Two Channel Multi impurity Kondo model

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We show that the Ruderman-Kittel-Kasua-Yoisida interaction in two channel Kondo impurity systems is a relevant perturbation when the number of impurities N is greater than 3. We find a new critical point with anomalous dimensions $\frac{1}{(N+1)}$ for the spin operator and the Sommerfeld coefficient of the specific heat scales as $\gamma \sim T^{-\frac{3}{N+1}}$ Interchannel hybridization plays the role of the most relevant operator at this fixed point having anomalous dimensions $\frac{N-1}{2N+2}$. The critical point universal properties are relevant to many strong correlation problems, such as impurity placed in a Majorana metal and the multichannel Kondo lattice model of heavy fermion materials. We discuss relevance of our results for cluster DMFT studies of quantum criticality.

I. INTRODUCTION

It has been observed that several notable correlated electron systems exhibit a phase diagram with an unconventional critical point, which is not well described by the Hertz-Millis-Morya theory of quantum criticality. This quantum critical point exhibits charge or spin susceptibilities which are weakly momentum dependent, and their frequency dependence in a broad frequency range is logarithmic, similar to the two channel Kondo impurity model. The specific heat also behaves as $T\ln(T)$. The term local quantum criticality[1]is widely used to describe this type of criticality, and there have been many efforts to understand it using quantum impurity models. Surprisingly, while the two channel Kondo impurity model describes the critical behavior in a broad frequency range, a more singular behavior emerges at much lower frequencies or temperatures. It was first noticed in the the heavy fermion materials, the logarithmic dependence of the specific heat, turns into a stronger divergence at lower temperatures[2]. The normal state of the cuprates at optimal doping has many similar characteristics in the charge sector as described in the successful marginal Fermi liquid phenomenology[3] which has very strong experimental support. For example, weakly momentum dependent charge response, was measured using EELS[4]. Here also, this response function develops a stronger divergence at lower frequencies, which suprisingly is conformally invariant as expected from a quantum impurity model[5]. Given the local character of these QCP's, there have been many efforts to describe them using quantum impurity models (QIM), but no consistent picture of the crossover described above has emerged.

In this paper we introduce a QIM which exhibits a non trivial fixed point in the universality class of the topological Kondo model. This model exhibits two channel Kondo behavior at intermediate energies and crosses over to a more singular behavior at lower energies making it a possible candidate for the local quantum critical point governing the heavy fermions and the cuprates.

This QIM has several other possible physical realizations. In particular it describes the Kondo effect in clusters of impurities embedded in a Majorana metal such as Yao Lee model[6]. This version of the model possesses certain advantages allowing one to avoid some unnecessary complications which only clouds the main idea.

The QIM we consider is also potentially relevant to the two channel *lattice* model. There is renewed interest in this model, as it may describe an increasing number of heavy fermion systems, starting with UBe_{13}[7], as proposed long ago and more recently for PrV_2Al_{20}[8]. This paradigmatic model may also display interesting phenomena such as topological order, channel symmetry breaking[9] and composite order[10,11]. Single site DMFT studies describe some properties of these materials, but at lower temperatures, cluster DMFT studies bring in new physics. Studies of this kind are not possible for the two channel case at this time, and the impurity model introduced in this paper, may be relevant for the description of its non-Fermi liquid critical phase, as well as provide indications for the type of long range order that would emerge at even lower temperatures in frustrated lattices.

The plan of the paper is as follows. In Section II we remind the reader about multichannel Kondo lattice model. In Section III we describe its simplified version, namely the Majorana metal with magnetic impurities. This model lacks the charge sector which somewhat simplifies the calculations.

Section IV discusses multi impurity Quantum Critical points generated by the RKKY interaction. Here we describe interesting features of the new fixed point such as the anomalous dimensions which depend on the number of impurities indicating that indeed at larger cluster sizes it indicates stronger tendencies to order. Section V contains conclusions. The Appendix contains Bethe ansatz calculations.

II. MULTICHANNEL KONDO LATTICE MODEL

The two channel Kondo lattice is an archetypical model in the theory of correlated electron systems. It was proposed as relevant to the heavy fermion materials such as UBe_{13}, which exhibits a large resistivity down to its superconducting transition temperature. Not much is known about this model, even in limiting cases. Within single site DMFT which is exact in infinite dimensions, its normal phase realizes a non fermi liquid incoherent metal state[11]. Slave particle studies reveal even more exotic properties[10]

