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Worldsheet fermion correlators, modular tensors and higher genus integration kernels

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Abstract

The cyclic product of an arbitrary number of Szegö kernels for even spin structure δ on a compact higher-genus Riemann surface Σ may be decomposed via a descent procedure which systematically separates the dependence on the points $z_i \in \Sigma$ from the dependence on the spin structure δ. In this paper, we prove two different, but complementary, descent procedures to achieve this decomposition. In the first procedure, the dependence on the points $z_i \in \Sigma$ is expressed via the meromorphic multiple-valued Enriquez kernels of e-print 1112.0864 while the dependence on δ resides in multiplets of functions that are independent of z_i, locally holomorphic in the moduli of Σ and generally do not have simple modular transformation properties. The δ-dependent constants are expressed as multiple convolution integrals over homology cycles of Σ, thereby generalizing a similar representation of the individual Enriquez kernels. In the second procedure, which was proposed without proof in e-print 2308.05044, the dependence on z_i is expressed in terms the single-valued, modular invariant, but non-meromorphic DHS kernels introduced in e-print 2306.08644 while the dependence on δ resides in modular tensors that are independent of z_i and are generally non-holomorphic in the moduli of Σ. Although the individual building blocks of these decompositions have markedly different properties, we show that the combinatorial structure of the two decompositions is virtually identical, thereby extending the striking correspondence observed earlier between the roles played by Enriquez and DHS kernels. Both decompositions are further generalized to the case of linear chain products of Szegö kernels.

Contents

1 Introduction

Recent years have witnessed an increasing symbiosis between progress in string perturbation theory and mathematical developments at the interface of number theory and algebraic geometry (see for example [1, 2]). Some of these results are intimately related to advances in the modern methods of quantum field theory amplitudes [3, 4, 5, 6].

The perturbative expansion of string amplitudes involves conformal field theory correlators on compact Riemann surfaces Σ of arbitrary genus, suitably integrated over multiple copies of Σ and over the complex structure moduli of Σ. The study of genus zero superstring amplitudes (see [7] for a review) has advanced in fruitful exchange with number-theoretic progress on multiple zeta values [8, 9] and twisted de Rham theory [10, 11]. The low energy expansion of closed superstring amplitudes at genus one may be organized in terms of modular graph functions and forms [12, 13], which naturally generalize Eisenstein series and were reformulated in algebraic geometry terms in [14, 15]. Open-string amplitudes at genus one in turn offer applications of and new perspectives on elliptic polylogarithms [16], elliptic multiple zeta values [17] and iterated integrals of holomorphic modular forms [18, 19]. The low energy expansion of closed superstring amplitudes at genus two were found to involve the number-theoretic invariants of Kawazumi [20] and Zhang [21] and to produce an infinite family of natural generalizations thereof [22, 23]. These results have further motivated the generalization to arbitrary genus of modular graph functions in [24] and of modular graph tensors in [25].

The overarching goal of the present project is to disentangle, organize and formalize the structure of the various spaces of functions out of which string amplitudes are built. In the Ramond-Neveu-Schwarz formulation of superstrings, the worldsheet fields are scalars, spin $\frac{1}{2}$ fermions, ghosts and super ghosts. The summation over all possible 2^{2h} spin structures of the spin $\frac{1}{2}$ fermions and super ghosts at genus h implements the Gliozzi-Scherk-Olive projection which, amongst other roles, ensures the presence of space-time supersymmetry in the different superstring theories. In practice, the summation over spin structures can be carried out fairly effectively at genus one where well-known identities between Jacobi ϑ-functions suffice, see for instance [26, 27, 28, 29] for external NS states and [30, 31, 32] for external R states. At higher genus, however, carrying out the spin structure summations requires a major effort (see for example [33, 34] for the calculations of the genus two four- and five-point amplitudes of massless NS-NS states, whose results were matched with calculations in the pure spinor formulation [35, 36]).

In this work, we shall focus on disentangling the correlators of the spin $\frac{1}{2}$ worldsheet fermions for even spin structure δ and generic moduli. They enter string amplitudes whose external states are all NS through cyclic products of Szegö kernels $S_\delta(x,y)$,[1]

$$C_\delta(z_1,\cdots,z_n) = S_\delta(z_1,z_2)S_\delta(z_2,z_3)\cdots S_\delta(z_{n-1},z_n)S_\delta(z_n,z_1) \tag{1.1}$$

as well as through linear chain products of Szegö kernels,

$$L_\delta(x;z_1,\cdots,z_n;y) = S_\delta(x,z_1)S_\delta(z_1,z_2)\cdots S_\delta(z_{n-1},z_n)S_\delta(z_n,y) \tag{1.2}$$

and products thereof. The Szegö kernel $S_\delta(x,y)$ is a meromorphic $(\frac{1}{2},0)$ form in $x,y \in \Sigma$ with a single pole at $x=y$, so that C_δ and L_δ are meromorphic $(1,0)$ forms in the points $z_1,\cdots,z_n \in \Sigma$ while L_δ is a meromorphic $(\frac{1}{2},0)$ form in $x,y \in \Sigma$. The cyclic products C_δ result from the correlators of spin $\frac{1}{2}$ fermions that occur in the NS vertex operators, while linear chain products L_δ are needed for correlators that end on a worldsheet supercurrent or on a worldsheet stress tensor [37, 38, 39].

In this paper we shall establish, for a Riemann surface of arbitrary genus, a decomposition of C_δ which completely separates the dependence on the points $z_1,\cdots,z_n$ from that on the spin structures δ. More specifically, C_δ will be expressed as a sum of binary products in which one factor contains all the dependence on $z_1,\cdots,z_n$ but is independent of δ, while the other factor contains all the dependence on δ but is independent of $z_1,\cdots,z_n$. An analogous decomposition will hold for L_δ but in this case both factors will depend on the end points x,y as well. In both cases, the dependence on $z_1,\cdots,z_n$ will be expressed in a well-controlled function space whose mathematical significance for integration on Riemann surfaces will be elaborated on below.

For the case of genus one, the spin structure independent components admit a natural formulation in terms of the Kronecker-Eisenstein coefficients [40, 41] that serve as integration kernels for elliptic polylogarithms [42, 16, 43]. More general correlators that enter heterotic string amplitudes may similarly be decomposed in terms of Kronecker-Eisenstein kernels [44, 45]. Accordingly, Kronecker-Eisenstein kernels furnish a universal function space out of which the integrands of genus one superstring amplitudes [46, 47, 48, 49, 50] and ambitwistor-string theories [51, 52, 53] may be built. An attractive property of the Kronecker-Eisenstein kernels and their associated polylogarithms is that they form a space that is closed under addition, multiplication, differentiation, and integration.

For the case of genus two, a complete solution to the decomposition problem for C_δ was obtained in [54] using the fact that every genus two Riemann surface is hyperelliptic, and that every point in moduli space is generic. An analogous decomposition was obtained for the linear chain products L_δ in unpublished work by the authors. Actually, all the spin

[1]For odd spin structures and for even spin structures at genus $h \geq 3$ and non-generic moduli there exist holomorphic $(\frac{1}{2},0)$ form zero modes which modify the correlators of the worldsheet fermions.

structure dependence, for any number n of points, may be reduced to that of the cases $n \leq 4$, thereby permitting a systematic summation over all even spin structures against the genus two superstring measure obtained in [55]. For genus greater than two, however, hyperelliptic surfaces are not generic and the above results do not generalize.

For arbitrary genus, it is the structure provided by integration kernels and their associated polylogarithms, already discussed above for the case of genus one, that systematically provides the proper ingredients for the decomposition of both the cyclic and linear chain products of Szegö kernels. Integration kernels and polylogarithms for Riemann surfaces of higher genus are conveniently constructed from flat connections that take values in certain freely generated Lie algebras. On general grounds, any two such flat connections may be related to one another by the composition of a gauge transformation and an automorphism of the freely generated Lie algebra [56]. Different flat connections may present themselves, however, in different guises depending on their analyticity, monodromy, and modular properties. On a compact Riemann surface Σ of genus h, we shall consider,

- the meromorphic integration kernels $g^{I_1 \cdots I_r}{}_J(x,y)$ with $I_1, \cdots, I_r, J \in \{1, \cdots, h\}$ for $r \geq 1$, which are multiple-valued in $x, y \in \Sigma$ with prescribed monodromies and are not modular tensors of $\mathrm{Sp}(2h, \mathbb{Z})$. They were introduced by Enriquez in [57] through their functional properties, which will be reviewed in section 2.1. They may be expressed as multiple $\mathfrak{A}$ periods of combinations of Abelian differentials and the prime form [58] or, on hyperelliptic surfaces, as Poincaré series [59].

- the real-analytic integration kernels $f^{I_1 \cdots I_r}{}_J(x,y)$ with $I_1, \cdots, I_r, J \in \{1, \cdots, h\}$ for $r \geq 1$ are single-valued in $x, y \in \Sigma$ and transform as tensors under the modular group $\mathrm{Sp}(2h, \mathbb{Z})$. They were introduced by D'Hoker-Hidding-Schlotterer (DHS) in [60] and will be reviewed in section 6.1. They may be expressed as multiple integrals over Σ involving the Arakelov Green function and Abelian differentials.

The relation between these families of integration kernels and their associated polylogarithms was exhibited and proven in [56].

1.1 Summary of results and organization

A first main result of this work is the construction of the decomposition of the cyclic product of Szegö kernels $C_\delta(z_1, \cdots, z_n)$ of (1.1) on a Riemann surface of arbitrary genus h

with even spin structure δ into a sum of a finite number of terms,[2]

$$C_\delta(z_1, \cdots, z_n) = \mathcal{W}(z_1, \cdots, z_n) + \sum_{r=2}^{n} \mathcal{W}_{I_1 \cdots I_r}(z_1, \cdots, z_n)\, D_\delta^{I_1 \cdots I_r} \tag{1.3}$$

where the various components have the following properties.

- The functions $\mathcal{W}_{I_1 \cdots I_r}(z_1, \cdots, z_n)$ are independent of the spin structure δ; single-valued and meromorphic $(1,0)$ forms in the points $z_1, \cdots, z_n$; expressed solely in terms of the Enriquez kernels; and cyclically symmetric in the indices $I_1, \cdots, I_r$.
- The functions $D_\delta^{I_1 \cdots I_r}$ are independent of the points $z_1, \cdots, z_n$; depend non-trivially on the spin structure δ; are locally holomorphic in the complex structure moduli of Σ; and cyclically symmetric in the indices $I_1, \cdots, I_r$.

In general, neither $\mathcal{W}_{I_1 \cdots I_r}(z_1, \cdots, z_n)$ nor $D_\delta^{I_1 \cdots I_r}$ transforms as a tensor under the modular group $\mathrm{Sp}(2h, \mathbb{Z})$ or under one of its congruence subgroups. For this reason we shall refer to them simply as *multiplets*, allowing for the possibility that they are not tensors.

The proof of this result will be given in Theorem 5.1 of section 5 with the help of the descent procedure stated in Theorem 2.3 of section 2, and subsequently proven in section 2 through 4, using the tools of generating functions developed in section 3. Various technical proofs are relegated to appendices B and C. Furthermore, it will be shown in section 4 that the coefficients $D_\delta^{I_1 \cdots I_r}$ may be expressed as multiple $\mathfrak{A}$ periods, thereby generalizing the analogous representation for Enriquez kernels obtained in [58].

A second main result is to provide a proof of the corresponding decomposition of $C_\delta(z_1, \cdots, z_n)$ of (1.1) on a Riemann surface of arbitrary genus h with even spin structure δ into DHS kernels f,

$$C_\delta(z_1, \cdots, z_n) = \mathcal{V}(z_1, \cdots, z_n) + \sum_{r=2}^{n} \mathcal{V}_{I_1 \cdots I_r}(z_1, \cdots, z_n)\, C_\delta^{I_1 \cdots I_r} \tag{1.4}$$

which had already been proposed without proof in an earlier paper [61], and where the various components have the following properties,

- The multiplets $\mathcal{V}_{I_1 \cdots I_r}(z_1, \cdots, z_n)$ are independent of the spin structure δ; single-valued and meromorphic $(1,0)$ forms in the points $z_1, \cdots, z_n$; cyclically symmetric in $I_1, \cdots, I_r$; expressed solely in terms of DHS kernels; and tensors under $\mathrm{Sp}(2h, \mathbb{Z})$.

