Electromagnetic momentum in the Aharonov-Bohm quantum interference experiment from a physical perspective

Ashok K. Singal[⋆]

Astronomy and Astrophysics Division, Physical Research Laboratory, Navrangpura, Ahmedabad - 380 009, India

⋆ ashokkumar.singal@gmail.com

Abstract

In the Aharonov-Bohm setup, a double-slit experiment, coherent beams of electrons passing through two slits form an interference pattern on the observing screen. However, when a long but thin solenoid of current is introduced behind the slits between the two electron beams, an extra phase difference between them appears, as shown by a shift in the interference pattern. This mysterious effect, purportedly arises owing to an electromagnetic momentum, attributed to the presence of a vector potential at the location of either beam, due to the solenoid of current even when there exists no magnetic field outside the solenoid. It has remained a puzzle, how just potential, thought to be a mere mathematical tool for calculating electromagnetic field, can give rise to an electromagnetic momentum in a system, in lieu of field itself. Experimentally the effect has been amply verified, with hardly any doubts that the observed effect is real. A satisfactory physical explanation of the existence of momentum, at least under the aegis of classical electromagnetism, is still missing since inception of the idea more than half a century back. We here show that a subtle momentum can be seen to lie in the product of the drift velocities of the current carrying charges and the mass equivalent of their non-localized potential energies in the electric field of the interfering electrons which manifests, from a classical perspective, a linear momentum in the system. This elusive, additional momentum is what reflected through an extra phase difference between the interfering electron beams.

Publication information to appear upon publication.

Received Date
Accepted Date
Published Date

Contents

1 Introduction

In the Aharonov-Bohm double-slit interference experiment of electron beams [1], when a long but thin solenoid of electric current is introduced between the two slits, a shift in the interference pattern is observed. This effect is sometimes called Ehrenberg-Siday-Aharonov-Bohm effect as the idea was mooted apparently a decade earlier [2]. An extra phase difference between the two interfering beams purportedly arises owing to an electromagnetic momentum, attributed to the presence of a vector potential at the location of either beam, due to the solenoid of current even when there exists no magnetic field outside the solenoid. The first experimental confirmation came soon after [3] and it has since been amply verified using clever experimental setups [4,5], leaving hardly any doubts that the observed effect is real. However, on the theoretical side the picture is not so clear and a satisfactory physical explanation of the existence of momentum, at least under the aegis of classical electromagnetism, is still missing since inception of the idea more than sixty years back. It has remained an enigma, how just potential, thought to be a mere mathematical tool for calculating electromagnetic field, can give rise to an electromagnetic momentum in a system, in the absence of the field itself.

In the experimental setup, a long thin solenoid comprising circular loops of current placed between the two slits (Fig. 1), generates at a distance R from the central axis of the solenoid a magnetic field (in cgs units) as [6]

$$\mathrm{B} = \frac{4\pi I M}{c}\hat{z}\,, \qquad R<r \tag{1}$$

$$\mathrm{B} = 0\,. \qquad R>r \tag{2}$$

where I is the constant electric current flowing in the solenoid, with M current loops per unit length along the z-axis, and r is the radius of each circular loop.

Thus there is no magnetic field outside the solenoid.

Vector potential, defined at a field point by $\nabla\times\mathrm{A}=\mathrm{B}$, is easily determined for such a long solenoid of current, using Stokes' theorem, to get [7]

$$\oint \mathrm{A}\cdot\mathrm{dx} = \int(\nabla\times\mathrm{A})\cdot\mathrm{da} = \int \mathrm{B}\cdot\mathrm{da}\,, \tag{3}$$

which, from the cylindrical symmetry, possesses a finite component only along the azimuthal direction, with $\mathrm{A}=A_\phi\hat{\phi}$, having a value at R as

$$A_\phi = \frac{2\pi R I M}{c} \qquad R<r \tag{4}$$

$$A_\phi = \frac{2\pi r^2 I M}{cR} \qquad R>r \tag{5}$$

An electric charge Q, of mass m and moving with a velocity v, and accordingly having a mechanical momentum $\mathrm{p_m}=m\mathrm{v}$ (we consider here only non-relativistic cases), when passing through a region with a finite vector potential A, gets associated with it, in addition, an

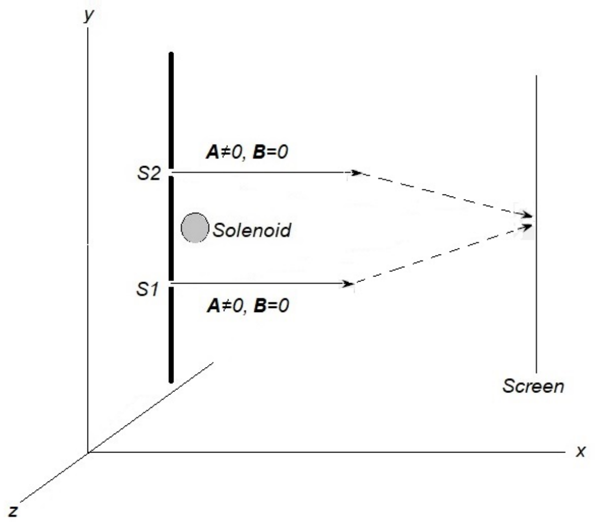

Figure 1: A schematic of the Aharonov-Bohm experimental setup. Coherent beams of electrons, passing through two slits, $S1$ and $S2$, separated along the y-axis, form an interference pattern on the screen. When a long but thin solenoid of electric current is introduced behind the two slits, midway between the two beams, an extra phase between the two electron beams appears, as shown by a shift in the interference pattern. In the region of electron beams, the magnetic field is nil ($\mathrm{B} = 0$) though the vector potential is finite ($\mathrm{A} \neq 0$).

