BV Formalism and Partition Functions

P. A. Grassi [a,b,c,1] and O. Hulik [d,2],

[(a)] *DiSIT, Università del Piemonte Orientale, viale T. Michel, 11, 15121 Alessandria, Italy*
[(b)] *Theoretical Physics Department, CERN, 1211 Geneva 23, Switzerland*
[(c)] *INFN, Sez. Torino, via P. Giuria, 1, 10125, Torino, Italy*
[(d)] *Institute for Mathematics Ruprecht-Karls-Universitat Heidelberg, 69120 Heidelberg, Germany*

Abstract

The BV formalism is a well-established method for analyzing symmetries and quantizing field theories. In this paper, we use BV formalism to derive partition functions and the space of gauge invariant operators implementing the equations of motions and their redundancies for selected theories. We discuss various interpretations of the results, some dualities, and relation to the first quantized models.

[1]pietro.grassi@uniupo.it
[2]ondra.hulik@gmail.com

1 Introduction

Some time ago, interesting analyses were put forth by A. Connes and M. Dubois-Violette and collaborators in a series of mathematical papers [1, 2, 3]. They provided a "background independent" analysis of interacting gauge theories based on the so-called "Yang-Mills" algebras, considering the Euler-Lagrangian equations and their redundancies. In [1, 2, 3] the complete *partition function* [1] for Yang-Mills and self-dual Yang-Mills are obtained using the algebraic techniques developed therein. Those works have been further analyzed in a series of papers [5, 6, 7, 8, 9] [2]and extended to the supersymmetric cases. Moreover, Nekrasov pointed out how homogeneous algebras emerge as limits of algebras appearing in the gauge theory of D-branes and open string theory.

Given such the partition function for YM and self-dual YM, one can easily construct all possible gauge invariant operators using the standard techniques as illustrated by A. Hanany *et al.* [11, 12, 13, 14, 15, 16]. There, a long list of applications and computations of partition functions has been derived and discussed. Recently, a surge of interest emerged in the computation of the space of gauge invariant operators in Effective Field theories [17, 18, 19, 20, 21, 22] all of which are based on the Hilbert series techniques listing all operators with multiple fields and their derivatives.

In the present work, we start from a different perspective. We notice that the partition function described in the quoted papers can be understood from a different point of view, namely from BV formalism for gauge theories [23, 24]. The BV formalism is useful for studying the complete structure based on symmetries and equations of motions of a given quantum field theory model. It provides a compact, efficient, and elegant way to deal with gauge symmetries, rigid symmetries, open algebras, and field-dependent symmetries with equations of motion. To each physical field of theory and for each ghost or ghost-for-ghost there is a corresponding antifield (in the forthcoming sections, we will show that it is not necessarily true, see also [25, 26]), then the gauge symmetries and the equations of motion, together with their redundancies, are translated into the BV-BRST symmetry for the fields and antifields, and everything is nicely encoded into an action (BV action), a symplectic form or, and into an anti-bracket.

The quantum numbers of the antifields are fixed in terms of the corresponding fields, and the BV-BRST variation of the antifield implements at the cohomological level the equations of motion and their redundancies (see [24]). Moreover, we assign suitable charges (scale) to each field and each ghost field, fix the charges of the antifields using the BV-BRST symmetry, and derive the partition function by counting the independent degrees of freedom. Perform-

[1] In the present work, we use the term partition function to denote what, mathematically more precisely, is known as *Hilbert-Poincaré series* (see [4]) where different degrees of freedom are counted with different powers of a fugacity t.

[2] For a nice review of index techniques, see [10].

ing such a counting for Yang-Mills (or the Maxwell for simplicity), one immediately sees the correspondence with the partition function computed by the homogeneous algebra in [1].

Note that the antifields are not quantum propagating fields, and therefore, the computation of the partition function listed in the forthcoming section cannot be obtained by quantum field theory path integral but rather by particle/superparticle models or even superstrings. In that respect, we quote the recent work [27] where N=2 superparticle in D=4, where the free model allows us to construct the partition function explicitly revealing indeed that it was describing, in a fixed sector of R-symmetry, Yang-Mills theory with the complete BV structure. Older works on the subject are based on pure spinors [4, 28] where the partition function is discussed and constructed from the algebraic analysis of the pure spinor spaces. When a worldline/worldsheet computation is possible, we find agreement with the computation in [1].

Once we establish such a procedure to determine the partition function, we have a powerful instrument to compute it for several models. In addition, in those cases where self-dual p-forms are present – for which no BV-BRST formalism is known – we can be reversed engineered from the partition function, discovering the BV spectrum of fields and antifields and their BV-BRST transformation rules. A very interesting consequence in those cases is that the BV formalism does not have the canonical structure: the pairing between fields and antifield is missing, the symplectic structure is missing, and the BV bracket is finally absent. Nonetheless, this is perfectly consistent since the antifield of the gauge field, in the case of Maxwell/Yang-Mills, is needed to implement the corresponding equations of motion, but those are already imposed by the self-duality condition. Therefore, a part of the antifield spectrum is unnecessary. This original observation has guided us to construct several examples of partition functions with self-dual p-forms to check their on-shell degrees of freedom and to obtain the set of gauge invariant operators. On the other hand, we discover a new antifield implementing self-duality at the level of BV-BRST cohomology. At the moment, we have not explored the consequence of this new BV spectrum, but this is certainly consistent with the common lore on the absence of a covariant, polynomial, finite-number of fields action for self-dual forms (see [29, 30]).

In order to check that the proposed partition function correctly describes the proposed models, we provide a second interpretation leading to the identification of the physical degrees of freedom. This is verified for each model described in the paper. Finally, where this construction is valid, under which assumptions work in an asymptotic free theory, we are neglecting the quantum corrections and even the interactions [31].

In sec. 2, we review the partition function construction's main ingredients through BV correspondence. In sec. 3, we study several examples of bosonic models with/without self-dual p-forms. It is shown how to identify the physical degrees of freedom and compute the gauge invariant operators using the plethystic polynomials. In addition, it is shown that in the presence of a non-minimal sector of the BV formalism, the partition function does not change.

In subsection. 3.2 The self-dual Maxwell/YM is studied, and the reverse engineering of the BV formalism is obtained. In sec. 3.4 also, Einstein's gravity is taken into account. In sec. 4, supersymmetric models are studied along the same lines. Also, the supersymmetric self-dual cases are considered, providing both the partition function and the BV spectrum. In the case of supersymmetric models, the auxiliary fields are taken into account, and how they emerge in our framework is shown. In sec. 5, the duality (BV duality) is shown; finally, in sec.6, a CFT perspective is provided. Some appendices conclude the paper.

