Emergence of Light Cones in Long-range Interacting Spin Chains Is Due to Destructive Interference

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Despite extensive research on long-range interacting quantum systems, the physical mechanism responsible for the emergence of light cones remains unidentified. This work presents a novel perspective on the origins of locality and emergent light cones in quantum systems with long-range interactions. We identify a mechanism in such spin chains where effective entanglement light cones emerge due to destructive interference among quantum effects that entangle spins. Although long-range entangling effects reach beyond the identified light cone, due to destructive interference, entanglement remains exponentially suppressed in that region, ultimately leading to the formation of the light cone. We demonstrate that this interference not only drives but is also necessary for the emergence of light cones. Furthermore, our analysis reveals that reducing the interaction range weakens this interference, surprisingly increasing the speed of entanglement transport—an effect that opens new experimental opportunities for investigation.

I. INTRODUCTION

Understanding the propagation of entanglement in many-body quantum systems is a cornerstone of modern quantum science, with critical implications for both fundamental research and practical applications [1–12]. Despite decades of progress in mapping light cones and constraining information propagation velocities, one pivotal question remains unanswered: what is the *physical mechanism* underlying locality—that is, the emergence of light cones—in quantum systems with long-range interactions? It has recently been posed as an open question how different notions of locality manifest themselves, for instance through entanglement dynamics [13]. In this work, we identify that quantum destructive interference is the driving force behind light cone formation in such systems.

In relativistic quantum field theory, the spread of correlations is strictly limited by a light cone [14], governed by the speed of light. However, in non-relativistic quantum systems, the speed of light does not play such a role. Despite the latter circumstance, the Lieb-Robinson (LR) bound establishes an effective light cone structure in short-range interacting quantum systems, where correlations decay exponentially beyond this boundary [15]. The word effective emphasizes the existence of tunneling effects beyond this boundary [16]. Over the past few decades, the LR theorem has been refined [17–28] and validated [29–33] across a wide range of quantum systems, including Long-Range Interacting (LRI) systems. In this class of systems, the long-range nature of the interactions intuitively suggests the absence of any sharp light cones, since all subsystems are directly coupled through nonlocal terms. Nevertheless, light-cone-like structures with surprisingly sharp boundaries continue to manifest in various LRI chains, even when the interactions decay slowly with distance [23, 27, 28, 34–41], suggesting the presence of a universal underlying mechanism.

The goal of this paper is not to refine existing LR bounds or validate established knowledge. Instead, our objective is to uncover the physical phenomenon that underpins the emergence of light cones, which, to the best of our knowledge, has been overlooked in the literature. Specifically, we demonstrate that quantum destructive interference drives the emergence of light cones in long-range interacting quantum systems, underscoring the role of quantum phases in the emergence of locality in quantum systems. This new insight provides not only a foundational understanding of the concept of locality in quantum systems but also a deeper perspective on the *emergent* nature of relativistic principles within quantum dynamics—an area of ongoing significant effort [42].

We show that destructive interference is both necessary and sufficient for the emergence of light cones in a wide class of LRI quantum chains. This conclusion is substantiated by several key observations. First, we demonstrate that light cones arise explicitly as a consequence of destructive interference (sufficiency). Second, we show that in the absence of destructive interference, light cones lose a fundamental property—observed and well-documented in the literature (necessity)—namely, their structural independence from the interaction length in LRI spin chains [13, 43]. Third, we establish that the breakdown of destructive interference coincides with the disappearance of light cones, aligning with prior theoretical and numerical findings. Finally, we present both analytical and numerical evidence of a counterintuitive phenomenon: the acceleration of entanglement propagation—and the corresponding diminishing of light cones—when the interaction range is reduced, which weakens destructive interference. This finding is particularly surprising, as one would intuitively expect shorter interaction ranges to slow down entanglement propagation. These results collectively highlight the central role of destructive interference in shaping the dynamics of long-range quantum systems, which has novel implication to quantum infor-

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mation processing [28].

II. SPIN CHAIN MODEL

We consider one-dimensional (1D) spin chains with the Hamiltonian

$$\mathbf{H} = \sum_{j<l}^{N} \mathbf{J}(|j-l|) H_{j,l}, \tag{1}$$

where $H_{j,l}$ (satisfying $||H_{j,l}|| = 1$) is a Hermitian operator acting on sites j and l. The interaction function $\mathbf{J}(n)$ is assumed to be real, symmetric ($\mathbf{J}(n) = \mathbf{J}(-n)$), and absolutely convergent, i.e.,

$$\sum_n |\mathbf{J}(n)| < \infty. \tag{2}$$

A special case is the power-law decaying interaction function, given by

$$\mathbf{J}(n) = \mathbf{J}(|j-l|) = \frac{J}{|j-l|^p}, \tag{3}$$

where J represents the interaction strength and p determines the interaction length. For simplicity, we set $\mathbf{J}(0) = 0$ and $\mathbf{J}(1) = 1$, implying $J = 1$. Additionally, we set $\hbar = 1$ throughout our formulation. The absolute convergence of the interaction function requires $p > 1$.

In this setting, an arbitrary local observable O ($||O|| = 1$) evolves as

$$O(t) = U^\dagger(t) O U(t), \tag{4}$$

where $U(t) = e^{-i\mathbf{H}t}$ is the unitary time evolution operator generated by the Hamiltonian (1). The Taylor expansion of $U(t)$,

$$U(t) = \sum_r \frac{(-it\mathbf{H})^r}{r!}, \tag{5}$$

contains hopping terms embedded in $\mathbf{H}^r$, which are induced by the interaction function $\mathbf{J}(n)$. These terms are given by the repeated convolution of $\mathbf{J}(n)$, expressed as (see also Figure 1):

$$\mathbf{J}^{*r}(q) = \sum_{d_1, d_2, \cdots, d_r \in \mathbb{Z}-\{0\}, \sum d_i = q} \prod_i \mathbf{J}(d_i). \tag{6}$$

Here, q denotes the total displacement along the chain resulting from r successive hopping events, with individual hopping distances given by $d_1, \ldots, d_r$.

Motivated by this observation, and considering the presence of both U and $U^\dagger$ in the expression for $O(t)$, we study the evolution of the following quantity:

$$\mathcal{Q}_q(t) = \sum_{r=0} \alpha_r t^{2r},$$
$$\alpha_r \equiv \frac{1}{(2r)!} \sum_{r_1+r_2=2r} \binom{2r}{r_1} (-1)^{r_1} \mathbf{J}^{*r_1}(q) \mathbf{J}^{*r_2}(q). \tag{7}$$

Since $||H_{j,l}|| = ||O|| = 1$, the function $\mathcal{Q}_q(t)$ serves as an upper bound for variations in the expectation value of O, evaluated with respect to an arbitrary wavefunction, when transported to distance q under the unitary evolution dictated by $\mathbf{H}$ [15, 44], while taking into account contributions from both U and $U^\dagger$ at distance q.

Notably, in Section II of [45], we prove that $\mathcal{Q}_q(t)$ exactly quantifies entanglement production at distance q, as a result of a local quench, in the free-particle limit. Hence, in the remainder of this paper, we will refer to $\mathcal{Q}_q(t)$ as entanglement produced at distance q.

Furthermore, the series expansion in (7) is restricted to even powers of t. This follows from the fact that $\mathbf{J}(n)$ is a real function, which ensures that we can always take $\mathcal{Q}_q(t) = \mathcal{Q}_q(-t)$, as discussed in [46].

III. CHARACTERIZING LIGHT CONES

In the short-range limit, i.e., Nearest-Neighbor (NN) interactions, we have $\alpha_r = 0$ for $r < q$, and the first nonzero term in the Taylor series expansion (7) is given by [46, 47]:

$$\mathcal{Q}_q(t) \propto \left(\frac{t}{q}\right)^{2q}. \tag{8}$$

This term dominates the entanglement expansion for $t < q$ [48] and explicitly dictates a linear light cone structure for $\mathcal{Q}_q(t)$, as expected [15]. (Note that since we set $\hbar = 1$ and $\mathbf{J}(1) = 1$, time t and distance q can be directly compared, e.g., $t \leq q$.)

Thus, (8), which corresponds to $r = q$ in the expansion (7), governs early entanglement growth at distance q (where *early* refers to $t \leq q$). Due to the interplay between r, q, and t, we can identify and define the emergent *linear light cone* by the term $r = q$ in the Taylor series (7). Similarly, terms with $r < q$ ($r > q$) correspond to regions *outside* (*inside*) the light cone [49].

Analogously, when the coefficients α_r vanish (or are exponentially suppressed) for $r < \frac{q}{2}$, and the early entanglement dynamics ($t < \frac{q}{2}$) satisfy

$$\mathcal{Q}_{\mathcal{M}}(t) \propto \left(\frac{2t}{q}\right)^{q}, \tag{9}$$

we can identify the emerging linear light cone with $2r \approx q$. The rationale for this characterization will be clarified in subsection IV C. Moreover, the conditions $2r < q$ ($2r > q$) define the regions outside (inside) the light cone in this scenario. We will use this approach to rigorously prove the existence of a light cone in Section IV. This proof is established by demonstrating that $\alpha_r \approx 0$ for $2r < q$, despite the extensive hopping effects— a direct consequence of destructive interference.

IV. MAIN RESULT: EMERGENCE OF LIGHT CONES VIA DESTRUCTIVE INTERFERENCE

For long-range interacting systems, the hopping process in (6) leads to an accumulation of entangling effects even on short time scales. This occurs due to the exponential increase in the number of possible hopping paths that can connect any two points on the chain for $r > 2$. In fact, for $r > 2$, there exist infinitely many r-step hopping pathways that connect any two sites (for infinitely long chains).

Despite this accumulative feature, one might expect that the formation of light cones results from the insignificance of the accumulated effects, attributed to the attenuation of the interaction function at long distances. However, this cannot be the underlying reason, as it does not explain the observation [13, 43] that the structure of light cones is largely independent of the interaction length (p) in power-law interactions when $p > 3$. We will further discuss this issue in Subsection IV D.

