + 11\chi_{5,1} + \cdots \tag{28}$$

where the coefficient in front of the Virasoro character represents the degeneracy of conformal fields and the subscript denotes the Kac index for its representation under Virasoro algebra. We also presented the small q expansions of several Virasoro characters as:

$$\begin{aligned}
\chi_{1,1} &= q^{-c/24+h_{1,1}}(1+q^2+q^3+2q^4+2q^5+4q^6+\cdots) \\
\chi_{2,1} &= q^{-c/24+h_{2,1}}(1+q+q^2+2q^3+3q^4+4q^5+6q^6+\cdots) \\
\chi_{3,1} &= q^{-c/24+h_{3,1}}(1+q+2q^2+2q^3+4q^4+5q^5+8q^6+\cdots) \\
\chi_{4,1} &= q^{-c/24+h_{4,1}}(1+q+2q^2+3q^3+4q^4+6q^5+9q^6+\cdots) \\
\chi_{5,1} &= q^{-c/24+h_{5,1}}(1+q+2q^2+3q^3+5q^4+6q^5+10q^6+\cdots)
\end{aligned} \tag{29}$$

where the central charge c and scaling dimension $h_{r,s}$ could be evaluated through Eq.(5).

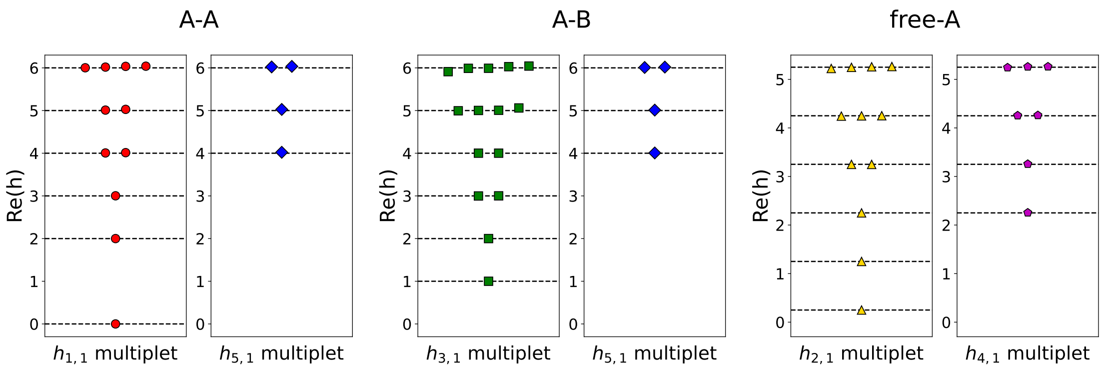

Figure 5: Boundary conformal spectrum of A-A, A-B and free-A boundary conditions. The numerical results are labeled by solid symbols, and the dashed lines represents the theoretical values of scaling dimensions within each Virasoro character.

4.2.2 Fixed b.c.

Next, we considered fixed b.c. (A-A, A-B, free-A), which means the boundary spin is polarized into a fixed direction among Q possible values [52,53,56]. We verified the emergent boundary criticality associated with these b.c.s by the vanishing of energy gaps and extracted the underlying operator content through an extrapolation in parallel to the discussion for free-free b.c.. Let's consider the finite-size correction within A-A b.c. as an example, the boundary transverse field break the full S_5 symmetry into its subgroup S_4 symmetry. Thus only S_4 singlet operator with one-fold degeneracy will modify the scaling form. From the numerical results, the Kac operator $\phi_{5,1}$ acquires S_4 vector representation, while the other primary with dimension smaller than 5 is the identity operator. Thus the finite-size correction also takes the same form as free-free b.c. and the extrapolation formula Eq.(27) is still valid in this case. For the A-B or free-A b.c.s, the identity multiplet is not included within the boundary operator content, which is a generic consequence when the b.c.s applied to the two ends are different. This means the whole spectrum would be shifted by an overall constant corresponding to the lowest scaling dimensions. However, the boundary perturbation could still be safely excluded due to the absence of irrelevant boundary singlet operators up to $\mathrm{Re}(h) = 5$. For these 2 cases, we could rescale the whole spectrum by setting the gap between the lowest singlet field ψ with its first-order descendant to be 1 and perform the extrapolation as:

$$\frac{E_n - E_0}{E_{L_{-1}\psi} - E_\psi} \approx h_n + \frac{\alpha}{L^2} + \cdots \tag{30}$$

Once again, we could align each boundary operators with its irreducible representation under Virasoro algebra from the existence of null states and the theoretical prediction for scaling dimensions. The numerical results are presented in Fig. 5, where the spectrum also displays boundary conformal tower structure, coming from the fusion between different conformal boundary states [8].