electromagnetic (EM) momentum

$$\mathrm{p}_e = \frac{Q\mathrm{A}}{c}. \tag{6}$$

This EM momentum is independent of either mass m or velocity v of the charge Q, which may thus even be stationary. This form of momentum in classical electromagnetism, suggested by Maxwell [8], has been discussed at some length in the literature [9–11]. From Eq. (6) the momentum would be finite for a non-zero A at the location of Q outside the solenoid of current, even if the magnetic field $\mathrm{B} = 0$ there. Two equal charges, Q_1 and Q_2, placed symmetrically on two opposite sides of the solenoid, and thus having equal and opposite vector potentials, A_1, A_2 with $\mathrm{A}_1 = -\mathrm{A}_2$ at their locations, will accordingly, possess equal and opposite EM momentum vectors, $Q_1\mathrm{A}_1/c = -Q_2\mathrm{A}_2/c$.

In quantum mechanics, the wave function associated with the electric charge Q, because of the additional EM momentum p_e, develops an extra phase shift

$$\varphi = \frac{1}{\hbar}\int \mathrm{p_e}\cdot \mathrm{dx} = \frac{Q}{c\hbar}\int \mathrm{A}\cdot\mathrm{dx}. \tag{7}$$

Then the two charge beams moving along two different paths, but each having the same start and end points as the other, will acquire accordingly, a phase difference

$$\Delta\varphi = \frac{Q}{c\hbar}\oint \mathrm{A}\cdot\mathrm{dx} = \frac{Q}{c\hbar}\int \mathrm{B}\cdot\mathrm{da} = \frac{Q\,\Phi}{c\hbar}, \tag{8}$$

which is thus determined by the magnetic flux Φ through the area enclosed between the paths, even though at the location of either beam the magnetic field is nil. The choice of gauge for

A does not affect $\Delta\varphi$ [12]. Once Eq. (6), from classical electromagnetism, is accepted to be providing a measure of EM momentum in the system, everything else seems to follow from quantum mechanics.

Such a phase difference has actually been inferred in the Aharonov-Bohm setup, from an observed shift in the interference pattern [3–5]. From a physical perspective, however, it is not clear from where does such a mysterious EM momentum appear, whose presence has been verified experimentally, implying that there is some interaction between the solenoid and the charge beams even in the absence of any electromagnetic field at the locations of the beams. This interaction, ostensibly through the vector potential at the location of either beam, is responsible for EM momenta that get reflected in the observed quantum mechanical phase shift between two beams. But to date it still remains a mystery how the solenoid influences the beams in the absence of EM fields at their locations, or how do the beams interact with the solenoid.

The presence of an EM momentum to have a classical origin in the system has been attempted for a *moving* charge [13]. However, according to Eq. (6) one should be able to account for the EM momentum in the system *even for a stationary charge* in a classical explanation, which somehow has not been successful. The literature on both experimental and theoretical fronts is so vast that we make reference to a recent review article [14]. The failure of an explanation within classical physics has led to quantum mechanical topological explanation where the Aharonov-Bohm phase of the wave function of a charged particle depends on the topology of the space it moves in, assuming that the presence of a solenoid of current makes the configuration space non-simply connected [15–17]. Such non-locality features of quantum mechanics may have deep philosophical implications [18, 19]; classical explanations of EM momentum are not in vogue in the contemporary literature.

2 EM momentum of a charge in a vector potential from classical perspective

Since Eq. (6), relating the momentum to the vector potential, has its genesis in classical electromagnetism, one should be able to resolve the issue of EM momentum, without invoking quantum effects, within the classical physics itself. A convincing argument that $Q\mathrm{A}/c$ in some respects does represent momentum within the classical electromagnetism, comes from the following.

Suppose the electric current I in the solenoid is slowly decreased at a constant rate, Then from Eq. (5), the vector potential A would decrease too, giving rise, in the absence of a scalar potential, to an electric field $c\mathrm{E} = -\partial \mathrm{A}/\partial t$. This electric field would exert a force on the stationary Q, changing its mechanical momentum

$$\begin{aligned} \frac{\mathrm{d}\mathrm{p}_\mathrm{m}}{\mathrm{d}t} &= m\frac{\mathrm{d}\mathrm{v}}{\mathrm{d}t} = Q\mathrm{E} \\ &= -\frac{Q}{c}\frac{\mathrm{d}\mathrm{A}}{\mathrm{d}t} = -\frac{2\pi Q r^2 M}{c^2 R}\frac{\mathrm{d}I}{\mathrm{d}t}\hat{\phi}\,, \end{aligned} \tag{9}$$

implying thereby that the mechanical momentum $m\mathrm{v}$ of Q increases at the cost of $Q\mathrm{A}/c$, with the latter decreasing for $\mathrm{d}I/\mathrm{d}t < 0$. This suggests $Q\mathrm{A}/c$ to be a form of momentum, called EM momentum p_e of the charge Q, along with conservation of $\mathrm{p}_\mathrm{m} + \mathrm{p}_\mathrm{e} = m\mathrm{v} + Q\mathrm{A}/c$, called generalized momentum [20], as

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{p}_\mathrm{m} + \mathrm{p}_\mathrm{e}] = 0\,, \tag{10}$$

all within classical electromagnetism. That the conservation of electromagnetic plus kinetic momentum is of interest in Aharonov-Bohm effect has recently been emphasized [21].