[2] Throughout we shall use the Einstein summation convention for a repeated pair of upper and lower indices whenever no confusion is expected to arise.

- The multiplets $C_\delta^{I_1\cdots I_r}$ are independent of $z_1,\cdots,z_n$; cyclically symmetric in $I_1\cdots,I_r$; depend non-trivially on δ; are tensors under the principal congruence subgroup $\Gamma_h(2)\subset \mathrm{Sp}(2h,\mathbb{Z})$; but are generally not locally holomorphic in the moduli of Σ.

The proof of these results is the subject of section 6. It is worth highlighting that the same modular tensors $C_\delta^{I_1\cdots I_r}$ and meromorphic multiplets $D_\delta^{I_1\cdots I_r}$ at rank r universally enter the decompositions (1.3) and (1.4) for any number $n\geq r$ of Szegö kernels.

A third main result is a remarkable correspondence between the decompositions of $C_\delta(z_1,\cdots,z_n)$ given in (1.3) and (1.4), despite the fact that the analyticity, monodromy, and modular properties of their respective building blocks are virtually opposite to one another. The precise correspondence may be formulated in terms of the following map,

$$\begin{aligned} D_\delta^{I_1\cdots I_r} \quad &\longleftrightarrow \quad C_\delta^{I_1\cdots I_r} \\ g^{I_1\cdots I_r}{}_J(x,y) \quad &\longleftrightarrow \quad f^{I_1\cdots I_r}{}_J(x,y) \\ \mathcal{W}_{I_1\cdots I_r}(z_1,\cdots,z_n) \quad &\longleftrightarrow \quad \mathcal{V}_{I_1\cdots I_r}(z_1,\cdots,z_n) \end{aligned} \tag{1.5}$$

The correspondence just between f and g was already observed to hold between various higher genus generalizations of the Fay identities and interchange lemmas in [62]. The correspondence may be further extended, modulo some subtle qualifiers related to regularization that will be addressed in section 4, to integral representations for various functions, including $C_\delta^{I_1\cdots I_r}$ and $D_\delta^{I_1\cdots I_r}$, by the following formal map,

$$\oint_{\mathfrak{A}^I} dt \quad \longleftrightarrow \quad \int_\Sigma d^2t\,\bar\omega^I(t) \tag{1.6}$$

between line integrals on homology cycles and surface integrals on Σ.

As our fourth main result, both types of decompositions are generalized in section 7 to the case of linear chain products $L_\delta(x;z_1,\cdots,z_n;y)$ of (1.2). Their entire dependence on the points $z_1,\cdots,z_n$ will then be carried by functions $\mathcal{W}_{I_1\cdots I_r}(x;z_1,\cdots,z_n;y)$ and $\mathcal{V}_{I_1\cdots I_r}(x;z_1,\cdots,z_n;y)$ that are single-valued and meromorphic $(1,0)$ forms in $z_1,\cdots,z_n$. The counterparts $M_\delta^{I_1\cdots I_r}(x,y)$ and $L_\delta^{I_1\cdots I_r}(x,y)$ of the constants $D_\delta^{I_1\cdots I_r}$ and $C_\delta^{I_1\cdots I_r}$ are spinors in x,y which compensate for the monodromies of $\mathcal{W}_{I_1\cdots I_r}(x;z_1,\cdots,z_n;y)$ in x,y and the non-meromorphic dependence of $\mathcal{V}_{I_1\cdots I_r}(x;z_1,\cdots,z_n;y)$ on x,y, respectively. The correspondence of (1.5) is also generalized to the case of linear chain products.

The tradeoff between meromorphicity and single-valuedness is familiar from the different constructions for genus one polylogarithms [42, 16, 40, 43, 63] and will be reviewed in section 8 to illustrate the case of arbitrary genus in the more familiar elliptic setting.

An expected virtue of both decompositions (1.3) and (1.4) of cyclic products and their counterparts for linear chain products is a significant simplification of spin structure sums in string amplitudes. For example, since the quantities $C_\delta^{I_1\cdots I_r}$ and $D_\delta^{I_1\cdots I_r}$ no longer depend on any points on the surface, their spin-structure sums for arbitrary chiral measures can be performed at the level of constants on Σ which depend only on the moduli of Σ. Moreover, the differentials $f^{I_1\cdots I_r}{}_J(x,y)$ and $g^{I_1\cdots I_r}{}_J(x,y)$ carrying the entire dependence on the points $z_1,\cdots,z_n$ in (1.3) and (1.4) are amenable to the integration techniques of higher-genus polylogarithms and modular graph tensors and thereby facilitate low-energy expansions of the associated string amplitudes. For chiral amplitudes at genus two, the combinations of integration kernels in (1.3) and (1.4) complete the classification of the admissible z_i dependences arising from the contributions of even spin structures for arbitrary multiplicity that was initiated in [64].

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2 Descent in terms of Enriquez kernels

The connection $d - \mathcal{K}_{\rm E}$ introduced by Enriquez on an arbitrary compact Riemann surface Σ of genus $h \geq 1$ is meromorphic with simple poles on the universal cover $\tilde\Sigma$ of Σ and takes values in the Lie algebra $\mathfrak{g}$ that is freely generated by $2h$ elements $a \cup b$ where $a = \{a^1, \cdots, a^h\}$ and $b = \{b_1, \cdots, b_h\}$ [57]. The Enriquez connection form $\mathcal{K}_{\rm E}$ depends on two points $x, y \in \tilde\Sigma$ and may be expressed as,

$$\mathcal{K}_{\rm E}(x,y;a,b) = \mathbf{K}_J(x,y;B)\, a^J \tag{2.1}$$

where B_I is defined by $B_I X = [b_I, X]$ for all $X \in \mathfrak{g}$ and $\mathbf{K}_J(x,y;B)$ may be expanded in powers of the generators B_I as follows,

$$\mathbf{K}_J(x,y;B) = \sum_{r=0}^{\infty} g^{I_1\cdots I_r}{}_J(x,y) B_{I_1} \cdots B_{I_r} \tag{2.2}$$

The coefficient functions $g^{I_1\cdots I_r}{}_J(x,y)$ will be referred to as *meromorphic integration kernels* or simply as *Enriquez kernels*.[3] A key motivation for the Enriquez connection and associated Enriquez kernels is their role in the construction of meromorphic polylogarithms on Riemann surfaces of arbitrary genus [59, 62, 65]. Alternative constructions of higher-genus polylogarithms can be derived from the meromorphic flat connections of [66, 67] or the modular connection [60] built from the DHS kernels of section 6.

After giving a summary of the properties of the Enriquez kernels in section 2.1, we shall devote the remainder of this section to constructing a descent procedure that expresses the cyclic product of Szegö kernels $C_\delta(1, 2, \cdots, n)$, already previewed in (1.1) of the Introduction, in terms of Enriquez kernels.

The descent procedure will be presented for the cases of $n = 2, 3$ and 4 in sections 2.2, 2.3 and 2.4, respectively, and for arbitrary n in section 2.5 as Theorem 2.3. This theorem is one of the core results of this paper, but its full proof is quite involved and for that reason will be carried out in two parts. The proof of the first part of Theorem 2.3 will be given in section 2.6 with the help of Lemmas 2.4 and 2.5 which, in turn, are proven in appendices B and C, respectively. The proof of the second part of Theorem 2.3 will be greatly facilitated through the use of the generating functions that will be introduced in section 3. For this reason, the second part of the proof of Theorem 2.3 is relegated to the subsequent section, namely section 4.

[3] They are related to those introduced in [57] by $g^{I_1\cdots I_r}{}_J(x,y) = (-2\pi i)^r \omega^{I_1\cdots I_r}{}_J(x,y)$ and we shall set $g^{\emptyset}{}_J(x,y) = \omega_J(x)$ throughout, where ω_J are the holomorphic Abelian differentials normalized in (A.1).

2.1 Properties of Enriquez kernels

The Enriquez kernel $g^{I_1\cdots I_r}{}_J(x,y)$, for $r \geq 0$ and $I_1, \cdots, I_r, J \in \{1, \cdots, h\}$, is a meromorphic $(1,0)$-form[4] in $x \in \tilde\Sigma$ and a $(0,0)$-form in $y \in \tilde\Sigma$ which is locally holomorphic in the moduli of Σ, with prescribed monodromies [57, 56]. Its monodromies in x and y around $\mathfrak{A}$ cycles are trivial, while those around $\mathfrak{B}$ cycles are given by,

$$
\begin{aligned}
\Delta_L^{(x)} g^{I_1\cdots I_r}{}_J(x,y) &= \sum_{k=1}^r \frac{(-2\pi i)^k}{k!}\, \delta_L^{I_1\cdots I_k}\, g^{I_{k+1}\cdots I_r}{}_J(x,y) \\
\Delta_L^{(y)} g^{I_1\cdots I_r}{}_J(x,y) &= \delta_J^{I_r} \sum_{k=1}^r \frac{(2\pi i)^k}{k!}\, g^{I_1\cdots I_{r-k}}{}_L(x,y)\, \delta_L^{I_{r-k+1}\cdots I_{r-1}}
\end{aligned} \tag{2.3}
$$

The generalized Kronecker symbol is defined by,

$$
\delta_L^{I_1 I_2\cdots I_k} = \delta_L^{I_1}\delta_L^{I_2}\cdots\delta_L^{I_k} \tag{2.4}
$$

while the monodromy of an arbitrary function $\phi(x)$ around a cycle $\mathfrak{B}_L$ is denoted by,

$$
\Delta_L^{(x)}\phi(x) = \phi(\mathfrak{B}_L\cdot x) - \phi(x) \tag{2.5}
$$

where $\mathfrak{B}_L \cdot x$ denotes the action of the element $\mathfrak{B}_L \in \pi_1(\Sigma, q)$ on the point $x \in \Sigma$.

Since the kernels $g^{I_1\cdots I_r}{}_J(x,y)$ have prescribed monodromies, we may define them for x, y in a fundamental domain D for Σ, which is obtained by cutting Σ open along $2h$ loops $\mathfrak{A}^I$ and $\mathfrak{B}_I$ with common base point q, as shown in figure 1.

One may choose a *preferred fundamental domain* D such that $g^{I_1\cdots I_r}{}_J(x,y)$ for x, y in the interior D^o of D is holomorphic in x and y for $r \geq 2$, $g^I{}_J(x,y)$ has a single simple pole in x at y with residue δ_J^I and a single simple pole in y at x with residue $-\delta_J^I$,

$$
g^I{}_J(x,y) = \frac{\delta_J^I}{x-y} + \text{regular} \tag{2.6}
$$

and is given by $g^{\emptyset}{}_J(x,y) = \omega_J(x)$ for $r=0$ as already stated in footnote 3. For x' and/or y' outside D^o, the kernels $g^{I_1\cdots I_r}{}_J(x',y')$ are obtained from $g^{I_1\cdots I_r}{}_J(x,y)$ with $x, y \in D^o$

[4]Throughout, we shall use the conventions of [60, 61, 62] in which a differential $(1,0)$-form ϕ is expressed in a local complex coordinate x on Σ or $\tilde\Sigma$ as $\phi = \phi(x)dx$. By a slight abuse of terminology, we shall refer also to the coefficient function $\phi(x)$ as a $(1,0)$-form. Similarly, a $(0,1)$-form will be denoted $\bar\phi = \bar\phi(x)d\bar x$. Thus, in form notation, the holomorphic Abelian differentials are $\omega_I = \omega_I(x)dx$; their complex conjugates $\bar\omega_I = \bar\omega_I(x)d\bar x$; the Enriquez kernels are $g^{I_1\cdots I_r}{}_J(x,y)dx$; the DHS kernels are $f^{I_1\cdots I_r}{}_J(x,y)dx$; and the Szegö kernel is $S_\delta(x,y)\sqrt{dx\,dy}$. Note, however, that in the conventions of [56, 58] the differentials dx or $d\bar x$ were included as part of the forms.

by mapping $(x',y') \to (x,y)$ by an element in $\pi_1(\Sigma,q)$ and then using the monodromy relations of (2.3). These cases where either x and/or y are in the boundary ∂D may be obtained by considering limits of interior points. As a result, the forms $g^{I_1\cdots I_r}{}_J(x,y)$ may acquire simple poles in x at $\pi^{-1}(y)$ for all $r \geq 1$, where π is the canonical projection $\pi : \tilde{\Sigma} \to \Sigma$.