The two channel Kondo lattice model Hamiltonian is given by

$$H = \sum_{k,m,\sigma} \epsilon_k \, c^\dagger_{km\sigma} c_{km\sigma} + J \sum_{i,m} \mathbf{S}_i \cdot \mathbf{s}_{im},$$
$$\mathbf{s}_{im} = \sum_{\sigma,\sigma'} c^\dagger_{im\sigma} \boldsymbol{\sigma}_{\sigma\sigma'} c_{im\sigma'},$$

where ϵ_k is the conduction electron dispersion σ and m spin and channel indices, running over two values, the index i runs over the sites of the lattice. Cluster DMFT maps the lattice problem into a cluster of N impurities embedded in a medium which obeys a self consistency condition. Now the index i, $i = 1, N$ with N runs over the sites of the cluster. The action is given by:

$$S_{imp} = S_{\text{conduction}} + S_{\text{impurity}} + S_{\text{interaction}}, \tag{1}$$
$$S_{\text{conduction}} = \sum_{m=1}^{2} \sum_{\sigma=1}^{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int d\tau \int d\tau' \, \psi^\dagger_{mj\sigma} (\partial_\tau + \Delta_{i,j}(\tau - \tau')) \psi_{mj\sigma}, \tag{2}$$
$$S_{\text{impurity}} = \sum_{\sigma=1}^{2} \int d\tau \, f^\dagger_\sigma (\partial_\tau + \lambda) f_\sigma - \lambda, \tag{3}$$
$$S_{\text{interaction}} = J \sum_{m=1}^{2} \sum_{j=1}^{N} \int d\tau \, \mathbf{S} \cdot \mathbf{s}_{jm}(0), \tag{4}$$
$$\mathbf{s}_j(0) = \psi^\dagger_{jm} \frac{\boldsymbol{\sigma}}{2} \psi_{jm} \qquad S = f^\dagger \frac{\boldsymbol{\sigma}}{2} f \tag{5}$$

$\Delta_{i,j}$ is a matrix in cluster site indices i j, and is diagonal in channel indices. It is determined by the cluster DMFT self consistency condition. Here we use the Dynamical Cluster Approximation (DCA)[12] which makes this matrix cyclical, and its Fourier transform diagonal in the cluster momenta K_i. The DCA can also be understood in terms of a coarse graining of the lattice Brillouin zone into N patches of the same size, which

$$\mathrm{i}\omega I - \Delta(\mathrm{i}\omega, K_i) = \Big[\sum_{k \in RBZ} \frac{1}{\mathrm{i}\omega - \hat{t}_{k+K_i} - \Sigma_{imp}(\mathrm{i}\omega,\ K_i)} \Big]^{-1} + \Sigma_{imp}(\mathrm{i}\omega,\ K_i) \tag{6}$$

Here Σ_{imp} is viewed as a functional of the hybridization function, and is obtained solving the impurity model. Cluster correlation functions can be computed from the action (1).It is easily checked that solution with a hybridization function which is regular (i.e. it has a finite density of states) at low energies is a self consistent solution of 6. Furthermore, in the context of single site studies of the Anderson lattice it is well established that in a broad range of energies and temperatures the self energy is local, single site DMFT is accurate and thus the hybridization function is local and at lower energies the RKKY interaction becomes relevant and gives rise to new physics.[13][14]. The subsequent discussion will heavily rely on the assumption of approximate locality of the hybridization matrix. We will assume that also holds in the two channel case.

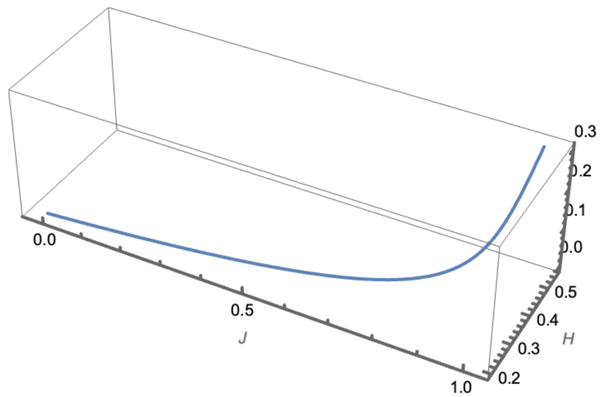

Figure 1: Renormalization Group Flow fixed points. $J = 0$: N decoupled two channel Kondo fixed point $J = 1$: new fixed point at $H = 0$. Finite H leads the RG trajectory towards Fermi liquid fixed point with large H.

The impurity solution displays the RG flow described in Fig. 1, and is forced to go from the single impurity two channel QCP_2 to the multi-impurity one QCP_N. The flow introduces its own energy scale T_{RKKY} which depends on the strength of the RKKY interaction between the impurities. In the presence of intersite tunneling H the RG trajectory bypasses it and continues to strong coupling regime identified as Fermi liquid. The degree of proximity of the trajectory to QCP_N depends on the ratio H/T_{RKKY}. As these are determined from the self consistency condition, this will be controlled by the lattice dispersion, and cannot be determined at this point. However, the fact the Fermi regime has not been reached in numerical studies of this model supports the idea that the coefficients of this relevant operators is small.

III. IMPURITIES IN THE MAJORANA METAL

By primary aim of introduction of the model of Majorana metal is simplification of the calculations. However, the model of Majorana metal with magnetic impurities discussed in this Section is not that different from the two channel model discussed in Section II provided one is interested only in its magnetic properties. Majorana fermions are real and hence are electrically neutral. The model carries certain advantage related to the properties of the hybridization matrix in its cluster DMFT formulation.