2 From BV-BRST Formalism to Partition Functions

In this section, we outline the main technique of this paper and recall basic facts about BV formalism. We denote by ϕ_A a set of fields, and the multi-index A stands for any quantum number needed to identify the field spanning a linear representation of the symmetry group. In addition, the BV-BRST symmetry for ϕ_A is parametrized by ghost $C_{A'}$ (again, we denote by A' the set of quantum numbers for the ghost fields). The BV-BRST transformation of the ghosts is parametrized by ghost-for-ghost $C_{A''}$ (again, we denote by A'' the set of quantum numbers for the ghost-for-ghost fields). Here, we provide only a very schematically description; for further details, we refer to the literature [23, 24]

$$s\,\phi_A = \mathcal{F}_A(C_B, \phi_A)\,, \quad s\,C_{A'} = \mathcal{G}_{A'}(C_{B''}, C_{A'}, \phi_A)\,, \quad s\,C_{A''} = \mathcal{H}_{A''}(C_{B'''}, C_{B''}, C_{A'}, \phi_A)\,, \quad \dots \tag{2.1}$$

where $\mathcal{F}_A, \mathcal{G}_{A'} \dots$ are functional-differential operators describing rigid or gauge symmetries. The closure of the algebra for the algebra is translated into the nilpontency of the BV-BRST differential s and follows from the Jabobi identities, equations of motion, and their relationships. Contextually to each field $\phi_A, C_{A'}, C_{A''}, \dots$ one introduces a corresponding antifield $\phi^{\star A}, C^{\star A'}, C^{\star A''}, \dots$ and they are paired by the symplectic form

$$\Omega = \int d^dx \left(\delta\phi_A \delta\phi^{\star A} + \delta C_{A'} \delta C^{\star A'} + \delta C_{A''} \delta C^{\star A''} + \dots\right) \tag{2.2}$$

Notice that the antifields carry both a ghost number such that the symplectic form has a global negative ghost number -1 and the antifield number $+1$. The integration in (2.2) is on the worldvolume of the theory. In terms of the symplectic form one can establish the antibracket defined as follows

$$(F, G) = \int d^dx \left(\frac{\partial F}{\partial \phi_A}\frac{\partial G}{\partial \phi^{\star A}} + \frac{\partial F}{\partial C_{A'}}\frac{\partial G}{\partial C^{\star A'}} + \dots - F \rightleftharpoons G\right) \tag{2.3}$$

and there exists a master action S, satisfying the classical BV master equations $(S,S)=0$.[3] The variation of the fields and the antifield are obtained by the linearized version of the antibracket as follows

$$\begin{aligned} s\phi_A &= (S,\phi_A)\,, \qquad sC_{A'}=(S,C_{A'})\,, \qquad sC_{A''}=(S,C_{A''})\,,\ldots \\ s\phi^{\star A} &= (S,\phi^{\star A})=\frac{\delta S}{\delta\phi_A}\,, \qquad sC^{\star A'}=(S,C^{\star A'})\,, \qquad sC^{\star A''}=(S,C^{\star A'})\,,\ldots \end{aligned} \tag{2.4}$$

where it is easy to observe that the variation of an antifield contains the equations of motion of the corresponding field and some covariantization terms. Therefore, as discussed in textbooks [23, 24], the equations of motion are implemented at the level of the BV-BRST cohomology under the name of Koszul-Tate differential. The nilpotency of the BV-BRST symmetry extended to the antifield implements the field equations and the relations among them. We must mention that open algebra implies quadratic or higher powers of antifields ϕ^* in the action. The symplectic structure has a twofold effect: firstly, if we assign a given charge to a field, the antifield necessarily carries a compatible charge, and the second effect is the field-antifield duality, which is manifest in the partition function (see also [32]).

The easiest way to assign a charge (which seems to work for several models) is by choosing the engineering dimension of the fields d_ϕ. The corresponding antifield will have the complementary dimension $d_{\phi^*}=d-d_\phi$, where d is the dimension of the spacetime. The same works for ghosts and ghost-for-ghosts. Note that the symplectic form respects such assignment because the variation of the fields and antifields are integrated over the spacetime. In addition, the spacetime derivative ∂_μ carries charge $+1$ with this simple assignment. Therefore, we have a simple way to build the partition function as follows

$$\mathbb{P}(t)=\frac{\sum_X(-1)^{\epsilon_X}\dim[X]t^{d_X}}{(1-t)^d} \tag{2.5}$$

where the sum is extended to $X=\phi,C,C',\ldots,\phi^*,C^*,C^{*'}\ldots$ to all fields and antifields of the theory. $\dim[X]$ is the dimension of the linear space spanned by the field X (for example, if $X=\phi_A$, $\dim[\phi_A]$ counts the dimension of the vector space spanned by the quantum number A of the fields ϕ_A. The powers t^{d_X} take care of the scale of each field, and $(-1)^{\epsilon_X}$ takes care of the parity of the fields. We recall that the BV-BRST differential is an antiderivative, and therefore at each generation, fields ϕ_A, ghost C_A, ghost-for-ghost $C_{A'}$ the parity flips. This is important since the partition function takes into account the parity of the set of fields (like in the Euler character in differential geometry). The denominator $(1-t)^d$ takes into account the derivatives of the gauge invariant operators.

The BV formalism is homological in its nature and, as such, highly nonunique. This means

[3]There is a quantum version of it, but we are not considering this here.

that two off-shell inequivalent formulations can have the same cohomology and, therefore, the same partition function. Even though in practice it might be hard to identify if two complexes are the same (without calculating the cohomology) upon field redefinition, they can only differ by the addition of Koszul pairs[4] $\Phi\,,\eta\,,\eta^*\,,\Phi^*$ with the BV-BRST trasformations

$$s\Phi = \eta \qquad s\eta = 0\,, \qquad s\Phi^* = 0 \qquad s\,\eta^* = \Phi^* \tag{2.6}$$

from where one can see that the scale of Φ, η must be equal, but the cohomological degree is opposite. So, the contributions in the partition function cancel. This will be shown explicitly in (3.9).

3 Non-Supersymmetric Models

3.1 Scalar Fields

As a warm-up exercise, we recall the partition function for $D = 4$ free scalar field

$$\mathbb{P}_\phi(t) = \frac{(t - t^3)}{(1 - t)^4} \tag{3.1}$$

where the numerator has the interpretation of in terms of the BV antifield formalism as the scalar ϕ and its antifield ϕ^*. We assign +1 dimension to the scalar fields and dimension +3 to the antifield ϕ^*. The difference between the signs is due to the opposite statistics of the field and antifields. Therefore by expanding in t the series (3.2) we have

$$\mathbb{P}_\phi(t) = t + 4t^2 + 9t^3 + 16t^4 + \mathcal{O}(t^5) \tag{3.2}$$

representing the operators $\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi - \frac{1}{4}\eta_{\mu\nu}\partial^2\phi, \partial_\mu\partial_\nu\partial_\rho\phi - \frac{1}{4}\eta_{(\mu\nu}\partial_{\rho)}\partial^2\phi\ldots$ modulo the equations of motion. We recall that the BV-BRST transformations are

$$s\,\phi = 0\,, \qquad s\,\phi^* = \partial^2\phi\,. \tag{3.3}$$

The partition function can be derived from a very simple worldline model invariant under reparametrization.

[4] Since we are working with BV formalism and each field has their antifield these are quadruplets of the fields.

3.2 Gauge Fields

We first recall the partition function for $d = 4$ Maxwell; we denote by t the scaling power associated with fields

$$\mathbb{P}_{MX}(t) = \frac{1 - 4t + 4t^3 - t^4}{(1-t)^4} = \frac{(1-t^2)}{(1-t)^4}(1 + t^2 - 4t) \tag{3.4}$$

There are several ways to compute this expression, but we refer to [1, 33] where the computation is based on the cubic algebra for YM.

The factor $(1 - t^2)$ denotes the on-shell condition, setting $t \to 1$ to put the states on-shell. The factor $(1 - t)^4$ denotes the derivative on the fields. Expanding to the first two orders that expression we have

$$\mathbb{P}_{MX}(t) = 1 - 6t^2 - 16t^3 + \mathcal{O}(t^3) \tag{3.5}$$

where the first terms correspond to the constants (present in the cohomology), the term $-6t^2$ denote the gauge invariant operators $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$, and the term $-16t^3$, correspond to the gauge invariant operators $\partial_\rho F_{\mu\nu}$, which are 24 independent gauge invariant fields. We have, however, to remove 4 because of Bianchi's identities and 4 because of the equations of motion, and we are left with 16 DOFs.

