Metric-induced nonhermitian physics

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Abstract

I consider the long-standing issue of the hermicity of the Dirac equation in curved spacetime metrics. Instead of imposing hermiticity by adding ad hoc terms, I renormalize the field by a scaling function, which is related to the determinant of the metric, and then regularize the renormalized field on a discrete lattice. I found that, for time-independent and diagonal metrics such as the Rindler, de Sitter, and anti-de Sitter metrics, the Dirac equation returns a hermitian or pseudohermitian ($\mathcal{PT}$-symmetric) Hamiltonian when properly regularized on the lattice. Notably, the $\mathcal{PT}$-symmetry is unbroken in the pseudohermitian cases, assuring a real energy spectrum with unitary time evolution. Conversely, considering a more general class of time-dependent metrics, which includes the Weyl metric, the Dirac equation returns a nonhermitian Hamiltonian with nonunitary time evolution. Arguably, this nonhermicity is physical, with the time dependence of the metric corresponding to local nonhermitian processes on the lattice and nonunitary growth or decay of the time evolution of the field. This suggests a duality between nonhermitian gain and loss phenomena and spacetime contractions and expansions. This metric-induced nonhermiticity unveils an unexpected connection between spacetime metric and nonhermitian phases of matter.

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1 Introduction

Reconciling general relativity with quantum field theory in a mathematically and physically consistent way is an open problem in high-energy physics. However, a well-studied semi-classical approximation of quantum gravity is the quantum field theory in curved spacetime, which is obtained when particle fields are treated as quantum-mechanical, and spacetime is treated as a classical background [1,2]. In this framework, e.g., the Dirac equation is extended from the flat Minkowski spacetime of quantum field theory to a curved spacetime by substituting e partial derivatives with covariant ones. Nevertheless, this theory itself is not free from apparent inconsistencies, such as the issue of the nonhermicity of the Dirac Hamiltonian in curved spacetime [3,4]. This nonhermicity is usually cured by adding extra terms that make the Hamiltonian hermitian [4,5]), or to consider sufficiently smoothly varying metrics, which makes the nonhermitian terms negligible. The issue of hermiticity or nonhermiticity of the Dirac Hamiltonian is also relevant in light of the recent advances in the study of nonhermitian quantum mechanics [6,7], and nonhermitian quantum field theories [8], and on the growing field of analog gravity, i.e., the study of quantum systems that simulate curved spacetime, such as Bose-Einstein condensates [9–12], optical metamaterials [13–17], cold atoms in optical lattices [18], graphene [19–25], and Weyl semimetals [26–29]. These systems are described by effective Hamiltonians corresponding to a Dirac equation regularized on a lattice and are usually derived perturbatively specifically for the condensed matter system considered [19–25].

Here, I consider the inverse approach and derive an effective lattice Hamiltonian by regularizing the continuum Dirac equation in curved spacetime, regardless of the specific of the condensed matter system considered. Generally, regularization on discrete lattices approximates the derivatives of the field with finite differences. However, this approach leads to nonhermitian lattice Hamiltonians, which can be made hermitian only by imposing hermiticity by hand [18]. Alternatively, one can rewrite the Hamiltonian in terms of the derivatives of the field times a scaling function, which is related to the determinant of the metric, and then approximate these "renormalized" derivatives with finite differences. This regularization approach returns a hermitian or pseudohermitian Hamiltonian for time-independent (static) metrics, such as Rindler [30], de Sitter and anti-de Sitter metrics [31,32], without arbitrarily adding extra terms to cancel out the nonhermitian terms. Notably, Rindler-like metrics yield hermitian Hamiltonians, while de Sitter and anti-de Sitter metrics yield pseudohermitian Hamiltonians with unbroken $\mathcal{PT}$-symmetry, corresponding to real energy spectra and unitary time evolutions. However, for metrics that depend explicitly on the time coordinate, such as the Weyl metric [33], the lattice Hamiltonian is, in general, neither hermitian nor pseudohermitian: I argue that this nonhermicity is physical and that it corresponds to local gain and loss nonhermitian processes on the lattice, corresponding to a nonunitary time evolution of the field. Finally, I speculate on some experimental realization of quantum analogs of curved spacetime using condensed matter systems simulating the lattice Hamiltonians derived here.