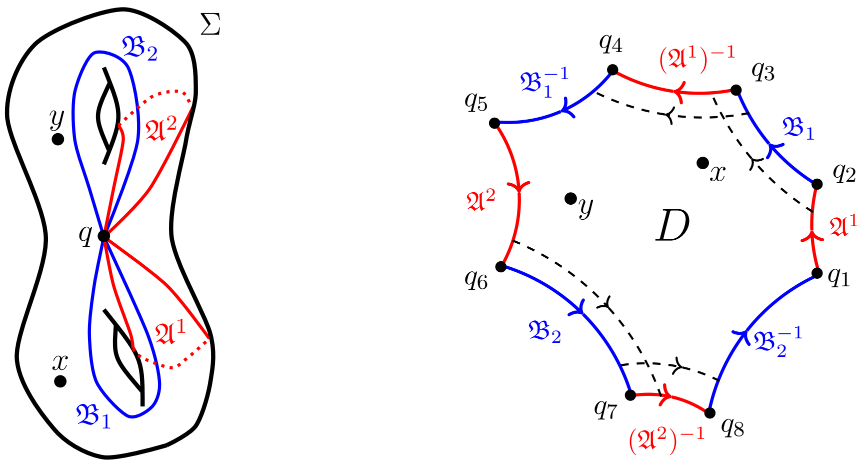

Figure 1: The left panel shows a compact genus two Riemann surface Σ and a choice of canonical homology cycles $\mathfrak{A}^1, \mathfrak{A}^2, \mathfrak{B}_1, \mathfrak{B}_2$ with a common base point q. A fundamental domain D, contained in the universal cover $\tilde{\Sigma}$ of Σ, for the action of $\pi_1(\Sigma,q)$ on Σ is obtained in the right panel by cutting Σ along the cycles in the left panel. The surface Σ may be recovered from D by pairwise identifying inverse boundary components with one another under the dashed arrows. The vertices $q_i \in \tilde{\Sigma}$ project to $q = \pi(q_i) \in \Sigma$ for $i = 1, \cdots, 8$ under the canonical projection $\pi : \tilde{\Sigma} \to \Sigma$.

The periods around $\mathfrak{A}^L$ cycles on the boundary of the fundamental domain D in figure 1 are given in terms of Bernoulli numbers Ber_r by,[5]

$$\oint_{\mathfrak{A}^L} dt\, g^{I_1\cdots I_r}{}_J(t,y) = (-2\pi i)^r \frac{\mathrm{Ber}_r}{r!}\, \delta_J^{I_1\cdots I_r L} \tag{2.7}$$

The y-dependence of $g^{I_1\cdots I_r}{}_J(x,y)$ is concentrated in the trace part with respect to the

[5]Throughout, we shall denote the Bernoulli numbers by Ber_r instead of the customary B_r in order to avoid confusion with the Lie algebra generators B_I used, for example, in (2.2). Recall that the Ber_r are generated by $\frac{x}{e^x-1} = \sum_{r=0}^{\infty} \frac{x^r}{r!}\, \mathrm{Ber}_r$. Furthermore, we shall systematically denote integration variables by t, or $t_1, \cdots, t_r$ in the case of multiple integrations.

last two indices [57], which leads us to introduce the following decomposition,

$$g^{I_1\cdots I_r}{}_J(x,y) = \varpi^{I_1\cdots I_r}{}_J(x) - \delta^{I_r}_J \chi^{I_1\cdots I_{r-1}}(x,y) \tag{2.8}$$

where ϖ is traceless, $\varpi^{I_1\cdots I_s J}{}_J(x) = 0$, independent of y, and holomorphic in $x \in \tilde\Sigma$. Hence, the simple pole (2.6) of $g^I{}_J(x,y)$ is entirely carried by $\chi(x,y) = -1/(x-y) + \text{regular}$. As a consequence of the unique double pole of $\partial_y \chi(x,y)$ in x at y, we deduce that the combination $\partial_y\chi(x,y) + \partial_y\partial_x \ln E(x,y)$ is holomorphic in x and y and single-valued for $x,y\in\Sigma$. Since its $\mathfrak{A}^L$ periods vanish, we conclude that the following equalities hold,

$$\begin{aligned} \partial_y\chi(x,y) &= -\partial_y\partial_x \ln E(x,y) \\ \chi(x,y) - \chi(x,z) &= -\partial_x \ln E(x,y) + \partial_x \ln E(x,z) \end{aligned} \tag{2.9}$$

A summary of definitions and useful properties of the prime form $E(x,y)$, the Szegö kernel and the Enriquez kernels is provided in appendix A.

2.2 The case $n=2$

By using the Fay trisecant identity,[6] the case $n=2$ may be recast in the following form,[7]

$$C_\delta(1,2) = \omega_I(1) D^I_\delta(2) + \partial_2\chi(1,2) \tag{2.10}$$

where $D^I_\delta(2)$ is given in terms of a multiplet D^{IJ}_δ that is constant on Σ,

$$D^I_\delta(2) = \omega_J(2) D^{IJ}_\delta \qquad\qquad D^{IJ}_\delta = -\frac{\partial^I\partial^J\vartheta[\delta](0)}{\vartheta[\delta](0)} \tag{2.11}$$

and we have used the relation $\partial_2\chi(1,2) = -\partial_1\partial_2 \ln E(1,2)$ of (2.9). As promised in the Introduction, all dependence on the spin structure δ is concentrated in the constant multiplet D^{IJ}_δ, while the remaining δ-independent part is expressed in terms of (the trace part of) an Enriquez kernel. We note that, while $C_\delta(1,2)$ is modular invariant when the spin structure $\delta=[\delta',\delta'']$ is transformed to $\tilde\delta = [\tilde\delta',\tilde\delta'']$ as follows [68],

$$\begin{pmatrix}\tilde\delta''\\ \tilde\delta'\end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D\end{pmatrix}\begin{pmatrix}\delta''\\ \delta'\end{pmatrix} + \frac{1}{2}\,\text{diag}\begin{pmatrix}AB^t\\ CD^t\end{pmatrix} \qquad\qquad \begin{pmatrix} A & B\\ C & D\end{pmatrix} \in \text{Sp}(2h,\mathbb{Z}) \tag{2.12}$$

neither $\partial_2\chi(1,2)$ nor $\omega_J(2)D^J_\delta(2)$ is modular invariant, and D^{IJ}_δ is *not* a modular tensor. As we shall see shortly, the symmetry $D^{JI}_\delta = D^{IJ}_\delta$, which is manifest in (2.11), will be of crucial importance for the descent procedure to succeed at multiplicity $n=3$ and beyond.

[6] The Fay trisecant identity between four points was introduced in equation (45) of [68]. The versions involving two or three points, used here, may be obtained therefrom by taking the limit of coincident points and has appeared, for example, in equations (A.26) and (A.27) of [33].

[7] Throughout, it will often be convenient to abbreviate the arguments $z_1,\cdots,z_n$ of various functions simply by their subscripts so that we set, for example, $C_\delta(1,\cdots,n) = C_\delta(z_1,\cdots,z_n)$.

2.3 The case $n = 3$

The case $n = 3$ may be handled by inspecting the poles of $C_\delta(1,2,3)$, collected in the following cyclic orbit of differential relations,

$$\begin{aligned}
\bar\partial_1 C_\delta(1,2,3) &= \pi\big(\delta(1,2) - \delta(1,3)\big) C_\delta(2,3) \\
\bar\partial_2 C_\delta(1,2,3) &= \pi\big(\delta(2,3) - \delta(2,1)\big) C_\delta(3,1) \\
\bar\partial_3 C_\delta(1,2,3) &= \pi\big(\delta(3,1) - \delta(3,2)\big) C_\delta(1,2)
\end{aligned} \tag{2.13}$$

Using the relation $\bar\partial_1 \chi(1,2) = -\pi\delta(1,2)$ and the fact that $\chi(1,2) - \chi(1,3)$ is single-valued in z_1, we see that the combination $C_\delta(1,2,3) + \big(\chi(1,2) - \chi(1,3)\big) C_\delta(2,3)$ is holomorphic and single-valued in z_1. Therefore, it must be a linear combination of the holomorphic Abelian differentials $\omega_I(1)$ with coefficients $D^I_\delta(2,3)$ that depend on the points z_2 and z_3 but not on z_1. As a result, we obtain the following decomposition,

$$\omega_I(1) D^I_\delta(2,3) = C_\delta(1,2,3) + \big(\chi(1,2) - \chi(1,3)\big) C_\delta(2,3) \tag{2.14}$$

Since $C_\delta(2,3)$ and $C_\delta(1,2,3)$ are single-valued in their arguments and χ has trivial $\mathfrak{A}$ monodromy the coefficients $D^I_\delta(2,3)$ have trivial $\mathfrak{A}$ monodromy, while their $\mathfrak{B}$ monodromy follows from the second line in (A.17) and is given by,[8]

$$\begin{aligned}
\Delta^{(2)}_L D^I_\delta(2,3) &= -2\pi i\, \delta^I_L C_\delta(2,3) \\
\Delta^{(3)}_L D^I_\delta(2,3) &= 2\pi i\, \delta^I_L C_\delta(2,3)
\end{aligned} \tag{2.15}$$

The $\bar\partial_2$ and $\bar\partial_3$ derivates of $D^I_\delta(2,3)$ are readily evaluated using (2.13), (2.14) and the derivatives of χ. For z_2 and z_3 in the same fundamental domain we find,

$$\begin{aligned}
\bar\partial_2 D^I_\delta(2,3) &= +\pi\delta(2,3) D^I_\delta(3) \\
\bar\partial_3 D^I_\delta(2,3) &= -\pi\delta(2,3) D^I_\delta(2)
\end{aligned} \tag{2.16}$$

with $D^I_\delta(3)$ given by (2.11). Using the first equation in (2.16) and $\bar\partial_2 \chi(2,3) = -\pi\delta(2,3)$, we see that the combination,

$$V^I_\delta(2,3) = D^I_\delta(2,3) + \chi(2,3) D^I_\delta(3) \tag{2.17}$$

[8] We note that the monodromies of $D^I_\delta(2,3)$ have a double pole in $z_2 - z_3$. This is due to the fact that, even though the prefactor of the second term on the right of (2.14) vanishes as $z_2 \to z_3$ when z_2 and z_3 are in the same fundamental domain, this factor does not vanish when z_2 and z_3 are in different fundamental domains, as brought about by performing a $\mathfrak{B}_L$ transformation on one of the points.

is holomorphic in z_2. Combining the first equation of (2.15) with the monodromy of χ given in (A.17), we find that this combination has non-trivial $\mathfrak{B}$ monodromy,

$$\Delta_L^{(2)} V_\delta^I(2,3) = -2\pi i\, \delta_L^I\, \partial_3 \chi(2,3) - 2\pi i\, \delta_L^I\, \omega_J(2) D_\delta^J(3) + \frac{2\pi i}{h} \omega_L(2) D_\delta^I(3) \tag{2.18}$$