For fixed b.c.s, we determine the expression of partition function over Virasoro characters as

$$Z_{\mathrm{A,A}} = \chi_{1,1} + 3\chi_{5,1} + \cdots \tag{31}$$

$$Z_{\mathrm{A,B}} = \chi_{3,1} + 2\chi_{5,1} + \cdots \tag{32}$$

$$Z_{\mathrm{free,A}} = \chi_{2,1} + 3\chi_{4,1} + \cdots \tag{33}$$

4.2.3 Free/Fixed-Mixed b.c.

In this subsection, we present more numerical results involving connecting free/fixed b.c. with mixed b.c.s. Due to the S_5 permutation symmetry, the free-mixed b.c. leads to three possible combinations with respect to two/three/four state mixed b.c.. The real parts for these boundary states are illustrated in Fig. 6. The extrapolated spectrum exhibits similar conformal structure as other boundary fixed points, which allows classification of different conformal multiplets. Within each conformal family, all states acquire the same representation under global symmetry, while their real parts deviate from each other with integer values and their imaginary parts are almost equal. However, it is difficult to align each multiplet with corresponding Virasoro characters. We suspect that it may come from the following reasons. Firstly, since the identity family is absent in most cases, the ground state has a non-zero conformal dimension and the exact spectrum should be shifted by the lowest one, which is an unknown complex number. Secondly, although the existence of null states is explicitly observed in several multiplets, the numerical results are incompatible with corresponding representations. For example, the level-2 null states are found in the lowest two conformal family marked by red and green symbols within free-AB b.c.. However, the numerical energy difference between these two states is approximately $0.257-0.046i$, different from $h_{2,1}-h_{1,2}\approx 0.017+0.227i$, excluding the identification of these two families as $\phi_{1,2}$ and $\phi_{2,1}$. This might be attributed to that the Kac indexes are allowed to take fractional numbers, similar to the bulk CFT. Additionally, the operator content for these boundary states is more complicated, containing many low-lying singlet boundary operators, which might induce more leading perturbation to the finite-size scaling behavior than the bulk correction from $T\bar{T}$. This also explains the large deviation from perfect integer-value spacing, compared with free/fixed b.c.s.

Next, we consider connecting fixed b.c. with two/three/four state mixed b.c.. There are 6 inequivalent boundary states in this case. We presented four of them in Fig.7. Similar as previous discussion, each conformal family exhibit apparent tower structure, while the correspondence between numerical results with Virasoro characters is currently unknown. Intriguingly, we identify a boundary operator having null states at its level-2 and level-3 descendants within A-BCD/ABCD/BCDE b.c.s. This phenomenon is usually occured within minimal models, where each primary operators corresponding to two Kac representations under Virasoro algebra. However, this is not expected to happen for any primary operators in the bulk logarithmic Potts CFTs in Eq.(6). All of these results urge a comprehensive investigation on the structure of conformal boundary states within irrational CFTs.

Principally, one could also verify these conformal boundary states by imposing different mixed b.c.s at the two ends of the spin chain. However, the symmetry of most configuration is much smaller than the original S_5 spin permutation symmetry, leaving the degeneracy of all states to be one-fold, and receives larger boundary perturbations. On the other hand, the results presented above is able to confirm the emergence of boundary fixed points for mixed b.c.s. We also checked the 'new' b.c. by polarizing the boundary spin along the direction of the transverse field [53]. Nevertheless, this b.c. does not lead to any novel boundary fixed point within our model.