Factoring out the $(1 - t^2)/(1 - t)^4$, we are left with the polynomial $1 + t^2 - 4t$ and in the limit, $t \to 1$ gives -2, which are the two degrees of freedom of the photon. Notice that $1 + t^2$ have a ghost/antighost interpretation, while $-4t$ are the four gauge boson DOFs.

Note that there is a dual interpretation of the first expression (3.4), the polynomial $1-4t+4t^3 - t^4$, can be read as counting the DOFs of the ghost field C, of the gauge field A_μ, of the antifield of the gauge field $A^*_{\mu\nu\rho}$ (which is a 3-form) and finally of the antifield of the ghost field C^* (which is a 4-form). The scale t corresponds to the form degree in the present case. The BRST transformations relating to the various fields are

$$sC = 0\,, \qquad sA_\mu = \partial_\mu C\,, \qquad sA^*_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma}\partial_\tau F^{\sigma\tau}\,, \qquad sC^*_{\mu\nu\rho\sigma} = \partial_{[\mu}A^*_{\nu\rho\sigma]}\,. \tag{3.6}$$

It is convenient to define $A^{*,\mu} = \epsilon_{\mu\nu\rho\sigma}A^*_{\nu\rho\sigma}$ and $C^* = \epsilon_{\mu\nu\rho\sigma}C^*_{\mu\nu\rho\sigma}$ to write the BRST symmetry as

$$sC = 0\,, \qquad sA_\mu = \partial_\mu C\,, \qquad sA^{*,\mu} = \partial_\tau F^{\mu\tau}\,, \qquad sC^* = \partial_\mu A^{*,\mu}\,. \tag{3.7}$$

The cohomology of this differential operator is captured by the partition function (3.4). Notice that in the cohomology, the antifield A^* is needed to impose the equations of motion for the gauge field (the dynamic is contained in that variation). The antifield C^* is needed in order to take into account the relations among the Maxwell equations. Finally, notice that using the potential A for expressing the field strength F in terms of it implies that Bianchi's identities are already solved.

In the paper, [27], a different BV-BRST formalism has been used. Namely, a non-minimal field ϕ (sometimes identified with the Nakanishi-Lautrup field) and its antifield ϕ^*. In particular, the BRST transformations are given by

$$\begin{aligned} & sC = 0\,, \qquad sA_\mu = \partial C\,, \qquad s\phi = \partial^2 C\,, \\ & sC^* = \partial^\mu A^*_\mu\,, \qquad sA^*_\mu = \partial^\nu F_{\mu\nu}\,, \qquad s\phi^* = -\partial^\mu A_\mu + \phi \end{aligned} \tag{3.8}$$

Adopting the charges as above, we see that the scale of ϕ and ϕ^* is the same and fixed by the BRST symmetry, and then we have

$$\mathbb{P}_{MX}(t) = \frac{1 - 4t + t^2 - t^2 + 4t^3 - t^4}{(1-t)^4} = \frac{(1-t^2)}{(1-t)^4}(1 - 4t + t^2) \tag{3.9}$$

the two central terms $t^2 - t^2$, associated with ϕ and ϕ^*, respectively cancel, giving the same expression as above. There is a small detail that is relevant: using the NL field ϕ, factor $(1-4t+t^2)$ emerges automatically, and this is because the addenda $1+t^2$ represents the ghost, and the antighost field. The latter becomes dynamic when the gauge fixing is added to the theory.

In BV formalism, adding a non-minimal sector to the theory to implement gauge fixing does not change the number of gauge invariant operators or the number of physical degrees of freedom. Indeed, adding the fields ϕ, ϕ^* does not alter the partition function, even though it is essential to perform Feynman diagram computations.

Another way to arrive at the (3.4) is from the world line model perspective discussed in [27] to which we refer for further details on the model. In that paper, the partition function has been constructed for the set of the fields $\{X, P, \psi, \gamma, \beta, c\}$ and their conjugates. The BRST symmetry of the model, implemented at the worldline level with the BRST charge Q, is translated at the target space theory by the equation

$$Q|\,\omega\rangle = s\,|\,\omega\rangle \tag{3.10}$$

where $|\omega\rangle$ are states in the Hilbert space whose coefficients are in the targetspace. (3.10) correctly encodes the BV fields and the BV-BRST symmetry discussed here.

The charges of the worldline fields are

Field	X^m	ψ^m	c	γ	β	P_m
r	0	1	0	1	1	0
s	0	0	+1	+1	–1	0
t	0	-1	2	0	-2	-1
	1	rt^{-1}	st^2	sr	$\frac{r}{st^2}$	t

the partition function then reads

$$\mathbb{P}_{N=2}(t,s) = \frac{1}{(1-t)^4}\frac{(1+\frac{r}{t})^4(1+st^2)}{(1-sr)(1-\frac{r}{st^2})} \tag{3.11}$$

Now, upon expanding this formula around $r=0$ to the second order and specializing $s=-1$ (namely by setting cohomological degree to its conventional value), we arrive at the same function as (3.4) up to prefactors $(r/t^2)^\alpha$

$$\begin{aligned}\mathbb{P}_{N=2}(t,s=-1) &= \frac{(1-t^2)}{(1-t)^4} - \left(\frac{r}{t^2}\right)\frac{(1-t^2)(t^2-4t+1)}{(1-t)^4} \\ &+ \left(\frac{r}{t^2}\right)^2\frac{(1-t^2)(t^4-4t^3+7t^2-4t+1)}{(1-t)^4} + O\Big(\left(\frac{r}{t^2}\right)^3\Big)\end{aligned} \tag{3.12}$$

Notice that the first term in (3.12) represents a scalar field φ and it antifields φ^*, the second term of the expansion to the Maxwell/YM sector, and finally, the third term corresponds to 2-form in $D=4$. It is also convenient to illustrate this case before discussing the self-dual cases.

3.2.1 2-Form in $D=4$

We illustrate this sector since the partition function computed in $N=2$ worldline (3.12) also describes a 2-form B. As is well-known, a 2-form in $D=4$ has only one DOF. This can be easily checked by observing that the field strength $H=dB$ can be dualized $\star H = d\varphi$. This is a consequence of the Poincaré lemma for flat $\mathbb{R}^{1,3}$, of the free equations of motion and Bianchi identities

$$d\star H = 0 \Longrightarrow \star H = d\varphi\,, \qquad dH = 0 \Longrightarrow d\star d\varphi = 0\,. \tag{3.13}$$

Solving the equations of motion by dualizing the field implies that the antifield B^* for the 2-forms is no longer needed, and as a consequence, also the ghost and the ghost-for-ghost, but there is the small remainder. Indeed, we can write the partition function in the following ways

$$
\begin{aligned}
(1-t^2)\frac{1-4t+7t^2-4t^3+t^4}{(1-t)^4} &= \frac{(1-t^2)}{(1-t)^4}\left((1+t^2+t^4)-4t(1+t^2)+6t^2\right) \\
&= (1-t^2)+\frac{(1-t^2)t^2}{(1-t)^4} \qquad (3.14)
\end{aligned}
$$

The second expression shows that DOF $(1+t^2+t^4)$ are the ghost-for-ghost, $-4t(1+t^2)$ are the ghost and the antighost and finally, $6t^2$ are the components of the 2-form in $D=4$. In the limit $t \to 1$, it gives one, which is the DOF of a scalar. The third expression indicates that the first term is the residual zero modes (no derivatives are acting on it since the usual denominator $(1-t^4)$ is missing) from the BV spectrum (the ghost-for-ghost zero mode and its antifield), while the second term stands for a scalar field (with an additional factor t^2, which is remnant of the partition function (3.12)) which is the dual field for the 2-form.