In this Section, having ruled out the above attenuation-based explanation, we demonstrate that light cone formation is entirely governed by the cancellation of terms in the expansion (7) for α_r. Specifically, we prove that destructive interference is not only sufficient (as shown in Subsection IV C) but also necessary (as shown in Subsection IV D) for the emergence of light cones.

A. Two Types of Interaction Functions

Before proceeding to the main proof of the role of destructive interference, it is essential to distinguish between two types of interaction functions (Type 1 and Type 2), as illustrated in Figure 1. Destructive interference occurs specifically for Type 1 functions.

- **Type 1**: The dominant terms in the expansion (6) arise when most d_i's are of order 1, with only a few d_i's of order q.
- **Type 2**: The dominant terms in expansion (6) correspond to a nearly uniform distribution of d_i's, where the d_i's are of order q/r.

In Section IV of [45], we prove that the classification of an interaction function $\mathbf{J}(n)$ is determined by the quantity $\mathbf{J}'' - \mathbf{J}'\mathbf{J}$, where $'$ denotes differentiation with respect to n, assuming the function is continuous. We show that as this quantity increases, Type 1 terms in (6) become more dominant. Furthermore, in Subsection III.D of [45], we demonstrate that for $p > 2$, the power-law interaction (3) falls under Type 1.

In the next subsection, we prove the existence of destructive interference for power-law interactions with $p > 2$. However, the proof can be readily extended to all Type 1 interaction functions.

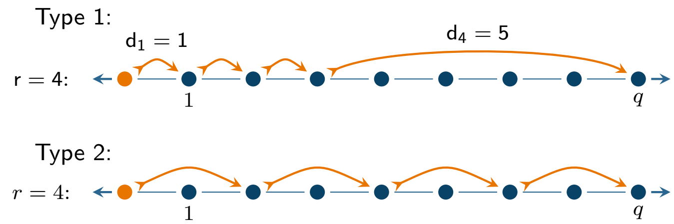

FIG. 1. Schematic illustration of the hopping process at distance q in r steps, given by the r-th order convolution of $\mathbf{J}(n)$, as described in (6). The figure distinguishes two different types of hopping, as defined in IV A. Type 1 consists of hoppings of order 1 accompanied by a long jump, while Type 2 corresponds to a more uniform distribution of hoppings.

B. Mechanism of Destructive Interference

Here we calculate α_r given by the series expansion (7). For this purpose, we first simplify the involved high-order convolutions ($\mathbf{J}^{*r}(q)$) of the power-law decaying interaction (3). Given that for $p > 2$, the power-law interaction function (3) is Type 1, the high-order convolution (defined in (6)) of this function can be expanded as

$$\mathbf{J}^{*r}(q) = \frac{\binom{r}{1}}{(q-r+1)^p} + \frac{\binom{r}{2}}{\big(2(q-r)\big)^p} + \cdots . \tag{10}$$

Here, the first term corresponds to $d_1 = d_2 = \cdots = d_{r-1} = 1$ and $d_r = q - r + 1$ in (6). Since any of the d_i's (among $d_1, d_2, \cdots, d_r$) can be chosen to equal $q - r + 1$, there are $\binom{r}{1}$ such terms, which gives the numerator of the first fraction in this expansion. The second term in (10) corresponds to $d_1 = d_2 \cdots = d_{r-2} = 1$, $d_{r-1} = 2$, and $d_r = q - r$ in the expansion (6). Simple combinatorial arguments dictate that there exist $\binom{r}{2}$ such terms.

We now focus on the first term in the expansion (10). A similar set of derivations holds for any other Type 1 term in this series, all of which have been taken into account in the full proof in Section III of [45]. When $0 < r < q$, the first term in (10) can be written as

$$\begin{aligned}\frac{\binom{r}{1}}{(q-r+1)^p} &= \frac{r}{q^p}\left(1 - \frac{r-1}{q}\right)^{-p} \\ &= \frac{r}{q^p}\Big(1 + p\frac{r-1}{q} + p(p+1)(\frac{r-1}{q})^2 + \cdots\Big).\end{aligned} \tag{11}$$

If the first two terms in this Taylor expansion are used for the high-order convolutions $\mathbf{J}^{*r_1}(q)$ and $\mathbf{J}^{*r_2}(q)$ involved in (7), then α_r can be expressed as follows for $0 < r_1 < q$ and $0 < r_2 < q$

$$\alpha_r = \frac{1}{(2r)!q^{2p}} \sum_{r_1+r_2=2r} \Bigg(\binom{2r}{r_1}(-1)^{r_1}\Big(r_1 + \frac{pr_1(r_1-1)}{q}\Big) \Big(r_2 + \frac{pr_2(r_2-1)}{q}\Big)\Bigg) = 0, \tag{12}$$

due to the binomial theorem.

Equation (12) shows that even though many hopping pathways contribute to transport entanglement through the chain, their effects cancel out due to accumulated phase differences, which as we will further demonstrate, ultimately enforces the emergence of light cones. Hence, equation (12) manifests the destructive interference phenomenon. It demonstrates that despite the extensive accumulation of hopping amplitudes in the high-order convolutions $\mathbf{J}^{*r_1}(q)$ and $\mathbf{J}^{*r_2}(q)$, the main effects contributing to the buildup of α_r cancels out due to accumulated phases during the hopping process. This cancellation of entangling effects at distance q leads to $\alpha_r = 0$ up to the considered order of approximations (see Figure 2).

To derive (12) and show that $\alpha_r = 0$, we truncated the expansions (10) and (11). Although such cancellations occur for other terms in both expansions, as demonstrated in the full proof [45], the cancellation is not complete for higher-order terms in (11) [50]. This results in a marginal contribution to α_r from these higher-order terms, which is factorially small in q and, for concreteness, is characterized in the proof [45] of the theorem in the next subsection. The theorem in the next subsection concisely accounts for all effects beyond the truncations used in this subsection to show that (12) ultimately leads to formation of the $2r < q$ light cone.

It is important to note that since (12) vanishes regardless of the value of p, the early entanglement dynamics and the structure of the emergent light cone become increasingly independent of the parameter p when $p > 2$, explaining prior observations [43]. We will use this feature in subsection (IV D) to prove the necessity of destructive interference in explaining the formation of light cones.

C. Emergence of the $2r \approx q$ light cone due to destructive interference

In the previous subsection, we showed that the main body of entangling effects traveling to distance q destructively interfere and cancel out when $r_1 < q$ and $r_2 < q$. To ensure the validity of these inequalities, and given that $2r = r_1 + r_2$, we consider the region $2r < q$. Therefore, for the $2r < q$ light cone, by taking into account all higher-order terms in the expansions (10) and (11), the following theorem is proved in Section III of [45].

Theorem:
For spin chains governed by the long-range interacting Hamiltonian (1)-(3), when $p > 2$, the entanglement measure $\mathcal{Q}_q(t)$, defined in (7), is constrained within the linear light cone $2r < q$, as defined in Section III. Specifically, for $2r < q$ with $q \gg 1$, due to destructive interference among hopping terms, the coefficients α_r scale as follows

$$\alpha_r \sim O\left(\frac{p^4}{q^{2p+4}(2r-2)!}\right). \tag{13}$$

See the proof in [45].

Notably, this scaling exhibits *exponential* suppression in q relative to the scaling of α_r required for an emergent light cone. We now elaborate. Consider the case of NN interactions, where the first nonzero α_r corresponds to $r = q$, and in this case, we have $\alpha_r = \frac{1}{(r!)^2}$ [46]. This leads to the formation of the $r = q$ light cone. Similarly, for the $2r \approx q$ light cone to emerge, the first nonzero α_r term (here $r \approx q/2$) must be of order $\hat{\alpha}_r = \frac{1}{(q/2)!^2}$. However, for $2r \approx q$, the scaling for α_r is given by:

$$\alpha_r \sim O\left(\frac{p^4}{q^{2p+4}(q-2)!}\right), \tag{14}$$

which is exponentially smaller than $\hat{\alpha}_r = \frac{1}{(q/2)!^2}$, as

$$\frac{\alpha_r}{\hat{\alpha}_r} = \frac{p^4}{q^{2p+4}}\frac{(q/2)!^2}{(q-2)!} \sim \frac{1}{q^{2p+2}\binom{q}{q/2}} \sim \frac{1}{q^{2p+\frac{3}{2}}2^q} \to 0, \tag{15}$$

for $q \gg 1$, where w have used the Stirling's approximation.

D. The necessity of destructive interference for emergent light cones

The destructive interference mechanism is not only sufficient (as described in subsection IV C) but also necessary for the consistency of observed results in the literature.

Prior studies have indicated a robust and universal emergence of *linear* light cones in LRI systems when the range of interactions exceeds a threshold that varies depending on the nature of the system [13, 23, 43, 51]. Here, the term robust signifies that the linearity of the light cone remains independent of p once it surpasses the threshold.

On the other hand, in Section (III), we demonstrated that for linear light cones to emerge, it is necessary for α_r to vanish for $r < q$ (or $2r < q$, depending on the light cone). These two statements together dictate that the condition $\alpha_r \approx 0$ must remain independent of p, meaning $\frac{d^s}{dp^s}\alpha_r = 0 \quad \text{for all } s = 1, 2, \ldots$.

Such a condition can arise only under one of two scenarios:

1. p is entirely absent from the formulation of entanglement—which is demonstrably not the case (see (7)), or

2. despite p appearing in the formulation, the value of α_r remains fully independent of p.