4.3 Duality between boundary conditions

Lastly, we identify some properties from the perspective of Kramers-Wannier duality which is an important property of the Potts model. For a periodic chain, the critical bulk Potts model is invariant under this duality transformation as discussed in Sec. 3. For open b.c.s, this trans-

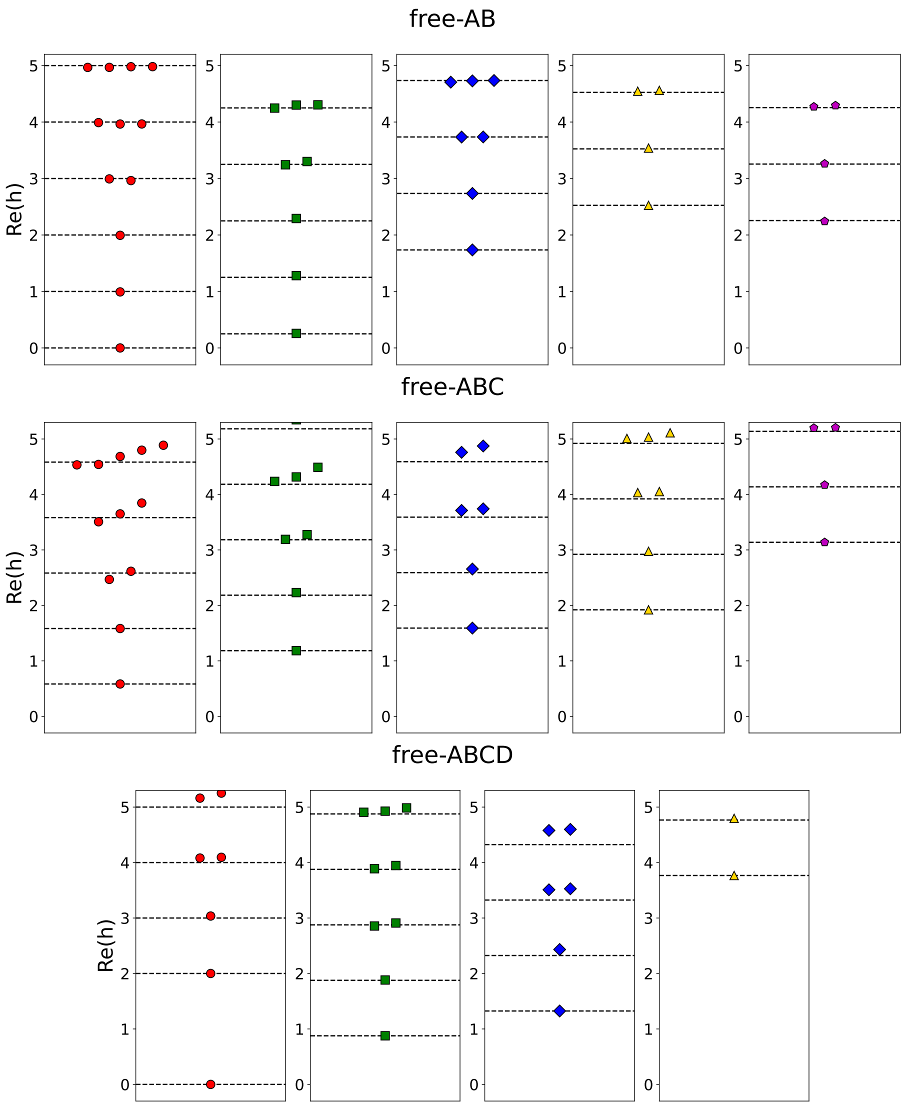

Figure 6: Boundary conformal spectrum of free-AB, free-ABC and free-ABCD boundary conditions. The solid symbols mark extrapolated spectrum from numerical computation. Different conformal multiplets are classified according to their degeneracy and tower structure. The dashed lines are the conformal tower generated from the lowest dimension within each multiplet (see main text).