A. Yao-Lee model

The Majorana metal is a system where elementary excitations are Majorana fermions with a Fermi surface. The appropriate model is the Yao-Lee (YL) model on a particular lattice. The model was introduced in[6]

$$H_{YL} = K/2 \sum_{<i,j>} \lambda_i^{\alpha_{ij}} \lambda_j^{\alpha_{ij}} (\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j), \tag{7}$$

and is a generalization of the famous Kitaev model of Z_2 spin liquid[15]. Each site of a lattice with coordination number three contains two kinds of spins 1/2 described by σ^a and λ^α operators respectively. σ^a operators are components of spin S=1/2 and λ^α are Pauli matrices describing the orbital degrees of freedom.

The anisotropic Ising coupling between orbitals induces Majorana fractionalization[6] of spins

$$\boldsymbol{\sigma}_j = -\frac{i}{2}[\boldsymbol{\chi}_j \times \boldsymbol{\chi}_j] \tag{8}$$

and orbitals $\boldsymbol{\lambda}_i = i\boldsymbol{b}_j \times \boldsymbol{b}_j$. In the physical Hilbert space, where $\sigma_j^a \lambda_j^\alpha = 2i\chi_j^a b_j^\alpha$, the fractionalized form of the Yao-Lee Hamiltonian H_{YL} (7) is

$$H_{YL} = \mathrm{i}K \sum_{<i,j>} \hat{u}_{ij} (\boldsymbol{\chi}_i \cdot \boldsymbol{\chi}_j). \tag{9}$$

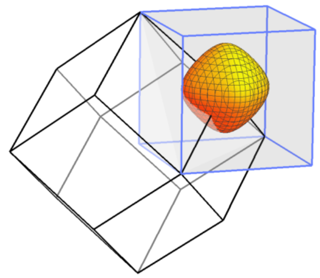

Figure 2: Majorana Fermi surface of the YL model on hyperoctagon lattice.

Here, $\hat{u}_{ij} = ib_i^{\alpha_{ij}} b_j^{\alpha_{ij}}$ are the static $\mathbb{Z}_2$ gauge fields, (i.e. $[H_{YL}, \hat{u}_{ij}] = 0$).

In three dimensions, $\mathbb{Z}_2$ gauge theories undergo a finite temperature Ising phase transition at T_c, into a deconfined phase, in which the visons (plaquettes with a π flux) are linearly confined. For example, in the Yao Lee model on a hyper-octagonal lattice, $T_c \sim 0.012K$[16–18], leading to a fractionalization of the spins into Majoranas at lower temperatures $T < T_c$.

In what follows we consider the model on 3D hyperoctogonal lattice where the Majorana spectrum has a Fermi surface (see Fig. 2). Below T_c this model describes thermal metal with coherent Majorana quasiparticles.

B. Magnetic impurities. Single impurity fixed point.

We start the discussion with a situation when a spin $\boldsymbol{\sigma}$ on one site of the hyperoctagon lattice is connected with an additional spin 1/2 $\mathbf{S}$ via an antiferromagnetic exchange interaction. This extra spin acts as an impurity in a 3D YL model with a Fermi surface. We assume that the temperature $T << T_c$ so that the visons are rare and can be neglected. In this temperature range the YL spin liquid acts as a thermal metal: a Majorana-Fermi liquid with well defined quasiparticles near the Fermi surface. Taking into account that the bulk spin operator is bilinear in Majorana fermions (8) we obtain the following single impurity Hamiltonian:

$$H_{ex} = \frac{\mathrm{i}}{2} J_K \Big(\mathbf{S}[\boldsymbol{\chi} \times \boldsymbol{\chi}]_{x=0} \Big) + \sum_k \epsilon(k) \Big(\boldsymbol{\chi}(-k) \boldsymbol{\chi}(k) \Big), \tag{10}$$

Since the commutation relations of the Majorana current operator coincide with the $\mathrm{SU}_2(2)$ Kac-Moody current, the dynamics of the impurity spin is the same as in the 2-channel Kondo model (see[19]). The exact solution can be generalized on the case of anisotropic exchange interaction as, for example, in[20]. Alternatively, the solution or the U(1) symmetric case $J_x = J_y = J_\perp \neq J_z$ can be obtained by means of bosonization as in[21].

In the standard (Dirac) formulation of the k-channel Kondo model, such as the one in Section II, the exchange interaction includes $\sum_{m=1}^k \psi^+_{m\alpha} \sigma^a_{\alpha\beta} \psi_{m\beta}$ combination which is $\mathrm{SU}_k(2)$ Kac-Moody current. The anisotropy in the spin space does not introduce currents with different symmetry and so is irrelevant. In contrast to that a channel anisotropy does introduce coupling to the currents with other symmetry and this destabilizes the critical point. The contrast between the Dirac and Majorana version (10) of the 2-channel Kondo model is that in the latter case there is no analogue of the channel anisotropy. So the QCP in the Majorana version is robust. If the YL model is robust, the QCP is robust too.