Since p appears explicitly in the hopping processes—through its role in the higher-order convolutions that define α_r—one would naturally expect α_r to depend on p. Therefore, for α_r to remain independent of p, all contributions involving p within these hopping terms

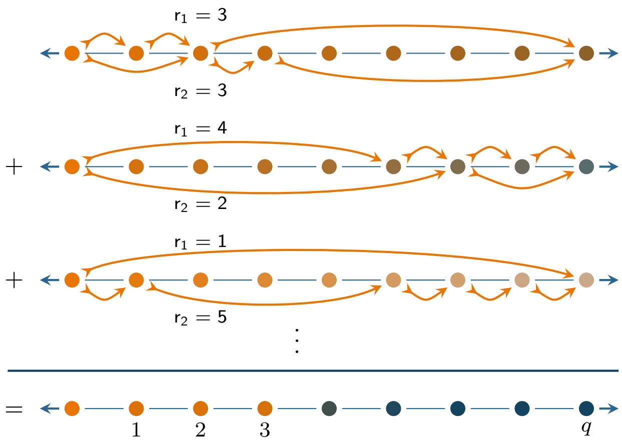

FIG. 2. Destructive interference as the mechanism behind light cone formation in long-range interacting spin chains. The arrows depict the long-range hopping process connecting the spins, as discussed in Figure 1. Each contributing term to entanglement production at distance q (illustrated in the top three spin chains) consists of two sequences of hoppings, shown above and below the chains, corresponding to $\mathbf{J}^{*r_1}(q)$ and $\mathbf{J}^{*r_2}(q)$ in (7). The order of hoppings (r_1 and r_2) in each example sums to $2r = r_1 + r_2 = 6$. These hoppings generate entanglement at distance q, represented by the intensity of the orange color in the spins at q, in contrast to dark blue (signifying no entanglement production). However, depending on r_1 and r_2, the accumulated entangling effects acquire a phase. Crucially, due to this phase accumulation, the total entanglement built up at distance q (when $q > r = 3$)—depicted in the lowest spin chain—is exponentially suppressed as a result of destructive interference between different hopping pathways. This suppression gives rise to an emergent effective light cone, restricting entanglement propagation despite the long-range nature of interactions. The abrupt transition from orange to blue in the lowest spin chain visually represents this suppression. The proof in Section IV establishes that destructive interference is both necessary and sufficient for the formation of light cones, underscoring the fundamental role of this phenomenon in explaining the persistence of locality in long-range interacting spin chains.

must undergo systematic cancellation, resulting in $\alpha_r \approx 0$ across all relevant values of p.

This situation can only occur through cancellation of terms (i.e., destructive interference) among the hopping processes (that involve p), as in (12). Consequently, destructive interference is not just an incidental effect but a necessary mechanism ensuring the independence of the light cone structure upon p, thereby reinforcing the robustness of the emergent linearity in the emergent light cone structure.

V. FASTER ENTANGLEMENT PROPAGATION FOR SHORTER-RANGED INTERACTIONS

The destructive interference phenomenon has a counter-intuitive implication. In particular, the entanglement edge must propagate faster when the interaction function is shortened, specifically when the interaction terms that extend beyond the range (η) are not present in the interaction function. This phenomenon is the direct consequence of destructive interference in the early entanglement region due to the following reason. The destructive interference occurs when *all* entangling effects are superposed (see (12)). Therefore, if part of the stream of quasi-particles is obstructed due to shortening the interaction function, then the destructive interference is weakened. Hence, the entanglement edge propagates faster. This phenomenon is mathematically proved in Section V of [45], and numerically demonstrated in the next subsection. Here, a shortened interaction refers to:

$$\mathbf{J}_\eta(n) = \begin{cases} \frac{J}{|n|^p}, & 0 < |n| \leq \eta \\ 0, & |n| > \eta, n = 0, \end{cases} \tag{16}$$

where $\eta = 1$ gives NN interaction.

A. Numerical Evidence

Figure 3 shows the Entanglement Edge Times (EETs) at a distance of $q = 10^3$ from the quenched spin in a ferromagnetic Heisenberg chain (when $H_{j,k} = \mathbf{S}_j \mathbf{S}_k$, with $\mathbf{S}_j$ being the spin operator at site j) as a function of the interaction range (η) and for different values of p. EET represents the time required for entangling effects, beyond quantum tunneling, to reach spins at the given distance. In this analysis, EET denotes the time when the entanglement measure $\mathcal{Q}_q(t)$ exceeded 10^{-5}. Prolonged EETs for the full-range interaction (denoted by $\eta = \infty$), compared to shortened interaction ranges demonstrate the destructive interference effect, which is apparent for $p \geq 2$ among the simulated cases. In Figure 3, the orange dashed lines show prolonged EETs when $\eta = \infty$ in contrast to $\eta = 14$. Destructive interference also manifests itself in the upward slopes of the curves, highlighted in orange, which indicate an increase in the EETs as the interaction range grows.

Another notable aspect in Figure 3 is the lack of dependence of the $\eta = \infty$-EET on the variable p for $p > 3$. This can be seen in the flattening of the curves as p increases, which is caused by destructive interference. To illustrate this phenomenon more clearly, the inset in figure 3 displays the EETs for the full-range interaction length as a function of p. Consequently, for $p > 3$, as η increases, the EETs quickly revert to the Nearest-Neighbor (NN) interaction time ($\eta = 1$), indicated by the horizontal dashed black line. Thus, in this region, the EETs are not only independent of p, but also become increasingly independent of η when it exceeds a certain value. Based on this feature, and given that $p \to \infty$ corresponds to NN interactions, for $p > 3$, light cones must be strictly linear (similar to the case of NN interactions), confirming the results in [23].

B. Practical Shortening of Interactions

As an alternative to the sharp shortening (16), which can be practically challenging to implement, the experimentally feasible [52, 53] softened shortening (characterized by the rate σ) is given by

$$\mathbf{J}_{\eta,\sigma}(n) = \begin{cases} \frac{J}{|n|^p}, & 0 < |n| \leq \eta \\ \frac{J}{|n|^p} e^{-\sigma(|n|-\eta)}, & |n| > \eta \end{cases} \tag{17}$$

This approach can also be used to experimentally observe the counterintuitive acceleration of entanglement transport in the chain. Table I shows the EET values for various softened interactions in a numerical simulation of ferromagnetic Heisenberg chains, described in the previous subsection. In this table, $\eta = 5$ represents the interaction range for shortened interactions, while $\eta = \infty$ denotes full long-range interaction. Additionally, $\sigma = \infty$, 1.5, and 0.5 correspond to sharp, semi-softened, and highly softened shortening of the interaction function, respectively.

As the simulation data clearly indicate, the EETs for shortened interactions are distinctly lower compared to the full long-range case ($\eta = \infty$). This difference becomes more pronounced with increasing sharpness of shortening. Therefore, even though sharp shortening might not be feasible in experimental settings, we propose that softened shortening can effectively be used to demonstrate the destructive interference phenomenon in the laboratory.

EETs for softened interaction shortening				
	$\eta = 5$ $\sigma = \infty$	$\eta = 5$ $\sigma = 1.5$	$\eta = 5$ $\sigma = 0.5$	$\eta = \infty$
EET ($\frac{\hbar}{J}$) ($p = 2.5$)	413.0	417.0	431.6	455.9

TABLE I. Entanglement Edge Time (EET) for softened interaction functions. Parameters η and σ represent the interaction range and shortening sharpness, respectively (see 17). $\eta = \infty$ denotes full long-range interaction, and $\sigma = \infty$ denotes sharp shortening. EETs are calculated for $p = 2.5$ and are given in units of $\frac{\hbar}{J}(\pm 0.1)$. The data in the table clearly demonstrate that even with soft shortening, which is expected to be experimentally feasible in an appropriate laboratory setting, the counterintuitive acceleration of entanglement persists.

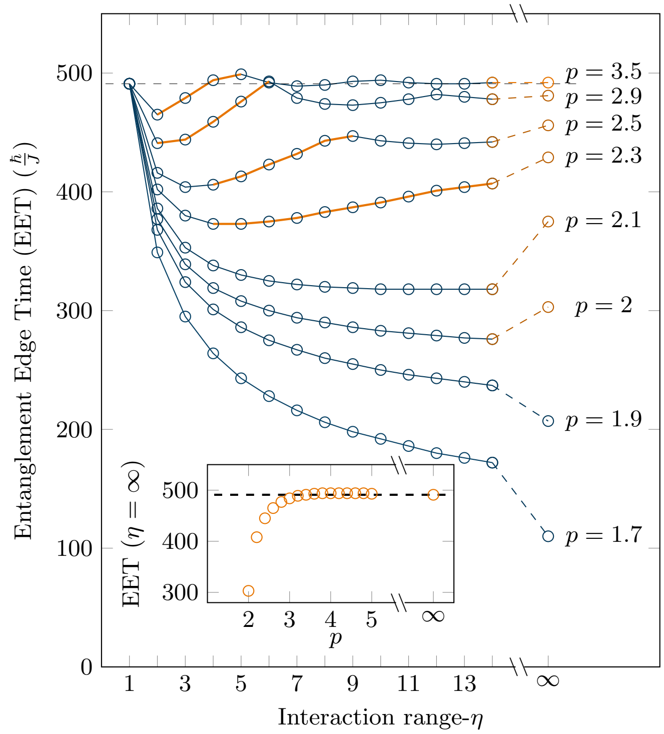

FIG. 3. Entanglement Edge Times (EETs) for various interaction ranges (η) and different values of p. The plot demonstrates how shortening interactions alters entanglement propagation. The orange-highlighted upward slopes of the curves illustrate the unexpected increase in EETs as the interaction range (η) grows for $p > 2$. Furthermore, the prolonged EETs for full-range interaction (represented by $\eta = \infty$) compared to shortened interaction values are shown by orange dashed lines. Both of these unusual phenomena result from the destructive interference between entangling effects, which leads to the formation of an effective light cone in long-range interacting spin chains. The inset shows the progress of the $\eta = \infty$-EETs versus p. This figures shows the quick convergence of EETs to the nearest-neighbor value for $p > 3$.

VI. CONCLUDING REMARKS

In this work, we showed that quantum destructive interference—a mechanism not previously recognized in the literature—is both necessary and sufficient for the emergence of light-cone-like structures in a broad class of long-range interacting quantum spin chains. Specifically, although long-range interactions enable non-local entangling effects, their propagation is significantly suppressed by destructive interference, enforcing the emergence of effective light cone structures.