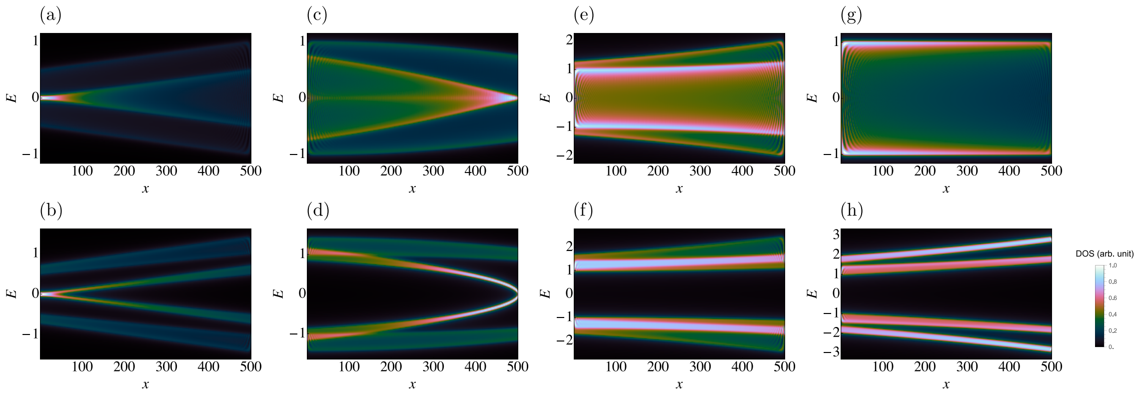

Figure 1: Local density of states (LDOS) of time-independent lattice Hamiltonians corresponding to the Dirac equation in curved and static spacetime calculated on a finite patch as a function of the energy and position. Different panels correspond to: Rindler metric [Eq. (10)] in the massless (a) and massive (b) cases; de Sitter metric in the massless (c) and massive (d) cases; anti-de Sitter metric [Eq. (18)] in the massless (e) and massive (f) cases; the static ($r=0$) Weyl metric [Eq. (22)] in the massless (e) and massive (f) cases. The Hamiltonian corresponding to the Rindler metric is hermitian, having real energy eigenvalues and unitary time evolution. Conversely, the Hamiltonians corresponding to the de Sitter and anti-de Sitter metrics and the static Weyl metric are pseudohermitian with unbroken $\mathcal{PT}$-symmetry, having real energy eigenvalues and unitary time evolution. The Rindler and de Sitter metrics exhibit an event horizon at $x=0$ and $x=1/q=N$, respectively. The event horizon corresponds to a localized zero-energy mode of the lattice Hamiltonian. The anti-de Sitter metric does not exhibit an event horizon, and the energy is gapless in the massless case and gapped in the massive case.

2 Results

2.1 Dirac equation in curved 1+1D spacetime

Let us start with the Dirac equation in curved 1+1D spacetime [34–36] to

$$\left[\mathrm{i}\gamma^a e_a{}^\mu \partial_\mu + \frac{\mathrm{i}}{2}\gamma^a \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, e_a{}^\mu) - M \right]\psi = 0, \tag{1}$$

where ψ is a two-spinor, γ^μ the flat spacetime gamma matrices satisfying $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta_{\mu\nu} = \mathrm{diag}(1,-1)$ the Minkowski metric, $\sqrt{-g}$ is the square root of the determinant of the metric, and the zweibein is related to the metric by the relations $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$, and $\eta_{ab} = e_a{}^\mu e_b{}^\nu g_{\mu\nu}$, with $g^{\mu\nu} = (g_{\mu\nu})^{-1}$. In the Weyl representation, $\gamma^0 = \sigma_x$ and $\gamma^1 = \mathrm{i}\sigma_y$. The conserved scalar product in curved spacetime is defined as [3,4] $(\phi,\varphi) = -\int \mathrm{d}x \sqrt{-g}\,\phi^\dagger\gamma^0 e_a{}^0\gamma^a\varphi$ which gives $(\phi,H\phi)-(H\phi,\phi) = \mathrm{i}\int \mathrm{d}x\,\phi^\dagger\gamma^0\partial_0(\sqrt{-g}\,e_a{}^0\gamma^a)\phi$ (see Ref. [4]). This mandates that, for time-dependent metrics, the Hamiltonian is in general nonhermitian.

2.2 Rindler-like metrics

First, consider the time-independent metric

$$\mathrm{d}s^2 = \alpha(x)^2\mathrm{d}t^2 - \mathrm{d}x^2, \tag{2}$$

with $\alpha(x) \geq 0$. When $\alpha(x) = qx$, this metric reduces to the Rindler metric in hyperbolic coordinates [30], which is the metric of a reference frame with constant proper acceleration q.

The nonvanishing components of the metric tensor are $g_{00} = \alpha(x)^2$ and $g_{11} = -1$, the square root of the determinant $\sqrt{-g} = \alpha(x)$, and the zweibein $e_0{}^0 = \alpha(x)^{-1}$ and $e_1{}^1 = 1$. Separating time and space components, the time evolution of the spinor field reads

$$\mathrm{i}\partial_0\psi = H\psi = -\gamma_0\left[\mathrm{i}\alpha(x)\gamma^1\partial_1 + \frac{\mathrm{i}}{2}\gamma^1\partial_1(\alpha(x)) - M\alpha(x)\right]\psi, \tag{3}$$

where $\gamma_0 = (\gamma^0)^{-1}$, and where the single-particle Hamiltonian density is identified as the operator acting on the spinor on the right side of the equation. This can be conveniently recast as

$$\mathrm{i}\partial_0\psi = H\psi = -\mathrm{i}\sqrt{\alpha(x)}\gamma_0\gamma^1\partial_1\left(\sqrt{\alpha(x)}\psi\right) + M\alpha(x)\gamma_0\psi. \tag{4}$$

This Hamiltonian is time-independent and thus hermitian with respect to the conserved scalar product defined above.

Remarkably, when the metric depends explicitly on time $\mathrm{d}s^2 = \alpha(x,t)^2\mathrm{d}t^2 - \mathrm{d}x^2$, one reobtains Eqs. (3) and (4) where $\alpha(x) \to \alpha(x,t)$, since $\partial_0(\sqrt{-g}\, e_0{}^0) = 0$ given that in this metric $\sqrt{-g}\, e_0{}^0 = 1$ even when $\partial_0\alpha(x,t) \neq 0$. Despite being time-dependent, the resulting Hamiltonian is hermitian with respect to the conserved scalar product also in this case, since $\partial_0(\sqrt{-g}\, e_a{}^0\gamma^a) = 0$.