However, to date, it still remains an unresolved enigma where after all could such a 'mysteriously hidden' momentum be residing since there is no obvious net linear motion in the system, especially when the charge is considered to be stationary. As the term "momentum" conjures up a vision of some kind of linear motion, a question arises where, after all such motion, if any, is lying in the system? The only non-random motion ostensibly present in the system is in the drift velocities of the current carrying charges in the steady current loop. However, a linear momentum cannot be solely due to the drift velocities, as any such momentum vector integrated over a closed circuit would be zero. Moreover, from Eq. (6), the momentum in question involves, not just the electric current that gives rise to A, but also the specification of charge Q and *its location* (where A is to be evaluated), although any movement of Q does not enter into picture.

We explore here accordingly, from a classical physics perspective, the mysterious momentum in the Aharonov-Bohm setup, endeavouring to possibly unravel wherein the momentum lies in the system. We shall demonstrate here explicitly how an electric charge stationary at a location, where there may be a finite vector potential A, though no magnetic field, does give rise to EM momentum, mysteriously latent in the system and which is reflected in the Aharonov-Bohm experiments. Moreover, as will be seen, the momentum in question is not confined to and localized at some specific location, like that of charge Q; the non-local characteristic of momentum is evident in such a case even within the classical physics picture itself.

In the case of N discrete charges q_{j}, with velocity vectors v_{j}, the vector potential A at the location of the charge Q is computed from the summation

$$\mathrm{A}(\mathrm{x}_0) = \frac{1}{c}\sum_{j=1}^{N}\frac{q_{\mathrm{j}}\,\mathrm{v}_{\mathrm{j}}}{|\mathrm{x}_{\mathrm{j}}-\mathrm{x}_0|}\,, \tag{11}$$

where $|\mathrm{x}_{\mathrm{j}}-\mathrm{x}_0|$ is the distance of charge q_{j}, moving with velocity v_{j}, from the location x_0 of the charge Q.

Then we have

$$\frac{Q\mathrm{A}}{c} = \frac{1}{c^2}\sum_{j=1}^{N}\frac{Q\,q_{\mathrm{j}}}{|\mathrm{x}_{\mathrm{j}}-\mathrm{x}_0|}\,\mathrm{v}_{\mathrm{j}}\,. \tag{12}$$

In order to comprehend the electromagnetic momentum in the system from a physical perspective, instead of the usual way of looking at it in terms of the vector potential A at the location x_0 of the charge Q, we can interpret Eq. (12) in terms of the scalar potential $\phi_{\mathrm{j}} = Q/|\mathrm{x}_{\mathrm{j}}-\mathrm{x}_0|$, due to Q at the location x_{j} of charge q_{j}. For that we rewrite Eq.(12) as

$$\frac{Q\mathrm{A}}{c} = \sum_{j=1}^{N}\frac{\phi_{\mathrm{j}}\,q_{\mathrm{j}}}{c^2}\,\mathrm{v}_{\mathrm{j}}\,. \tag{13}$$

We can use the energy-mass relation, to express the potential energy $q_{\mathrm{j}}\,\phi_{\mathrm{j}}$ of a charge q_{j}, owing to the presence of charge Q at x_0, in terms of its mass equivalent

$$\Delta m_{\mathrm{j}} = \frac{q_{\mathrm{j}}\,\phi_{\mathrm{j}}}{c^2} = \frac{1}{c^2}\frac{Q\,q_{\mathrm{j}}}{|\mathrm{x}_{\mathrm{j}}-\mathrm{x}_0|}\,. \tag{14}$$

Then we can write

$$\frac{Q\mathrm{A}}{c} = \sum_{j=1}^{N}\Delta m_{\mathrm{j}}\,\mathrm{v}_{\mathrm{j}}\,, \tag{15}$$

which can now be readily recognized as an electromagnetic momentum, p_e, in the system. It should be noted that Δm_j here has nothing to do with mass m_j of the jth charged particle and that the electromagnetic momentum in Eq. (15) is not sum of the kinetic momentum, $\Sigma m_\mathrm{j}\,\mathrm{v}_\mathrm{j}$, of moving charged particles. Also it is not possible to localize the potential energy $q_\mathrm{j}\,\phi_\mathrm{j}$ or the equivalent mass Δm_j at either of the charge locations, x_0 or x_j. Nor could one pinpoint the electromagnetic momentum p_e at the location x_0 of charge Q, one has to instead take a holistic view that the system comprises an electromagnetic momentum p_e, without localizing it, even from a classical physics perspective.

For a continuous distribution of moving charges or a current density $\mathrm{j}(\mathrm{x}) = \rho(\mathrm{x})\,\mathrm{v}(\mathrm{x})$, the vector potential A at a field point x_0 is determined from the volume integral [6,7,22,23]

$$\mathrm{A}(\mathrm{x}_0) \;=\; \int \frac{\mathrm{j}(\mathrm{x})}{c|\mathrm{x}-\mathrm{x}_0|}\,\mathrm{d}\tau = \frac{1}{c}\int \frac{\rho(\mathrm{x})\,\mathrm{v}(\mathrm{x})}{|\mathrm{x}-\mathrm{x}_0|}\,\mathrm{d}\tau\,, \tag{16}$$

where $|\mathrm{x}-\mathrm{x}_0|$ is the distance from the point x_0 of the charge element $\rho(\mathrm{x})\,\mathrm{d}\tau$, moving with a velocity $\mathrm{v}(\mathrm{x})$, here $\mathrm{d}\tau$ denotes an element of volume.