3.3 Self-dual Maxwell

In [33], the partition function of a self-dual model is discussed in some detail

$$
\mathbb{P}_{MX_+}(t) = \frac{1-4t+3t^2}{(1-t)^4} = \frac{(1-t^2)}{(1-t)^4}\left(1-\frac{4t}{1+t}\right) \qquad (3.15)
$$

where MX_+ stands for self-dual Maxwell. This partition function is computed in [1] based on a 2-homogeneous algebra directly related to the self-duality condition. In that paper, it is shown that due to the properties of the algebra, the coefficients in the polynomial in the numerator are fixed. In contrast, the numerator stands for the derivatives with gauge invariant operators.

The interpretation of the partition function is similar to the Yang-Mills case, factoring $(1-t^2)/(1-t)^4$, we are left with $(1-4t/(1+t))$ which in the limit gives -1 which is the single DOF of self-dual Maxwell (we are assuming Euclidean or $(2,2)$ signature). In the present case, one ghost is removed due to the self-duality condition, and the gauge DOFs are halved (the factor $1/(1+t)$).

How can one reconcile (3.4) with (3.15)? In order to provide a bridge between the two expressions, we notice that there is no power t^2 in (3.4), but we can add and subtract it as follows

$$
\mathbb{P}_{MX}(t) = \frac{1-4t+3t^2-3t^2+4t^3-t^4}{(1-t)^4} = \frac{1-4t+3t^2}{(1-t)^4} - t^4\frac{1-4t^{-1}+3t^{-2}}{(1-t)^4} \qquad (3.16)
$$

where the inserted and removed $3t^2$ and we separate $\mathbb{P}_{MX_+}(t)$ from the rest which has a "dual" form. By selecting only the self-dual part, we are left with the partition function's first piece, we forget about the second term. Following the interpretation above, we have the self-dual

DOF. However, we can also try to provide a BV interpretation of the partition function by observing that term $3t^2$ might correspond to an antifield (which has a negative ghost number), 2-form (which justifies the scale t^2), and self-dual (to have 3 DOFs as listed in the partition function). We denote it as Σ^*_+.

As above, we can describe the BRST symmetry in the present case. We have

$$sC = 0\,, \qquad sA_\mu = \partial_\mu C\,, \qquad s\Sigma^*_{\mu\nu,+} = F_{\mu\nu,+} \tag{3.17}$$

which is nilpotent since $F_{\mu\nu,+}$ is BRST invariant. The cohomology of this BRST differential is described by the partition function $\mathbb{P}_{MX_+}(t)$. Note the fact that the self-dual part of the field strength is BRST exact, which also implies the equations of motion using the Bianchi identities. The role of A^* in the non-self-dual case is superseded by Σ^*. To show this, we observe that the BRST variation of A^* can be rewritten as follows

$$sA^\star_\mu = \partial^\tau(F_{\tau\mu} + \epsilon_{\mu\tau\rho\sigma}F^{\rho\sigma}) \tag{3.18}$$

and the second piece is zero because of the Bianchi identities. Using (3.17), we can replace the r.h.s. of (3.18) with

$$sA^\star_\mu = \partial^\tau s\Sigma^*_{\mu\tau,+} \tag{3.19}$$

so we can observe that, up some exact term, we can write

$$A^*_\mu = \partial^\tau \Sigma^*_{\mu\tau} \tag{3.20}$$

and therefore the field A^*_μ becomes superfluous. In the same way, inserting (3.20) into the BRST variation of C^*, we see that it vanishes, and therefore, C^* is no longer needed and should be removed. This the reason that the fields A^*_μ is superseded by $\Sigma^*_{\mu\nu}$.

Essentially, this is also the content of the discussion in [33] where they show that the self-dual YM algebra is 2-homogeneous, and this implies the equations of motion.

3.4 2-Form in $D = 6$ and its Self-dual Counterpart

We consider the case of 2-form in $D = 6$. The partition function is

$$\begin{aligned}\mathbb{P}_{2F}(t) &= \frac{1 - 6t + 15t^2 - 15t^4 + 6t^5 - t^6}{(1-t)^6} \\ &= \frac{(1-t^2)}{(1-t)^6}\Big((1 + t^2 + t^4) - 6t(1+t^2) + 15t^2\Big)\end{aligned} \tag{3.21}$$

which describes a 2-form. The interpretation of the last factor is always as follows: $-6t(1+t^2)$ are the usual ghost for the primary gauge symmetry, $(1+t^2+t^4)$ are the ghost-for-ghosts and finally, $15t^2$ are the B-field DOF's. Taking the limit $t \to 1$, it gives the on-shell DOFs, which are six. On the other side, one can read from the first expression of (3.21) the BV content of the model.

Using the same trick as above, we add and subtract $-10t^3+10t^3$ to get the self-dual partition function

$$\begin{aligned}\mathbb{P}_{2F}(t) &= \frac{1-6t+15t^2-10t^3+10t^3-15t^4+6t^5-t^6}{(1-t)^6} \\ &= \frac{1-6t+15t^2-10t^3}{(1-t)^6} + t^3\frac{10-15t+6t^2-t^3}{(1-t)^6}\end{aligned} \tag{3.22}$$

where again, the first term in the last expression gives the the partition function of the self-dual 2-form

$$\mathbb{P}_{SD-2F}(t) = \frac{1-6t+15t^2-10t^3}{(1-t)^6} = \frac{(1-t^2)}{(1-t)^6}\frac{(1-5t+10t^2)}{1+t} \tag{3.23}$$

The last factor in the limit $t \to 1$ gives the correct DOFs of the self-dual form B, with $H = \star H$. Again, notice that the BV antified for $B_{\mu\nu}, C_\mu, C$ are unnecessary since the self-duality condition puts the theory on-shell. The BRST symmetry in the present case becomes

$$sC = 0\,, \qquad SC_\mu = \partial_\mu C\,, \qquad sB_{\mu\nu} = \partial_{[\mu}C_{\nu]}\,, \qquad s\Gamma^\star_{\mu\nu\rho} = H^-_{\mu\nu\rho} \tag{3.24}$$

where $H^-_{\mu\nu\rho}$ is the anti-self-dual part of the field strength H. $\Gamma^\star_{\mu\nu\rho}$ is the only antifield needed to impose the self-duality condition at the level of the cohomology described by the partition function (3.23). Note that the BRST symmetry is nilpotent since H^- is BRST invariant. As above, the antifields $B^*_{\mu\nu}, C^*_\mu, C^*$ are no longer needed, since the self-duality condition imposes the equations of motion.

We can write the last factor as follows

$$\frac{(1-5t+10t^2)}{1+t} = 1 - \frac{6t}{1+t} + \frac{10t^2}{1+t} \tag{3.25}$$

which has the following interpretation: the denominator $(1+t)$ always halves the DOFs; this implies that H (which has 20 DOFs since it is the field strength of a 2-form) contains only the self-dual part, and this gives the coefficient 10. Since the on-shell condition is imposed by self-duality, we do not need the antighost. Therefore, also for ghost C_μ, which, in principle, should have 6 DOFs, only 3 are indeed needed. Finally, we need only one ghost-for-ghost, as described in the first term.[5]

3.5 Gravity

We recall that Einstein's gravity can be formulated in two ways: metric formulation with a metric and the corresponding invariances or first-order formulation utilizing vielbein and spin-connection degrees of freedom with extra gauge symmetry. At the level of counting the fundamental DOF (perturbatively, at the quadratic level), they give the same answer, and here we present the implications of the partition functions constructed with our scheme. Let us start from the metric formulation: in that case, we have a symmetric tensor g, a ghost vector ξ parametrizing the diffeomorphisms in $D = 4$, the antifield of the metric g^* and finally the antifield ξ^* of the ghost ξ. Putting all these data together, we end up with the expression

$$\mathbb{P}_{metric}(t) = \frac{4 - 10t + 10t^3 - 4t^4}{(1-t)^4} = \frac{(1-t^2)}{(1-t)^4}(4(1+t^2) - 10t) \tag{3.26}$$

Again, by factoring out $(1 - t^2)/(1 - t)^4$, we can recognize the two physical degrees of freedom of the perturbative graviton on flat space.