Single particle Hamiltonians can be discretized [37] on a lattice $x = na$ by substituting the spatial derivatives with finite differences $\partial_1\psi \approx \frac{1}{2a}(\psi_{n+1} - \psi_{n-1})$. This approach, however, leads to a nonhermitian lattice Hamiltonian unless one imposes the hermiticity by hand using $\psi^\dagger H\psi \to (H\psi)^\dagger\psi + \psi^\dagger H\psi$ [see, e.g., Eq. (15) in Ref. [18]]. Here, one can use more conveniently

$$\partial_1\left(\sqrt{\alpha(x)}\psi\right) \approx \frac{1}{2a}\left(\sqrt{\alpha_{n+1}}\psi_{n+1} - \sqrt{\alpha_{n-1}}\psi_{n-1}\right), \tag{5}$$

with $\alpha_n = \alpha(na)$ and $\psi_n = \psi(na)$ in Eq. (4), yielding

$$H\psi_n = -\frac{\mathrm{i}\sqrt{\alpha_n}}{2a}\gamma_0\gamma^1\left(\sqrt{\alpha_{n+1}}\psi_{n+1} - \sqrt{\alpha_{n-1}}\psi_{n-1}\right) + M\alpha_n\gamma_0\psi_n. \tag{6}$$

Hence, the Hamiltonian is $\mathcal{H} = \int \mathrm{d}x\psi^\dagger H\psi \approx a\sum_n \psi_n^\dagger H\psi_n$, which reads

$$\mathcal{H} = a\sum_n -\frac{\mathrm{i}\sqrt{\alpha_n\alpha_{n+1}}}{2a}\left(\psi_n^\dagger\gamma_0\gamma^1\psi_{n+1} - \psi_{n+1}^\dagger\gamma_0\gamma^1\psi_n\right) + M\alpha_n\psi_n^\dagger\gamma_0\psi_n, \tag{7}$$

which is hermitian on the lattice (i.e., $\mathcal{H}^\dagger = \mathcal{H}$) since the factor $\mathrm{i}\gamma_0\gamma^1$ is anti-hermitian. Hence, even when the metric is explicitly time-dependent $\alpha(x) \to \alpha(x,t)$, the Hamiltonian is hermitian when regularized on the lattice given that in this metric $\sqrt{-g}\, e_0{}^0 = 1$ even when $\partial_0\alpha(x,t) \neq 0$. This is a peculiar property of the metric considered, and it is not true in general, as discussed below.

The quantity

$$\frac{\sqrt{\alpha_n\alpha_{n+1}}}{2a}, \tag{8}$$

is proportional to the geometric average of the square root of the determinant of the metric on two contiguous lattice sites and can be identified as the "hopping" term of a tight-binding model in Eqs. (6) and (7). Introducing

$$\alpha(x) = e^{-\delta(x)}, \qquad \alpha_n = e^{-\delta_n}, \tag{9}$$

it is natural to interpret the quantity $(\delta_n + \delta_{n+1})/2$ as the distance (in some characteristic units) between the lattice sites n and $n+1$, at least on patches of the spacetime where $\delta(x) = -\log(\alpha(x)) \geq 0$. Generally, hopping amplitudes in condensed matter describe overlap integrals between modes localized on a single lattice site (such as Wannier functions). These overlap integrals typically scale exponentially with the distance as $\propto e^{-d_n/l}$, where d_n is the distance between the modes and l is a characteristic length scale describing the localization of the modes (e.g., end modes in topological insulators or superconductors or atomic modes in optical lattices). Note that the metric also renormalizes the mass term in Eqs. (6) and (7), which corresponds to a spatially dependent on-site energy of the tight-binding model.

In the Rindler metric [30] defined for $x \geq 0$, as

$$ds^2 = (qx)^2 dt^2 - dx^2, \tag{10}$$

one has $\alpha(x) = qx$ in Eq. (2) and

$$\alpha_n = qn, \qquad \frac{\sqrt{\alpha_n \alpha_{n+1}}}{2a} = \frac{q\sqrt{n(n+1)}}{2a}, \tag{11}$$

into Eqs. (6) and (7). The metric tensor has zero determinant $x = 0$, describing an event horizon. Remarkably, the Rindled metric approximates the Schwarzschild metric near a black hole horizon. In general, the metric in Eq. (2) describes an event horizon at $x = x_0$ when $\alpha(x_0) = 0$, i.e., when the distance $\delta(x)$ diverges for $x \to x_0$ with $\delta(x) \geq 0$ near the horizon. Figures 1(a) and 1(b) show the density of states of the time-independent and hermitian lattice Hamiltonian in the Rindler metric as a function of energy and position in the massless and massive cases, respectively.