The first term arises in the monodromy of $\partial_3 \chi^I(2,3)$ thanks to the third line in (A.17). The remaining terms in (2.18) are proportional to $D_\delta^J(3)$ and arise in the monodromy of $\varpi^I{}_J(2)$ thanks to (A.18). The combination $D_\delta^I(2,3) + \chi(2,3) D_\delta^I(3) - \partial_3 \chi^I(2,3) - \varpi^I{}_J(2) D_\delta^J(3)$ is holomorphic and single-valued in z_2 and therefore must be a linear combination of $\omega_J(2)$ with coefficients $D_\delta^{IJ}(3)$ that are independent of z_2,

$$\omega_J(2) D_\delta^{IJ}(3) = D_\delta^I(2,3) - \partial_3 \chi^I(2,3) - g^I{}_J(2,3) D_\delta^J(3) \tag{2.19}$$

where we have combined the contributions from $\chi(2,3) D_\delta^I(3)$ in (2.17) with $\varpi^I{}_J(2) D_\delta^J(3)$ into $g^I{}_J(2,3) D_\delta^J(3)$. Using the second line in (2.16), one verifies that $D_\delta^{IJ}(3)$ is holomorphic in z_3 while its $\mathfrak{A}$ monodromy is trivial and its $\mathfrak{B}$ monodromy is given by,

$$\Delta_L^{(3)} D_\delta^{IJ}(3) = 2\pi i \delta_L^I D_\delta^J(3) - 2\pi i \delta_L^J D_\delta^I(3) \tag{2.20}$$

The next step in the descent procedure consists in constructing a linear combination involving the holomorphic form $\varpi^A{}_B(3)$ and the constant multiplet D_δ^{MN} that matches the $\mathfrak{B}$ monodromy of $D_\delta^{IJ}(3)$ and allows us to express it in terms of a constant multiplet D_δ^{IJK}. Consulting the first line in (A.18), we observe that the combination $-\varpi^J{}_B(3) D_\delta^{IB} + \varpi^I{}_B(3) D_\delta^{JB}$ partially compensates the monodromy of $D_\delta^{IJ}(3)$,

$$\Delta_L^{(3)} \Big(D_\delta^{IJ}(3) - \varpi^J{}_B(3) D_\delta^{IB} + \varpi^I{}_B(3) D_\delta^{JB} \Big) = \frac{2\pi i}{h} \omega_L(3) \big(D_\delta^{JI} - D_\delta^{IJ} \big) \tag{2.21}$$

It is at this point that we use the symmetry $D_\delta^{JI} = D_\delta^{IJ}$, which is manifest from its expression in (2.11), to ensure that the $\mathfrak{B}$ monodromy of the above combination indeed cancels, so that its dependence on z_3 is holomorphic and single-valued and may be expressed as a linear combination of the holomorphic Abelian differentials $\omega_K(3)$. The results for the case $n=3$ are summarized by the proposition below.

Proposition 2.1. *The descent of the $n=3$ case is given by the following relations,*

$$\begin{aligned} \omega_I(1) D_\delta^I(2,3) &= C_\delta(1,2,3) + \big(\chi(1,2) - \chi(1,3)\big)\, C_\delta(2,3) \\ \omega_J(2) D_\delta^{IJ}(3) &= D_\delta^I(2,3) - \partial_3 \chi^I(2,3) - g^I{}_K(2,3) D_\delta^K(3) \\ \omega_K(3) D_\delta^{IJK} &= D_\delta^{IJ}(3) + \varpi^I{}_K(3) D_\delta^{JK} - \varpi^J{}_K(3) D_\delta^{IK} \end{aligned} \tag{2.22}$$

where the multiplet D_δ^{IJK} is cyclically symmetric in I, J, K, constant on Σ, and locally holomorphic in the moduli of Σ.

The validity of the three relations in (2.22) was already established in the paragraphs that precede the proposition, so it only remains to prove the cyclic symmetry relation,

$$D_\delta^{IJK} = D_\delta^{JKI} \tag{2.23}$$

One way to proceed is to take the difference between the first relation of (2.22) and its version for cyclically permuted points, then eliminate $D_\delta^I(2,3)$, $D_\delta^{IJ}(3)$ and their cyclic permutations in the points z_1, z_2, z_3 using the second and third relations. The difference may be simplified using the cyclic invariance of $C_\delta(1,2,3)$ in the points z_1, z_2, z_3; the z-derivative of the Fay identity in (9.24) of [62] for $x = z_1, y = z_2, z = z_3$; the interchange lemma in (9.11) of [62]; and Theorem 9.4 for $r = 1$ of [62]. The resulting relation reduces to the vanishing of $(D_\delta^{IJK} - D_\delta^{JKI})\omega_I(1)\omega_J(2)\omega_K(3)$, which implies the cyclic property of D_δ^{IJK} and completes the proof of the proposition. In section 4 a more streamlined proof of (2.23) will be presented that applies to the case of arbitrary n with equal ease. Similar computations lead to the reflection relation $D_\delta^{IJK} = -D_\delta^{KJI}$.

2.4 The case $n = 4$

The result is given by the following proposition.

Proposition 2.2. *The descent equations for the case $n = 4$ are given by,*

$$\begin{aligned}
\omega_I(1) D_\delta^I(2,3,4) &= C_\delta(1,2,3,4) + \big(\chi(1,2) - \chi(1,4)\big) C_\delta(2,3,4) \\
\omega_J(2) D_\delta^{IJ}(3,4) &= D_\delta^I(2,3,4) + \big(\chi^I(2,3) - \chi^I(2,4)\big) C_\delta(3,4) - g^I{}_J(2,3) D_\delta^J(3,4) \\
\omega_K(3) D_\delta^{IJK}(4) &= D_\delta^{IJ}(3,4) - \partial_4 \chi^{JI}(3,4) - g^J{}_K(3,4) D_\delta^{IK}(4) - g^{JI}{}_K(3,4) D_\delta^K(4) \\
\omega_L(4) D_\delta^{IJKL} &= D_\delta^{IJK}(4) - \varpi^K{}_L(4) D_\delta^{IJL} + \varpi^I{}_L(4) D_\delta^{JKL} + \varpi^{KI}{}_L(4) D_\delta^{JL} \\
&\quad - \varpi^{KJ}{}_L(4) D_\delta^{IL} + \varpi^{IK}{}_L(4) D_\delta^{JL} - \varpi^{IJ}{}_L(4) D_\delta^{KL}
\end{aligned} \tag{2.24}$$

The multiplet D_δ^{IJKL} is cyclically symmetric in I, J, K, L, constant on Σ and locally holomorphic in the moduli of Σ.

The proof of this proposition proceeds as for the case $n = 3$ and starts with,

$$\bar\partial_1 C_\delta(1,2,3,4) = \pi\big(\delta(1,2) - \delta(1,4)\big) C_\delta(2,3,4) \tag{2.25}$$

and its three cyclic permutations. The term $\omega_I(1) D_\delta^I(2,3,4)$ is obtained by verifying that the poles and monodromies in z_1 of the right side on the first line of (2.24) cancel, so it must be a linear combination of $\omega_I(1)$ with z_1-independent coefficients $D_\delta^I(2,3,4)$. The differential equations for $D_\delta^I(2,3,4)$ are obtained by differentiating the first line of (2.24) using (2.25) and its cyclic permutations, as well as the relations of (2.22) and we find,

$$\begin{aligned}
\bar\partial_2 D^I_\delta(2,3,4) &= \pi\delta(2,3) D^I_\delta(3,4) \\
\bar\partial_3 D^I_\delta(2,3,4) &= \pi\big(\delta(3,4)-\delta(3,2)\big) D^I_\delta(2,4) \\
\bar\partial_4 D^I_\delta(2,3,4) &= -\pi\delta(4,3) D^I_\delta(2,3)
\end{aligned} \tag{2.26}$$

with $D^I_\delta(3,4)$ defined by the $n=3$ case in (2.14). The $\mathfrak{A}$ monodromies vanish while the $\mathfrak{B}$ monodromies may be computed by evaluating the $\mathfrak{B}$ monodromies of the first equation in (2.24) and are found to be given by,

$$\begin{aligned}
\Delta^{(2)}_L D^I_\delta(2,3,4) &= -2\pi i\,\delta^I_L C_\delta(2,3,4) \\
\Delta^{(3)}_L D^I_\delta(2,3,4) &= 0 \\
\Delta^{(4)}_L D^I_\delta(2,3,4) &= 2\pi i\,\delta^I_L C_\delta(2,3,4)
\end{aligned} \tag{2.27}$$

Next, we show that the poles and monodromies in z_2 of the right side of the second equation in (2.24) cancel so that it must be a linear combination of $\omega_J(2)$ whose z_2-independent coefficients $D^{IJ}_\delta(3,4)$ in turn satisfy the differential equations,

$$\begin{aligned}
\bar\partial_3 D^{IJ}_\delta(3,4) &= +\pi\delta(3,4) D^{IJ}_\delta(4) \\
\bar\partial_4 D^{IJ}_\delta(3,4) &= -\pi\delta(4,3) D^{IJ}_\delta(3)
\end{aligned} \tag{2.28}$$

(see (2.19) for $D^{IJ}_\delta(4)$), have vanishing $\mathfrak{A}$ monodromies and the following $\mathfrak{B}$ monodromies (see (2.4) for the generalized Kronecker delta δ^{IJ}_L),

$$\begin{aligned}
\Delta^{(3)}_L D^{IJ}_\delta(3,4) &= -2\pi i\,\delta^J_L\, D^I_\delta(3,4) - 2\pi^2\,\delta^{IJ}_L\, C_\delta(3,4) \\
\Delta^{(4)}_L D^{IJ}_\delta(3,4) &= 2\pi i\,\delta^I_L\, D^J_\delta(3,4) - 2\pi^2\,\delta^{IJ}_L\, C_\delta(3,4)
\end{aligned} \tag{2.29}$$

Using (2.28) and (2.29) we show that the right side of the third equation of (2.24) is holomorphic and has vanishing monodromies in z_3. The coefficients $D^{IJK}_\delta(4)$ of $\omega_K(3)$ are holomorphic in z_4, have vanishing $\mathfrak{A}$ monodromies and their $\mathfrak{B}$ monodromies are given by,

$$\begin{aligned}
\Delta^{(4)}_L D^{IJK}_\delta(4) = &-2\pi i\,\delta^K_L\Big(\omega_M(4) D^{IJM}_\delta - \varpi^I{}_M(4) D^{JM}_\delta + \varpi^J{}_M(4) D^{IM}_\delta\Big) \\
&+2\pi i\,\delta^I_L\Big(\omega_M(4) D^{JKM}_\delta - \varpi^J{}_M(4) D^{KM}_\delta + \varpi^K{}_M(4) D^{JM}_\delta\Big) \\
&-2\pi^2\omega_M(4)\Big(\delta^{JK}_L\, D^{IM}_\delta + \delta^{IJ}_L\, D^{KM}_\delta - 2\delta^{IK}_L\, D^{JM}_\delta\Big)
\end{aligned} \tag{2.30}$$

The next step in the descent procedure consists in constructing a linear combination involving the holomorphic forms $\varpi^A{}_B(4)$, $\varpi^{AB}{}_C(4)$ and the constant multiplets D^{MN}_δ and

D_δ^{MNP} that matches the $\mathfrak{B}$ monodromy of $D_\delta^{IJK}(4)$ and allows us to express it in terms of a constant multiplet D_δ^{IJKL}. Consulting (A.18), we observe that the combination on the right side of the last equation of (2.24) has the following monodromy,

$$\begin{aligned}
\Delta_L^{(3)} \Big(& D_\delta^{IJK}(4) - \varpi^K{}_L(4) D_\delta^{IJL} + \varpi^{KI}{}_L(4) D_\delta^{JL} - \varpi^{IJ}{}_L(4) D_\delta^{KL} \\
& + \varpi^I{}_L(4) D_\delta^{JKL} + \varpi^{IK}{}_L(4) D_\delta^{JL} - \varpi^{KJ}{}_L(4) D_\delta^{IL} \Big) \\
= & \frac{2\pi i}{h} \omega_L(4) \big(D_\delta^{JKI} - D_\delta^{IJK} \big) + \frac{2\pi^2}{h} \Big(\delta_L^K \big(D_\delta^{JI} - D_\delta^{IJ} \big) - \delta_L^I \big(D_\delta^{KJ} - D_\delta^{JK} \big) \Big)
\end{aligned} \tag{2.31}$$

It is at this point that we use the cyclic symmetries $D_\delta^{JI} = D_\delta^{IJ}$ and $D_\delta^{JKI} = D_\delta^{IJK}$, established earlier, to ensure that the $\mathfrak{B}$ monodromy of the above combination indeed cancels, so that its dependence on z_4 is holomorphic and single-valued and may be expressed as a linear combination of the holomorphic Abelian differentials $\omega_L(4)$ in the last equation of (2.24). Finally, cyclic invariance of D_δ^{IJKL} itself may be established using the interchange lemmas and Fay identities of [62] in analogy with the $n = 3$ case.