Let us consider the other formulation: Cartan formalism. In that case we still have diffeomorphism ghost ξ, but we have the Lorentz ghost $\Lambda^{[ab]}$ which carry 6 DOF ($a, b = 0, \ldots 3$). In order to keep track of the difference between the two approaches, we introduce a second parameter s as a secondary ghost number assigned to additional fields. Then, we have the vielbein e^a and the spin-connection $\omega^{[ab]}$ which are 1-forms. The spin connection also carries the charge s. Then we have the antifields $e^*_a, \omega^*_{[ab]}$ and finally the antifields for the ghosts. We can collect all these data in the expression

$$\mathbb{P}_{Cartan}(t) = \frac{(4 + 6s) - (4 + 6s)4t + (4 + \frac{6}{s})4t^3 - \left(4 + \frac{6}{s}\right) t^4}{(1-t)^4} \tag{3.27}$$

Notice that we assigned the charge of $1/s$ both to the antifield to $\omega^{[ab]}$ and to the antifield to the Lorentz ghost. This is needed to respect the anti-bracket since scaling the Lorentz ghost by s implies the scaling of its antifield Λ^* oppositely.

In order to reconcile the two expressions we notice the following. The ghost $\Lambda^{[ab]}$ is needed to remove the DOF from the vielbein. That reduces the DOF of e^a from 16 to 10. However, since the BRST ghost $\Lambda^{[ab]}$ appears linearly in the transformation of the vielbein, to effectively

[5] As shown in the Maxwell case, there is yet again a direct way of deriving the partition function from the same world line model. This time, however, it is in the $R = 2$ subsector of the theory. Yet again, the spectrum is much bigger than the minimal BV content, but all the extra fields imposed on us by the structure of the $N = 2$ world line theory are auxiliary and, therefore, cancel out from the partition function.

cancel those DOF in the partition function, we have to choose $s = t$. In this way, we get

$$\mathbb{P}_{Cartan}(t) = \frac{(4+6t) - (4+6t)4t + (4+\frac{6}{t})4t^3 - \left(4+\frac{6}{t}\right)t^4}{(1-t)^4} = \frac{4 - 10t + 10t^3 - 4t^4}{(1-t)^4} \tag{3.28}$$

and the interpretation is the following. The $24t^2$ constraints coming from the BRST variation of the antifields ω^* impose the torsion condition, which can be solved by fixing ω in terms of e^a. Therefore, there is a cancellation $-24t^2 + 24t^2 = 0$. The BRST ghost Λ now has the right powers of t to cancel 6 DOF from the vielbein, and contextually, there are 6 antifield imposing the condition of the equations of motion $16t^3 - 6t^3 = 10t^3$. Then, finally, the two partition functions for $s = t$ do coincide.

4 Supersymmetric Models

In this section of the paper, we focus on models with rigid supersymmetry. As the partition function counts the shell degrees of freedom as an imprint of SUSY, we will see the matching of fermionic and bosonic parts of the spectrum. The examples on which we show this matching are the Wess-Zumino multiplet, $N = 1, 4$ super Maxwell, and $D = 4, N = 1$ supergravity.

In several models, the supersymmetry algebra closes on fermionic equations of motion (open algebra), preventing an off-shell description; in some cases, however, introducing auxiliary fields may solve those issues. The following sections show how to read the auxiliary fields in the partition functions. This will help only for the first issue, namely to achieve the off-shell matching, but it does not solve the issue of supersymmetry algebra closure.

4.1 Wess-Zumino Model

We now proceed to a discussion of the first supersymmetric model of this section, the Wes-Zumino model. The WZ multiplet is formed by a complex scalar ϕ, a Weyl spinor ψ and an auxiliary fields F. The partition function reads

$$\mathbb{P}_{WZ}(t) = \frac{-2t + 4t^{3/2} - 4t^{5/2} + 2t^3}{(1-t)^4} = \frac{(1-t^2)}{(1-t)^4}\left(-2t + \frac{4t^{3/2}}{1+t}\right) \tag{4.1}$$

Factoring out $\frac{(1-t^2)}{(1-t)^4}$, we are left with the expression

$$\left(-2t + \frac{4t^{3/2}}{1+t}\right) \tag{4.2}$$

and in the limit $t \to 1$ it reduces to $-2 + 2 = 0$, which is the on-shell matching of WZ DOFs. The first contribution to (4.2), namely $-2t$, is due to the scalars, which have dimension one for D=4 theory, and the second piece is the contribution of the fermions. The sign is reversed, and the coefficient is 4. Because of two complex DOFs, the scale is 3/2 as for the WZ model in $D = 4$, and the denominator halves the DOF going on-shell, which is what the Dirac equation does to physical DOF.

Another possible interpretation of (4.1) uses BV fields. The first two terms correspond to the scalar ϕ and to the fermionic ψ_α, the third term $4t^{5/2}$ stands for the antifield $\psi^{*,\dot\alpha}$ which can be considered as the antifield of ψ_α. Its sign is opposite to the previous term $-4t^{3/2}$, implying that it is a bosonic field, and the scale $t^{5/2}$ correctly matches its scale dimension. The last term, namely $2t^2$, represents the scalar's antifield ϕ^*. It is complex; it has the correct scale dimension and has opposite statistics with respect to the scalars $-2t$. The numerator denotes all possible derivatives of those fields. The BRST symmetry relates them is simply

$$s\phi = 0\,, \qquad s\phi^* = \partial^2\phi + \dots\,, \qquad s\psi_\alpha = 0\,, \qquad s\psi^{*,\dot\alpha} = \partial^{\dot\alpha\beta}\psi_\beta + \dots\,. \tag{4.3}$$

where the dots indicate we are neglecting the additional terms of the EOM for ϕ and ψ_α. The BRST differential is nilpotent. Note that by going from the first expression in (4.1) to the second one, we have put in evidence the factor $(1 - t^2)$, which can be interpreted as the on-shell condition (notice that it has the correct scale for the Laplacian ∂^2). It means that we have removed the antifields and have the theory on-shell (this is essentially Green's function construction of BV formalism with a gauge fixing).

Note that at the classical level, the presence of the antifields guarantees that we can deal with open algebras without auxiliary fields since their variation gives the equations of motion needed to close the algebra (see later for an algebraic discussion on this point). Nonetheless, at the quantum level (the BV formalism works only at the classical level since the BV fields are only sources for quantum operators and they are not propagating DOFs), we need the auxiliary fields to go off-shell, write an action, and construct the propagators of dynamical fields. We can modify the above expression as follows

$$\begin{aligned}\mathbb{P}_{WZ}(t) &= \frac{-2t + 4t^{3/2} - 2t^2 + 2t^2 - 4t^{5/2} + 2t^3}{(1-t)^4} = \\ &= \frac{-2t + 4t^{3/2} - 2t^2}{(1-t)^4} + \frac{2t^2 - 4t^{5/2} + 2t^3}{(1-t)^4}\end{aligned} \tag{4.4}$$

where we added and subtracted the term $-2t^2$. This stands for the auxiliary fields F; it has the correct sign, the multiplicity, and the scale dimension. Then we see the polynomial $-2t + 4t^{3/2} - 2t^2$ vanishes in the limit $t \to 1$ beacuse of the off-shell matching of DOFs. The second term in the last expression of (4.4) describes all antifields to fields and the auxiliary fields, usually denoted by $F^\star$.