The results of this paper not only provide new theoretical insights into the notion of locality (as a result of the emergent light cones) in quantum systems but also offer a concrete example of how causal structures, such as light cones, emerge within the quantum dynamics of many-body systems. This raises intriguing questions about the role of destructive interference in embedding relativistic principles within quantum many-body dynamics. Whether relativity itself is an emergent phenomenon remains one of the most fundamental questions in theoretical physics [42], explored in contexts such as holographic spacetime [54, 55], and other frameworks [56–58]. Our work introduces destructive interference as a candidate mechanism that could bridge quantum dynamics and emergent causal structures, a direction for further investigation.

This analysis also unifies prior findings in the literature, including the emergence of Lieb-Robinson bounds in power-law decaying interactions with $p > 2$ [13], which coincides with the breakdown of destructive interference. Furthermore, the presence of a strictly linear light cone for $p > 3$ [23, 39] aligns with our result that, in this regime, destructive interference constrains entanglement propagation to mimic that of nearest-neighbor interactions.

We predict that the destructive interference phenomenon, underlying the counterintuitive acceleration of entanglement growth, can be experimentally observed through the use of shortened interactions. While exact (sharp) shortening may pose experimental challenges, softened shortening yield similar acceleration effects. Notably, similar tunable shortening have already been implemented in Rydberg atom arrays [12], making them a promising platform for testing our predictions. Additional platforms for investigation include cold atoms [6, 29] and trapped ions [31, 59–61]. Future work should investigate whether destructive interference persists in the strong long-range interacting limit $p < 1$, potentially uncovering new mechanisms for entanglement suppression.

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[49] (), this identification is valid because if higher-order terms, e.g., $r = q + 1$, $r = q + 2$, etc., were to dominate the series for arbitrary time $t < q$, then the coeficients of these terms would need to grow exponentially with r to compensate for the exponentially smaller parts $(\frac{t}{q+1})^{2(q+1)}$, etc.
[50] (), when the term of order s in (11), i.e., corresponding to $(\frac{r_1-1}{q})^s$ is substituted for $\mathbf{J}^{*r_1}(q)$ in (7), all of the terms with $r_1 \geq s$ cancel out in the summation. Therefore, only the first $s - 1$ terms remain, which are exponentially (in s) suppressed outside the light-cone $r < q$.
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Supplementary Material: Emergence of light cones in long-range interacting spins chains Is Due to destructive interference

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This supplemental document includes the full proof of the destructive interference phenomenon in long-range interacting spin chains, expanding on the key findings presented in the main text. For concreteness of presentation, we will consider a specific model of Heisenberg chains, but the proof holds for the main model presented via eq (1) in the main text. Additionally, the document introduces foundational material, including the Quantum Correlation Transfer Function (QCTF) formulation and the Holstein-Primakoff transformation, to offer a comprehensive understanding of the methods and concepts used in this study.

CONTENTS

I. HEISENBERG MODEL AND THE FREE-PARTICLE APPROXIMATION

In this section, we will revisit the Holstein-Primakoff transformation to obtain a linear spin-wave approximation for the ferromagnetic Heisenberg model with long-range interactions. The Hamiltonian is given by

$$\mathbf{H} = \frac{1}{2}\sum_{j,0<n}^{N} \mathbf{J}(n)\mathbf{S}_j\mathbf{S}_{j+n}, \tag{S.1}$$

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where $\mathbf{S}_j$ represents the spin operator at the j-th site, N is the number of spins and the interaction function $\mathbf{J}(n)$ is defined by the subsequent power-law decay expression

$$\mathbf{J}(n) = \begin{cases} \frac{J}{|n|^p}, & n \in \mathbb{N} - \{0\} \\ 0, & n = 0. \end{cases} \tag{S.2}$$

The interaction function in Fourier space (distinguished by the symbol $\tilde{}$) is

$$\tilde{\mathbf{J}}(k) = \sum_n e^{-ikn}\mathbf{J}(n), \tag{S.3}$$

where the momenta (k) are multiples of $\frac{2\pi}{N}$. For spin-$\frac{1}{2}$ particles, Holstein-Primakoff bosons are given by Nakano and Takahashi (1994) and Holstein and Primakoff (1940),

$$S_j^+ = (1 - b_j^\dagger b_j)^{\frac{1}{2}} b_j, S_j^- = b_j^\dagger (1 - b_j^\dagger b_j)^{\frac{1}{2}}, S_j^z = \frac{1}{2} - b_j^\dagger b_j. \tag{S.4}$$

Thus, the representation of bosons in Fourier space is:

$$\tilde{b}_k^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-ikj} b_j^\dagger, \tilde{b}_k = \frac{1}{\sqrt{N}} \sum_k e^{ikj} b_j. \tag{S.5}$$

Applying this transformation allows for the expression of the Bosonized Hamiltonians in the following manner

$$\mathbf{H} = \mathbf{H}_{\mathrm{LSW}} + \mathbf{H}', \tag{S.6}$$

where the linear spin-wave Hamiltonian ($\mathbf{H}_{\mathrm{LSW}}$) encompasses quadratic terms involving bosonic annihilation and creation operators, and $\mathbf{H}'$ comprises both constant and higher-order terms that characterize magnon interactions. Consequently, it follows from Nakano and Takahashi (1994) that

$$\mathbf{H}_{\mathrm{LSW}} = \sum_k \underbrace{(\frac{\tilde{\mathbf{J}}(0) - \tilde{\mathbf{J}}(k)}{2})}_{\epsilon_k} \tilde{b}_k^\dagger \tilde{b}_k. \tag{S.7}$$

The resolvent for the LSW Hamiltonian, denoted as $\mathbf{G}(s) = (s + \frac{i}{\hbar}\mathbf{H}_{\mathrm{LSW}})^{-1}$, is expressed as follows

$$\mathbf{G}(s) = \sum_k (s + \frac{i}{\hbar}\epsilon_k)^{-1} \tilde{b}_k^\dagger \tilde{b}_k. \tag{S.8}$$

In the next subsection, we will utilize this resolvent for the QCTF analysis. Given that the resolvent incorporates only the LSW Hamiltonian and excludes the magnon interaction terms, this approximation is solely applicable within the free-particle limit. The free-particle approximation is used only in Subsection III A to prove that the measure (7) in the main text exactly quantifies entanglement production at distance q in this regime. Therefore, our main proof of the existence of the destructive interference mechanism, which is based on the consecutive convolution of the interaction function (6) and does not rely on the free-particle approximation, remains valid beyond the free-particle limit.

II. QUANTUM CORRELATION TRANSFER FUNCTION (QCTF) ENTANGLEMENT FORMULA

In this section, we briefly summarize the foundational principles of the Quantum Correlation Transfer Function (QCTF) method from Azodi and Rabitz (2025). This approach captures the bipartite entanglement of the subsystem $\mathcal{M}$ within a combined closed quantum system that also comprises the subsystem $\mathcal{R}$.

In this formulation, we use arbitrary basis sets for the subsystems $\mathcal{M}$ (dimension d_m) and $\mathcal{R}$ (dimension d_r), labeled as $\{|m\rangle\,|m=1,\cdots,d_m\}$ and $\{|r\rangle\,|r=1,\cdots,d_r\}$, respectively. Consequently, the set $\{|m\rangle\otimes|r\rangle\}$ forms a basis for the combined, isolated quantum system of dimension $d_m d_r$.

We represent the state of the composite system in both the Laplace and time domains as $|\Psi(s)\rangle$ and $|\psi(t)\rangle$, respectively, such that $|\Psi(s)\rangle=\mathcal{L}\{|\psi(t)\rangle\}$. Using the basis defined above, we have $|\Psi(s)\rangle=\sum_{m,r}\psi_{m,r}(s)\,|m\otimes r\rangle$. Consequently, $|\Psi(s)\rangle$ can be expressed as $\mathbf{G}(s)\,|\psi(0)\rangle$, with $\mathbf{G}=(s+\frac{i}{\hbar}\mathbf{H})^{-1}$ serving as the resolvent for the entire system. Using this notation, the *projected wave function* corresponding to the basis ket $|m\rangle$ can be defined in the following manner

$$|m\psi\rangle \doteq \Big(\langle m|\otimes I_{\mathcal{R}}\Big)\,|\Psi(s)\rangle=\Big(\sum_r\big(\,|r\rangle\,(\langle m|\otimes\langle r|)\big)\Big)\mathbf{G}(s)\,|\psi(0)\rangle=\sum_r\psi_{m,r}(s)\,|r\rangle\,. \tag{S.9}$$

Observe that each projected wave function, which depends on s, represents the unnormalized state of the subsystem $\mathcal{R}$, following a projective measurement of subsystem $\mathcal{M}$ in the $\{|m\rangle\}$ basis, with the measurement result being $|m\rangle$.

We use the following dynamical bipartite entanglement measure for the subsystem $\mathcal{M}$:

$$\tilde{\mathcal{Q}}_{\mathcal{M}}(s)=\sum_{1\le m_1<m_2\le n}\big(\langle m_1\psi|\wedge\langle m_2\psi|\big)\star\big(|m_1\psi\rangle\otimes|m_2\psi\rangle\big) \tag{S.10}$$

$$=\sum_{1\le m_1<m_2\le n}\langle m_1\psi|\star m_1\psi\rangle\star\langle m_2\psi|\star m_2\psi\rangle-\langle m_2\psi|\star m_1\psi\rangle\star\langle m_1\psi|\star m_2\psi\rangle \tag{S.11}$$

Here, the symbol $\wedge$ represents anti-symmetrization, and is defined as $\langle m_1\psi|\wedge\langle m_2\psi|=\langle m_1\psi|\otimes\langle m_2\psi|-\langle m_2\psi|\otimes\langle m_1\psi|$. Also, the operator $\star$ represents convolution in the domain s such that simple Laplace poles obey $(s+i\omega_1)^{-1}\star(s+i\omega_2)^{-1}=(s+i(\omega_1+\omega_2))^{-1}$. Consequently, $\langle|\star\rangle$ is the normal inner product, except for the convolution between the Laplace variables on the sides according to the mentioned rule. This entanglement measure is the linear entanglement entropy of the subsystem $\mathcal{M}$ in the Laplace domain, and it is associated with the second-order Rényi entropy of the subsystem, following Azodi and Rabitz (2025).