2.3 de Sitter and anti-de Sitter metrics

The most general time-independent and diagonal metric in 1+1D is given by

$$ds^2 = \alpha(x)^2 dt^2 - \beta(x)^2 dx^2, \tag{12}$$

with $\alpha(x), \beta(x) \geq 0$, yielding the metric tensor $g_{00} = \alpha(x)^2$ and $g_{11} = -\beta(x)^2$, square root of the determinant $\sqrt{-g} = \alpha(x)\beta(x)$, and zweibein ${e_0}^0 = \alpha(x)^{-1}$ and ${e_1}^1 = \beta(x)^{-1}$. This metric yields

$$\mathrm{i}\partial_0 \psi = H\psi = -\frac{\mathrm{i}\sqrt{\alpha(x)}}{\beta(x)} \gamma_0 \gamma^1 \partial_1 \left(\sqrt{\alpha(x)}\psi\right) + M\alpha(x)\gamma_0\psi, \tag{13}$$

which renormalizes the "hopping" terms, giving

$$\mathcal{H} = a\sum_n -\frac{\mathrm{i}}{2a}\frac{\sqrt{\alpha_n}}{\beta_n}\left(\sqrt{\alpha_{n+1}}\psi_n^\dagger \gamma_0\gamma^1\psi_{n+1} - \sqrt{\alpha_{n-1}}\psi_n^\dagger\gamma_0\gamma^1\psi_{n-1}\right) + M\alpha_n\psi_n^\dagger\gamma_0\psi_n, \tag{14}$$

with $\beta_n = \beta(na)$, which is now nonhermitian on the lattice (i.e., $\mathcal{H}^\dagger \neq \mathcal{H}$) unless $\partial_1\alpha(x) = 0$, in which case the Hamiltonian becomes

$$\mathcal{H} = a\alpha\sum_n -\frac{\mathrm{i}}{2a}\frac{1}{\beta_n}\left(\psi_n^\dagger\gamma_0\gamma^1\psi_{n+1} - \psi_n^\dagger\gamma_0\gamma^1\psi_{n-1}\right) + M\psi_n^\dagger\gamma_0\psi_n, \tag{15}$$

where $\alpha = \alpha(x)$, which is hermitian. However, the Hamiltonian in Eq. (14) is pseudohermitian and $\mathcal{PT}$-symmetric [38–40], i.e., $\mathcal{H} = \mathcal{PT}\mathcal{H}(\mathcal{PT})^{-1} = \mathcal{P}\mathcal{H}^*\mathcal{P}$, with $\mathcal{P}$ the unitary operator describing space inversion acting as $n+1 \to n-1$ on the lattice site indexes, and $\mathcal{T}$ the antiunitary complex conjugation operator. Using the similarity transformation

$$\psi_n \to \frac{1}{\sqrt{\beta_n}}\psi_n = e^{\mathrm{i}\theta_n}\psi_n, \qquad \psi_n^\dagger \to \sqrt{\beta_n}\psi_n^\dagger = e^{-\mathrm{i}\theta_n}\psi_n^\dagger, \tag{16}$$

which can be seen as a gauge transformation with imaginary angles [7,41,42] $\theta_n = \frac{i}{2}\log\beta_n$, returns the hermitian Hamiltonian

$$\mathcal{H} = a\sum_n -\frac{i}{2a}\sqrt{\frac{\alpha_n\alpha_{n+1}}{\beta_n\beta_{n+1}}}\left(\psi_n^\dagger\gamma_0\gamma^1\psi_{n+1} - \psi_{n+1}^\dagger\gamma_0\gamma^1\psi_n\right) + M\alpha_n\psi_n^\dagger\gamma_0\psi_n, \tag{17}$$

which is not unitarily equivalent, but only isospectral to Eq. (14). Hence, the Hamiltonian in Eq. (14) has unbroken $\mathcal{PT}$-symmetry, having real energy spectra corresponding to unitary time evolution. Notice also that this Hamiltonian is a generalization of the Hatano-Nelson model [41–43] with nonuniform hopping terms and nonuniform on-site energies. The metric in Eq. (12) describes an event horizon at $x = x_0$ when $\alpha(x = x_0) = 0$ or $\beta(x = x_0) = 0$.

The de Sitter and anti-de Sitter metrics [31,32] in static coordinates [44] are defined for $x \geq 0$ as

$$\mathrm{d}s^2 = \left(1 \mp (qx)^2\right)\mathrm{d}t^2 - \frac{1}{(1 \mp (qx)^2)}\mathrm{d}x^2, \tag{18}$$

respectively, where $\alpha(x) = \beta(x)^{-1} = \sqrt{1 \mp (qx)^2}$ in Eq. (12). The de Sitter metric has an event horizon at $x = 1/q$. Figures 1(c) to 1(f) show the density of states of time-independent and pseudohermitian lattice Hamiltonians in the de Sitter and anti-de Sitter metrics as a function of energy and position in the massless and massive cases.