Then we can write

$$\frac{Q\mathrm{A}}{c} = \frac{1}{c^2}\int \frac{Q\,\rho(\mathrm{x})\,\mathrm{v}(\mathrm{x})}{|\mathrm{x}-\mathrm{x}_0|}\,\mathrm{d}\tau\,, \tag{17}$$

Because of the scalar potential $\phi(\mathrm{x}) = Q/|\mathrm{x}-\mathrm{x}_0|$ at x due to Q, the system comprising a charge density $\rho(\mathrm{x})$, possesses $\rho(\mathrm{x})\,\phi(\mathrm{x})$ as potential energy per unit volume. Then in the expression

$$\frac{Q\mathrm{A}}{c} \;=\; \frac{1}{c^2}\int \phi(\mathrm{x})\,\rho(\mathrm{x})\,\mathrm{v}(\mathrm{x})\,\mathrm{d}\tau\,, \tag{18}$$

we could use the energy-mass relation, to express the potential energy density in terms of its equivalent mass density, $\mu(\mathrm{x}) = \rho(\mathrm{x})\phi(\mathrm{x})/c^2$, to write

$$\frac{Q\mathrm{A}}{c} \;=\; \int \mu(\mathrm{x})\,\mathrm{v}(\mathrm{x})\,\mathrm{d}\tau = \mathrm{p}_e\,. \tag{19}$$

Here $\mu(\mathrm{x})\mathrm{v}(\mathrm{x})$ is the momentum density in the system, whose volume integral yields the EM momentum p_e in the system, owing to the influence of charge Q, lying at x_0, on the charges moving in the volume.

3 Electromagnetic momentum of a charge outside a solenoid of current

We want now to examine the EM momentum of a charge outside a solenoid and we shall show here that Eqs. (17), (18), or equivalently (19), lead to an electromagnetic field momentum, latent in the system. We should clarify that this EM momentum is different from what sometimes is referred to as 'hidden momentum' [24] and which actually is a mechanical momentum [25], present in the system.

A long, thin solenoid, carrying a steady electric current I, can be considered as a superposition of large number of small planar current loops, each carrying current I, with, say, M loops stacked per unit length of the solenoid, plus a long straight wire carrying current I along the axis of the solenoid. Inside the solenoid the magnetic field would still be $B_\mathrm{z} = 4\pi IM/c$ (Eq. (5)). On the outside, however, there will be an azimuthal field $B_\phi = 2I/cR$ due to current I along the axis of the solenoid [6], which nonetheless, would not affect $\Delta\varphi$ (Eq. (8)), since flux Φ enclosed between the two beams will not change. Therefore we shall henceforth ignore the axial current in the solenoid

A small planar loop carrying a current I around area a of the loop constitutes, irrespective of its shape, a magnetic dipole $\mathrm{m} = I\mathrm{a}/c$, giving rise to a vector potential [6] $\mathrm{A} = \mathrm{m} \times \hat{\mathrm{R}}/R^2$ at R from the loop. This implies for the charge Q at R, from Eq. (6), an EM momentum

$$\mathrm{p_e} \;=\; \frac{Q\,\mathrm{A}}{c} = \frac{Q\,\mathrm{m}\times\hat{\mathrm{R}}}{cR^2} = \frac{(\mathrm{E}\times\mathrm{m})}{c} = \frac{(\mathrm{E}\times\mathrm{a})I}{c^2}, \tag{20}$$

where $\mathrm{E} = -Q\hat{\mathrm{R}}/R^2$ is the electric field due to the charge Q at the location of the small current loop. Thus Eq. (6), expressing EM momentum due to the vector potential of the current loop at the location of Q, represents implicitly a mutual interaction of the charge Q and the current loop since from Eq. (20), the EM momentum vector could as well be considered due to the cross product of electric field E of Q and magnetic moment m of the current loop. Of course this does not in any way resolve the issue of momentum in this apparently static system.

Since the current loop may consist of a conducting wire, the electric field of the charge Q will not extend inside the loop wire, as the induced surface charge density there would tend to cancel any external static electric field inside the wire, leaving only the perpendicular components at the surface, to make the loop equipotential. In that case the system, in the absence of mutual interaction of charge Q and the current loop, may not possess an EM momentum [26], contrary to what would have been otherwise expected from Eq. (6). However, the EM momentum inferred for the system from the experimentally observed shift in the fringe patterns [3–5] indicates that there may be something amiss in the above arguments.

Actually, there is a rather subtle issue involved here as in these experiments there are two coherent charged beams, emerging simultaneously from slits $S1$ and $S2$, assumed to be symmetrically placed on either side of the thin solenoid or its small current loops. Thus one has to consider the EM momentum simultaneously for a pair of equal charges, placed symmetrically, on two opposite sides of the current loop. To be specific, we designate the charges as Q_1 from slit $S1$ and Q_2 from $S2$, lying respectively at distances R_1 and R_2, with $\mathrm{R}_1 = -\mathrm{R}_2$, measured from the loop position. Thus, at least to a first order, the electric fields, say E_1 and E_2 at the loop location and the corresponding scalar potentials across the small loop will be equal and opposite for the two charges, making the loop effectively equipotential, even without the aid of induced surface charges that would have otherwise got formed there.

One might object that the assumed symmetric situation, where interfering electrons emerge from the two slits simultaneously may not always hold good as the interference patterns on the screen arise even if there is only one electron in the system at a time, and that assumption of electrons emerging from both slits 'simultaneously' may not be always justified. Here it may be pointed out that an electron does not go through either slit 1 or slit 2, and that the only way to generate an interference pattern is if the electron simultaneously goes through both slits [27, 28].