III. PROOF OF THE DESTRUCTIVE INTERFERENCE EFFECT AND THE THEOREM IN THE MAIN TEXT

In this section, we provide a mathematical proof of the destructive interference phenomenon, including the detailed proof of the Theorem presented in the main text. The derivations in the main text are a summarized form of the more comprehensive proof given here. The proof is organized into four main parts. In the first part (subsection III A), we derive the QCTF entanglement measure for the Heisenberg chain. The second part (subsection III B) demonstrates the cancellation of terms in the QCTF entanglement measure, which demonstrates the presence of destructive interference. In subsection III C, we prove the scaling for α_r as stated in the Theorem. Finally, subsection III D explains why the theorem holds for $p>2$.

A. Deriving the bipartite QCTF entanglement measure

In this section, we aim to derive the dynamical entanglement measure for a segment of the Heisenberg chain, while also illustrating the destructive interference occurring among quasi-particles within the entanglement measure. We are considering a chain of N spin-$\frac{1}{2}$

particles with Heisenberg interactions (S.1-S.3). This chain is split into two contiguous parts, called subsystems $\mathcal{M}$ and $\mathcal{R}$. Initially, from the ferromagnetic order $|F\rangle = |vac\rangle$, a group of spins in subsystem $\mathcal{R}$ are flipped: The initial state of the compound system is denoted by $|\psi(0)\rangle = b_{0_1}^\dagger \cdots b_{0_w}^\dagger |F\rangle$. Therefore, $\{0_i|i=1,\cdots w\}$ (also denoted by $\{0_i\}$) is the set of indices of the initially flipped spins. This set should satisfy the dilute excitation condition, namely w, the number of flipped spins should be much smaller than the size of $\mathcal{R}$. The overall density of excitations needs to be small, as a high concentration of excitations can lead to failure of the LSW approximation; therefore, the initial state should be *locally* dilute as well. For instance, adjacent flipped spins can lead to formation of bound states and strong magnon-magnon interactions Jurcevic *et al.* (2014); Kranzl *et al.* (2023).

The initial state $|\psi(0)\rangle$ is a member of a larger set of product states with exactly w flipped spins. For a clearer demonstration, we denote the collection of product states, each having precisely w flipped spins compared to the ferromagnetic order, by $\mathcal{E}_w$ i.e., $\mathcal{E}_w = \{E_i|E_i = b_{i_1}^\dagger \cdots b_{i_w}^\dagger |F\rangle\}$. Here, i is an arbitrary identifier for the members of this set. Also, in this definition, flipped spins can be anywhere in the chain. Therefore, in the framework of quench experiments, the system is initially quenched from ferromagnetic order to a member of $\mathcal{E}_w$, denoted by $|\psi(0)\rangle = E_0 = b_{0_1}^\dagger \cdots b_{0_w}^\dagger |F\rangle$. Each member of the set $\mathcal{E}_w$ can be rewritten as $|E_l\rangle = |E_l^{\mathcal{R}}\rangle \otimes |E_l^{\mathcal{M}}\rangle$, such that the kets $|E_l^{\mathcal{R}}\rangle$ and $|E_l^{\mathcal{M}}\rangle$ correspond to the subsystems $\mathcal{R}$ and $\mathcal{M}$ respectively. Given this decomposition, in our proof, we will use the following definitions of subsets of pairs of distinct members in E_w,

$$
\begin{aligned}
S &= \{(|E_l\rangle, |E_k\rangle)| \langle E_l^{\mathcal{R}}|E_k^{\mathcal{R}}\rangle = 1, \langle E_l^{\mathcal{M}}|E_k^{\mathcal{M}}\rangle = 0\} \\
D &= \{(|E_l\rangle, |E_k\rangle)| \langle E_l^{\mathcal{R}}|E_k^{\mathcal{R}}\rangle = 0, \langle E_l^{\mathcal{M}}|E_k^{\mathcal{M}}\rangle = 0\}.
\end{aligned}
\tag{S.12}
$$

The set S (D) consists of pairs of product states $(|E_l\rangle, |E_k\rangle)$, each with exactly w flipped spins, for which the flipped spins in the subsystem $\mathcal{R}$ are the same (distinct), and the flipped spins in the subsystem $\mathcal{M}$ are always distinct. Given this definition, $S \bigcup D$ includes all pairs $(|E_l\rangle, |E_k\rangle)$ such that $\langle E_l^{\mathcal{M}}|E_k^{\mathcal{M}}\rangle = 0$. This differentiation is advantageous for calculating the QCTF entanglement measure, as these sets correspond to different terms in (S.11).

Let us proceed to calculate the dynamical entanglement measure $\tilde{\mathcal{Q}}_{\mathcal{M}}(s)$ given by (S.10-S.11). As we will see, structures of the form $\langle E_0|G^\dagger(s^*)|E_l\rangle \star \langle E_k|G(s)|E_0\rangle$ are the building blocks of the entanglement measure (S.11). To simplify these forms, note that the system's resolvent in the LSW approximation, denoted as $\mathbf{G}(s) = (s + \frac{i}{\hbar}\mathbf{H}_{\mathrm{LSW}})^{-1}$, is specified in (S.8). Using (S.5), the resolvent can be rewritten as follows in the lattice domain

$$
\begin{aligned}
\mathbf{G}(s) &= \frac{1}{N}\sum_{k,n,m}(s + \frac{i}{\hbar}\epsilon_k)^{-1} e^{ik(m-n)} b_m^\dagger b_n, \\
\mathbf{G}^\dagger(s^*) &= \frac{1}{N}\sum_{k,n,m}(s - \frac{i}{\hbar}\epsilon_k)^{-1} e^{ik(m-n)} b_m^\dagger b_n.
\end{aligned}
\tag{S.13}
$$

Let us use the notation $|E_k\rangle = b_{k_1}^\dagger \cdots b_{k_w}^\dagger |\mathrm{vac}\rangle$ and $|E_l\rangle = b_{l_1}^\dagger \cdots b_{l_w}^\dagger |\mathrm{vac}\rangle$. Therefore, we have the following expansion

$$
\begin{aligned}
\langle E_0|G^\dagger(s^*)|E_l\rangle \star \langle E_k|G(s)|E_0\rangle = \frac{1}{N^2} \sum_{\substack{k_1,k_2 \\ n_1,n_2,m_1,m_2 \\ l_1,\cdots l_w, k_1 \cdots k_w}} & (s + \frac{i}{\hbar}(\epsilon_{k_2} - \epsilon_{k_1}))^{-1} e^{i\left(k_1(m_1-n_1) - k_2(m_2-n_2)\right)} \\
& \langle \mathrm{vac}| b_{0_1} \cdots b_{0_w} b_{m_1}^\dagger b_{n_1} b_{l_1}^\dagger \cdots b_{l_w}^\dagger |\mathrm{vac}\rangle \\
& \langle \mathrm{vac}| b_{k_1} \cdots b_{k_w} b_{n_2}^\dagger b_{m_2} b_{0_1}^\dagger \cdots b_{0_w}^\dagger |\mathrm{vac}\rangle .
\end{aligned}
\tag{S.14}
$$

Using the basic properties of the Holstein-Primakoff boson creation and annihilation operators, namely $b_i |\mathrm{vac}\rangle = \mathbf{0}$) and ${b_i^\dagger}^2 |\mathrm{vac}\rangle = \mathbf{0}$ (which is a consequence of (S.4)), we have the following conditions on the indices in Equation (S.14)

$$\left(\mathbf{A}=\begin{cases}\text{A1. } m_1\in\{0_i\},\\ \text{A2. } n_1\notin\{0_i\}\setminus\{m_1\},\\ \text{A3. } l_1=n_1,\\ \text{A4. } \{l_2,\cdots,l_w\}=\{0_i\}\setminus\{m_1\}\end{cases}\right)\text{AND}\left(\mathbf{B}=\begin{cases}\text{B1. } m_2\in\{0_i\},\\ \text{B2. } n_2\notin\{0_i\}\setminus\{m_2\},\\ \text{B3. } k_1=n_2,\\ \text{B4. } \{k_2,\cdots,k_w\}=\{0_i\}\setminus\{m_2\}\end{cases}\right). \tag{S.15}$$

We note that A3-A4 and B3-B4 are written without loss of generality, meaning that, for example, one could alternatively choose $l_2 = n_1$ for A3 and have $\{l_1, l_3, \cdots, l_w\} = \{0_i\} - \{m_1\}$ for A4, without loss of generality.

Conditions $A3-4, B3-4$ fully determine the kets $(|E_k\rangle, |E_l\rangle)$. As a result, since, according to (S.12), these kets are required to be different in the sublattice $\mathcal{M}$, as at least one of $n_1(= l_1)$ or $n_2(= k_1)$ must be inside $\mathcal{M}$, otherwise all labels $(\{k_i\},\{l_i\})$ would be inside $\mathcal{R}$ and the kets would be identical inside $\mathcal{M}$. Therefore, when (I.) n_1 and $n_2(\neq n_1)$ are both inside $\mathcal{M}$, and (II.) $m_1 = m_2$, (and only when both I and II are satisfied) then $\{l_2, \cdots, l_w\} = \{k_2, \cdots, k_w\}$, and therefore we have $\langle E_l^{\mathcal{R}} | E_k^{\mathcal{R}}\rangle = 1$, hence $(|E_k\rangle, |E_l\rangle) \in S$. In contrast, when only one of n_1, n_2 is in $\mathcal{M}$ (negation of I), or $m_1 \neq m_2$ (negation of II) then $\langle E_l^{\mathcal{R}} | E_k^{\mathcal{R}}\rangle = 0$, and therefore $(|E_k\rangle, |E_l\rangle) \in D$. Given this analysis, $\langle E_0|G^\dagger(s^*)|E_l\rangle \star \langle E_k|G(s)|E_0\rangle$ in equation (S.14) simplifies to the following

$$\text{for } (|E_l\rangle, |E_k\rangle) \in S: \sum_{\substack{k_1,k_2\\ m\in\{0_i\}\\ n_1\neq n_2\in\mathcal{M}}} (s+\frac{i}{\hbar}(\epsilon_{k_1}-\epsilon_{k_2}))^{-1}\frac{e^{i(k_1(m-n_1)-k_2(m-n_2))}}{N^2}, \tag{S.16}$$

$$\text{for } (|E_l\rangle, |E_k\rangle) \in D: {\sum_{\substack{k_1,k_2\\ m_1,m_2\in\{0_i\}\\ n_1\in\mathcal{M}\\ n_2\notin\{n_1\}\cup(\{0_i\}\setminus\{m_2\})}}}' (s+\frac{i}{\hbar}(\epsilon_{k_1}-\epsilon_{k_2}))^{-1}\frac{e^{i(k_1(m_1-n_1)-k_2(m_2-n_2))}}{N^2}. \tag{S.17}$$

The prime ($'$) on the second summation emphasizes that if $m_1 = m_2$, then $n_2 \in \mathcal{R}$.