As usual, in the isotropic case the Kondo temperature T_K is exponential in $1/J$. The effective low energy $E << T_K$ Hamiltonian is[19]

$$H_{eff} = \sum_k \epsilon(k) \Big(\boldsymbol{\chi}(-k) \boldsymbol{\chi}(k) \Big) + T_K \xi (\chi_1 \chi_2 \chi_3)_0 \tag{11}$$

where ξ is the local Majorana zero mode. In what follows I will find it convenient to switch to the action formalism. As far as the impurity is concerned one can replace the 3D fermions with 1D chiral ones. Then the effective action corresponding to the Hamiltonian (11) is

$$A_{2ch} = \int \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{2} \Big(\chi^a \partial_\tau \chi^a - \mathrm{i} \chi^a \partial_x \chi^a \Big) + \frac{1}{2} \xi \partial_\tau \xi + T_K \xi (\chi_1 \chi_2 \chi_3)_0 \tag{12}$$

where x is a fictitious coordinate. The impurity spin operator undergoes transmutation so that at energies $<< T_K$ we have

$$\boldsymbol{S}(x=0) \to (\Lambda/T_K)^{1/2}\xi\boldsymbol{\chi}(x=0) + ... \tag{13}$$

where χ^a is the band Majorana fermion located at the impurity site i and $\Lambda \sim K$ is the UV cut-off.

In the conventional Kondo model the Majorana fermion is nonlocal in terms of the observables. However, in the present case it is local, for example, the YL spin field σ is a bilinear in the Majoranas (8). Hence it makes sense to calculate its self energy. It is determined by the quartic term in (11):

$$\Sigma \sim n_i\rho(\epsilon_F)T_K^2\int \mathrm{d}\tau\frac{\sin(\omega\tau)}{\tau^2} \sim n_i T_K^2\rho(\epsilon_F)\omega\ln\omega. \tag{14}$$

where n_i is the impurity concentration and $\rho(\epsilon_F) \sim 1/K$ is the density of states at the Fermi surface. It enters in the expression for the thermal conductivity. Hence in the limit of small impurity concentration the YL model with spin impurities behaves as marginal FL with linear T dependence in the thermal resistivity.

IV. MULTI IMPURITY CASE

Now let us consider a multi-impurity case with the number of impurities N. Integrating out all fermionic degrees of freedom except ones located on the impurity sites we arrive to the following effective action valid for energies below the single impurity temperature T_K:

$$A = \sum_i A_{2ch}\Big[\boldsymbol{\chi}_i, \boldsymbol{S}_i\Big] + I_{ij}(\boldsymbol{S}_i\boldsymbol{S}_j) + \frac{1}{2}\sum_{i\neq j}\int \mathrm{d}\tau\mathrm{d}\tau'\boldsymbol{\chi}_i(\tau)G^{-1}(i,j)(\tau-\tau')\boldsymbol{\chi}_j(\tau'), \tag{15}$$

where A_{2ch} is given by (12), the spin operators are defined in (13) and the last term is introduced to take into account the correlations between Majorana fermions on different sites (in the cluster DMFT formulation this would be the hybridization matrix). The advantage of model (15) is that the intersite tunneling contains only Majorana fermions and hence the model is self contained.

It will be demonstrated that the direct RKKY exchange interaction I_{nm} drives the model to a new critical point and the intersite tunneling (the last term) destroys the criticality. So, contrary to the conventional wisdom, the criticality is destroyed not by the RKKY interaction *per se*, but by the interference between the fermionic wave packets from different sites.

Substituting (13) into (15) we obtain

$$I_{ij}(\boldsymbol{S}_i\boldsymbol{S}_j) \to I_{ij}(\Lambda/T_K)(\xi_i\xi_j)(\boldsymbol{\chi}_i\boldsymbol{\chi}_j) = -I_{ij}J_{ij}s_{ij}. \tag{16}$$

The model with this interaction is a multichannel version of the model for the topological Kondo effect studied in[22–25]. Now $J_{ij} = \mathrm{i}(\boldsymbol{\chi}_i\boldsymbol{\chi}_j)$ are $O_3(N)$ Kac-Moody currents (N is the number of impurities) and $s_{ij} = \mathrm{i}\xi_i\xi_j$ are generators of the $o(N)$ algebra. Hence we have $n = 3$ channel Kondo problem with orthogonal symmetry.

The RKKY interactions pulls the system towards fixed point with the maximal symmetry. The intersite tunneling violates this symmetry playing a role of magnetic field. The Fermi surface of the YL model on the hyperoctogonal lattice is almost spherical (see Fig.2). Then an elementary analysis yields the following estimates:

$$G^{-1}_{ij}(\omega=0) \approx G_{ij}/G^2_{ii} = \frac{1}{\pi\rho(\epsilon_F)}\frac{\sin(k_F R_{ij})}{(k_F R_{ij})}. \tag{17}$$

$$I_{ij}(\Lambda/T_K) \sim \frac{\cos(2k_F R_{ij})}{(k_F R_{ij})^3}(\Lambda/T_K), \tag{18}$$

where k_F is the Fermi wave vector and R_{nm} is the distance between the corresponding sites. The RKKY interaction decays with distance faster than G^{-1} which may make it difficult to reach critical points with high symmetry, but it is strongly enhanced by the two-channel Kondo effect. In addition to that, since both matrix elements are oscillatory functions of distance with different periods, $G^{-1}(R)$ may vanish for particular values for $k_F R$ leaving room high symmetry criticality. As we mentioned in Section II, the numerical calculations for cluster DMFT indicate that the off-diagonal elements of the hybridization matrix are small. This fact provides us with assurance that the current results may be relevant for cluster DMFT.