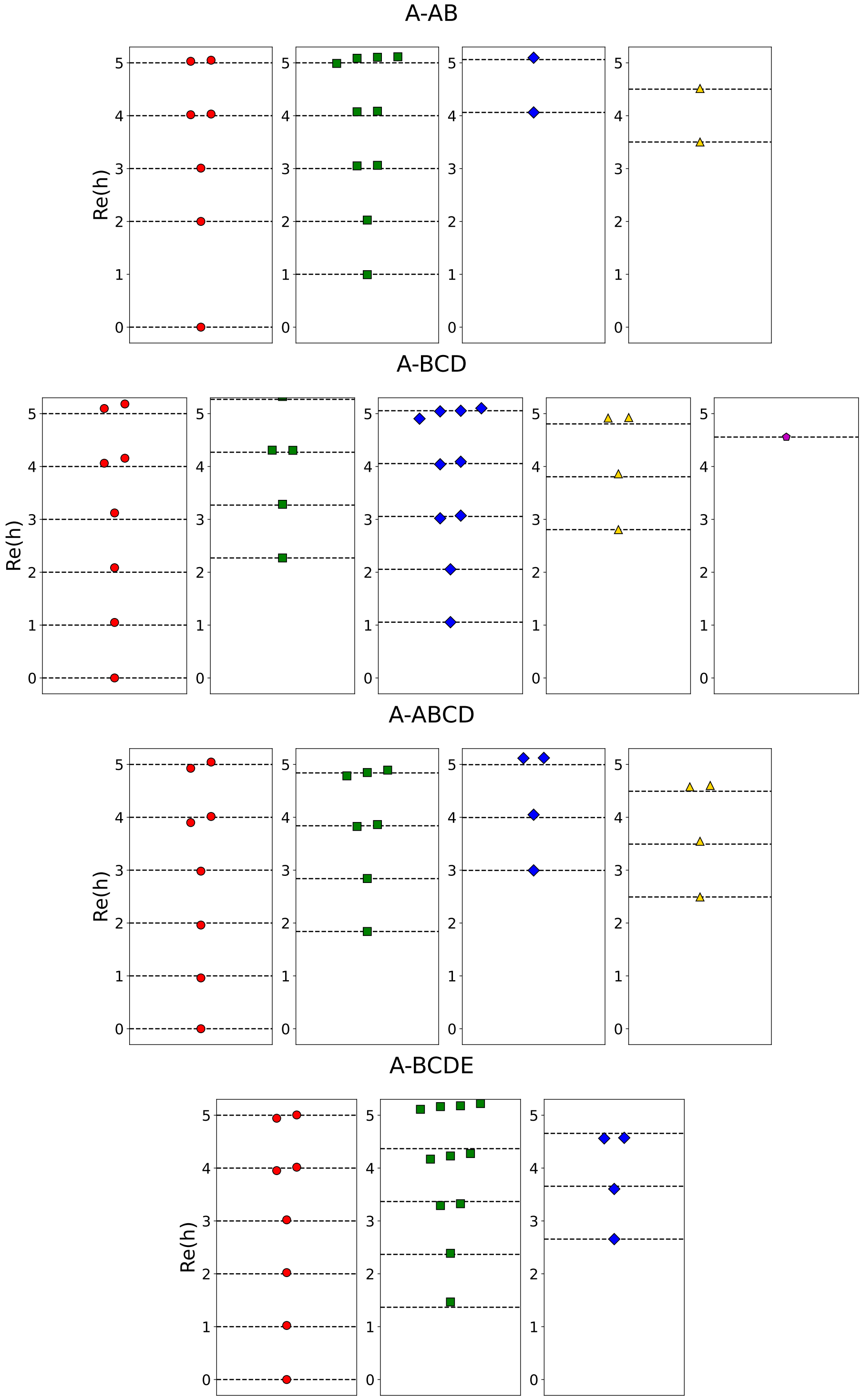

Figure 7: Boundary conformal spectrum for different boundary conditions: (from top to bottom) A-AB, A-BCD, A-ABCD and A-BCDE. The typical conformal tower structures can be well identified, but they are not directly connected with the Virasora algebra (see main text).

formation can be explicitly written as

$$\begin{aligned} \sigma'_{i+\frac{1}{2}} &= \Pi_{j=1}^{i}\tau_j \\ \tau'_{i+\frac{1}{2}} &= \sigma_i^{\dagger}\sigma_{i+1} \end{aligned} \tag{34}$$