Now, we can put together all the pieces and find the entanglement measure $\tilde{\mathcal{Q}}_{\mathcal{M}}(s)$ given by (S.10-S.11). Let us clarify the relation between $\langle E_0|G^\dagger(s^*)|E_l\rangle \star \langle E_k|G(s)|E_0\rangle$ and the terms in the entanglement measure. First, consider $(|E_l\rangle, |E_k\rangle) \in S$. According to (S.12), this set (S) includes <u>all</u> pairs such that the partial wave functions corresponding to the subsystem $\mathcal{R}$ are identical, and partial wave functions corresponding to the subsystem $\mathcal{M}$ are different. The identical partial wave functions corresponding to subsystem $\mathcal{R}$ are characterized via variable m in summation (S.16), and the different partial wave functions corresponding to subsystem $\mathcal{M}$ are characterized via variables n_1, n_2. Therefore, if for each pair n_1, n_2, we take the summation over m, this operation will perform the inner product in the sub-Hilbert space of subsystem $\mathcal{R}$, and the pair n_1, n_2, characterizing the different partial wave function of subsystem $\mathcal{M}$, play the role of the variables m_1 and m_2 in (S.11). The summation over k_1, k_2 in (S.16) gives the evolution (dynamics in the Laplace domain) of this quantity. Given this exposition, for each pair n_1, n_2, the norm of $\langle E_0|G^\dagger(s^*)|E_l\rangle \star \langle E_k|G(s)|E_0\rangle$ gives the second term in (S.11). Finally, if we sum over all such terms for all pairs n_1, n_2, i.e., equivalent to the summations over all m_1, m_2 in (S.11), then the second part of this entanglement measure is given as

$$\sum_{\substack{k_1,k_2,k_3,k_4\\ m_1,m_2\in\{0_i\}\\ n_1\neq n_2\in\mathcal{M}}} (s+\frac{i}{\hbar}(\epsilon_{k_1}-\epsilon_{k_2}-\epsilon_{k_3}+\epsilon_{k_4}))^{-1}\frac{e^{i(k_1(m_1-n_1)-k_2(m_1-n_2)-k_3(m_2-n_1)+k_4(m_2-n_2))}}{N^4}. \tag{S.18}$$

Analogously, we can relate the first term in the dynamical entanglement measure (S.11) with the $\langle E_0|G^\dagger(s^*)|E_l\rangle \star \langle E_k|G(s)|E_0\rangle$ terms when $(|E_l\rangle, |E_k\rangle) \in D\bigcup S$. To show this, consider the expression

$$\sum_{(|E_k\rangle,|E_l\rangle)\in D\bigcup S} \langle E_0|G^\dagger(s^*)|E_l\rangle \star \langle E_k|G(s)|E_0\rangle \star \langle E_0|G^\dagger(s^*)|E_k\rangle \star \langle E_l|G(s)|E_0\rangle\,. \tag{S.19}$$

We recall that $S\bigcup D$ includes *all* basis kets for which the partial states corresponding to subsystem $\mathcal{M}$ are different. Therefore, the first and last terms can form an inner product in the sub-Hilbert space $\mathcal{R}$ due to the summation over $|E_l\rangle$, and the second and third terms can form a similar inner product due to the summation over $|E_k\rangle$. Now, using (S.16-S.17), the summation (S.19) is

$$\sum_{\substack{k_1,k_2,k_3,k_4\\ m_1,m_2\in\{0_i\}\\ n_1\in\mathcal{M}\\ n_2\notin\{n_1\}\cup(\{0_i\}\setminus\{m_2\})}} (s+\frac{i}{\hbar}(\epsilon_{k_1}-\epsilon_{k_2}-\epsilon_{k_3}+\epsilon_{k_4}))^{-1}\frac{e^{i(k_1(m_1-n_1)-k_2(m_2-n_2)-k_3(m_1-n_1)+k_4(m_2-n_2))}}{N^4}. \tag{S.20}$$

In summary, using (S.10-S.11), in this part of the proof we showed that the dynamical entanglement measure of subsystem $\mathcal{M}$, is given by the difference between equations (S.20) and (S.18),

$$\tilde{\mathcal{Q}}_{\mathcal{M}}(s) = \text{(S.20)} - \text{(S.18)}. \tag{S.21}$$

Here, we identify three main categories of the QCTF Laplace poles (i.e., linear combinations of ϵ's) in the entanglement measure (S.21), more specifically in part (S.20) as follows. I: when $k_1 = k_3, k_2 = k_4$, these poles contribute only to the zero frequency poles ($s = 0$)), II: when $k_1 \neq k_3, k_2 = k_4$, called *dominant poles*, and III: $k_1 \neq k_3, k_2 \neq k_4$, which are highly ($\sim \frac{1}{N^4}$) suppressed, compared to the dominant poles, as explained below. The reason for this classification is also explored in detail in the analysis of short-range interacting chains in Azodi and Rabitz (2024). Dominant poles lead to the characteristic behavior of early entanglement evolution, and therefore we will only consider the effect of these poles. To see how dominant poles emerge, let us keep all the variables fixed in the summation in (S.20), except n_2. Since the summation over n_2 includes all lattice labels $1,\cdots,N$ except n_1, for each term in the summation (over all variables except n_2) we have:

$$(s+\frac{i}{\hbar}(\epsilon_{k_1}-\epsilon_{k_2}-\epsilon_{k_3}+\epsilon_{k_4}))^{-1}e^{i(k_1(m_1-n_1)-k_3(m_1-n_1))}\sum_{n_2\notin\{n_1\}\cup(\{0_i\}\setminus\{m_2\})}\frac{e^{i(k_4(m_2-n_2)-k_2(m_2-n_2))}}{N^4}. \tag{S.22}$$

When $k_2 \neq k_4$, and n_2 is summed over all possible numbers, then the inner summation is zero due to orthogonality of exponential functions. Also, note that for $n_2 = \{0_i\}\setminus\{m_2\}$, this term is zero, due to (S.15). Therefore, when $k_2 \neq k_4$, the inner summation in (S.22) is equal to negative the value of the summand for $n_2 = \{n_1, m_2\}$,

$$\sum_{n_2\notin\{n_1\}\cup(\{0_i\}\setminus\{m_2\})}\frac{e^{i(k_4(m_2-n_2)-k_2(m_2-n_2))}}{N^4} = \begin{cases} -\frac{1+e^{i(k_4(m_2-n_1)-k_2(m_2-n_1))}}{N^4}, & k_2\neq k_4\\ \frac{N-w}{N^4}\approx\frac{1}{N^3}, & k_2 = k_4. \end{cases} \tag{S.23}$$

For the case of $k_2 = k_4$, the summation is proportional to the number of elements in the set $\{n_1\}\cup(\{0_i\}\setminus\{m_2\})$ which is $N-w\approx N$. Hence, dominant poles, corresponding to $k_2 = k_4$ are much stronger (on the order of N) than the other poles, due to the N^{-3} factor. Note that the second term in the entanglement measure (S.18) does not have dominant poles because of the absence of a similar complete summation. In what follows, we will only consider the effects of the dominant poles, which characterize the early entanglement dynamics in the $N \to \infty$ regime. Consequently, by incorporating the contribution from dominant poles, entanglement dynamics can be effectively approximated as follows:

$$\sum_{\substack{k_1,k_3 \\ m_1,m_2\in\{0_i\} \\ n_1\in\mathcal{M}}} (s+\frac{i}{\hbar}(\epsilon_{k_1}-\epsilon_{k_3}))^{-1}\frac{e^{i(k_1-k_3)(m_1-n_1))}}{N^2}. \tag{S.24}$$

In this equation, the denominator is N^2, instead of N^4, for two reasons: First since $N - w \approx N$ in the numerator in (S.23), and second, the summation over $k_2 = k_4$ in (S.20). We will continue the analysis in the time domain and accordingly denote the dynamical entanglement measure by $\mathcal{Q}_\mathcal{M}(t) = \mathcal{L}^{-1}\{\tilde{\mathcal{Q}}_\mathcal{M}(s)\}$. For this purpose, it is convenient to use the symmetrized (in the time domain around $t = 0$) version of the entanglement measure, which is $\frac{1}{2}(\mathcal{Q}_\mathcal{M}(t)+\mathcal{Q}_\mathcal{M}(-t))$. This symmetrization is valid because we are only concerned about $t > 0$, and entanglement dynamics is invariant when $J \to -J$. Therefore, the time-domain equivalent of (S.24) is

$$\mathcal{Q}_\mathcal{M}(t) = \frac{1}{N^2}\sum_{\substack{m\in\{0_i\} \\ n\in\mathcal{M} \\ k_1,k_3}} \cos\big((\epsilon_{k_1}-\epsilon_{k_3})t\big)e^{i(k_1-k_3)(m-n)}. \tag{S.25}$$

Thus, upon finding the derivatives at $t = 0$, one can form the Taylor expansion of $\mathcal{Q}_\mathcal{M}(t) = \sum_{r=0}\alpha_r t^{2r}$ as follows (Note that the odd derivatives are equal to zero),

$$\mathcal{Q}_\mathcal{M}(t) = \frac{1}{N^2}\sum_{\substack{r=0 \\ k_1,k_3 \\ m\in\{0_i\},n\in\mathcal{M}}} (-1)^r(\epsilon_{k_1}-\epsilon_{k_3})^{2r}e^{i(k_1-k_3)(m-n)}\frac{t^{2r}}{(2r)!}. \tag{S.26}$$