A. Conformal field theory

Now let us imagine that we have a cluster of N impurities with approximately equal RKKY couplings and negligible tunneling matrix elements. Then the system will scale to a new critical point. The bulk Hamiltonian consists of $3N$ Majorana modes with symmetry $O_1(3N)$. Meanwhile the interaction involves $O_3(N)$ Kac-Moody currents which means that only a part of the bulk is involved in the impurity screening.

Below we will consider more general number of Majorana species n and consider the conformal embedding:

$$O_1(nN) = O_n(N) \times O_N(n). \tag{19}$$

Hence the $O_n(N)$ sector coupled to impurities becomes quantum critical and the $O_N(n)$ sector of the bulk remains idle. The resulting critical theory is the $O_n(N)$ Wess-Zumino-Novikov-Witten model subject to a conformally invariant boundary condition. Its basic properties are described in the paper by Kimura[26]. The conformal dimensions of the primary fields of the $O_n(N)$ Wess-Zumino-Novikov-Witten model are

$$h_i = \frac{C_i}{n+N-2}, \tag{20}$$

where C_i are $O(N)$ group Casimirs in the corresponding representations.

The most important operators is the operator from the adjoint representation which first Kac-Moody descendant determines a flow towards the ground state. Its conformal dimension is

$$h_{adj} = \frac{N-2}{n+N-2}. \tag{21}$$

The corresponding contributions to the impurity free energy are

$$F_{imp}^{irrel} \sim -TS(0) - aT(T/T_{RKKY})^{h_{adj}}, \tag{22}$$

where $S(0)$ is the ground state entropy and a is a numerical constant.

Another important operator is the impurity spin (13). It is a composite operator which includes both sectors. The Majorana fermion χ_n^a has conformal dimension 1/2 which is a sum of conformal dimensions of the $O_n(N)$ and $O_N(n)$ WZNW models:

$$1/2 = \frac{(N-1)/2}{N+n-2} + \frac{(n-1)/2}{N+n-2}. \tag{23}$$

It is reasonable to assume that the $O_3(N)$ part becomes a singlet with the conformal dimension zero. Then the remaining part has conformal dimension

$$h_S = \frac{1}{N+1}. \tag{24}$$

The correlation function

$$\langle S_n^a(\tau) S_m^a(0)\rangle = F_{nm}\Big|\sin\Big(\pi T\tau\Big)\Big|^{-2/(N+1)} \tag{25}$$

is nonlocal in space.

Yet another operator is $\xi_i\xi_j$. It is a relevant perturbation coupled to the intersite tunneling. Its flow towards the critical point can be figured out from the Bethe ansatz (see the Appendix). The conformal dimension of $\xi_i\xi_j$ becomes h_1, this much is clear.

$$\langle \xi_i\xi_j\rangle \sim H^\beta, \quad \beta = \frac{h_1}{1-h_1}, \quad h_1 = (N-1)/2(N+1). \tag{26}$$

As far as the intersite tunneling is concerned, we can distinguish several regimes. First of all there is a regime of $H > T_K$ when the system does not even scale to 2-channel critical point. Then there is a regime $T_{RKKY} < H < T_K$ when the system behaves as independent critical 2-channel impurities. At last, there is the most interesting region $H < T_{RKKY}$ when there is a competition between two scales: temperature T and the field induced scale $T_h = H(H/T_{RKKY})^\beta$, $\beta = (N-1)/(N+3)$. Below this scale the impurity is a Fermi liquid. The crossover temperature can be determined as the temperature at which the relevant perturbation removes the ground state entropy:

$$S(0) \sim \frac{H^2}{T^{2-2h_1}}. \tag{27}$$

B. Two impurities problem. Spin-spin interactions

The term in the expansion quadratic in the impurity concentration n_i comes from the interaction of two impurities. The two-impurity models considered in[27,28] include many couplings related to scattering between the impurities. We consider the simplest model which takes into account only a direct interaction between the spins:

$$H = H_{2ch1} + H_{2ch2} + I^a S_1^a S_2^a. \tag{28}$$

Already this model is sufficient to describe a crossover from the 2-channel QCP to a QCP with more singular magnetic response.