However, under this duality transformation, the boundary Hamiltonian is transformed into different boundary conditions. We illustrate this duality using free-free b.c.. When $h_L = h_R = 0$, the model Eq.(25) is mapped to a new Hamiltonian:

$$\begin{aligned} H'(J,h,\lambda) = &-J\sum_{i=1}^{L-1}\sum_{k=1}^{Q-1}\tau'^{k}_{i+\frac{1}{2}} - h\sum_{i=2}^{L}\sum_{k=1}^{Q-1}(\sigma'^{\dagger}_{i-\frac{1}{2}}\sigma'_{i+\frac{1}{2}})^{k} \\ &+\lambda\sum_{i=2}^{L-1}\sum_{k_1,k_2=1}^{Q-1}[(\sigma'^{\dagger}_{i-\frac{1}{2}}\sigma'_{i+\frac{1}{2}})^{k_1}\tau'^{k_2}_{i+\frac{1}{2}} + \tau'^{k_2}_{i+\frac{1}{2}}(\sigma'^{\dagger}_{i-\frac{1}{2}}\sigma'_{i+\frac{1}{2}})^{k_1}] \\ &+\lambda\sum_{i=1}^{L-1}\sum_{k_1,k_2=1}^{Q-1}[(\sigma'^{\dagger}_{i+\frac{1}{2}}\sigma'_{i+\frac{3}{2}})^{k_1}\tau'^{k_2}_{i+\frac{1}{2}} + \tau'^{k_2}_{i+\frac{1}{2}}(\sigma'^{\dagger}_{i+\frac{1}{2}}\sigma'_{i+\frac{3}{2}})^{k_1}] \\ &-h\sum_{k=1}^{Q-1}\sigma'^{k}_{\frac{3}{2}} + \lambda\sum_{k_1,k_2=1}^{Q-1}(\sigma'^{k_1}_{\frac{3}{2}}\tau'^{k_2}_{\frac{3}{2}} + \tau'^{k_2}_{\frac{3}{2}}\sigma'^{k_1}_{\frac{3}{2}}) \end{aligned} \tag{35}$$

where the site $\{1, 2, \cdots, L\}$ are rearranged into site $\{\frac{3}{2}, \frac{5}{2}, \cdots, L + \frac{1}{2}\}$ within the new Hamiltonian. At the boundary critical point $J_c = h_c = 1$ and $\lambda_c \approx 0.079 + 0.060i$, the transformed model H' has transverse field, longitudinal field and transverse field-longitudinal field interaction terms at its left end: $i = \frac{3}{2}$, while there is not any on-site interaction at the opposite side: $i = L + \frac{1}{2}$. Since $|\lambda| \ll |J| = |h|$, the transverse field-longitudinal field interaction is negligible compared with the other two terms. Additionally, the longitudinal field tends to polarize the left end into the A direction and this b.c. is expected to be renormalized into fixed b.c.. The Potts spin at the other end is free to take any values among Q states, corresponding to a sum of fixed b.c. with A, B, C, D and E states [14, 54]. This means the free b.c. is dual to the fixed b.c. and the partition function is quantitatively related via:

$$Z_{\text{free,free}} = Z_{A,A} + Z_{A,B} + Z_{A,C} + Z_{A,D} + Z_{A,E} = Z_{A,A} + 4Z_{A,B} \tag{36}$$

Up to $\text{Re}(h) \leq 5$, we find (36) numerically holds by using Eq.(28) and Eq.(31) and (32). So we conclude that the duality between free and fixed b.c.s Eq.(36) is verified within our computation. In particular, new boundary conditions can be constructed by fusing topological defect lines (TDLs), which was studied systematically in Ref. [61] with focusing on rational CFTs. The duality transformation here also falls into this picture, where the TDL is now chosen in a non-trivial way that corresponds to the Kramers-Wannier duality as a non-invertible symmetry [62, 63] of the Potts model. The duality between different boundary conditions is then established by fusing the duality defect line. While most existing literature focus on fusing TDLs in rational CFTs [61, 64–67] , here for the first time we consider such duality relation for an irrational CFT with complex conformal data.