As was already noted for the case $n = 3$, a more streamlined proof of the cyclic invariance of D_δ^{IJKL} will be presented in section 4 which applies to the case of arbitrary n with equal ease. Similar computations lead to the reflection relation $D_\delta^{IJKL} = D_\delta^{LKJI}$, see appendix D.1 for a proof by direct computation and appendix D.3 for a proof based on general arguments that applies to arbitrary n.

2.5 Formulation of the case of arbitrary n

The case of arbitrary n may be constructed by extending the pattern observed for the cases $n = 2, 3, 4$. The lowest rank cases may be calculated *by hand* and are given by,

$$\begin{aligned}
\omega_J(1) D_\delta^J(2, \cdots, n) &= C_\delta(1, \cdots, n) + \big(\chi(1,2) - \chi(1,n) \big) C_\delta(2, \cdots, n) \\
\omega_J(2) D_\delta^{I_1 J}(3, \cdots, n) &= D_\delta^{I_1}(2, \cdots, n) + \big(\chi^{I_1}(2,3) - \chi^{I_1}(2,n) \big) C_\delta(3, \cdots, n) \\
& \quad - g^{I_1}{}_J(2,3) D_\delta^J(3, \cdots, n) \\
\omega_J(3) D_\delta^{I_1 I_2 J}(4, \cdots, n) &= D_\delta^{I_1 I_2}(3, \cdots, n) + \big(\chi^{I_2 I_1}(3,4) - \chi^{I_2 I_1}(3,n) \big) C_\delta(4, \cdots, n) \\
& \quad - g^{I_2}{}_J(3,4) D_\delta^{I_1 J}(4, \cdots, n) - g^{I_2 I_1}{}_J(3,4) D_\delta^J(4, \cdots, n)
\end{aligned} \tag{2.32}$$

The first line is valid for $n \geq 3$, the second for $n \geq 4$ and the third for $n \geq 5$. To extend the pattern to arbitrary rank, it will be convenient to rearrange the relations so that the number of points $z_i \in \Sigma$ involved in each relation equals n. Throughout, we set,

$$D_\delta^{\emptyset}(1, \cdots, n) = C_\delta(1, \cdots, n) \tag{2.33}$$

The resulting *meromorphic descent equations* are given by the theorem below.

Theorem 2.3. *The multiplet $D_\delta^{I_1\cdots I_{s+1}}(2,\cdots,n)$ is determined uniquely in terms of the multiplets $D_\delta^{I_1\cdots I_r}(1,\cdots,m)$ with $0\leq r\leq s$ and $m\leq n$ as well as Enriquez kernels by the following descent equations.*

(a) For $n\geq 3$,

$$\omega_J(1)D_\delta^{I_1\cdots I_r J}(2,\cdots,n)=D_\delta^{I_1\cdots I_r}(1,\cdots,n)+\Big(\chi^{I_r\cdots I_1}(1,2)-\chi^{I_r\cdots I_1}(1,n)\Big)C_\delta(2,\cdots,n)$$
$$-\sum_{i=0}^{r-1}g^{I_r\cdots I_{i+1}}{}_J(1,2)D_\delta^{I_1\cdots I_i J}(2,\cdots,n) \tag{2.34}$$

(b) For $n=2$,

$$\omega_J(1)D_\delta^{I_1\cdots I_r J}(2)=D_\delta^{I_1\cdots I_r}(1,2)-\partial_2\chi^{I_r\cdots I_1}(1,2)-\sum_{i=0}^{r-1}g^{I_r\cdots I_{i+1}}{}_J(1,2)D_\delta^{I_1\cdots I_i J}(2) \tag{2.35}$$

(c) The constant multiplets, defined for $r\geq 2$ by,

$$D_\delta^{I_1\cdots I_r}=\sum_{k=0}^{r-1}\sum_{\ell=0}^{r-1-k}\frac{(-)^k(2\pi i)^{k+\ell}}{(k+\ell+1)k!\ell!}\sum_J\delta_J^{I_1\cdots I_\ell}\,\delta_J^{I_{r-k}\cdots I_r}\oint_{\mathfrak{A}^J}dt\,D_\delta^{I_{\ell+1}\cdots I_{r-1-k}}(t) \tag{2.36}$$

and for $r=0,1$ to vanish, are invariant under cyclic permutations of the indices,

$$D_\delta^{I_1 I_2\cdots I_r}=D_\delta^{I_2\cdots I_r I_1} \tag{2.37}$$

(d) The constant multiplets $D_\delta^{I_1\cdots I_r}$ defined by (2.36) satisfy the following descent equation

$$\omega_J(1)D_\delta^{I_1\cdots I_r J}=D_\delta^{I_1\cdots I_r}(1)-\sum_{\substack{0\leq i<j\leq r\\(i,j)\neq(0,r)}}(-1)^i\varpi^{I_1\cdots I_i\,\sqcup\, I_r\cdots I_{j+1}}{}_J(1)D_\delta^{I_{i+1}\cdots I_j J} \tag{2.38}$$

which extends (2.34) and (2.35) to the case of $n=1$.

The proof of items (a) and (b) of the theorem will be carried out in section 2.6 below, while the proof of items (c) and (d) are relegated to section 4. As will be detailed in section 5, iterative use of items (a), (b) and (d) of the theorem leads to the advertised decomposition (1.3) of cyclic products of Szegö kernels that separates their spin structure dependence from their dependence on the points.

2.6 Proof of items (a) and (b) of Theorem 2.3

The proof of items (a) and (b) of Theorem 2.3 proceeds by induction in the rank r for $n\geq 2$. We shall show that the descent equations hold for $r=0$ as the initial step in the induction procedure in r. The second step in the induction procedure is to show that the descent equations for $r = s$ follow from the assumption that the descent equations (2.34) and (2.35) hold for all $r \leq s-1$.

For $r = 0$ and $n \geq 3$, the recursion relation of (2.34) coincides with the first relation in (2.32). To establish its validity, we use the $r = 0$ relation,

$$\bar\partial_k D^{\emptyset}_{\delta}(1,\cdots,n) = \pi\left(\delta(k,k+1) - \delta(k,k-1)\right) D^{\emptyset}_{\delta}(1,\cdots,\hat k,\cdots,n) \tag{2.39}$$

for the cyclic product (2.33) to verify that the right side is holomorphic in z_1 (since the residues of the poles in z_1 at z_2 and z_n cancel) and single-valued in z_1 since the difference $\chi(1,2) - \chi(1,n)$ is single-valued in z_1 in view of the first equation of (A.17). Therefore the left side must be a linear combination of the holomorphic Abelian differentials $\omega_J(1)$ whose coefficients are independent of z_1 and are denoted by $D^{I_1\cdots I_r J}_{\delta}(2,\cdots,n)$. For $r = 0$ and $n = 2$, the recursion relation (2.35) coincides with (2.10) whose validity was already proven in section 2.2. Thus, items (a) and (b) of Lemma 2.3 hold true for $r = 0$.

The induction step from ranks $r \leq s-1$ to rank $r = s$ will be proven in sections 2.6.1 and 2.6.2 using two lemmas for the properties of $D^{I_1\cdots I_r}_{\delta}(1,\cdots,n)$: the first giving differential equations and the second giving monodromies.

Lemma 2.4. *Assuming the relations (2.34) and (2.35) for all $r \leq s-1$, the multiplets $D^{I_1\cdots I_r}_{\delta}(1,\cdots,n)$ satisfy the following equations for all $1 \leq r \leq s$ and $2 \leq k \leq n-1$,*

$$\begin{aligned}
\bar\partial_1 D^{I_1\cdots I_r}_{\delta}(1,\cdots,n) &= \pi\,\delta(1,2) D^{I_1\cdots I_r}_{\delta}(2,\cdots,n)\\
\bar\partial_k D^{I_1\cdots I_r}_{\delta}(1,\cdots,n) &= \pi\Big(\delta(k,k+1) - \delta(k,k-1)\Big) D^{I_1\cdots I_r}_{\delta}(1,\cdots,\hat k,\cdots,n)\\
\bar\partial_n D^{I_1\cdots I_r}_{\delta}(1,\cdots,n) &= -\pi\,\delta(n,n-1) D^{I_1\cdots I_r}_{\delta}(1,\cdots,n-1)
\end{aligned} \tag{2.40}$$

where the middle equation above is absent when $n = 2$. For $1 \leq r \leq s$ and $n = 1$ we have,

$$\bar\partial_1 D^{I_1\cdots I_r}_{\delta}(1) = 0 \tag{2.41}$$

The proof of this lemma is relegated to appendix B.

Lemma 2.5. *The cyclic product $C_\delta(1,\cdots,n)$ is single-valued in $z_1,\cdots,z_n$ for all n. Assuming that the relations (2.34) and (2.35) hold for all $r \leq s-1$, the multiplets $D^{I_1\cdots I_r}_{\delta}(1,\cdots,n)$ have vanishing $\mathfrak{A}$-monodromies in $z_1,\cdots,z_n$, while their $\mathfrak{B}$-monodromies* *are given as follow for all values* $1 \leq r \leq s$ *and* $n \geq 2$ *with* $2 \leq k \leq n-1$,

$$\begin{aligned}
\Delta_L^{(1)} D_\delta^{I_1 \cdots I_r}(1, \cdots, n) &= \sum_{\ell=1}^{r} \frac{(-2\pi i)^\ell}{\ell!} \, \delta_L^{I_r \cdots I_{r+1-\ell}} D_\delta^{I_1 \cdots I_{r-\ell}}(1, \cdots, n) \\
\Delta_L^{(k)} D_\delta^{I_1 \cdots I_r}(1, \cdots, n) &= 0 \\
\Delta_L^{(n)} D_\delta^{I_1 \cdots I_r}(1, \cdots, n) &= \sum_{\ell=1}^{r} \frac{(2\pi i)^\ell}{\ell!} \, \delta_L^{I_1 \cdots I_\ell} D_\delta^{I_{\ell+1} \cdots I_r}(1, \cdots, n)
\end{aligned} \tag{2.42}$$

where the middle equation is absent for $n = 2$.

The expression for $\Delta_L^{(1)} D_\delta^{I_1 \cdots I_r}(1)$ will be derived in compact form in terms of generating functions in section 3 below, and the proof of Lemma 2.5 is relegated to appendix C.

2.6.1 Proof of item (a) of Theorem 2.3

It remains to prove the induction step giving the derivation of the descent equations of (2.34) and (2.35) for $r = s$, assuming the validity of the descent equations for $0 \leq r \leq s-1$. To this end, we define the combination $\tilde{D}_\delta^{I_1 \cdots I_s}(1, \cdots, n)$ for arbitrary $n \geq 3$ by,

$$\begin{aligned}
\tilde{D}_\delta^{I_1 \cdots I_s}(1, \cdots, n) = D_\delta^{I_1 \cdots I_s}(1, \cdots, n) &+ \Big(\chi^{I_s \cdots I_1}(1,2) - \chi^{I_s \cdots I_1}(1,n) \Big) C_\delta(2, \cdots, n) \\
&- \sum_{i=0}^{s-1} g^{I_s \cdots I_{i+1}}{}_J(1,2) D_\delta^{I_1 \cdots I_i J}(2, \cdots, n)
\end{aligned} \tag{2.43}$$

where we have chosen the right side to be the combination that enters the descent equation (2.34) for $r = s$ and the corresponding value of n. To prove the induction step, it will suffice to prove that $\tilde{D}_\delta^{I_1 \cdots I_s}(1, \cdots, n)$ is single-valued and holomorphic in z_1, so that it must a linear combination of the holomorphic Abelian differentials $\omega_J(1)$ and defines the corresponding coefficients $D_\delta^{I_1 \cdots I_s J}(2, \cdots, n)$ of rank $s+1$ by,

$$\tilde{D}_\delta^{I_1 \cdots I_s}(1, \cdots, n) = \omega_J(1) D_\delta^{I_1 \cdots I_s J}(2, \cdots, n) \tag{2.44}$$

To prove these properties of $\tilde{D}_\delta^{I_1 \cdots I_s}(1, \cdots, n)$, we make use of Lemmas 2.4 and 2.5, as we shall now show in the remainder of this subsection.