Since the induced surface charge on the conducting coil due to either charge may be nil, the mutual interaction between the current loop and each charge would still be present, with the resulting momentum associated with either charge being equal and opposite to that associated with the other charge, This can be seen from the conservation of generalized momentum. Suppose the electric current I in the solenoid is slowly reduced at some constant rate. Then the induced electric field vector (Eq. (9)) at R_1 would exert force on Q_1, changing its mechanical momentum. Now any such change in the mechanical momentum of a charge is possible, from the conservation of generalized momentum (Eq. (10)), only at the cost of its EM momentum, $\Delta\mathrm{p_{m1}} = -\Delta\mathrm{p_{e1}}$, implying the presence of p_{e1} in the system. Similar is the argument for p_{e2} in the case of Q_2.

Thus one could proceed with the investigation of EM momentum associated with either charge, without considering any induced surface charges in the current loop that otherwise

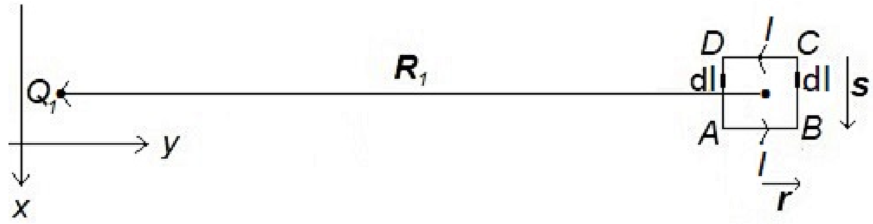

Figure 2: A charge Q_1 lies a distance R_1 from a small current loop $ABCD$, carrying a uniform current I. Two equal current elements, dl, separated by distance $2r$, on two opposite sides of the current loop, are shown.

might have formed to cancel the electric fields, thus leaving the mutual interaction between corresponding charge and the current loop unperturbed.

3.1 Electromagnetic momentum of a charge outside a small current loop

We apply our above results to show, from a physical perspective, that in the presence of an external charge, a current loop does give rise to an EM momentum. For this, we consider a small rectangular loop, carrying a steady current, $I = j\sigma = nev_\mathrm{d}\sigma$, where n is the number density of charges in the circuit, e is the electric charge of conducting charged particles, v_d is their drift velocity and σ is the cross-section of the current-carrying wire. Then from Eq. (11) or (16), the vector potential A at the location x_0 of Q is determined from the integral over the circuit length

$$\mathrm{A}(\mathrm{x}) = \frac{1}{c}\oint \frac{ne\sigma}{|\mathrm{x}-\mathrm{x}_0|}\mathrm{v}_\mathrm{d}\,\mathrm{d}l\,, \tag{21}$$

where dl is an element of the circuit, with v_d as the drift velocity along the direction of the current at the location x of the circuit. Then the EM momentum of the system, from Eq. (17), can be written as

$$\mathrm{p}_e = \oint \frac{Qne\sigma\mathrm{v}_\mathrm{d}}{c^2|\mathrm{x}-\mathrm{x}_0|}\mathrm{d}l = \oint \frac{\mathrm{v}_\mathrm{d}}{c^2}\mathrm{d}\mathcal{E} = \oint \mathrm{v}_\mathrm{d}\,\mathrm{d}m. \tag{22}$$

Here $\mathrm{d}\mathcal{E} = Qne\sigma\mathrm{d}l/|\mathrm{x}-\mathrm{x}_0|$ is the electric potential energy of current carrying charges in the volume element $\sigma\mathrm{d}l$ in the presence of charge Q at x_0, while $\mathrm{d}m = \mathrm{d}\mathcal{E}/(c^2)$ is the equivalent electric mass element. The momentum in Eq. (22) is not the same as the kinetic momentum, $\oint mn\sigma\mathrm{v}_\mathrm{d}\,\mathrm{d}l$, of the electric current carriers, each supposedly of individual mass m. Such a kinetic momentum of the electric current carriers for a steady current, in any case, yields a nil value when summed over the whole current loop ($mn\sigma\oint \mathrm{v}_\mathrm{d}\,\mathrm{d}l = 0$), however, $\oint \mathrm{v}_\mathrm{d}\,\mathrm{d}m$ over the loop does result in a finite value, as will be shown below.

3.2 Electromagnetic momentum of charges placed symmetrically on opposite sides a small current loop

We have still to show that the pair of equal charges, Q_1 and Q_2, placed symmetrically, on two opposite sides of the current carrying solenoid will give rise to equal and opposite EM momentum in the system, something not so readily apparent from Eqs. (15), (19) or (22).

Let us consider charge Q_1 at a distance R_1 along the $-y$ direction, from a small rectangular current loop (Fig. 2). We can calculate the EM momentum of the system by considering pairs of current elements, each of infinitesimal length dl, placed symmetrically on two opposite sides of the loop. The drift velocity in the arm DA of the current loop is along $\hat{\mathrm{x}}$ direction and the current element is nearer to the charge Q_1, thus making a higher contribution to the EM momentum which is along $\hat{\mathrm{x}}$ direction, while the drift velocity in the arm BC is along $-\hat{\mathrm{x}}$ direction and the current element being farther from Q_1, makes a contribution which is relatively lower and is along $-\hat{\mathrm{x}}$ direction. The EM momentum of the whole system, for a small current loop, with $r \ll R_1$ (Fig. 2), is

$$\mathrm{p}_{e1} = \frac{Q_1\, ne\sigma\, s\, v_{\rm d}}{c^2(R_1 - r)}\hat{\mathrm{x}} - \frac{Q_1\, ne\sigma\, s\, v_{\rm d}}{c^2(R_1 + r)}\hat{\mathrm{x}}. \tag{23}$$

The first term on the right hand side is the contribution of the current loop from the arm DA while the second term is from the arm BC, each arm of length s. The contributions from the arms AB and CD to the EM momentum, being equal and opposite, cancel.