When $I^a << T_K$ for the spins we can adopt their expressions at the 2-channel QCP (13). Then the interacting term becomes

$$I^a S_1^a S_2^a = I^a(\Lambda/T_K)\xi_1\xi_2\chi_1^a\chi_2^a = I^a(\Lambda/T_K)(d^+d - 1/2) : \psi_a^+\psi_a : \tag{29}$$

where $\psi = \chi_1 + \mathrm{i}\chi_2$. It is strictly marginal and hence affects continuous changes in the scaling dimensions of the 2-channel Kondo model operators.

C. Three impurities

Now we consider three impurities at energies much smaller than the single impurity Kondo temperature T_K such that the UV cut-off for the theory is T_K. An equivalent method to describe the bulk is to use chiral Majorana fermions located on a line. The corresponding chiral Kondo Hamiltonian is

$$H = \sum_{n=1}^{3}\sum_{a=1}^{3}\int \mathrm{d}x(\mathrm{i}/2)\chi_n^a\partial_x\chi_n^a + \mathrm{i}\bar{I}^n\Big[\epsilon_{nmk}\sigma^n\Big(\boldsymbol{\chi}_m\boldsymbol{\chi}_k\Big)_{x=0}\Big] + \mathrm{i}H_{nm}\Big(\boldsymbol{\chi}_n(0)\boldsymbol{\chi}_m(0)\Big), \tag{30}$$

where $\sigma^n = \frac{\mathrm{i}}{2}\epsilon_{nmk}\xi_m\xi_k$ and $\bar{I}$ contains factor (Λ/T_K) as in (18). We replaced the dynamic Green's functions G_{nm}^{-1} by their zero frequency limit H_{nm} acting as static "magnetic" fields.

In contrast with the $N = 2$ case the interaction is not marginal. Depending on the sign of I_3 the interaction in (30) is either marginally relevant or irrelevant. We consider the relevant case as the most interesting. Then the interaction will generate its own crossover scale T_{RKKY}.

In the isotropic case we have

$$T_{RKKY} \sim T_K \exp[-1/2\rho(\epsilon_F)\bar{I}] \tag{31}$$

The model (30) is exactly solvable by Bethe ansatz when $|\bar{I}_1| = |\bar{I}_2| < \bar{I}_3$ and $H_{13} = H_{23} = 0$. The solution is similar to the one for spin S XXZ Heisenberg Hamiltonian[29] and has been discussed in[20,30]. The Bethe ansatz equations are written in the Appendix.

In the absence of H the system flows from 2-channel QCP at high energies to 6-channel one at low energies since $O_3(3) \equiv SU_6(2)$, it is the 6-channel Kondo. The thermodynamics can be extracted from[31]. In the absence of magnetic field (intersite tunneling) it is quantum critical. The conformal dimension of the σ-operator at the critical point is $h_{j=1} = j(j+1)/(k+2) = 1/4$. Hence the magnetic field is relevant and generates energy scale $H^{4/3}T_{RKKY}^{-1/3}$. It must be much smaller than T_{RKKY} for the criticality to be noticable.

The impurity contribution to the ground state entropy

$$S(0) = \ln\sqrt{2+\sqrt{2}}. \tag{32}$$

The irrelevant operator which determines the temperature dependence of the impurity contribution is the first Kac-Moody descendant of the adjoint one $J_{-1}\Phi_{adj}$. So the impurity free energy scales as $F_{imp} \sim T^{2h_1+1}$ and the specific heat is

$$C_{imp} \sim (T/T_{RKKY})^{1/2}. \tag{33}$$

D. Three impurities. The role of intersite tunneling

The Bethe ansatz solution enables us to analyze in greater detail the role of the intersite tunneling. The model remains exactly solvable when only the sites with the greatest exchange are connected: $H \equiv H_{12} \neq 0$. The details of the solution are given in the Appendix.

We have to distinguish several regimes. First of all there is a regime of $H > T_K$ when the system does not even scale to 2-channel critical point. Then there is a regime $T_{RKKY} < H < T_K$ when the system behaves as independent critical 2-channel impurities. At last, there is the most interesting region $H < T_{RKKY}$ when there is a competition between two scales: temperature T and field induced scale $T_h = H(H/T_{RKKY})^3$.

First we show the zero temperature "magnetization" $\mathcal{M} = \mathrm{i}\langle\xi_1\xi_2\rangle$ as it is extracted from the exact solution:

$$\mathcal{M} = \mathrm{i}\frac{G^{(+)}(0)}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega G^{(-)}(\omega)\tanh(\omega/2)}{(\omega - \mathrm{i}0)\sinh(3\omega)}\mathrm{e}^{\mathrm{i}\frac{\omega}{\pi}\ln(H/T_{RKKY})}, \tag{34}$$

(35)

where $G^{(-)}(\omega)$ is analytic in the lower half plane. The explicit expression is given in the Appendix; here it is sufficient to know that it is the only part of the integrand which depends on the anisotropy. The crossover from weak coupling to strong coupling regime occurs around $H = T_{RKKY}$. When $H < T_{RKKY}$ one can bend the integration contour in (34) to the lower half plane where $G^{(-)}(\omega)$ is analytic and the singularities come from the poles of the hyperbolic functions. Since the only function depending on the anisotropy is $G^{(-)}$, the anisotropy reveals itself only in the numerical coefficients in the expansion in

$$(H/T_{RKKY})^{n/3}, \quad n = 1, 2, ...; \quad (H/T_{RKKY})^{(1+2n)}, \quad n = 0, 1, ... \tag{36}$$

In the leading order we have $\mathcal{M} \sim (H/T_{RKKY})^{1/3}$ which corresponds to $h_M = 1/4$ (26).