To prove that $\tilde{D}_\delta^{I_1 \cdots I_s}(1, \cdots, n)$ is holomorphic in z_1 we use the fact that $0 \leq r \leq s-1$ implies that $1 \leq s$ so that the combination $\chi^{I_s \cdots I_1}(1,2) - \chi^{I_s \cdots I_1}(1,n)$ is holomorphic inside the preferred fundamental domain D. Furthermore, only the contribution from $i = s-1$ to the sum on the second line has a pole, so that,

$$\bar{\partial}_1 \tilde{D}_\delta^{I_1 \cdots I_s}(1, \cdots, n) = \bar{\partial}_1 D_\delta^{I_1 \cdots I_s}(1, \cdots, n) - \pi \delta(1,2) D_\delta^{I_1 \cdots I_s}(2, \cdots, n) \tag{2.45}$$

In view of of the first line of (2.42) in Lemma 2.4, which holds for $r = s$, the right side vanishes, so that $\tilde D_\delta^{I_1\cdots I_s}(1,\cdots,n)$ is indeed holomorphic in z_1.

To prove that $\tilde D_\delta^{I_1\cdots I_s}(1,\cdots,n)$ is single-valued in z_1, we use the fact that its $\mathfrak{A}$ monodromy vanishes by construction while its $\mathfrak{B}$ monodromy is given as follows,

$$\Delta_L^{(1)}\tilde D_\delta^{I_1\cdots I_s}(1,\cdots,n) = \Delta_L^{(1)} D_\delta^{I_1\cdots I_s}(1,\cdots,n) - \sum_{i=0}^{s-1}\Delta_L^{(1)} g^{I_s\cdots I_{i+1}}{}_J(1,2) D_\delta^{I_1\cdots I_i J}(2,\cdots,n)$$
$$+\Big(\Delta_L^{(1)}\chi^{I_s\cdots I_1}(1,2)-\Delta_L^{(1)}\chi^{I_s\cdots I_1}(1,n)\Big)C_\delta(2,\cdots,n) \tag{2.46}$$

The first term on the right side may be evaluated using the first equation in (2.42) of Lemma 2.5, while the monodromies of g and χ are given by the first equations in (2.3) and (A.14), respectively. Assembling all contributions and rearranging the double sum that emerges in the last line above, we obtain,

$$\Delta_L^{(1)}\tilde D_\delta^{I_1\cdots I_s}(1,\cdots,n) = \sum_{k=1}^{s}\frac{(-2\pi i)^k}{k!}\,\delta_L^{I_s\cdots I_{s+1-k}}\bigg[D_\delta^{I_1\cdots I_{s-k}}(1,\cdots,n)$$
$$+\Big(\chi^{I_{s-k}\cdots I_1}(1,2)-\chi^{I_{s-k}\cdots I_1}(1,n)\Big)C_\delta(2,\cdots,n)$$
$$-\sum_{i=0}^{s-k} g^{I_{s-k}\cdots I_{i+1}}{}_J(1,2) D_\delta^{I_1\cdots I_i J}(2,\cdots,n)\bigg] \tag{2.47}$$

The combination inside the square brackets vanishes by the descent equation of (2.34) for $r = s-k \leq s-1$. This concludes the proof of item (a) of Theorem 2.3.

2.6.2 Proof of item (b) of Theorem 2.3

In item (b) we have $n = 2$ and we define the corresponding combination,

$$\tilde D_\delta^{I_1\cdots I_s}(1,2) = D_\delta^{I_1\cdots I_s}(1,2) - \partial_2\chi^{I_s\cdots I_1}(1,2) - \sum_{i=0}^{s-1} g^{I_s\cdots I_{i+1}}{}_J(1,2) D_\delta^{I_1\cdots I_i J}(2) \tag{2.48}$$

in (2.35) at $r = s$. Its holomorphicity in z_1 follows from the facts that $\partial_2\chi^{I_s\cdots I_1}(1,2)$ is holomorphic in $z_1 \in D$ for $s \geq 1$ combined with the first line of (2.40) of Lemma 2.4. The multiplet $\tilde D_\delta^{I_1\cdots I_s}(1,2)$ is single-valued in z_1 since its $\mathfrak{A}$ monodromy vanishes by construction while its $\mathfrak{B}$ monodromy in z_1 may be evaluated using the first equation in (2.42) of Lemma 2.5, while the monodromies of g and χ are given by the first lines in

(2.3) and (A.14), respectively. After some simplifications, we obtain,

$$\Delta_L^{(1)} \tilde{D}_\delta^{I_1 \cdots I_s}(1,2) = \sum_{k=1}^{s} \frac{(-2\pi i)^k}{k!} \, \delta_L^{I_s \cdots I_{s+1-k}} \Big[D_\delta^{I_1 \cdots I_{s-k}}(1,2) + \partial_2 \chi^{I_{s-k} \cdots I_1}(1,2) - \sum_{i=0}^{s-k} g^{I_{s-k} \cdots I_{i+1}}{}_J(1,2) D_\delta^{I_1 \cdots I_i J}(2) \Big] \tag{2.49}$$

The combination inside the square brackets vanishes by the descent equation of (2.34) for $r = s-k \leq s-1$. Therefore, $\tilde{D}_\delta^{I_1 \cdots I_s}(1,2)$ must be a linear combination of the holomorphic Abelian differentials $\omega_J(1)$ and we shall define the corresponding coefficients $D_\delta^{I_1 \cdots I_s J}(2)$ of rank $s+1$ by,

$$\tilde{D}_\delta^{I_1 \cdots I_s}(1,2) = \omega_J(1) D_\delta^{I_1 \cdots I_s J}(2) \tag{2.50}$$

This completes the proof of item (b) of Theorem 2.3.

3 Generating functions

In this section, the multiplets D_δ will be collected in a generating function $\mathbf{D}_\delta$ analogously to how the Enriquez kernels g were collected in the form $\mathbf{K}_J$ given in (2.2). Both generating functions are valued in an infinite-dimensional Lie algebra $\mathfrak{g}_b$ that is freely generated by h elements $b = \{b_1, \cdots, b_h\}$. The recursion relations of Theorem 2.3, the differential equations of Lemma 2.4, and the monodromies of Lemma 2.5 will be compactly expressed in terms of these generating functions and used to provide the proof of items (c) and (d) of Theorem 2.3 in section 4, thereby completing the proof of the theorem.

3.1 Generating functions for the Enriquez kernels

In this first subsection, we begin by reviewing and elaborating on the properties of the connection form $\mathbf{K}_J$, viewed as a generating function for the Enriquez kernels $g^{I_1\cdots I_r}{}_J$. Recall that the Taylor series expansion of the form $\mathbf{K}_J(x,y;B)$ is given by equation (2.2) which we repeat here for convenience,

$$\mathbf{K}_J(x,y;B) = \sum_{r=0}^{\infty} g^{I_1\cdots I_r}{}_J(x,y) B_{I_1}\cdots B_{I_r} \tag{3.1}$$

with $g^{\emptyset}{}_J(x,y) = \omega_J(x)$ as was stated already in footnote 3 and where the h elements B_I generate the free Lie algebra $\mathfrak{g}_b$ by adjoint action $B_I X = [b_I, X]$ for all $X \in \mathfrak{g}_b$.[9] It will be useful to introduce separate generating functions that capture the decomposition (2.8) of Enriquez kernels $g^{I_1\cdots I_r}{}_J(x,y)$ into their trace and traceless parts, given by $\chi^{I_1\cdots I_r}(x,y)$ and $\varpi^{I_1\cdots I_r}{}_J(x)$, respectively, as follows,

$$\begin{aligned} \mathbf{X}(x,y;B) &= \sum_{r=0}^{\infty} \chi^{I_1\cdots I_r}(x,y) B_{I_1}\cdots B_{I_r} \\ \mathbf{W}_J(x;B) &= \sum_{r=0}^{\infty} \varpi^{I_1\cdots I_r}{}_J(x) B_{I_1}\cdots B_{I_r} \end{aligned} \tag{3.2}$$

with $\varpi^{\emptyset}{}_J(x) = \omega_J(x)$. The decomposition of (2.8) is then equivalent to,

$$\mathbf{K}_J(x,y;B) = \mathbf{W}_J(x;B) - \mathbf{X}(x,y;B) B_J \tag{3.3}$$

The basic differential equations are stated in a *preferred fundamental domain* D for Σ (see figure 1) in which $g^I{}_J(x,y)$ has a simple pole while $g^{I_1\cdots I_r}{}_J(x,y)$ for $r=0$ and for

[9] Henceforth, the generators $a^1, \cdots, a^h$ will no longer play a role. The generators b_I will enter only through their adjoint action via B_I. The algebra freely generated by the B_I is isomorphic to $\mathfrak{g}_b$.

$r \geq 2$ are non-singular. For $x, y \in D$ we then have,

$$\begin{aligned}
\partial_{\bar x}\mathbf{K}_J(x,y;B) &= \pi\delta(x,y)B_J & \partial_{\bar x}\mathbf{X}(x,y;B) &= -\pi\delta(x,y) \\
\partial_{\bar y}\mathbf{K}_J(x,y;B) &= -\pi\delta(x,y)B_J & \partial_{\bar y}\mathbf{X}(x,y;B) &= \pi\delta(x,y) \\
& & \partial_{\bar x}\mathbf{W}_J(x;B) &= 0
\end{aligned} \tag{3.4}$$

Beyond the fundamental domain we will use the monodromy equations, to be derived below, to extend the above formulas to all of $\tilde\Sigma$ (see the discussion preceding figure 1). As a result, $\mathbf{K}_J(x,y;B)$ and $\mathbf{X}(x,y;B)$ will have simple poles at the points $\pi^{-1}(y)$ with residues that involve all powers of B so that $g^{I_1\cdots I_r}{}_J(x,y)$ will have poles for all $r \geq 1$.

The monodromy relations for g in (2.3), for χ in (A.14), and for ϖ in (A.13) translate into the following monodromy relations for $\mathbf{K}_J$,

$$\begin{aligned}
\mathbf{K}_J(\mathfrak{B}_L\cdot x, y; B) &= e^{-2\pi i B_L}\,\mathbf{K}_J(x,y;B) \\
\mathbf{K}_J(x, \mathfrak{B}_L\cdot y; B) &= \mathbf{K}_J(x,y;B) + \mathbf{K}_L(x,y;B)\,\frac{e^{2\pi i B_L}-1}{B_L}\,B_J
\end{aligned} \tag{3.5}$$

for the trace part $\mathbf{X}$,

$$\begin{aligned}
\mathbf{X}(\mathfrak{B}_L\cdot x, y; B) &= e^{-2\pi i B_L}\,\mathbf{X}(x,y;B) - \frac{1}{h}\,\frac{e^{-2\pi i B_L}-1}{B_L}\,\omega_L(x) \\
\mathbf{X}(x, \mathfrak{B}_L\cdot y; B) &= \mathbf{X}(x,y;B) - \mathbf{K}_L(x,y;B)\,\frac{e^{2\pi i B_L}-1}{B_L}
\end{aligned} \tag{3.6}$$

and for the traceless part $\mathbf{W}_J$,

$$\mathbf{W}_J(\mathfrak{B}_L\cdot x; B) = e^{-2\pi i B_L}\mathbf{W}_J(x;B) - \frac{1}{h}\,\frac{e^{-2\pi i B_L}-1}{B_L}\,B_J\omega_L(x) \tag{3.7}$$

Finally, the integral relations for g in (2.7) translate into the following integral of $\mathbf{K}_J$,

$$\oint_{\mathfrak{A}^L} dt\,\mathbf{K}_J(t,y;B) = \frac{-2\pi i B_L}{e^{-2\pi i B_L}-1}\,\delta^L_J \tag{3.8}$$

using the generating function for the Bernoulli numbers given in footnote 5.