B. Destructive interference as cancellation of terms in the QCTF entanglement measure

In this subsection, we show how the destructive interference occurs through cancellation of terms in the coefficients of the entanglement measure (S.26). Given this measure, the coefficients α_r in the expansion $\mathcal{Q}_\mathcal{M}(t) = \sum_{r=0}\alpha_r t^{2r}$ are given by (using (S.7)):

$$\begin{aligned}\alpha_r =& \frac{(-1)^r}{N^2(2r)!}\sum_{m\in\{0_i\},n\in\mathcal{M}}\sum_{k_1,k_3}(\tilde{\mathbf{J}}(k_3)-\tilde{\mathbf{J}}(k_1))^{2r}e^{i(k_1-k_3)(m-n)} \\ =& \frac{(-1)^r}{N^2(2r)!}\sum_{\substack{m\in\{0_i\},n\in\mathcal{M} \\ r_1+r_3=2r}}\binom{2r}{r_1}\sum_{k_1,k_3}\tilde{\mathbf{J}}(k_3)^{r_3}(-\tilde{\mathbf{J}}(k_1))^{r_1}e^{i(k_1-k_3)(m-n)}.\end{aligned} \tag{S.27}$$

The inner summations (over k_1, k_3) give the inverse Fourier transforms of powers of $\tilde{\mathbf{J}}_{k_3}$ and $\tilde{\mathbf{J}}_{k_1}$, which, in the lattice domain correspond to (respectively r_3 and r_1 times) consecutive convolutions of $\mathbf{J}_n$, evaluated at $m - n$. This statement is a consequence of the following property of the inverse Fourier transform Wong (2011) (we have used the fact that $\mathbf{J}(n)$ is an even function.)

$$\frac{1}{N}\sum_k \tilde{\mathbf{J}}(k)^r e^{\pm ikm} = \underbrace{\mathbf{J}*\mathbf{J}*\cdots*\mathbf{J}}_{r \text{ times}}\Big|_m \doteq \mathbf{J}^{*r}(m). \tag{S.28}$$

Therefore α_r is

$$\frac{-1^r}{(2r)!}\sum_{\substack{r_1+r_3=2r \\ m\in\{0_i\},n\in\mathcal{M}}}\binom{2r}{r_1}(-1)^{r_1}\mathbf{J}^{*r_1}(m-n)\mathbf{J}^{*r_3}(m-n). \tag{S.29}$$

We will return to this equation after simplifying the terms involving the convolutions. For $p > 1$ the convolution is convergent (see Section III E for the proof). In this region, using the basic properties of the convolution operator, and given the definition of $\mathbf{J}(n)$ in (S.2), we have:

$$\mathbf{J}^{*r}(m) = \sum_{\substack{d_1,\cdots,d_r \\ \sum d_i = m, d_i \neq 0}} \frac{J^r}{\prod |d_i|^p}. \tag{S.30}$$

Note that although the d_i's can be negative numbers, it is evident that the main contribution comes from the positive d_i's. (For each negative d_i, other positive d_i's must compensate to keep the total summation equal to m, therefore, larger numbers appear in the denominator). We expand the summation (S.30) (when $r > 1$ and $p > 1$) as

$$\frac{\mathbf{J}_n{}^{*r}(m)}{J^r} = \frac{\binom{r}{1}}{(m-r+1)^p} + \frac{\binom{r}{2}}{\big(2(m-r)\big)^p} + \cdots + \frac{\binom{r}{l}}{g(m,r,l)^p}, + \cdots . \tag{S.31}$$

Here the first term corresponds to the case where all $d_i = 1$, except for one of them being equal to $m - r + 1$, the second term corresponds to one d_i being $m - r$, another equal to 2, and the rest being 1, and so on. Each term in this series corresponds to the case where l of the d_is are not equal to 1 (and the remaining d_is are equal to 1), such that the criteria on the d_is are satisfied (see (S.31)). The contribution from such a term is $\frac{f(m,r,l)}{\big(g(m,r,l)\big)^p}$ where $g(m,r,l)$ is the product of d_is, and $f(m,r,l)$ counts the number of possible combinations of d_is. For $p \gg 1$, these terms approach 0 very fast, because the denominator is the pth power of a number that grows polynomialy with l. Therefore, as p grows, the first term progressively dominates the series. In the next section, we identify the class of interaction functions ($\mathbf{J}(n)$) for which this property holds.

We now return to equation (S.29), and consider the case for $2r \ll \min(|m-n|)$, which corresponds to *transient* entanglement growth. In this regime, given that the first term in the series (S.31) dominates for $p \gg 1$, let us start by approximating the convolution terms in (S.29) only using the first term in their corresponding expansion (S.31). Therefore, (S.29) becomes

$$\alpha_r \approx \frac{(-J)^r}{(2r)!} \sum_{\substack{m\in\{0_i\}, n\in\mathcal{M}, \\ r_1+r_3=2r \\ r_1\geq 1, r_3\geq 1}} \binom{2r}{r_1}\binom{r_1}{1}\binom{r_3}{1}(-1)^{r_1}\big((m-n-r_1+1)(m-n-r_3+1)\big)^{-p}. \tag{S.32}$$

Given the early entanglement assumption, $r_1, r_3 \leq 2r \ll |m-n|$, for all possible m's and n's, let us use the following expansion:

$$\begin{aligned}\big((m-n-r_1+1)(m-n-r_3+1)\big)^{-p} &= (m-n)^{-2p}\Big(1 + p\frac{r_1-1}{m-n} + \mathcal{O}\big(p(p+1)(\frac{r_1-1}{m-n})^2\big)\Big) \\ &\quad \Big(1 + p\frac{r_3-1}{m-n} + \mathcal{O}\big(p(p+1)(\frac{r_3-1}{m-n})^2\big)\Big) \\ &= (m-n)^{-2p}(1 + p\frac{2r-2}{m-n} + p^2\frac{r_1-1}{m-n}\frac{r_3-1}{m-n} + \mathcal{O}\big(p^2(\frac{r-1}{m-n})^2\big)).\end{aligned} \tag{S.33}$$

It is easy to see that the first three terms in this expansion lead to vanishing α_r. Here, the contribution of the first two terms is given by

$$\begin{aligned}\alpha_r &\approx \frac{(-J)^r}{(2r)!} \sum_{m\in\{0_i\}, n\in\mathcal{M}} (m-n)^{-2p}(1 + p\frac{2r-2}{m-n}) \sum_{\substack{r_1+r_3=2r \\ r_1\geq 1, r_3\geq 1}} \binom{2r}{r_1}\binom{r_1}{1}\binom{r_3}{1}(-1)^{r_1} \\ &\propto \sum_{\substack{r_1'+r_3'=2r-2 \\ r_1', r_3'\geq 0}} \binom{2r-2}{r_1'}(-1)^{r_1'} = 0,\end{aligned} \tag{S.34}$$

where in the last line, we have changed the variables, $r_1' = r_1 - 1$, $r_3' = r_3 - 1$. Similarly, the summation corresponding to the term $p^2 \frac{r_1-1}{m-n}\frac{r_3-1}{m-n}$ (the third term in (S.31)) vanishes, since

$$\sum_{\substack{r_1+r_3=2r \\ r_1\geq 1, r_3\geq 1}} \binom{2r}{r_1}\binom{r_1}{1}\binom{r_3}{1}(r_1-1)(r_3-1)(-1)^{r_1} \propto \sum_{\substack{r_1'+r_3'=2r-4 \\ r_1',r_3'\geq 0}} \binom{2r-4}{r_1'}(-1)^{r_1'} = 0. \quad \text{(S.35)}$$

The fact that the summations (S.34-S.35) are zero demonstrates the main objective of the paper, namely, the destructive interference of quasi-particles in the early entanglement growth regime.

So far we have demonstrated the particle hopping cancellations corresponding to the first (dominant) term in the expansion of $\mathbf{J}^{\star r}$ in (S.31). Let us take a step further by showing that the quasi-particle hopping cancellation happens in other terms as long as the dependence of $g(m,r,l)$ on r is weak. These terms are negligibly small in the $p \gg 1$ region, as discussed earlier. In addition to their insignificance, similar particle hopping cancellations can happen as well. If the dependence of $g(m,r,l)$ on r is weak, or when $g(m,r,l) \approx \hat{g}(m,l)$, upon replacing $\mathbf{J}_n{}^{*r_1}(m-n)$ and $\mathbf{J}_n{}^{*r_3}(m-n)$ by their corresponding expansions (S.31), a series of pairs of the form $\frac{\binom{r_1}{l_1}}{g_1(m-n,r_1,l_1)}\frac{\binom{r_3}{l_3}}{g_1(m-n,r_3,l_3)}$ will be produced. For each of these pairs (corresponding to the l_1th and l_3th terms in the expansion (S.31)), we have the following inner summation for α_r, according to (S.32)

$$\begin{aligned}
&\frac{(-J)^r}{(2r)!}\sum_{m\in\{0_i\},n\in\mathcal{M}}\ \sum_{\substack{r_1+r_3=2r \\ r_1\geq l_1, r_3\geq l_3}} \binom{2r}{r_1}\binom{r_1}{l_1}\binom{r_3}{l_3}(-1)^{r_1}(g_1(m-n,r_1,l_1)g_3(m-n,r_3,l_3))^{-p} \\
&\approx \frac{(-J)^r}{(2r)!}\sum_{m\in\{0_i\},n\in\mathcal{M}} (\hat{g}_1(m-n,l_1)\hat{g}_3(m-n,l_3))^{-p} \sum_{\substack{r_1+r_3=2r \\ r_1\geq l_1, r_3\geq l_3}} \binom{2r}{r_1}\binom{r_1}{l_1}\binom{r_3}{l_3}(-1)^{r_1} = 0.
\end{aligned} \quad \text{(S.36)}$$

Therefore, the quasi-particle hopping cancellation happens at higher order approximations in (S.31), as long as the approximation $g(m,r,l) \approx \hat{g}(m,l)$ holds. This criterion is further examined in the next Section. We emphasize that, these higher-order approximation are highly suppressed compared to the first term in this series, when one takes the early entanglement approximation.