5 Summary, Discussion and Outlook

In this work, we have investigated the boundary critical phenomena of a non-Hermitian quantum 5-state Potts model which exhibits complex confomal invariance in the bulk. We propose a list of potential conformally invariant boundary conditions (b.c.s) inherited from the existing

$Q \leq 4$ Potts boundary conformal field theory (BCFT), and we verify them using characteristic conformal tower structures of operator spectra through numerical calculations. We further extract conformal dimension of boundary operators that connected with the irreducible representation of underlying Virasoro algebra. These findings unambiguously demonstrate that the conformal invariant boundary fixed points exists even in the complex conformal field theory. We hope this work could stimulate future exploration along several directions as follows.

First, it is of great importance to develop novel theoretical tools to analytically determine the operator content for boundary complex fixed points. For example, the conformal feature and content for mixed b.c.s (Fig. 6 and Fig. 7) is awaiting a good understanding. The existing 2d BCFT approach is mainly applicable to rational CFTs, with little known for irrational cases. The underlying reason is that the well-established Verlinde formula is only applied to rational CFTs, while for irrational CFTs there is no general approach to construct the complete set of boundary states. Taking the Potts model studied in this work as an example. On the one hand, the Coulomb Gas formalism offers a complete classification for all of the bulk primary fields in 2d Potts/O(n) CFT [32–34, 68], which potentially provides a feasible way to tackle this problem. On the other hand, since the Potts CFT is logarithmic for generic Q [35, 36], the boundary conformal spectrum there might share totally the same Virasoro structure as the complex case $Q = 5$ within this work. It might be promising to derive its boundary spectrum from the perspective of logarithmic BCFTs, where some earlier exploration has been achieved in several simple cases [69–75]. The result of this work stands as valuable benchmarks for future computation.

Second, in principle, given the conformal b.c.s, it is supposed to be able to evaluate the bulk-boundary OPE coefficients from wavefunction overlaps and obtain more conformal data through an extrapolation to the thermodynamic limit [76, 77]. However, to interpret the physical meaning of these results in this model requires a deep understanding for the emergent boundary criticality in complex CFTs. From a different perspective, these data could not only enable a more thorough computation for the complex boundary criticality, but also facilitate much insight into boundary pseudo-criticality, where a conformal b.c. is applied to a bulk exhibiting pseudo-critical behavior induced by nearby complex fixed points.

Third, the free, fixed and mixed b.c.s might only constitute a small fraction of potential conformal b.c.s within this model, since the bulk CFT contains infinite primary fields. It is meaningful to work out more boundary fixed points, such as the 'new' b.c. for 3 or 4 state Potts model [54]. On the one hand, the structure of Cardy states for the complex BCFT could be further inspected from these results. On the other hand, since the Verlinde formula could not be directly applied to this scenario [8, 27, 78, 79], one might ask whether some modified or generalized version governs the conformal structure for complex boundary criticality, and this should facilitate a straightforward verification or construction of novel conformal boundary states. An alternate approach is to consider adding relevant perturbations onto a subsystem of the CFT, which is expected to be converted to certain conformal boundary conditions (which are expected to be Cardy states) for the unperturbed bulk [80–88]. This setup is also helpful for studying the boundary RG flows, for which the irreversibility of coarse-graining degrees of freedom of the boundary theories is characterized by a g function [89]. While it is conjectured that there is not any c theorem for the bulk complex fixed points [90], the question about generalizing the g function and corresponding irreversibility theorem to complex boundary fixed points remains unexplored.

Moreover, as we have mentioned in the discussion of duality between different boundary conditions, another systematic approach to construct boundary states is by fusing topological defect lines (TDLs), which has achieved great progress for rational CFTs [61, 64–67]. Applying this approach could provide insights into investigating conformal boundary conditions for the complex CFT. Meanwhile, the lattice realization of TDLs [91–95], within our studied non-

Hermitian 5 state Potts model is also an interesting and promising problem for gaining a wealth of knowledge for the complex CFT.