2.4 Weyl-like metrics

The most general diagonal metric in 1+1D is obtained simply by allowing the metric to depend explicitly on the time coordinate with $\alpha(x) \to \alpha(x,t)$, $\beta(x) \to \beta(x,t)$. In this case, the time derivatives of the metric do not vanish, and Eq. (13) contains an additional nonhermitian term, reading

$$i\partial_0\psi = H\psi = -\frac{i\sqrt{\alpha(x,t)}}{\beta(x,t)}\gamma_0\gamma^1\partial_1\left(\sqrt{\alpha(x,t)}\psi\right) + M\alpha(x,t)\gamma_0\psi - \frac{i}{2}\frac{\partial_0(\beta(x,t))}{\beta(x,t)}\psi. \tag{19}$$

Regularizing on a discrete lattice yields

$$H\psi_n = -\frac{i\sqrt{\alpha_n(t)}}{2a\beta_n(t)}\gamma_0\gamma^1\left(\sqrt{\alpha_{n+1}(t)}\psi_{n+1} - \sqrt{\alpha_{n-1}(t)}\psi_{n-1}\right) + M\alpha_n(t)\gamma_0\psi_n - \frac{i}{2}\frac{\partial_0(\beta_n(t))}{\beta_n(t)}\psi_n, \tag{20}$$

and

$$\mathcal{H} = a\sum_n -\frac{i}{2a}\frac{\sqrt{\alpha_n(t)}}{\beta_n(t)}\left(\sqrt{\alpha_{n+1}(t)}\psi_n^\dagger\gamma_0\gamma^1\psi_{n+1} - \sqrt{\alpha_{n-1}(t)}\psi_n^\dagger\gamma_0\gamma^1\psi_{n-1}\right) + M\alpha_n(t)\psi_n^\dagger\gamma_0\psi_n - \frac{i}{2}\frac{\partial_0(\beta_n(t))}{\beta_n(t)}\psi_n^\dagger\psi_n, \tag{21}$$

which is neither hermitian nor pseudohermitian on the lattice when $\partial_0(\beta_n(t))/\beta_n(t) \neq 0$. The nonhermitian term is proportional to the logarithmic derivative of $\beta(x,t)$ with respect to time.

Consider now the special case where $\alpha(x,t) = \beta(x,t)$, in particular the Weyl metric [33]

$$\mathrm{d}s^2 = e^{2rt+2qx}\left(\mathrm{d}t^2 - \mathrm{d}x^2\right), \tag{22}$$

where $\alpha(x,t) = e^{rt+qx}$. One obtains

$$H\psi_n = -\frac{i}{2a}\gamma_0\gamma^1\left(e^{\frac{1}{2}qa}\psi_{n+1} - e^{-\frac{1}{2}qa}\psi_{n-1}\right) + Me^{rt+qna}\gamma_0\psi_n - \frac{ir}{2}\psi_n, \tag{23}$$

and

$$\mathcal{H} = a\sum_n -\frac{i}{2a}\left(e^{\frac{1}{2}qa}\psi_n^\dagger\gamma_0\gamma^1\psi_{n+1} - e^{-\frac{1}{2}qa}\psi_n^\dagger\gamma_0\gamma^1\psi_{n-1}\right) + Me^{rt+qna}\psi_n^\dagger\gamma_0\psi_n - \frac{ir}{2}\psi_n^\dagger\psi_n, \tag{24}$$

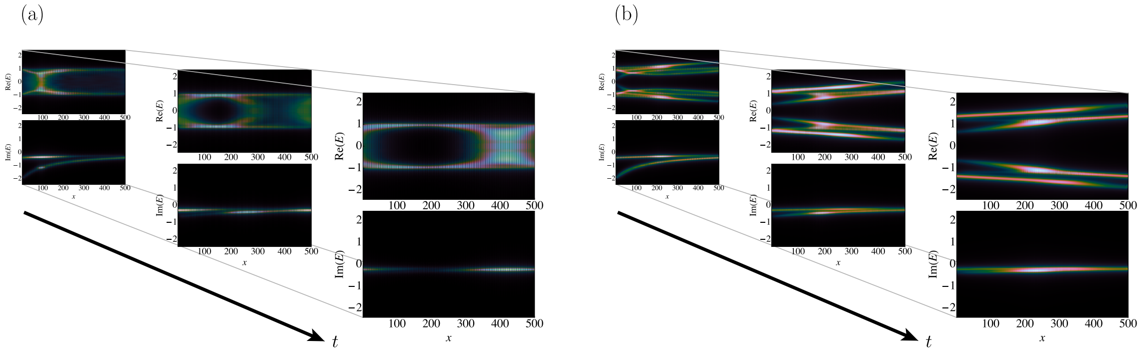

Figure 2: Local density of states (LDOS) on the real and imaginary axes of time-dependent and nonhermitian lattice Hamiltonian corresponding to curved spacetime with the metric in Eq. (25), as a function of position and at different time slices $t = 0.25, 0.5, 0.75$ with $r = 0.5$. Different panels correspond to the massless (a) and massive (b) cases.

which is not pseudohermitian for $r \neq 0$, and time-dependent for $M \neq 0$ but time-independent for $M = 0$. Hence, by regularizing a time-dependent metric on a lattice, one obtains a discrete Hamiltonian that is time-independent for $M = 0$. In the massless case with time-independent metric $M = r = 0$, the Hamiltonian in Eq. (24) coincides with that of the Hatano-Nelson model [41–43] (without disorder), which is the archetype of nonhermitian models in condensed matter physics. In this case, applying the imaginary gauge transformation in Eq. (16), which becomes $\psi_n \to e^{-\frac{1}{2}qna}\psi_n$, $\psi_n^\dagger \to e^{\frac{1}{2}qna}\psi_n^\dagger$, returns simply $\mathcal{H} = a\sum_n -\frac{\mathrm{i}}{2a}\left(\psi_n^\dagger\gamma_0\gamma^1\psi_{n+1} - \psi_{n+1}^\dagger\gamma_0\gamma^1\psi_n\right)$. Figures 1(g) and 1(h) show the density of states of time-independent and pseudohermitian lattice Hamiltonians in the static Weyl metric ($r = 0$) as a function of energy and position in the massless and massive cases. In the case $r \neq 0$ (not shown), the mass term becomes time-dependent, and the imaginary term becomes nonzero.