3.2 Generating functions for D_δ

The multiplets $D_\delta^{I_1\cdots I_r}(1,\cdots,n)$ may be collected in a generating function that takes values in the Lie algebra $\mathfrak{g}_b$. Setting $D_\delta^{\emptyset}(1,\cdots,n) = C_\delta(1,\cdots,n)$ as in (2.33), we introduce the following closely related generating functions for arbitrary $n \geq 0$,[10]

$$\mathbf{D}_\delta(1, \cdots, n; B) = \sum_{r=0}^{\infty} D_\delta^{I_1 \cdots I_r}(1, \cdots, n) B_{I_r} \cdots B_{I_1}$$
$$\mathbf{D}_\delta^J(1, \cdots, n; B) = \sum_{r=0}^{\infty} D_\delta^{I_1 \cdots I_r J}(1, \cdots, n) B_{I_r} \cdots B_{I_1} \tag{3.9}$$

These relations combined with (2.33) for $n \geq 2$ imply,

$$\mathbf{D}_\delta(1, \cdots, n; B) = C_\delta(1, \cdots, n) + B_J \mathbf{D}_\delta^J(1, \cdots, n; B) \qquad n \geq 2 \tag{3.10}$$

while for $n = 0, 1$ we have,

$$D_\delta^\emptyset(\emptyset; B) = 0 \qquad \mathbf{D}_\delta(\emptyset; B) = B_J \mathbf{D}_\delta^J(\emptyset; B)$$
$$D_\delta^\emptyset(1; B) = 0 \qquad \mathbf{D}_\delta(1; B) = B_J \mathbf{D}_\delta^J(1; B) \tag{3.11}$$

The generating function $\mathbf{D}_\delta^J$ may be deduced from $\mathbf{D}_\delta$ by the operation *Iota* on elements in the Lie algebra $\mathfrak{g}_b$, defined by,

$$\mathcal{I}^J 1 = 0 \qquad \mathcal{I}^J \big(B_I X\big) = \delta_I^J \, X \tag{3.12}$$

for any $X \in \mathfrak{g}$. The relation in (3.10) then gives,

$$\mathbf{D}_\delta^J(1, \cdots, n) = \mathcal{I}^J \mathbf{D}_\delta(1, \cdots, n) \tag{3.13}$$

Thus, the operation $\mathcal{I}^J$ *removes* the first factor of B in any Taylor series in powers of B to which it is being applied and generates the extra upper index J as a result.

3.3 Recursion relations for D_δ

The recursion relations on the components $D_\delta^{I_1 \cdots I_r}(1, \cdots, n)$ for $n \geq 3$ in item (a) and for $n = 2$ in item (b) of Theorem 2.3 may be reformulated compactly in terms of the generating functions $\mathbf{D}_\delta$ and $\mathbf{D}_\delta^J$ in (3.9) via the following corollary.

Corollary 3.1. *The generating functions $\mathbf{D}_\delta$ and $\mathbf{D}_\delta^J$ satisfy the following relations.*
(a) For $n \geq 3$,

$$\mathbf{D}_\delta(1, \cdots, n; B) = \mathbf{K}_J(1, 2; B) \mathbf{D}_\delta^J(2, \cdots, n; B)$$
$$- \big(\mathbf{X}(1, 2; B) - \mathbf{X}(1, n; B)\big) C_\delta(2, \cdots, n) \tag{3.14}$$

[10]The reversal of the order in which the contractions with the generators B_I are performed results from our choice for the order of the indices on D_δ and is purely a matter of (perhaps unfortunate) convention.

(b) For $n=2$,

$$\mathbf{D}_\delta(1,2;B) = \mathbf{K}_J(1,2;B)\mathbf{D}_\delta^J(2;B) + \partial_2\mathbf{X}(1,2;B) \tag{3.15}$$

Parts (a) and (b) of the corollary are proven by reformulating items (a) and (b), respectively, of Theorem 2.3 into generating functions.

3.4 Differential relations

The system of differential equations of (2.40) and (2.41) of Lemma 2.4 on the components $D_\delta^{I_1\cdots I_r}(1,\cdots,n)$ may be readily reformulated in terms of the differential equations for the generating functions given in the corollary below. Equivalently, they may be derived directly from the relations (3.14) and (3.15) on the generating functions together with the antiholomorphic derivatives (3.4) of the generating series for Enriquez kernels.

Corollary 3.2. *The generating functions* $\mathbf{D}_\delta$ *satisfy the following differential relations.*
(a) For $n\geq 3$ *and* $2\leq k\leq n-1$,

$$\begin{aligned}
\bar\partial_1\mathbf{D}_\delta(1,\cdots,n;B) &= \pi\,\delta(1,2)\mathbf{D}_\delta(2,\cdots,n;B) - \pi\,\delta(1,n)C_\delta(2,\cdots,n) \\
\bar\partial_k\mathbf{D}_\delta(1,\cdots,n;B) &= \pi\Big(\delta(k,k+1)-\delta(k,k-1)\Big)\mathbf{D}_\delta(1,\cdots,\hat k,\cdots,n;B) \\
\bar\partial_n\mathbf{D}_\delta(1,\cdots,n;B) &= \pi\,\delta(n,1)C_\delta(1,\cdots,n-1) - \pi\,\delta(n,n-1)\mathbf{D}_\delta(1,\cdots,n-1;B)
\end{aligned} \tag{3.16}$$

(b) For $n=1,2$,

$$\begin{aligned}
\bar\partial_1\mathbf{D}_\delta(1,2;B) &= \pi\,\delta(1,2)\,\mathbf{D}_\delta(2;B) + \pi\,\partial_1\delta(1,2) \\
\bar\partial_2\mathbf{D}_\delta(1,2;B) &= -\pi\,\delta(1,2)\,\mathbf{D}_\delta(1;B) + \pi\,\partial_2\delta(1,2) \\
\bar\partial_1\mathbf{D}_\delta(1;B) &= 0
\end{aligned} \tag{3.17}$$

The differential relations for the generating functions $\mathbf{D}_\delta^J$ may be easily obtained by applying the operator $\mathcal{I}^J$ introduced in (3.12) to the relations of Corollary 3.2. For $n\geq 2$ and $2\leq k\leq n-1$ we obtain,

$$\begin{aligned}
\bar\partial_1\mathbf{D}_\delta^J(1,\cdots,n;B) &= \pi\,\delta(1,2)\mathbf{D}_\delta^J(2,\cdots,n;B) \\
\bar\partial_k\mathbf{D}_\delta^J(1,\cdots,n;B) &= \pi\big(\delta(k,k+1)-\delta(k,k-1)\big)\mathbf{D}_\delta^J(1,\cdots,\hat k,\cdots,n;B) \\
\bar\partial_n\mathbf{D}_\delta^J(1,\cdots,n;B) &= -\pi\,\delta(n,n-1)\mathbf{D}_\delta^J(1,\cdots,n-1;B)
\end{aligned} \tag{3.18}$$

where the middle equation is absent for $n=2$, while for $n=1$ we have,

$$\bar\partial_1\mathbf{D}_\delta^J(1;B) = 0 \tag{3.19}$$

3.5 Monodromy relations for $n \geq 2$

The monodromy relations of (2.42) in Lemma 2.5 on the components $D_\delta^{I_1 \cdots I_r}(1, \cdots, n)$ may be reformulated in terms of monodromy relations for the generating functions given in the corollary below. Equivalently, they may be derived directly by evaluating the monodromy of the recursion relations (3.14) and (3.15) on the generating functions through the monodromies (3.5) and (3.6) of the contributing series in Enriquez kernels.

Corollary 3.3. *The generating functions* $\mathbf{D}_\delta$ *satisfy the following monodromy relations for* $n \geq 2$ *and* $2 \leq k \leq n-1$,

$$\begin{aligned} \mathbf{D}_\delta(\mathfrak{B}_L \cdot 1, \cdots, n; B) &= e^{-2\pi i B_L} \mathbf{D}_\delta(1, \cdots, n; B) \\ \mathbf{D}_\delta(1, \cdots, \mathfrak{B}_L \cdot k, \cdots n; B) &= \mathbf{D}_\delta(1, \cdots, n; B) \\ \mathbf{D}_\delta(1, \cdots, \mathfrak{B}_L \cdot n; B) &= \mathbf{D}_\delta(1, \cdots, n; B)\, e^{2\pi i B_L} \end{aligned} \tag{3.20}$$

where the middle equation is absent when $n = 2$.

The monodromy relations for the generating functions $\mathbf{D}_\delta^J$ may be obtained by applying the operator $\mathcal{I}^J$ of (3.12) to the monodromy relations of Corollary 3.3. For $n \geq 2$ and $2 \leq k \leq n-1$ we obtain,

$$\begin{aligned} \mathbf{D}_\delta^J(\mathfrak{B}_L \cdot 1, \cdots, n; B) &= \mathbf{D}_\delta^J(1, \cdots, n; B) + \delta_L^J \, \frac{e^{-2\pi i B_L} - 1}{B_L} \, \mathbf{D}_\delta(1, \cdots, n; B) \\ \mathbf{D}_\delta^J(1, \cdots, \mathfrak{B}_L \cdot k, \cdots, n; B) &= \mathbf{D}_\delta^J(1, \cdots, n; B) \\ \mathbf{D}_\delta^J(1, \cdots, \mathfrak{B}_L \cdot n; B) &= \mathbf{D}_\delta^J(1, \cdots, n; B)\, e^{2\pi i B_L} + \delta_L^J \, \frac{e^{2\pi i B_L} - 1}{B_L} C_\delta(1, \cdots, n) \end{aligned} \tag{3.21}$$

3.6 Monodromy relation for $\mathbf{D}_\delta(1; B)$

The generating function $\mathbf{D}_\delta(2; B)$ is related to the generating function $\mathbf{D}_\delta(1, 2; B)$ by equation (3.15) with the help of the series $\mathbf{K}_J(1, 2)$ and $\mathbf{X}(1, 2)$ in Enriquez kenels. This relation allows us to evaluate the $\mathfrak{B}$ monodromy of $\mathbf{D}_\delta(2; B)$ in the variable z_2 in terms of the $\mathfrak{B}$ monodromy of $\mathbf{D}_\delta(1, 2; B)$ in the same variable, which is given by the last equation of (3.20) of Corollary 3.3 and is stated in the lemma below. This lemma will serve as an intermediary step in the proof of items (c) and (d) of Theorem 2.3.

Lemma 3.4. *The monodromy relations for the one-point functions are given by,*

$$\begin{aligned} \mathbf{D}_\delta(\mathfrak{B}_L \cdot 1; B) &= e^{-2\pi i B_L} \, \mathbf{D}_\delta(1; B) \, e^{2\pi i B_L} \\ \mathbf{D}_\delta^J(\mathfrak{B}_L \cdot 1; B) &= \mathbf{D}_\delta^J(1; B) \, e^{2\pi i B_L} + \delta_L^J \, \frac{e^{-2\pi i B_L} - 1}{B_L} \, \mathbf{D}_\delta(1; B) \, e^{2\pi i B_L} \end{aligned} \tag{3.22}$$

To prove the lemma, we apply $\mathfrak{B}_L$ to the point z_2 in (3.15),

$$\mathbf{D}_\delta(1,\mathfrak{B}_L\cdot 2;B)=\mathbf{K}_J(1,\mathfrak{B}_L\cdot 2;B)\mathbf{D}_\delta^J(\mathfrak{B}_L\cdot 2;B)+\partial_2\mathbf{X}(1,\mathfrak{B}_L\cdot 2;B) \tag{3.23}$$

and then use the last equation of (3.20) for $n=2$ along with the monodromy relations given by the second equations of (3.5) and (3.6). The resulting identity,

$$\begin{aligned}\mathbf{K}_J(1,2;B)\mathbf{D}_\delta^J(2;B)e^{2\pi i B_L}=\mathbf{K}_J(1,2;B)\mathbf{D}_\delta^J(\mathfrak{B}_L\cdot 2;B)&\\ +\mathbf{K}_L(1,2;B)\,\frac{e^{2\pi i B_L}-1}{B_L}\,\mathbf{D}_\delta(\mathfrak{B}_L\cdot 2;B)&\end{aligned} \tag{3.24}$$

implies the first line of (3.22) by matching the residues of the pole in z_1 at z_2 where $\mathbf{K}_J(1,2;B)$ contributes via B_J and the contractions $\mathbf{K}_J(1,2;B)\mathbf{D}_\delta^J(i;B)$ thereby simplify to $\mathbf{D}_\delta(i;B)=B_J\mathbf{D}_\delta^J(i;B)$ (for $i=2$ or $i=\mathfrak{B}_L\cdot 2$). The second equation of (3.22) then follows from applying the operator $\mathcal{I}^J$ to the first equation.