To a first order (for $r/R_1 \ll 1$), we get

$$\mathrm{p}_{e1} = \frac{2Q_1\, ne\sigma\, s\, v_{\rm d}\, r}{c^2 R_1^2}\hat{\mathrm{x}} = \frac{(\mathrm{E}_1 \times \mathrm{a})\, I}{c^2}, \tag{24}$$

where $\mathrm{E}_1 = Q_1\hat{\mathrm{y}}/R_1^2$, is the electric field of the charge Q_1 at the location of the small current loop and $\mathrm{a} = 2\,\mathrm{s} \times \mathrm{r} = 2s\, r\, \hat{\mathrm{z}}$ is the area vector of the current loop. The EM momentum in the system, being directly proportional to the drift velocity ($p_{\rm e} \propto v_{\rm d}$) of current carriers, is thus zero to start with for $v_{\rm d} = 0$ (when $I = 0$) and increases as $v_{\rm d}$ increases with I. It is, however, interesting that a finite linear momentum exists in the system owing to the presence of charge Q_1 outside the current loop, even when Q_1 as well as the current loop are both stationary, and momentum vector, if any, due to current carrying charges adds to zero over the closed loop ($mn\sigma \oint \mathrm{v}_{\rm d} dl = 0$).

As for the charge Q_2, lying on opposite side of the current loop at distance R_2 along the y direction, the contribution of EM momentum from the arm BC of the current loop will be higher than that of the arm DA, as a result the net EM momentum will be along $-x$ direction. It will be so even though the potential energy and the mass equivalent for Q_2 is similar as for Q_1, the opposite directions of drift velocities in arms DA and BC will make $\mathrm{p}_{e2} = -\mathrm{p}_{e1}$.

The electric current in the loop could be due to electrons instead of positive charges, essentially making no difference to any of our arguments. Also, at the locations of charge Q_1 or Q_2, the electric field as well as the scalar potential of the lattice of positive ions, is equal and opposite to that of negative current carrying electrons for an over all charge neutral current loop, however, that does not cancel the EM momenta p_{e1} and p_{e2}, arising from the drift velocities of current carrier electrons as the positive ions, fixed in the lattice, do not move with any drift velocity. There is though no net electric potential energy in the system due to Q_1 or Q_2, nonetheless it gives rise to a net EM momentum.

We could replace the rectangular shape of the loop with a polygon comprising a larger number of sides or in the limit even with a circular loop without changing the final result. In fact, by a side by side superposition of a sufficiently large number of small rectangular loops, any such shape of the current loop could be realized. Further, the electric field E now need not necessarily be only in the plane of the current loop. Equation (24), therefore, is a fairly general result as long as the current loop is small enough for E to be considered uniform over its extent.

3.3 An alternate computation of electromagnetic momentum for current loop in electric field of charge

The presence of EM momentum (Eq. (24) in the system, can be worked out in an alternate way, without starting from Eq. (6). In fact, along with the computation of the EM momentum,

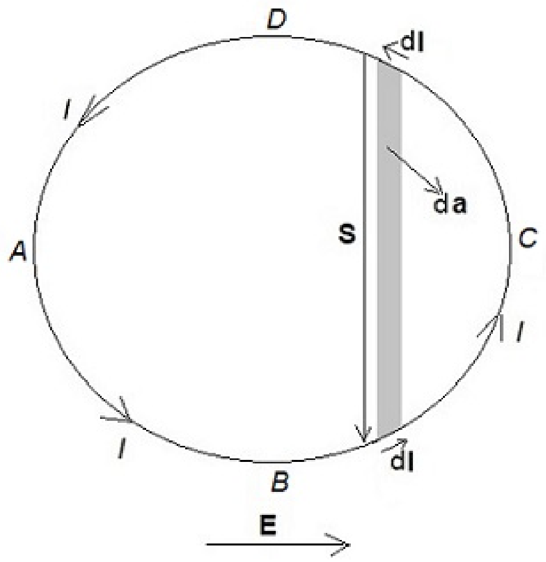

Figure 3: A current loop $ABCD$ is carrying current I. There is an electric field E, uniform over the current loop. The electric field E does positive work, $I\mathrm{E}\cdot\mathrm{dl}$ on a current element dl in the arc ABC of the current loop and an equal and opposite work on a similar current element dl in the arc CDA, the two current elements separated by a distance S, on two opposite sides of the current loop. This implies transfer of an infinitesimal electromagnetic energy, between two current elements across the shaded area da of the circuit, which represents an element of electromagnetic momentum in the system.

one could even derive Eq. (6) as well, that way.

We consider a small circular loop, $ABCD$, carrying a constant current $I = j\sigma$, with j as the current density and σ the cross-section of the current-carrying wire. There is an electric field E, uniform over the dimensions of the current loop (Fig. 3). The electric field does work on the current density j at a rate $\mathrm{j}\cdot\mathrm{E}$ per unit volume [7, 22, 23]. Accordingly, on a current element dl of the loop, the electric field does work at a rate $\mathrm{j}\cdot\mathrm{E}\,\sigma\,\mathrm{d}l = I\,\mathrm{dl}\cdot E$. In section ABC of the loop, the electric field E is doing positive work while in section CDA, the work done is negative.

Thus, power is fed through the electric field into the system through the side ABC while a similar amount of power is being drained off from the side CDA. Effectively, a continuous transfer of electric energy per unit time is taking place from arm CDA to ABC across the current loop due to the presence of the electrical field (Fig. 3). Even Though in this process there is no net change in the total energy content of the system, there is nevertheless a linear momentum associated with this energy flux across the loop from side CDA to ABC.