C. Scaling of α_r in the Theorem of the main text

Cancellations similar to (S.34) occur for higher order terms in (S.33). A general higher order term in this expansion is proportional to $(\frac{r_1-1}{m-n})^{s_1}(\frac{r_3-1}{m-n})^{s_3}$. The contribution of these terms to α_r is proportional to:

$$\sum_{\substack{r_1+r_3=2r \\ r_1\geq 1, r_3\geq 1}} \binom{2r}{r_1}\binom{r_1}{1}\binom{r_3}{1}(-1)^{r_1}(r_1-1)^{s_1}(r_3-1)^{s_3}. \quad \text{(S.37)}$$

When $s_1 \gg 1$, in the $r_1 - 1 < m-n$ regime, the terms $(\frac{r_1-1}{m-n})^{s_1}$ and $(\frac{r_3-1}{m-n})^{s_3}$ are exponentially suppressed. Also, we already showed that for $s_1, s_2 = 0, 1$ the destructive interference mechanism is complete (see S.34-S.35). What remains is $s_1, s_3 = 2, 3, \cdots$, and $1 \ll r_1, r_3 < m-n$ (this later region is where the strongest contributions exist). In this regime we can write $(r_1-1)^{s_1} \approx \frac{(r_1-1)!}{(r_1-1-s_1)!}$. Therefore, these contributions to α_r are proportional to (when $r_1 - 1 \geq s_1$ and $r_3 - 1 \geq s_3$)

$$\sum_{\substack{r_1+r_3=2r \\ r_1-1\geq s_1, r_3-1\geq s_3}} \binom{2r}{r_1}\binom{r_1}{1}\binom{r_3}{1}(-1)^{r_1}(r_1-1)^{s_1}(r_3-1)^{s_3}$$
$$\approx \sum_{\substack{r_1+r_3=2r \\ r_1-1\geq s_1, r_3-1\geq s_3}} \binom{2r}{r_1}\binom{r_1}{1}\binom{r_3}{1}(-1)^{r_1}\frac{(r_1-1)!}{(r_1-1-s_1)!}\frac{(r_3-1)!}{(r_3-1-s_3)!}. \tag{S.38}$$

Now, with the change of variables $r'_1 = r_1 - 1 - s_1$ and $r'_3 = r_3 - 1 - s_3$, this equation can be rewritten as

$$\propto \sum_{\substack{r'_1+r'_3=2r-s_1-s_3-2 \\ r'_1\geq 0, r'_3\geq 0}} \binom{2r}{r'_1}(-1)^{r'_1} = 0. \tag{S.39}$$

This equation shows the perfect cancellation amongst contributions from higher order terms in the expansion (S.33). However, here we used the approximation that the summations in (S.38-S.39) are over $r_1 \geq s_1 + 1$ and $r_3 \geq s_3 + 1$, therefore the contribution from the complementary terms is not included. This complimentary set of indices include $1 < r_1 \leq s_1$ and $1 < r_3 \leq s_3$. Given the analysis above (namely that the effects are exponentially weak in s_1, s_3), the strongest contributions correspond to $r_1 = s_1 = 2$ and $r_3 = s_3 = 2$. Using this fact, and based on (S.32- S.33), then α_r must be of the following order of magnitude

$$\alpha_r \sim O\big(\frac{(-J)^4}{(2r)!}\sum_{m\in\{0_i\}, n\in\mathcal{M}} \binom{2r}{2} 2\times 2(m-n)^{-2p-4}(p(p+1))^2\big). \tag{S.40}$$

We define $q \doteq \min|m-n|$ for $m \in \{0_i\}$ and $n \in \mathcal{M}$. Then in the early entanglement regime

$$\alpha_r \sim O(\frac{(-J)^4 p^4}{(2r-2)!q^{-2p-4}}|\mathcal{M}|). \tag{S.41}$$

When $\mathcal{M}$ includes one spin, $|\mathcal{M}| = 1$, and $J = 1$, this equation gives the scaling in the Theorem in the main text.

D. Explanation of $p > 2$ in the Theorem of the main text

Equations (S.34-S.36) prove the main finding in this paper, which is the fact that in the early entanglement regime quasi-particles cancel through destructive interference despite the long-range interactions among spins.

In the last part of this proof, we now show that for $p > 2$, the Type I series of hoppings dominate the expansion (S.31). As discussed earlier, the general form of terms in this expansion is $\frac{f(m,r,l)}{\big(g(m,r,l)\big)^p}$ where $g(m,r,l)$ is the product of d_is, the numerator counts the possible combinations of $\{d_1, \cdots, d_r\}$. Therefore, for large p, the denominator of this fraction grows factorially (to the power p) fast for Type 2 hoppings. (We recall that Type 2 includes hoppings of mostly order r, as defined in the main text). Given this, we identify p^* such that for $p > p*$, the numerator of the general term $\frac{f(m,r,l)}{\big(g(m,r,l)\big)^p}$ grow slower than its denominator. For this to happen, notice that the most significant contribution from terms of Type 2 comes from the term with $d_1 = 2, d_2 = 2, \cdots, d_l = 2, d_{l+1} = 1, d_{l+2} = 1, \cdots d_{r-1} = 1, d_r = q - l - r + 1$. This combination ensures maximal combinatorial scaling for the numerators, and minimal scaling in the denominator (by minimizing the product of d_i's). Including all similar combinations of d_i's, these terms show up as

$$\frac{\binom{r}{l+1}(l+1)}{(2^l(q-l-r+1))^p}. \tag{S.42}$$

We will analyze the features of this term, in particular the scaling of the numerator versus the denominator of this fraction as a function of p. The fastest rate for the growth of the numerator is when $l \approx r/2$, namely when about half of the d_i's equal 2. As a result, we should consider the scaling of

$$\binom{r}{r/2} \text{ versus } 2^{lp}(q - \frac{3l}{2} + 1)^p. \tag{S.43}$$

Using Stirling's approximation $r! \approx \sqrt{2\pi r}(\frac{r}{2})^r$, the binomial term in (S.43) is approximately $\frac{1}{\sqrt{2\pi r}}2^r \sim 2^{r-\frac{1}{2}\log_2 r}$. On the other hand, since $l \sim r/2$, the second term in (S.43) scales as $2^{rp/2}$. Therefore, for the denominator of (S.42) to grow faster than its numerator (when $r \gg 1$), we require $r < rp/2$ or $p > 2$.

E. Convergence of the high-order convolution expansion for $\mathbf{J}^{*r}(q)$ in (S.31)

Here, we show that for $p > 1$, the higher order convolution $\mathbf{J}^{*r}(q)$ exists and is bounded, therefore expansion (S.31) must be convergent. To prove the boundedness of the high-order convolution, we will use the following theorem (refer to Bogachev and Ruas (2007) for the proof):
If $f, g \in \mathcal{L}_1(\mathbb{R})$, then $f * g \in \mathcal{L}_1(\mathbb{R})$. Here, $\mathcal{L}_1$ denotes the first $\mathcal{L}_p$, or Lebesgue, space. Thus, the space $\mathcal{L}_1(\mathbb{R})$ is closed with respect to the convolution operation. This implies that, when $\mathbf{J}(n) \in \mathcal{L}_1(\mathbb{R})$, then all the higher orders of convolutions of this function must be bounded too, for finite r. Given that for $p > 1$ the interaction functions $\mathbf{J}(n)$ are members of $\mathcal{L}_1(\mathbb{R})$, then the convolution exists and is bounded.

IV. NECESSARY CONDITION FOR THE DESTRUCTIVE INTERFERENCE PHENOMENON

Here, we identify the class of interactions that allow for the destructive interference effect. As described in the main text, as well as in the proof of the destructive interference phenomenon, the weak dependence of the denominators in (S.31) on r results in the separation of terms in (S.36), which leads to $\alpha_r \approx 0$ in (S.35-S.36). Also, as explained earlier, this weak dependence is a result of the dominance of certain groups of quasi-particle hoppings, namely one long jump accompanied by a series of shorter jumps, in the high-order convolution (S.30). Here, we will examine the criterion required for these type of jumps to dominate the high-order convolution $\mathbf{J}^{*r}(m)$. For this purpose, note that for a generic interaction function, i.e., $\mathbf{J}(n) = \mathbf{J}(-n)$, one can rewrite their high-order convolution as a summation over all possible combinations as:

$$\mathbf{J}^{*r}(m) = \sum_{\sum d_i = m} \prod_{i=1}^{r} \mathbf{J}(d_i). \tag{S.44}$$

Therefore, the criterion mentioned above requires that (for large r) $\mathbf{J}(1)\mathbf{J}(r-1) > \mathbf{J}(2)\mathbf{J}(r-2) > \mathbf{J}(3)\mathbf{J}(r-3), \cdots$. Thus, this criterion can be examined by comparing $\mathbf{J}(m)\mathbf{J}(n)$ and $\mathbf{J}(m-\delta)\mathbf{J}(n+\delta)$, when $m - n > 2\delta > 0$. Considering n as a continuous variable, this behavior is determined by the sign of the infinitesimal variation

$$\frac{d}{d\delta^+}\mathbf{J}(n-\delta)\mathbf{J}(m+\delta), \tag{S.45}$$

which is equal to (up to a first order variation in δ)

$$\delta(\mathbf{J}(m)\mathbf{J}'(n) - \mathbf{J}(n)\mathbf{J}'(m)), \tag{S.46}$$

where $'$ denotes the derivative with respect to n. Hence, for monotonic functions (i.e., $\mathbf{J}'(n)\mathbf{J}'(m) > 0$), the sign of (45) is given by

$$(\mathbf{J}'(n))^2 - \mathbf{J}(n)\mathbf{J}''(n) > 0; \forall n. \tag{S.47}$$

This quantity determines whether the aforementioned series of jumps *can* dominate the high-order convolution of the interaction function $\mathbf{J}(n)$, or not.

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