To illustrate a case with a more interesting time evolution, consider the metric

$$\mathrm{d}s^2 = (rt + qx)^2\left(\mathrm{d}t^2 - \mathrm{d}x^2\right), \tag{25}$$

for $x, t \geq 0$, which yields

$$H\psi_n = -\frac{\mathrm{i}}{2a}\gamma_0\gamma^1\left(\sqrt{\tfrac{w_n(t)+qa}{w_n(t)}}\psi_{n+1} - \sqrt{\tfrac{w_n(t)-qa}{w_n(t)}}\psi_{n-1}\right) + Mw_n(t)\gamma_0\psi_n - \frac{\mathrm{i}}{2}\frac{r}{w_n(t)}\psi_n, \tag{26}$$

where $w_n(t) = rt + qna$, and

$$\mathcal{H} = a\sum_n -\frac{\mathrm{i}}{2a}\left(\sqrt{\tfrac{w_n(t)+qa}{w_n(t)}}\psi_n^\dagger\gamma_0\gamma^1\psi_{n+1} - \sqrt{\tfrac{w_n(t)-qa}{w_n(t)}}\psi_n^\dagger\gamma_0\gamma^1\psi_{n-1}\right) + Mw_n(t)\psi_n^\dagger\gamma_0\psi_n - \frac{\mathrm{i}}{2}\frac{r}{w_n(t)}\psi_n^\dagger\psi_n, \tag{27}$$

which is not hermitian nor pseudohermitian, and it is time-dependent for $r \neq 0$ in both the massless and massive cases. Figures 2(a) and 2(b) show the density of states on the real and imaginary axes of the time-dependent and nonhermitian lattice Hamiltonian with the metric in Eq. (25) as a function of position at different time slices, in the massless and massive cases, respectively.

2.5 Time evolution and nonhermiticity

Finally, we want to unveil the physical meaning of the nonunitary time evolution ensuing from the nonhermiticity induced by the spacetime curvature. Considering the metric in Eq. (12),

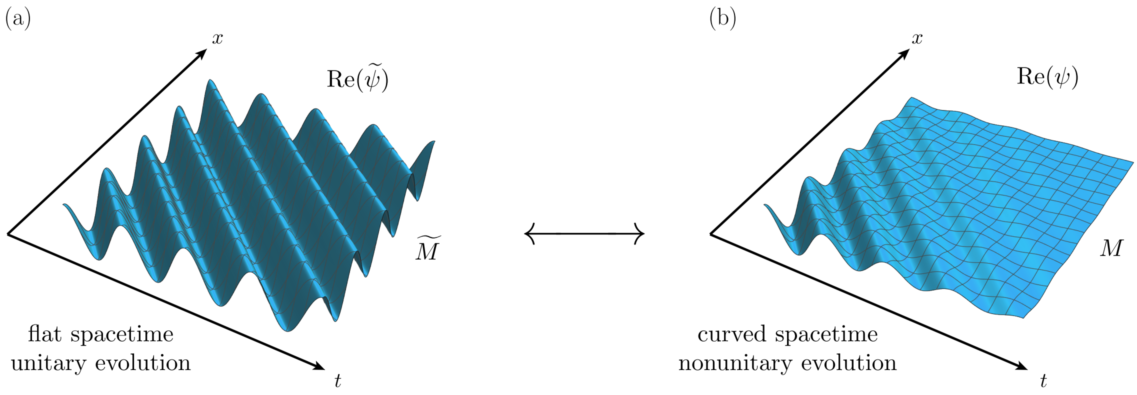

Figure 3: The duality between the unitary evolution of the field $\widetilde{\psi}$ with mass $\widetilde{M}$ in flat spacetime and the nonunitary evolution of the field ψ with mass M in curved spacetime. The relation between the two fields is determined by the metric tensor by $\widetilde{\psi} = \sqrt{\alpha(x,t)}\,\psi$ and $\widetilde{M}(x,t) = M\alpha(x,t)$. For simplicity, the plots show only the real part of the field in the case $\widetilde{M} = M = 0$, with a curved spacetime corresponding to the Weyl metric in Eq. (22), giving $\widetilde{\psi} = e^{rt+qx}\,\psi$ and $\psi \propto e^{\mathrm{i}(\omega t+kx)}$ (plane waves). In the massive case $M \neq 0$, the renormalized mass $\widetilde{M}(x,t)$ depends explicitly on the spacetime coordinates.