Note that the first term of (4.4) can be recast in the following form

$$\mathbb{P}_{WZ,fields}(t) \;=\; \frac{-2t+4t^{3/2}-2t^2}{(1-t)^4} = -2t\frac{(1-\sqrt{t})^2}{(1-t)^4} \tag{4.5}$$

This expression has also a very intuitive (worldline) Description: the factor $(1-\sqrt{t})^2$ is the partition function of an anticommuting variable θ_α where $\alpha = 1,2$, which scale as $\sqrt{t}$, which is the correct engineering dimension of the target space super coordinates. In fact, the superfield Φ built with these coordinates is

$$\Phi(x,\theta) = \phi(x) + \psi_\alpha(x)\theta^\alpha + F(x)\frac{\epsilon_{\alpha\beta}}{2}\theta^\alpha\theta^\beta \tag{4.6}$$

where $\phi(x), \psi_\alpha(x)$ and $F(x)$ are the component fields, and these provide the coefficients of the polynomial $(1-\sqrt{t})^2$. Then, the factor $(1-t)^4$ in the denominator stands for the bosonic components x^μ of target space coordinates. So, the partition function of the WZ multiplet can be obtained by considering a Green-Schwarz superparticle whose target space is D=4 chiral superspace. It remains to explain the factor $-2t$. The scale t is appropriate for the interpretation, the coefficient 2 stands for complex DOF, and the negative sign is needed since the plethystic exponential indicates that those are bosonic superfields and, therefore, they provide positive coefficients in the expansion.

4.2 $N = 1$ Super Maxwell

Let us consider the Super-Maxwell case $N = 1$ in $D = 4$. The partition function is

$$\mathbb{P}_{SM}(t) = \frac{1-4t+4t^{3/2}-4t^{5/2}+4t^3-t^4}{(1-t)^4} = \frac{(1-t^2)}{(1-t)^4}\left(1+t^4-4t+\frac{4t^{3/2}}{1+t}\right) \tag{4.7}$$

Notice that switching off the $t^{3/2}, t^{5/2}$ terms, it reduces to the Maxwell case (3.4). The additional contributions are the fermions DOFs. In the last expression, we have a factor out the on-shell condition and the derivatives, and we are left with the expression

$$\left(1+t^4-4t+\frac{4t^{3/2}}{1+t}\right) \tag{4.8}$$

with the following interpretation: $(1+t^2)$ are the ghost and the antighost field (as is well known, one can assign zero scaling dimension to the ghost C and the complement to the antighost B).Note that fermionic DOFs come with a positive sign in the partition function. The term $-4t$ stands for the gauge bosons A_μ; the dimension, the sign, and the coefficient follow the rules we discussed. The fourth piece describes the fermions: they have scale dimensions $t^{3/2}$, four are DOFs, the sign is positive, and the factor $1/(1+t)$ halves the DOFs on-shell (in the limit

$t \to 1$). Note that in that limit there is a matching of DOFs, which is the well-known on-shell matching.

Let us check the second interpretation. We notice that the number can be interpreted as follows: the first constant 1 stands for the ghost field C (it has a vanishing dimension, it is a fermion, and it is a Lorentz scalar), the next one: $-4t$ stands for the gauge field A_μ: four DOF's, bosonic and scale dimension one. The next one is the fermion $\psi_\alpha, \bar{\psi}_{\dot{\alpha}}$. Then we have $4t^{5/2}$ has to be compared with the antifield to the fermions $\psi^*_\alpha, \bar{\psi}^*_{\dot{\alpha}}$ matching again the quantum numbers. The term $+4t^3$ corresponds to the antifield of the gauge field $A^*\mu$, and, finally, $-t^4$ reads as the antifield of the ghost field C^*. The BRST symmetry relating to those fields contains the equations of motion as variations of the antifield and the usual gauge symmetry. The factor $1/(1-t)^4$ represents the derivatives acting on these fields.

4.2.1 Auxiliary Fields

In order to study the auxiliary fields of this model, we can perform the same analysis as for the WZ model. We notice that we can also add and subtract a term as $-t^2$ to the numerator of (4.7) to get

$$\begin{aligned} \mathbb{P}_{SM}(t) &= \frac{1 - 4t + 4t^{3/2} - t^2 + t^2 - 4t^{5/2} + 4t^3 - t^4}{(1-t)^4} \\ &= \frac{1 - 4t + 4t^{3/2} - t^2}{(1-t)^4} + \frac{t^2 - 4t^{5/2} + 4t^3 - t^4}{(1-t)^4} \end{aligned} \tag{4.9}$$

where we have separated the expression into two parts that we are calling $\mathbb{P}_{SM,fields}(t)$ and $\mathbb{P}_{SM,antif}(t)$. The first one represents the field part of the partition function where we added the term $-t^2$. This can be read as the auxiliary field D needed in $N = 1$ $D = 4$ case. The polynomial

$$p(t) = 1 - 4t + 4t^{3/2} - t^2 \tag{4.10}$$

has the property to vanish at $t = 1$, and that corresponds to the off-shell matching (we have not extract the coefficient $(1 - t^2)$) for super Maxwell theory. The additional piece of the partition function collects the antifield together with D^*, the antifield of the auxiliary field D.

In order to find out if there is a superparticle model that might have such a partition function, we make the following ansatz: we need at least 4 supercoordinates θ^α and $\bar{\theta}^{\dot{\alpha}}$ and their partition function is $(1-\sqrt{t})^4$ as above. Now, we use the simple identity

$$(1-\sqrt{t})^4 = 2(1-\sqrt{t})^2 - (1 - 4t + 4t^{3/2} - t^2) \tag{4.11}$$

to relate it to the polynomial in (4.10). Note that there is a contribution of the piece $2(1-\sqrt{t})^2$, but that corresponds (as seen above) to a chiral multiplet. Indeed, the super Maxwell multiplet is obtained by removing from a real general superfield $\Phi(x, \theta, \bar{\theta})$ a chiral multiplet unessential for physics by means of the WZ gauge. Therefore, provided this interpretation, we can conclude that the partition function of $N = 1$ $D = 4$ can be constructed by means a GS superparticle with $D = 4$ complete superspace $(x, \theta, \bar{\theta})$, or equivalenty with $N = 2$ supersymmetric particle as in [27].

4.3 Self-Dual $N = 1$ Super Maxwell

From the N=1 super Maxwell case, we can extract the self-dual counterpart. As done in the previous section, we add and subtract some contributions as follows

$$\begin{aligned} \mathbb{P}_{SM}(t) \quad &= \quad \frac{1 - 4t + 4t^{3/2} + 3t^2 - 2t^{5/2} - 3t^2 - 2t^{5/2} + 4t^3 - t^4}{(1-t)^4} \\ &\longrightarrow \quad \mathbb{P}_{SDSM}(t) = \frac{1 - 4t + 2t^{3/2} + 3t^2 - 2t^{5/2}}{(1-t)^4} \end{aligned} \tag{4.12}$$

where the antifield part of the partition function is removed, and we are left with the self/anti-self part of the partition function. Notice that we have added the contribution $+3t^2$ as in the non-susy case, we keep half of the fermion with half of the antifield ψ^* of the fermion to implement its equation of motion (in the fermionic case, the chirality condition is not sufficient to impose the equations of motion) Again factoring out $(1-t^2)/(1-t)^4$ we are left with

$$\left(1 - \frac{4t}{(1+t)} + \frac{2t^{3/2}}{(1+t)}\right) \tag{4.13}$$

and in the limit $t \to 1$, there is a matching between the self-dual photon DOF and the on-shell chiral fermion. The denominators $1/(1+t)$ have the duty to halve the DOF's.