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A More numerical results

In this section, more numerical data of the boundary operator content for fixed, free-mixed and fixed-mixed b.c.s are presented. Since the descendant fields are related to the primaries with integer-deviation, we only exhibit the scaling dimensions of boundary primaries in the following. For the fixed b.c.s, the Kac index for the low-lying fields has been identified within the main text. Thus we compare the theoretical prediction and numerical results in Tab. 3. For the later two cases, the whole spectrum is shifted by the lowest scaling dimension within each b.c.. So we present the numerical evaluation of the difference between the scaling dimensions in Tab. 4 and Tab. 5.

Table 3: The scaling dimensions of boundary primaries for fixed boundary conditions. Different boundary conditions are listed in the first column. The second column presents the representation of different boundary operators $\phi_{r,s}$ specified by Kac index r,s. Then the theoretical prediction and numerical results after extrapolation are presented within the last two column. For the A-B and free-A cases, the identity multiplet is not included within the boundary operator content and we shift the whole spectrum by the lowest scaling dimensions.

Boundary Condition	Operators	theoretical	numerical
A-A	$\phi_{1,1}$	0	0
	$\phi_{5,1}$	$4+0.9190i$	$4.0216+0.9241i$
A-B	$\phi_{3,1}$	$1+0.3063i$	$1+0.3063i$
	$\phi_{5,1}$	$4+0.9190i$	$4.0063+0.9233i$
free-A	$\phi_{2,1}$	$0.25+0.1149i$	$0.25+0.1149i$
	$\phi_{4,1}$	$2.25+0.5744i$	$2.2542+0.5768i$

References

[1] J. Cardy, *Scaling and Renormalization in Statistical Physics*, Cambridge University Press, Cambridge, England, ISBN 9781316036440 (1996).

Table 4: The scaling dimensions of boundary conformal primaries for free-mixed boundary conditions. Different boundary conditions are listed in the first column. For these cases, we label the scaling dimension of the i-th lowest boundary operator as h_i and its difference with respect to the lowest scaling dimensions $h_i - h_0$ could be evaluated from the extrapolation. The numerical results are presented within the last column.

Boundary Condition	Operators	numerical
free-AB	$h_1 - h_0$	$0.2568 - 0.0460i$
	$h_2 - h_0$	$1.7354 + 0.4008i$
	$h_3 - h_0$	$2.2422 + 0.2891i$
	$h_4 - h_0$	$2.5247 + 0.3923i$
	$h_5 - h_0$	$3.0351 + 0.3072i$
free-ABC	$h_1 - h_0$	$0.5831 + 0.0566i$
	$h_2 - h_0$	$1.1837 - 0.0386i$
	$h_3 - h_0$	$1.5908 + 0.2114i$
	$h_4 - h_0$	$1.9206 + 0.0016i$
	$h_5 - h_0$	$3.1374 + 0.4052i$
free-ABCD	$h_1 - h_0$	$0.8766 + 0.2817i$
	$h_2 - h_0$	$1.3239 + 0.1931i$
	$h_3 - h_0$	$3.7666 + 0.8687i$

Table 5: The scaling dimensions of boundary conformal primaries for fixed-mixed boundary conditions. Different boundary conditions are listed in the first column. For these cases, we label the scaling dimension of the i-th lowest boundary operator as h_i and its difference with respect to the lowest scaling dimensions $h_i - h_0$ could be evaluated from the extrapolation. The numerical results are presented within the last column.

Boundary Condition	Operators	numerical
A-AB	$h_1 - h_0$	$0.9931 + 0.2152i$
	$h_2 - h_0$	$3.5023 + 0.8565i$
	$h_3 - h_0$	$4.0618 + 0.6801i$
A-BCD	$h_1 - h_0$	$1.0549 + 0.1170i$
	$h_2 - h_0$	$2.2698 - 0.1090i$
	$h_3 - h_0$	$2.8058 + 0.0655i$
	$h_4 - h_0$	$4.5570 + 0.6225i$
A-ABCD	$h_1 - h_0$	$1.8394 + 0.4112i$
	$h_2 - h_0$	$2.4927 + 0.2089i$
	$h_3 - h_0$	$2.9952 + 0.0970i$
A-BCDE	$h_1 - h_0$	$1.4675 + 0.4090i$
	$h_2 - h_0$	$2.6565 - 0.1240i$
	$h_3 - h_0$	$5.0068 + 1.0670i$

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