Eq. (19) can also be recast as

$$\mathrm{i}\sqrt{\beta(x,t)}\partial_0\left(\sqrt{\beta(x,t)}\psi\right) = -\mathrm{i}\sqrt{\alpha(x,t)}\gamma_0\gamma^1\partial_1\left(\sqrt{\alpha(x,t)}\psi\right) + M\alpha(x,t)\beta(x,t)\gamma_0\psi, \quad (28)$$

where the nonunitary evolution becomes explicit. Moreover, in the metric in Eq. (12) with $\alpha(x,t) = \beta(x,t)$ one gets

$$\mathrm{i}\partial_0\left(\sqrt{\alpha(x,t)}\,\psi\right) = -\mathrm{i}\gamma_0\gamma^1\partial_1\left(\sqrt{\alpha(x,t)}\,\psi\right) + M\alpha(x,t)\gamma_0\left(\sqrt{\alpha(x,t)}\,\psi\right), \quad (29)$$

which describes the unitary evolution of the field $\widetilde{\psi} = \sqrt{\alpha(x,t)}\,\psi$ in flat spacetime, given by $\mathrm{i}\partial_0\widetilde{\psi} = -\mathrm{i}\gamma_0\gamma^1\partial_1\widetilde{\psi} + \widetilde{M}(x,t)\gamma_0\widetilde{\psi}$, where the renormalized mass $\widetilde{M}(x,t) = M\alpha(x,t)$ depends on the spacetime coordinates.

In the massless case, the field in flat spacetime evolves as plane waves as $\widetilde{\psi} = e^{\omega t+kx}\phi$ with momentum k, energy $\omega = \pm k$, and with ϕ the eigenvector of $\gamma_0\gamma^1$ with eigenvalues ± 1. Hence, a massless Dirac field in curved spacetime evolves like

$$\psi = \frac{e^{\omega t+kx}}{\sqrt{\alpha(x,t)}}\,\phi, \quad (30)$$

which is nonunitary for $\partial_0\alpha(x,t) \neq 0$. In the massive case $M \neq 0$, the renormalized mass depends explicitly on the spacetime coordinates, and thus the evolution of the Dirac field is not universal but depends on the details of the metric.

The $\widetilde{\psi}$ describes a Dirac field in flat spacetime and, as such, exhibits unitary evolution and hermitian Hamiltonian. Conversely, the field ψ describes a Dirac field in curved spacetime and exhibits nonunitary evolution with a Hamiltonian that is neither hermitian nor pseudohermitian. The relation between the two fields is determined by the metric tensor. Figure 3 shows the duality between the field $\widetilde{\psi}$ with mass $\widetilde{M}$ the field ψ with mass M, in the simplest case where the fields are massless $\widetilde{M} = M = 0$. Note that the duality between the Dirac field in flat spacetime and curved spacetime expressed by Eq. (29) is a general property of the metric Eq. (12) with $\alpha(x,t) = \beta(x,t)$ which holds in the massless and massive cases.

3 Discussion

Both the Rindler and de Sitter metrics exhibit an event horizon, respectively at $x = 0$ and $x = 1/q$. These event horizons sit respectively at the lattice sites $n = 0$ and $n = 1/(aq)$, assuming $1/(aq) \in \mathbb{N}$ in the second case. A defining property of event horizons is that, when a particle reaches the horizon, there is no way for the particle to escape, no matter the energy of the particle. In the language of the lattice, this point-of-no-return property translates to the fact that the "hopping" amplitude defined in Eq. (8) for the Rindler metric vanishes between the lattice sites $n = 0$ and $n = 1$, and correspondingly the "hopping" amplitude in Eq. (14) for the de Sitter metric vanishes between the lattice sites $n = 1/(aq) - 1$ and $n = 1/(aq)$. Hence, the lattice points corresponding to the event horizon become completely uncoupled from the rest of the lattice. This corresponds to the presence of a localized zero-energy mode for both the massless and the massive cases. The localized zero-energy mode is visible in Fig. 1(a) and Fig. 1(b) at $x = 0$ for the Rindler metric, and in Fig. 1(c) and Fig. 1(d) at $x = N$ for the de Sitter metric (we choose the parameter q in order to match the length of the lattice $q = 1/N$). The anti-de Sitter metric does not exhibit an event horizon. The energy spectrum is gapless in the massless case, and gapped in the massive case, as visible in Fig. 1(d) and Fig. 1(e). The Rindler metric and generalized time-dependent Rindler-like metrics yield hermitian Hamiltonians with unitary time evolutions. Similarly, the de Sitter and anti-de Sitter metrics yield pseudohermitian Hamiltonians with unbroken $\mathcal{PT}$-symmetry, again with unitary time evolutions.

On the other hand, the Weyl metric describes a conformal scale expansion or contraction, i.e., a transformation changing the proper distances at each point by the factor $e^{2rt+2qx}$, which depends on both space and time. This corresponds to a nonhermitian loss for time expansions $r > 0$ and a nonhermitian gain for time contractions $r < 0$. Hence, the nonhermitian loss and gain correspond to the expansion or contraction of the spacetime background. Furthermore, there is thus a duality between two different interpretations of the same phenomena: In one interpretation, the field ψ evolves nonunitarily due to nonhermitian loss or gain effects, and the spacetime is fixed; In the other interpretation, the rescaled field $\widetilde{\psi}$ evolves unitarily due to the expansion or contraction of spacetime. The same considerations can be made for the metric defined in Eq. (25), and for any time-dependent metric in the form of Eq. (12) with $\alpha(x,t) = \beta(x,t)$, where the time evolution takes the form of Eq. (29). This duality between nonhermitian gain/loss and spacetime contractions/expansions also motivates the development of rigorous nonhermitian quantum field theories [8].