Let us return to the discussion on antifields and their BRST symmetry. As we have already noticed, the term $3t^3$ has been understood as the presence of an antifield, a 2-form whose variation is the self-dual field strength. On the other hand, we have to take into account the term $-2t^{5/2}$, these are the antifields for the fermion ψ_α needed to implement the equations of motion, we finally have

$$\begin{aligned} &sC = 0\,, \qquad sA_\mu = \partial_\mu C\,, \qquad s\Sigma^+_{\mu\nu} = F^+_{\mu\nu}\,, \\ &s\psi_\alpha = 0\,, \qquad s\psi^*_{\dot\alpha} = \partial_{\dot\alpha\alpha}\psi^\alpha \end{aligned} \tag{4.14}$$

In the case of self-dual Maxwell, removing the anti-chiral part of the fermion, there is no way to write the Dirac action [34]. However, the equations of motion are easily implemented at the level of the cohomology by the variation of the antifields. Note again the asymmetry between fields and antifields, which prevents us from getting the BRST transformations from a master action S leading to the antibracket. Again, the partition function suggests an asymmetrical treatment of fields and antifields.

4.4 $N = 4$ Super Maxwell

In this section, we now discuss the most interesting example: the $N = 4, D = 4$ super Maxwell.[6] In that case, the partition function is

$$\mathbb{P}_{SM4}(t) = \frac{1 - 10t + 16t^{3/2} - 16t^{5/2} + 10t^3 - t^4}{(1-t)^4} = \frac{(1-t^2)}{(1-t)^4}\left(1 + t^2 - 10t + \frac{16t^{3/2}}{1+t}\right) \quad (4.15)$$

Again, we have twofold interpretations, and we are not repeating them. Note that the treatment of the super Maxwell with antifields has been done, and, at the classical level, it provides a way out for open supersymmetry algebra without introducing auxiliary fields.

The matching of on-shell DOFs is achieved by computing the limit $t \to 1$ of the last factor of the last expression, leading to $-8 + 8$ cancellation. Note that in the first sector, we have $1 + t^2 - 10t$, which correspond to the ghost C, the antighost B, the gauge fields A_μ and the scalar fields ϕ_{IJ} (six real scalar fields whose scale dimension is the same as of the gauge fields).

Let us introduce the auxiliary fields. We can insert a term $-7t^2$ in the expression and subtract simultaneously. That leads to

$$\mathbb{P}_{SM4}(t) = \frac{1 - 10t + 16t^{3/2} - 7t^2}{(1-t)^4} + \frac{7t^2 - 16t^{5/2} + 10t^3 - t^4}{(1-t)^4} \quad (4.16)$$

where we separated the expression into the field and antifield parts again. The additional term $-7t^2$ is needed in order to have the off-shell matching, and the coefficient stands for one auxiliary field D for the gauge multiplet plus a set of 6 auxiliary fields F^{IJ} for the scalars. In that case, we can achieve the matching off-shell of DOF's.

The number 7 is not a choice, but we can recall that the $N = 4$ on-shell spectrum can be viewed as a gauge boson plus 6 scalars [35, 36, 37]. Therefore, we should also be able to get it from the partition-function point of view. This can be done by summing the contribution of $N = 1$ super Maxwell computed in (4.10) to 6 WZ multiplets (4.4)

$$(1 - 4t + 4t^{3/2} - t^2) + \left(-6t(1 - 2\sqrt{t} + t)\right) = 1 - 10t + 16t^{3/2} - 7t^2 \quad (4.17)$$

leading to the correct expression. The plus takes into account also the representations of the fields, a more refined version of (4.17) can be exploited using Young tableaux.

[6] Everything in the present section also works for $N = 2, D = 4$ super Maxwell theory with appropriate changes, and we do not discuss it here.

4.5 D=6 N=1,2 Self-Dual Supersymmetric 2-form

As a next example, we consider the case of 2-form in $D = 6$ with $N = 1, 2$ supersymmetric theory. For higher forms, there is no derivation of the partition functions using the algebraic techniques (see for example [38]) and therefore we Use the knowledge of the previous sections to construct the partition function as follows. We start from the on-shell degrees of freedom, which are 3 DOF from the self-dual 2-form, 4 DOF for the fermonic DOF, and 1 from a scalar completing the multiplet (see [39]). Then we have

$$\mathbb{P}_{2FN1}(t) \;=\; \frac{(1-t^2)}{(1-t)^6}\Big(1 - \frac{6t}{(1+t)} + \frac{10t^2}{(1+t)} + t^2 - \frac{8t^{5/2}}{(1+t)}\Big) \tag{4.18}$$

where we added the scalar field t^2 and the fermions. Note that in order to achieve the correct DOF, we have provided the correct signs. Taking the limit $t \to 1$, this gives $+4 - 4 = 0$ matching of the on-shell DOF. Expanding the overall factors we finally get the final expression

$$\mathbb{P}_{2FN1}(t) = \frac{1 - 6t + 16t^2 - 8t^{5/2} - 10t^3 + 8t^{7/2} - t^4}{(1-t)^6} \tag{4.19}$$

In terms of BV fields, we find that the first terms $(1, -6, 15+1, -8)$ are the physical fields with the ghost and the ghost-for-ghost, while the additional terms $(-10, 8, -1)$ are the antifields. Note that $-10t^3$ represent the antifields DOF for the self-duality (as discussed above), $+8t^{7/2}$ the antifield for the fermions (implementing the equations of motion), and finally $-t^4$ is the antifield for the scalar implementing its equation of motion.

For the case N=2, we have

$$\mathbb{P}_{2FN2}(t) = \frac{1 - 6t + 18t^2 - 16t^{5/2} - 10t^3 + 16t^{7/2} - 3t^4}{(1-t)^6} \tag{4.20}$$

representing the multiplet of N=2 self-dual form in D=6.

4.6 N=1 D=4 Supergravity

In the last section, we depart from rigid supersymmetric models and give an analysis of the linearized supergravity partition function

$$\begin{aligned}\mathbb{P}_{SG}(t) \;&=\; \frac{4 - 4\sqrt{t} - 10t + 16t^{3/2} - 16t^{5/2} + 10t^3 + 4t^{7/2} - 4t^4}{(1-t)^4} \\ &=\; \frac{(1-t^2)}{(1-t)^4}\left(4(1+t^2) - 10t + \frac{\sqrt{t}(16t - 4(1+t+t^2))}{1+t}\right)\end{aligned} \tag{4.21}$$

The interpretation of the last factor is the following $4(1+t^2)$ are the ghost and antighost of diffeomorphisms, and $-10t$ are the independent components of the metric tensor. In the second term, $16t^{3/2}$ are the components of the gravitino, and $-4(1+t+t^2)$ are the supersymmetric ghosts and the on-shell traceless condition. The minus sign is needed since they are commuting ghosts. Taking the limit $t \to 1$, one can observe the on-shell matching condition for $-2+2$ of the two graviton states and those of gravitino.