The scaling function

$$F_{imp} = -H^{1+2/k} f\Big(\frac{T}{H^{1+2/k}}\Big), \tag{37}$$

where $f(x = 0) =$const $+S(0)x$, and $f(x >> 1) \sim x^2$ (see the Appendix). The latter means that in the presence of intersite tunneling the system becomes Fermi liquid at temperatures below $T_h \sim H^{4/3}$.

V. CONCLUSIONS. QUALITATIVE PICTURE

The study of single impurity models played an important role in describing heavy fermion systems, and then other strongly correlated electron system with the introduction of Dynamical Mean Field Theory. The study of two impurity models, starting with the work of Jones and Varma[32], brought new features arising from the RKKY interaction. This work was generalized by Georges and Sengupta[21,28] to the two channel situation. Our work is a natural next step beyond these ones and reveals qualitatively new features when the number of impurities $N \geq 3$. We showed, using a combination of conformal field theory and Bethe ansatz methods, that in this case, there is a new quantum critical point in the universality class $O_3(N)$. The scaling dimension of the spin operator at this point is $1/(N+1)$ and is universal. The intersite tunneling acts as relevant operators reducing the effective size of the impurity cluster.

This critical point is a natural candidate, to explain the criticality found in cluster DMFT studies of the Hubbard model at optimal doping[33,34]. In this case, the tuning needed to reach criticality would be responsible for zeroing the relevant operator H. This critical point is also realized when magnetic impurities are embedded in a Majorana metal. In this case, since the tunneling matrix elements decay slower with the distance then the RKKY interaction, and they have oscillatory character, they can be tuned to zero by adjusting the location of the impurities. Finally, this model describes the two particle local correlator in the normal state of the cluster DMFT solution of the two channel Kondo lattice model. Further work is needed to compute the one particle Greens function of this problem.

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VI. APPENDIX. BETHE ANSATZ FOR $N = 3$

The Bethe ansatz equations for the XXZ 6-channel Kondo model are

$$[e_6(\lambda_a)]^L e_1(\lambda_a - 1/g) = \prod_{b\neq a}^{M} e_2(\lambda_a - \lambda_b),$$
$$E = \frac{1}{2\mathrm{i}}\sum_a^M \ln[e_6(\lambda_a)] - h(6L - M), \quad e_n(x) = \frac{\sinh[\gamma(x - \mathrm{i}n/2)]}{\sinh[\gamma(x + \mathrm{i}n/2)]}. \tag{38}$$

where γ is a parameter related to the exchange anisotropy. In the isotropic limit $\gamma \to 0$

We consider the simplest case $\gamma = \pi/\nu$, where $\nu > 6$ is an integer. The equations for the ground state is

$$\rho(u) + \int_B^\infty K_\gamma(u - v)\tilde\rho(v)\mathrm{d}v = \mathrm{e}^{-\pi u}\mathrm{e}^{-\pi B} + \frac{1}{L} f[u - 1/\pi \ln(\Lambda/T_{RKKY})], \tag{39}$$
$$\epsilon^{(-)}(u) + \int_B^\infty K_\gamma(u - v)\epsilon^{(+)}(v)\mathrm{d}v = -\Lambda\mathrm{e}^{-\pi u}\mathrm{e}^{-\pi B} + H, \tag{40}$$
$$K_\gamma(\omega) = \frac{\tanh(\omega/2)\sinh(\nu\omega/2)}{2\sinh[(\nu - 6)\omega/2]\sinh(3\omega)}, \quad \nu = \pi/\gamma$$
$$f(\omega) = \frac{\tanh(\omega/2)}{2\sinh(3\omega)}. \tag{41}$$

where $\rho, \tilde\rho$ are densities of rapidities in the $k = 6$-th string and holes respectively and $\epsilon^{(-)} < 0, \;\; \epsilon^{(+)} > 0$ are negative and positive parts of the dispersion.

The "magnetic moment" $\mathcal{M} = -\partial F/\partial H$ is

$$\mathcal{M} = \int_B^\infty \tilde\rho(u)\mathrm{d}u. \tag{42}$$

The integration limit B is determined by the condition that the bulk magnetization is $\mathcal{M}_{bulk} = H/\Lambda$ or equivalently by the condition $\epsilon(B) = 0$. Equations (39,40) are the Wiener-Hopf ones and can be solved analytically as, for example, in[35]. The result for the magnetic moment is (34) where

$$K_\gamma(\omega) = [G^{(+)}(\omega)G^{(-)}(\omega)]^{-1}, \tag{43}$$
$$G^{(-)}(\omega) = G^{(+)}(-\omega) = [12\pi(\nu - 6)/\nu]^{1/2}\frac{\Gamma(1 + \mathrm{i}\omega/2\pi)\Gamma(1 + \mathrm{i}\nu\omega/2\pi)}{\Gamma(1/2 + \mathrm{i}\omega/2\pi)\Gamma(1 + 3\mathrm{i}\omega/\pi)\Gamma(1 + \mathrm{i}(\nu - 6)\omega/2\pi)}, \tag{44}$$

where $G(\pm)(\omega)$ are analytic in upper (lower) ω-plane respectively.