The electromagnetic momentum can be calculated easily if we consider a pair of current elements placed symmetrically on two opposite sides of the loop (Fig. 3). The power being fed into the circuit element dl in the side ABC through the electric field is $I\mathrm{E}\cdot\mathrm{dl}$, while a similar amount of power is being drained off from a similar circuit elementdl in the side CDA. Effectively, an energy $I\mathrm{E}\cdot\mathrm{dl}$ is being transported per unit time along **s**, the vector joining the current element in side CDA to that in ABC, implying a momentum

$$\mathrm{dP} = I(\mathrm{E}\cdot\mathrm{dl})\frac{\mathrm{s}}{c^2} = \mathrm{E}\times\mathrm{da}\frac{I}{c^2}, \tag{25}$$

where $\mathrm{da} = \mathrm{s}\times\mathrm{dl}$ is the element of area vector contained between these two current elements across the loop. An integration over the entire loop gives the total momentum

$$\mathrm{P_e} = \frac{(\mathrm{E}\times\mathrm{a})I}{c^2} = \frac{\mathrm{E}\times\mathrm{m}}{c} = \frac{Q\mathrm{m}\times\hat{\mathrm{R}}}{R^2 c} = \frac{Q\mathrm{A}}{c}\,, \tag{26}$$

Thus in this way we get not only the EM momentum, which is the same as in Eq. (24), we also derive Eq. (6).

There are other, similar examples in physics where momentum is present in the system due to equal and opposite work being done on spatially separated parts of the system, implying a flux of energy between these spatially separated parts, implying momentum in the system. For instance, a similar continuous transport of electromagnetic energy per unit time across an electric circuit between its opposite arms, but with no change in the net energy of the system, has been shown elsewhere [25] to explain the presence of linear electromagnetic momentum in a stationary system comprising a pair of crossed electric and magnetic dipoles, where nothing obviously is moving or no temporal changes are occurring in the system. A perfect fluid under pressure, having a bulk motion even with non-relativistic velocities has finite momentum proportional to pressure that does work on two opposite ends of a fluid element giving rise to momentum in the system [29]. There is also an opposite example where a charged parallel plate capacitor, moving parallel to the plate separation, has finite electromagnetic energy, but in spite of its motion, has zero electromagnetic momentum in the system [30, 31]. A moving spherical charge distribution, representing a classical electron model, where an equal and opposite work done by the opposite forces of the leading and trailing hemispheres, gives rise to an energy flux and thereby an electromagnetic momentum in the system that explains the more than a century-old famous factor of 4/3 in the electromagnetic momentum of such a system [30, 31].

One would normally expect the power difference between arms ABC and CDA to be compensated by the agency tending to maintain a uniform and steady electric current in the loop, with an equal amount of mechanical energy transfer rate from arm ABC to CDA in the circuit that itself might entail a mechanical momentum [25],

However, in the present case, with equal charges, Q_1 and Q_2, on opposite sides of the current loop, a constant current in both arms will be maintained because of their equal and opposite electric fields, E_1 and E_2. Moreover, the energy flux from arm CDA to ABC because of Q_1 will be compensated by an equal energy flux from arm ABC to CDA due to Q_2. However, as was discussed earlier, associated with individual charges there would still be present EM momenta, p_{e1}, p_{e2}, source of the phase difference, $\Delta\varphi$, experimentally observed between the two charge beams.

Now, we can compute from Eq. (24) the total electromagnetic linear momentum associated with the charge Q_1 and the solenoid of current by summing over Ml current loops of the solenoid, and using $B_z = 4\pi IM/c$ inside the solenoid (Eq. (1)), to get

$$\mathrm{p_{e1}} = \sum_{1}^{Ml} \frac{(\mathrm{E}_1\times\mathrm{a})I}{c^2} = \int \frac{(\mathrm{E}_1\times\mathrm{B})}{4\pi c}\mathrm{d}\tau\,, \tag{27}$$

which is the volume integral of the EM momentum density, $(\mathrm{E}\times\mathrm{B})/4\pi c$ over the solenoid. For the charge Q_2, with $\mathrm{E}_2 = -\mathrm{E}_1$, Eq. (27) implies $\mathrm{p_{e2}} = -\mathrm{p_{e1}}$.

The seat of the field momentum (Eq. (27)) might appear to be within the solenoid, however, one has to take the holistic view that the EM momentum actually lies in the *composite system* of the charge plus solenoid, as it has been emphasized [32] that the composite system is represented by one state. Accordingly the system acts as a whole, giving rise to the momentum, that gets reflected in the Aharonov-Bohm quantum interference experiment. This non-localized interaction seems to be the explanation of this intriguing phenomenon from a classical physics perspective. The vector potential A may be still considered in this case as a convenient, intermediary mathematical step with QA/c presenting a façade of the mutual electric interaction of the conducting current carriers in the solenoid and an external charge Q, which gives rise to an EM momentum in the system.

4 Conclusions

It was shown how a small current loop, source of a finite vector potential at the location of an external electric charge, gives rise to a non-localized electromagnetic momentum, manifested in the product of the drift velocities of the current-carrying charges and the mass equivalent of their potential energies in the electric field of the external charge. A pair of equal charges lying at symmetrical opposite locations outside the current loop gives rise to an equal and opposite momentum in the system. This elusive, equal and opposite momentum for a pair of charges is reflected through an extra phase difference between the interfering electron beams in the double-slit Aharonov-Bohm experiment, when a long but thin solenoid of current, equivalent to a stack of large numbers of current loops, is introduced between the electron beams. A viable explanation of this curious phenomenon in terms of such a subtle momentum under the aegis of classical electromagnetism, has been coveted since inception of the idea more than 60 years back.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the present study.