5 Dualities

All the models- aside from those cut down by self-duality- admit a duality at the level of the partition function. Here, we list these models alongside the duality transformation $t \to 1/t$

Fields	Partition Function	Duality
MX	$\mathbb{P}_{MX}(t) = \frac{1-4t+4t^3-t^4}{(1-t)^4}$	$\mathbb{P}_{MX}(1/t) = -\mathbb{P}_{MX}(t)$
2-FORM	$\mathbb{P}_{2FO}(t) = \frac{1-6t+15t^2-15t^4+6t^5-t^6}{(1-t)^6}$	$\mathbb{P}_{2FO}(1/t) = -\mathbb{P}_{2FO}(t)$
WZ-MULT	$\mathbb{P}_{WZ}(t) = \frac{-2t+4t^{3/2}-4t^{5/2}+2t^3}{(1-t)^4}$	$\mathbb{P}_{WZ}(1/t) = -\mathbb{P}_{WZ}(t)$
N=1 SMX	$\mathbb{P}_{SMX-1}(t) = \frac{1-4t+4t^{3/2}-4t^{5/2}+4t^3-t^4}{(1-t)^4}$	$\mathbb{P}_{SMX-1}(1/t) = -\mathbb{P}_{SMX-1}(t)$
N=4 SMX	$\mathbb{P}_{SMX-4}(t) = \frac{1-10t+16t^{3/2}-16t^{5/2}+10t^3-t^4}{(1-t)^4}$	$\mathbb{P}_{SMX-4}(1/t) = -\mathbb{P}_{SMX-4}(t)$
N=1 SG	$\mathbb{P}_{SG}(t) = \frac{4-4t^{1/2}-10t+16t^{3/2}-16t^{5/2}+10t^3+4t^{7/2}-4t^4}{(1-t)^4}$	$\mathbb{P}_{SG}(1/t) = -\mathbb{P}_{SG}(t)$

Physical interpretation of this duality is just a relation of fields to antifields. The minus sign is the change of the statistic going from one to the other.

6 Conformal field theories

In the case of Maxwell and of higher forms in sec. 3.1, we have shown how the partition function can be computed from the worldvolume fields starting from a simple worldline model. One might wonder whether this can be done in any case studied in the present paper. We do not have any conclusive answer, but we can observe that if we can reproduce those expressions for a set of 2d fields, with a $(1,0)$ CFT of the fermionic type b_I, c_I or of the bosonic type β_I, γ_I by computing the partition function of their zero modes. The partition functions discussed above allow us to extract the dimensions of the representation for the 2D fields, using which we can compute some of the CFT parameters that characterize the model.

Once we consider $(1,0)$ CFT systems, we can establish their conformal algebra: the energy-momentum tensor $T(z), \bar{T}(\bar{z})$ and the current $J(z), \bar{J}(\bar{z})$ are spin 2 and spin 1 fields, respectively.

Computing the algebra, we expect the general form

$$\begin{aligned} T(z)T(w) &\sim \frac{c_{Vir}}{(z-w)^4}+\frac{T(w)}{(z-w)^2}+\frac{\partial T}{(z-w)}+\dots \\ T(z)J(w) &\sim \frac{c_g}{(z-w)^3}+\frac{J(w)}{(z-w)^2}+\frac{\partial J}{(z-w)}+\dots \\ J(z)J(w) &\sim \frac{a_g}{(z-w)^2}+\dots \end{aligned} \tag{6.1}$$

and analogously for the anti-holomorphic sector. The computation of numbers c_{Vir}, c_g, a_g can be computed in terms of the fields' representation, charges, and conformal spin. This is usually very difficult since the dimension of representations grows very rapidly. Nonetheless, we can adopt a different route.

We first compute the number of independent fields at each stage. This can be done by recasting the above partition functions in terms of infinite products

$$\mathbb{P}(t)=\frac{P(t)}{Q(t)}=\prod_{n=1}^{\infty}(1-t^n)^{-N_n} \tag{6.2}$$

where N_n are integers. The above expression stands for the contribution to the zero mode partition function computed with conformal field theory techniques on the torus Riemann surface. [4]. To compute those terms, we replace $t \to e^x$, and we compute

$$-\log\frac{P(e^x)}{Q(e^x)}=\sum_{n=1}^{\infty}N_n\log(1-e^{nx}) \tag{6.3}$$

since

$$\log(1-e^x)=\log(-x)+\frac{x}{2}+\sum_{g=1}^{\infty}\frac{B_{2g}}{2g(2g)!}x^{2g}=\log(-x)+\frac{x}{2}+\frac{x^2}{24}+\dots \tag{6.4}$$

where B_k are the Bernoulli numbers and then, inserting it into (6.3), we have

$$\log(-x)\sum_n N_n+\sum_n\log(n)N_n+\frac{x}{2}\sum_n nN_n+\sum_{g=1}^{\infty}\frac{B_{2g}}{2g(2g)!}x^{2g}\sum_n n^{2g}N_n=-\log\frac{P(e^x)}{Q(e^x)}$$

and therefore, we can extract the the sums of coefficients, which can be finally identified with the central charges in (6.1) as follows

$$c_{Vir}=2\sum_n N_n\,, \quad L=\sum_n\log(n)N_n\,, \quad a_g=-\sum_n nN_n \quad c_g=-\sum_n n^2N_n\,, \tag{6.5}$$

which are the coefficients in front of $\log(-x), 1, x/2$ and $x^2/24$ in the expansion with Bernoulli numbers.

We computed this number, and we list it here We will return to this problem in the future.

Fields	C_{Vir}	L	a_g	c_g
MX	6	$\log(-4)$	0	48
SD-MX	6	$\log(-2)$	0	48
2-FORM	10	$\log(12)$	0	$-5/4$
SD-2-FORM	10	$\log(6)$	0	$-5/4$
WZ-MULT	2	$\log(-1/2)$	0	$-5/2$
N=1 SMX	2	$\log(3/2)$	0	$-7/2$
N=2 SMX	2	0	0	$-13/2$
N=4 SMX	2	$\log(1/2)$	0	1

For now, we leave the question open and provide only these observations.

7 Plethystic exponential

The last step to obtain the full flagged multi-word partition function is to use a plastic exponential on the single letter word partition function; these techniques were studied, for instance, in [11].

As mentioned above, we now only consider a specialized partition function with one fugacity and cohomological degree taken into account by specializing the appropriate fugacity to -1. Given such a partition function in the expanded form

$$\mathbb{P}(t) = \sum_{n=1}^{\infty} C_n t^n \tag{7.1}$$

where C_n are integer numbers, the sign informs us about the bosonic/fermionic nature of the given space. Then, we can build the multifield expression using the plethystic exponential

$$PE[\mathbb{P}(t)x] = \prod_{n=1}^{\infty} (1 - xt^n)^{C_n} \tag{7.2}$$

the parameter x counts the number of letters in each word of a fixed fugacity. Notice that (7.1) is a series because of the multiple derivatives of the fields. Therefore, (7.2) contains all possible monomials with fields and their derivatives. Furthermore, the signature of C_n produces either appropriate fermionic/bosonic contribution.

Let us consider the simplest case of Mawxell $D = 4$. The partition function is depicted in (3.4) if we put the formula in the following form

$$\mathbb{P}_{MX}(t) = \frac{1+t}{(1-t)^3}(1 + t^2 - 4t) \tag{7.3}$$

then, we can use the identity

$$\frac{1+t}{(1-t)^3} = \sum_{n=1}^{\infty} n^2 t^{n-1}$$

which inserted in (7.3) gives

$$\mathbb{P}_{MX}(t) = 1 + 2\sum_{n\geq 2}(n^2-1)t^n \tag{7.4}$$

which is in the form useful for extracting the Plethystic Exponential

$$\begin{aligned} PE[\mathbb{P}(t)u] &= \frac{1}{1-u\,t}\prod_{n\geq 2}\frac{1}{(1-u\,t^n)^{2(n^2-1)}} \\ &= 1 + \left(48t^5 + 30t^4 + 16t^3 + 6t^2 + t\right)u \\ &+ \left(1176t^{10} + 1440t^9 + 1233t^8 + 768t^7 + 364t^6 + 126t^5 + 37t^4 + 6t^3 + t^2\right)u^2 + O\left(u^3\right) \end{aligned} \tag{7.5}$$

where u counts the number of fields. The linear term in u is the single letter operator, the second line is the double letter operator, and so forth.

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