The thermodynamic equations in magnetic field.

In order to have a better idea of what happens when the relevant "magnetic" field perturbs the critical point, we consider finite temperature thermodynamics. We consider the smallest temperatures, the ones which are much smaller than the energy scale generated by the field. The impurity contribution to the free energy comes from the real rapidities. The ground state string is $j = k$ (in the present case $k = 6$, but we keep it generic). So we need the thermodynamic equations only for the first k strings. They are

$$\epsilon_j = Ts * \ln(1 + \mathrm{e}^{\epsilon_{j-1}/T})(1 + \mathrm{e}^{\epsilon_{j+1}/T}) - \Lambda\delta_{j,k}\mathrm{e}^{-\pi\lambda}, \quad j = 1, ...k. \tag{45}$$

where

$$s * f(x) = \int \mathrm{d}y \frac{f(y)}{2\cosh[\pi(x - y)]}$$

In finite H the function ϵ_k is positive at $\lambda > B = \frac{1}{\pi}\ln(\Lambda/H)$. We can rewrite equations (45) for $j < k$ as

$$\epsilon_j = Ts * \ln(1+\mathrm{e}^{\epsilon_{j-1}/T})(1+\mathrm{e}^{\epsilon_{j+1}/T}) + \delta_{j,k-1} s * \epsilon_k^{(+)}, \quad j = 1, ...k-1. \tag{46}$$

Inverting the kernel we get

$$\ln(1+\mathrm{e}^{\epsilon_j/T}) - \mathcal{A}_{jm} * \ln(1+\mathrm{e}^{-\epsilon_m/T}) = \frac{1}{T} s * \mathcal{A}_{j,k-1} * \epsilon_k^{(+)}. \tag{47}$$

$$\mathcal{A}_{jm}(\omega) = \frac{2\coth(\omega/2)\sinh[(k-\max(j,m))\omega/2]\sinh[\min(j,m)\omega/2]}{\sinh(k\omega/2)}. \tag{48}$$

To calculate the leading contribution to the specific heat we take the asymptotic of the r.h.s. of (47):

$$\frac{1}{T} s * \mathcal{A}_{j,k-1} * \epsilon_k^{(+)} \approx \frac{2}{kT}\sin(\pi j/k)\mathrm{e}^{2\pi(\lambda-B)/k}\epsilon_k^{(+)}(\omega = 2\mathrm{i}\pi/k) = \frac{T^*}{T}\sin(\pi j/k)\mathrm{e}^{2\pi\lambda/k}$$
$$T^* \sim H^{1+2/k}\Lambda^{-2/k}. \tag{49}$$

After a shift of variables

$$\lambda + \frac{k}{2\pi}\ln(H^{1+2/k}/T) = \lambda'.$$

the TBA equations become dimensionless:

$$\ln(1+\mathrm{e}^{\phi_j}) - \mathcal{A}_{jm} * \ln(1+\mathrm{e}^{-\phi_m}) = \sin(\pi j/k)\mathrm{e}^{2\pi\lambda/k}. \tag{50}$$

and the free energy is

$$F_{imp}(T) - F_{imp}(0) = -T\sum_{j=1}^{k-1}\int \mathrm{d}\lambda \Big[\frac{\sinh(k-j)\omega/2}{2\cosh(\omega/2)\sinh(k\omega/2)}\Big]_{\lambda-(k/2\pi)\ln(T_h/T)} \ln(1+\mathrm{e}^{-\phi_j(\lambda)}). \tag{51}$$

where $T_h = H(H/T_{RKKY})^{2/k}$. To extract the low temperature asymptotic we expand

$$\Big[\frac{\sinh(k-j)\omega/2}{2\cosh(\omega/2)\sinh(k\omega/2)}\Big]_{\lambda-(k/2\pi)\ln(T_h/T)} \approx \frac{\sin(\pi j/k)}{k\cos(\pi/k)}\mathrm{e}^{2\pi\lambda/k}(T/T_h).$$

Substituting it into (51) we get the Fermi liquid form of the free energy:

$$F_{imp}(T) - F_{imp}(0) = -bT(T/T_h), \quad b = \sum_{j=1}^{k-1}\frac{\sin(\pi j/k)}{k\cos(\pi/k)}\int \mathrm{d}\lambda \mathrm{e}^{2\pi\lambda/k}\ln(1+\mathrm{e}^{-\phi_j(\lambda)}). \tag{52}$$

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