Author Declarations

The author declares no conflicts/competing interests and no funds, grants, or other support of any kind for this article.

References

[1] Aharonov, Y., Bohm, D.: Significance of electromagnetic potential in the quantum theory. Phys. Rev. 115, 485-491 (1959)

[2] Ehrenberg, W., Siday, R.: The refractive index in electron optics and the principles of dynamics. Proceedings of the Physical Society. Section B, 62, 8-21 (1949)

[3] Chambers, R.G.: Shift of an Electron Interference Pattern by Enclosed Magnetic Flux. Physical Review Letters. 5, 3-5 (1960).

[4] Tonomura, A, Osakabe, N., Matsuda, T., Kawasaki, T., Endo, J., Yano, S., Yamada, H.: Evidence for Aharonov-Bohm effect with magnetic field completely shielded from electron wave. Phy. Rev. Let. 56, 792-795 (1986).

[5] Osakabe N., Matsuda, T., Kawasaki, T., Endo, J., Tonomura, A., Yano, S., Yamada, H. Experimental confirmation of Aharonov-Bohm effect using a toroidal magnetic field confined by a superconductor. Phys. Rev. A 34, 815-822 (1986).

[6] Purcell, E.M.: Electricity and Magnetism - Berkeley Phys. Course vol. 2, 2nd ed. McGraw-Hill, New York (1985).

[7] Griffiths, D.J.: Introduction to Electrodynamics, 3rd edn. Prentice, New Jersey (1999)

[8] Maxwell, J.C.: A Dynamical Theory of the Electromagnetic Field. Phil. Trans. Roy. Soc. London 155, 459-512 (1865).

[9] Konopinski, E.J.: What the electromagnetic vector potential describes. Am. J. Phys. 46, 499-502 (1978)

[10] Semon, M.D., Taylor, J. R.: Thoughts on the magnetic vector potential. Am. J. Phys., 64, 1361-1369 (1996)

[11] Griffiths, D.J.: Resource letter EM-1: Electromagnetic momentum. Am. J. Phys. 80, 7-18 (2012).

[12] Feynman, R.P., Leighton, R.B., Sands, M.: The Feynman Lectures on Physics, Vol. II Addison-Wesley, Reading (1964)

[13] Boyer, T.: Comment on experiments Related to the AharonovBohm phase shift. Found. Phys. 38, 498-50 (2008)

[14] Batelaan, H., Tonomura, A.: The Aharonov-Bohm effects: Variations on a Subtle Theme. Physics Today. 62, 38-43 (2009)

[15] Aharonov, Y.: Proc. Int. Symp. Foundations of Quantum Mechanics and their Technical Implications, Tokyo, 10-19 (1983)

[16] Aharonov, Y., Cohen E., Rohrlich, D.: Nonlocality of the Aharonov-Bohm effect. Phys. Rev. A 93, 042110 (2016).

[17] Aharonov, Y., Cohen, E,, Colombo, F., Landsberger, T., Sabadini, I., Struppa, D.C., Tollaksen, J.: Proc. Nat. Acad. Sci., Finally making sense of the double-slit experiment. 114, Issue 25, 6480-6485 (2017).

[18] Popescu, S.: Nonlocality beyond quantum mechanics. Nat. Phys. 6 (3), 151-153 (2010).

[19] Healey, R.: Nonlocality and the aharonov-Bohm effect. Phil. Sci. 64, 1, 18-41 (1997).

[20] Goldstein, H.: Classical Mechanics. Addison-Wesley, Reading (1950)

[21] Essén, H.: Equations of motion of interacting classical charged particles and the motion of an electron outside a long solenoid. Prog. Elect. Res. B 100, 39-53 (2023).

[22] Jackson, J.D.: Classical Electrodynamics, 2nd edn. Wiley, New York (1975)

[23] Panofsky, W.K.H., Phillips, M.: Classical Electricity and Magnetism, 2nd edn. Addison-Wesley, Reading (1962)

[24] Shockley W., James, R.P.: "Try simplest cases" discovery of "hidden momentum" forces on "magnetic currents". Phys. Rev. Lett. 18, 876-879 (1967).

[25] Singal, A.K.: Do the standard expressions for the electromagnetic field momentum need any modifications?. Am. J. Phys. 84, 780-785 (2016)

[26] Johnson, F.B., Cragin, B.L., Hodges, R.R.: Electromagnetic momentum density and the Poynting vector in static fields. Am. J. Phys. 62, 33-41 (1994),

[27] Feynman, R.P., Leighton, R.B., Sands, M.: The Feynman Lectures on Physics, Vol. 3, Addison-Wesley, Reading (1965)

[28] Wichmann, E.H.: Quantum Physics - Berkeley Phys. Course, vol. 4., McGraw-Hill, New York (1971).

[29] Singal, A.K.: Contribution of pressure to the energy-momentum density density in a moving perfect fluid: a physical perspective. Found. Phys. 51, 4 (2021)

[30] Singal, A.K.: Energy-momentum of the self-fields of a moving charge in classical electromagnetism. J. Phys. A 25 1605-1620 (1992)

[31] Singal, A.K.: Contribution of electric self-forces to electromagnetic momentum in a moving system. arXiv.2206.00431 (2022)

[32] Vaidman, L.: Role of potentials in the Aharonov-Bohm effect. Phys. Rev. A. 86, 040101 (2012).