There are several physical systems that simulate the Dirac equation in curved spacetime regularized on a lattice. In essence, the regularized Hamiltonians are discrete tight-binding Hamiltonians with space-dependent and/or time-dependent hopping amplitudes and on-site energies, in combination with nonhermitian effects such as dissipation [6, 7]. Tight-binding Hamiltonians with controllable hoppings and on-site energies are effectively simulated by arrays of lattices of atoms deposited on a surface [45, 46], arrays of quantum dots [47], cold atoms in optical lattices [18, 48–50], photonic crystals [51], superconducting quantum circuits [52–54], topologically nontrivial stripes [55], and exciton-polariton condensates in artificial lattices [56]. Controlling the hopping amplitudes corresponds to controlling the distances and overlaps between contiguous localized states on the lattice, while on-site energies correspond to the presence of a potential.

4 Conclusion

In conclusion, I considered the Dirac equation in curved spacetime regularized on a lattice in different metrics, including Rindler, de Sitter, anti-de Sitter, and Weyl metrics. I found that for time-independent metrics, the lattice Hamiltonian is always hermitian or pseudohermitian with unitary time evolutions. In particular, Rindler-like metrics yield hermitian Hamiltonians, while the de Sitter and anti-de Sitter metrics yield pseudohermitian Hamiltonians. For time-dependent metrics, I found that for a Rindler-like with explicit time dependence, the lattice Hamiltonian turns out to be hermitian after proper regularization. For more general classes of metrics describing spacetime expansions or contractions, such as the Weyl-like metrics, however, the lattice Hamiltonian turns out to be nonhermitian with nonunitary time evolutions. In these cases, nonhermitian loss and gain can be considered the counterparts of expansion and contraction of spacetime. This unveils an unexpected connection between classical gravity and nonhermitian physics. Hence, hermiticity and unitary time evolution intrinsically depend on the spacetime metric.

Finally, we remark that this work suggests a broad implication on the nature of the physical reality: Nature may not necessarily be hermitian on a local scale, and the whole universe may, in fact, be a closed nonhermitian system.

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Appendix

To calculate the local density of states (LDOS) of a given lattice Hamiltonian with real energy spectrum as a function of energy, space, and time, one can first diagonalize the Hamiltonian in position basis and then calculate the LDOS via

$$\mathrm{LDOS}(x,t,E) = \sum_j \left| \left\langle \psi_j(t) \middle| \hat{x} \middle| \psi_j(t) \right\rangle \right|^2 \delta(E_j(t) - E), \tag{31}$$

where $E_j(t)$ is the j-th energy eigenvalue and $\left|\psi_j(t)\right\rangle$ the corresponding eigenstate at time t, $\hat{x}$ the position operator, and where the delta function is approximated by

$$\delta(x) = \frac{1}{\pi} \mathrm{Im}\left(\frac{1}{x - \mathrm{i}\Gamma} \right), \tag{32}$$

by taking a conveniently small $\Gamma \to 0^+$. For time-dependent Hamiltonians, the LDOS describes the adiabatic evolution of the spectra in time. These equations apply to Hamiltonians with real energy eigenvalues, i.e., hermitian or pseudohermitian Hamiltonians with unbroken $\mathcal{PT}$-symmetry. The results for the Rindler metric [Eq. (10)] and de Sitter and anti-de Sitter metrics [Eq. (18)] are shown in Fig. 1.

For nonhermitian Hamiltonians with complex energy spectrum, one can consider the LDOS as a function of the real and imaginary parts of the energy [57] by taking

$$\mathrm{LDOS}(x,t,E) = \sum_j \left| \left\langle \psi_j(t) \middle| \hat{x} \middle| \psi_j(t) \right\rangle \right|^2 \delta(\mathrm{Re}(E_j(t) - E))\delta(\mathrm{Im}(E_j(t) - E)), \tag{33}$$

where $E_j(t)$ is the j-th complex energy right eigenvalue and $\left|\psi_j(t)\right\rangle$ the corresponding right eigenstate at time t. On the real line, integrating on the imaginary line, one gets

$$\mathrm{rLDOS}(x,t,E) = \sum_j \left|\left\langle \psi_j(t) \middle| \hat{x} \middle| \psi_j(t) \right\rangle\right|^2 \delta(\mathrm{Re}(E_j(t) - E)), \tag{34}$$

which recovers the usual expression valid for hermitian Hamiltonians. Conversely, on the imaginary line, integrating on the real line yields

$$\mathrm{iLDOS}(x,t,E) = \mathrm{Im} \sum_j \left|\left\langle \psi_j(t) \middle| \hat{x} \middle| \psi_j(t) \right\rangle\right|^2 \delta(\mathrm{Im}(E_j(t) - E)). \tag{35}$$

These equations apply to Hamiltonians with complex energy eigenvalues, i.e., to pseudohermitian Hamiltonians with broken $\mathcal{PT}$-symmetry, or Hamiltonians which are neither hermitian nor pseudohermitian. The results for the metric defined in Eq. (25) are shown in Fig. 2.

In all cases considered, the calculations are performed on a finite lattice of 500 sites with open boundary conditions and lattice parameter $a = 1$, taking $q = 1/N$, $M = 0$ in the massless and $M = 1$ in the massive case, respectively. The LDOS is plotted in arbitrary units, normalized to its maximum value (separately on the real and imaginary line), and using the "cubehelix" color scheme [58].

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