4 Convolution periods

To complete the proof of Theorem 2.3, it remains to establish its items (c) and (d). We shall do so in this section with the help of the generating functions introduced in section 3 and the use of $\mathfrak{A}$ periods of the multiplets $D_\delta^{I_1\cdots I_r}(1,\cdots,n)$ which we shall introduce in this section. In the process we shall establish a recursion relation for the periods of the multiplets $D_\delta^{I_1\cdots I_r}(1,\cdots,n)$; show that the operation of taking $\mathfrak{A}$ periods closes in the space of multiplets; and prove that the multiplets can be represented by multiple $\mathfrak{A}$ period integrals of cyclic products of Szegö kernels.

4.1 Recursion relation for $\mathfrak{A}$ periods

The starting points for the recursion relations between the $\mathfrak{A}$ periods of the multiplets $D_\delta^{I_1\cdots I_r}(1,\cdots,n)$ are the recursion relations (2.34) and (2.35) of items (a) and (b) of Theorem 2.3, respectively, which were already proven in section 2.6. To compute the $\mathfrak{A}$ periods, we make use of the crucial property, shown in (2.7) and (3.8), that the $\mathfrak{A}$ periods of $g^{I_1\cdots I_r}{}_J(x,y)$ and $\mathbf{K}_J(x,y)$ in the variable x are independent of y. As a result, when taking the integrals of (2.34) and (2.35) in the variable z_1 over a cycle $\mathfrak{A}^L$, the contributions from the terms $\chi^{I_r\cdots I_1}(1,2)-\chi^{I_r\cdots I_1}(1,n)$ and $\partial_2\chi^{I_r\cdots I_1}(1,2)$ vanish, and we obtain the following recursion relations for all $n\geq 2$,

$$\oint_{\mathfrak{A}^L} dt\, D_\delta^{I_1\cdots I_r}(t,2,\cdots,n)=\sum_{k=0}^{r}(-2\pi i)^k\,\frac{\mathrm{Ber}_k}{k!}\,\delta_L^{I_r\cdots I_{r+1-k}}\,D_\delta^{I_1\cdots I_{r-k}L}(2,\cdots,n) \tag{4.1}$$

where we recall that Ber_k are the Bernoulli numbers (see footnote 5). Reformulated in terms of generating functions, the relation (4.1) is equivalent to,

$$\oint_{\mathfrak{A}^L} dt\,\mathbf{D}_\delta(t,2,\cdots,n;B)=\frac{-2\pi i B_L}{e^{-2\pi i B_L}-1}\,\mathbf{D}_\delta^L(2,\cdots,n;B) \tag{4.2}$$

The prefactor on the right is invertible, since its series expansion starts with the identity. Its inverse gives the generating function for the multiplets $D_\delta^{I_1\cdots I_r}(2,\cdots,n)$ in terms of an $\mathfrak{A}$ period of the generating function with an extra point t,

$$\mathbf{D}_\delta^L(2,\cdots,n;B)=\frac{e^{-2\pi i B_L}-1}{-2\pi i B_L}\oint_{\mathfrak{A}^L} dt\,\mathbf{D}_\delta(t,2,\cdots,n;B) \tag{4.3}$$

which translates into the following relation between the components,

$$D_\delta^{I_1\cdots I_r L}(2,\cdots,n)=\sum_{\ell=0}^{r}\frac{(-2\pi i)^{r-\ell}}{(r-\ell+1)!}\,\delta_L^{I_{\ell+1}\cdots I_r}\oint_{\mathfrak{A}^L} dt\,D_\delta^{I_1\cdots I_\ell}(t,2,\cdots,n) \tag{4.4}$$

For example, to low rank we have,

$$
\begin{aligned}
D_\delta^L(2,\cdots,n) &= \oint_{\mathfrak{A}^L} dt\, C_\delta(t,2,\cdots,n) \\
D_\delta^{IL}(2,\cdots,n) &= \oint_{\mathfrak{A}^L} dt \left\{ D_\delta^I(t,2,\cdots,n) - i\pi\delta_L^I C_\delta(t,2,\cdots,n) \right\}
\end{aligned}
\tag{4.5}
$$

One may now proceed by eliminating the multiplet $D_\delta^I(2,\cdots,n)$ between the first and second equations in (4.5) in order to obtain a formula for $D_\delta^{IL}(2,\cdots,n)$ in terms of $\mathfrak{A}$ periods of the cyclic product of Szegö kernels only,

$$
D_\delta^{IL}(2,\cdots,n) = \oint_{\mathfrak{A}^L} dt_2 \oint_{\mathfrak{A}^I} dt_1\, C_\delta(t_1,t_2,2,\cdots,n) - i\pi\delta_L^I \oint_{\mathfrak{A}^L} dt\, C_\delta(t,2,\cdots,n) \tag{4.6}
$$

Actually, the double integral on the right side is well-defined only when $L \neq I$. When $L = I$ the pole of $C_\delta(t_1,t_2,2,\cdots,n)$ in t_2 at t_1 sits on the contour of the integration in t_2. A careful contour prescription is required for how the t_2 integration across this pole should proceed, which will be formulated in the next subsection.

4.2 Prescription for multiple $\mathfrak{A}$ period integrations

The Enriquez kernels $g^{I_1\cdots I_r}{}_J(t,y)$ are defined for t,y in the (open) interior D^o of the fundamental domain D. To evaluate their $\mathfrak{A}^L$-periods in the variable t using (2.7), we take their integral over a cycle $\mathfrak{A}^L_\varepsilon$ that is homotopic to $\mathfrak{A}^L$ and is entirely contained in D^o (barring the end points), as shown in the left panel of figure 2, and then take the limit of this integral as $\varepsilon \to 0$ and $\mathfrak{A}^L_\varepsilon \to \mathfrak{A}^L$ on the boundary of D, while leaving the point y fixed in the interior $y \in D^o$. The small displacement $\mathfrak{A}^L_\varepsilon$ of $\mathfrak{A}^L$ is understood to not cross the point $y \in D^o$ as we take $\varepsilon \to 0$. This limiting procedure was implicit in our earlier manipulations of the $\mathfrak{A}$ period integrals where the $\mathfrak{A}$ cycles lie on the boundary of D as shown in figure 1.

When multiple integrations over $\mathfrak{A}$ cycles are considered, the above limiting procedure provides a unique prescription for how the nested integrations over multiple contours should be defined. Consider, for example, the double integral encountered in (4.6). When the indices take different values $L \neq I$, the integral is well-defined as it stands. When $L = I$, we introduce two curves $\mathfrak{A}^L_\varepsilon$ and $\mathfrak{A}^L_{2\varepsilon}$ which are homotopic to $\mathfrak{A}^L$, entirely contained in D^o, such that $\mathfrak{A}^L_{2\varepsilon}$ lies *deeper inside* D^o than $\mathfrak{A}^L_\varepsilon$ and all other arguments of the integrand, namely $z_2,\cdots,z_n$, lie deeper inside D^o than the curve $\mathfrak{A}^L_{2\varepsilon}$. This set-up is shown in the left panel of figure 2. The double integral is then defined by,

$$
\oint_{\mathfrak{A}^L} dt_2 \oint_{\mathfrak{A}^L} dt_1\, C_\delta(t_1,t_2,2,\cdots,n) = \lim_{\varepsilon\to 0} \oint_{\mathfrak{A}^L_{2\varepsilon}} dt_2 \oint_{\mathfrak{A}^L_\varepsilon} dt_1\, C_\delta(t_1,t_2,2,\cdots,n) \tag{4.7}
$$

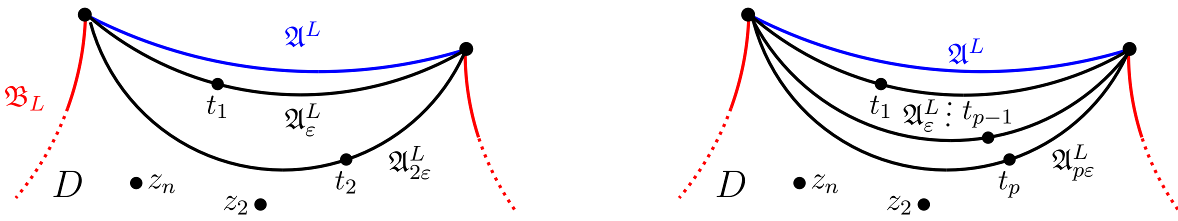

Figure 2: The left panel depicts the cycles $\mathfrak{A}^L_\varepsilon$ and $\mathfrak{A}^L_{2\varepsilon}$ for $\varepsilon > 0$ as a small homotopic deformation of $\mathfrak{A}^L$ contained in the interior D^o of D (barring the end points). The right panel depicts the integration contours $\mathfrak{A}^L_{k\varepsilon}$ for $k = 1, \cdots, p$ for the multiple integrals in (4.8) and coincident indices $I_{i_1} = I_{i_2} = \cdots = I_{i_p} = L$, with the associated integration variables $t_{i_1}, \cdots, t_{i_p}$. In both cases, any other arguments of the integrand, including $z_2, \cdots, z_n$, are assumed to be *deeper inside* D^o than any of the curves $\mathfrak{A}^L_{k\varepsilon}$.

To define the r-fold $\mathfrak{A}$ period integral of the product of Szegö kernels,

$$\mathfrak{D}_\delta^{I_1\cdots I_r}(1,\cdots,n) = \oint_{\mathfrak{A}^{I_r}} dt_r \cdots \oint_{\mathfrak{A}^{I_1}} dt_1 \, C_\delta(t_1,\cdots,t_r,1,\cdots,n) \tag{4.8}$$

we proceed analogously. When the indices I_k and I_ℓ for $k \neq \ell$ take different values, the integrals are well-defined as they stand. When p of the indices $I_{i_1}, \cdots, I_{i_p}$, with $2 \leq p \leq r$, are all equal to one another (and different from all other indices) we shall denote this common value by L. To define the integral over the variables $t_{i_1}, \cdots, t_{i_p}$, we introduce a sequence of contours $\mathfrak{A}^L_{k\varepsilon}$ for $k = 1, \cdots, p$ which are all homotopic to $\mathfrak{A}^L$, as shown in the right panel of figure 2. Each contour is entirely contained in D^o; the contours are ordered so that $\mathfrak{A}^L_{k\varepsilon}$ is *deeper inside* D^o than $\mathfrak{A}^L_{k'\varepsilon}$ for all $k' < k$; and all other arguments of the integrand lie *deeper inside* D^o than $\mathfrak{A}^L_{p\varepsilon}$. The p-fold integral is then defined by the following limit,

$$\lim_{\varepsilon\to 0} \oint_{\mathfrak{A}^L_{p\varepsilon}} dt_{i_p} \oint_{\mathfrak{A}^L_{(p-1)\varepsilon}} dt_{i_{p-1}} \cdots \oint_{\mathfrak{A}^L_{2\varepsilon}} dt_{i_2} \oint_{\mathfrak{A}^L_{\varepsilon}} dt_{i_1} \, C_\delta(t_1,\cdots,t_r,1,\cdots,n) \tag{4.9}$$

When several other indices in the multiple integral of (4.8) have the same value $K \neq L$, the above prescription is to be applied to their $\mathfrak{A}$ cycle integrations. We note that the same prescription to displace integration contours in iterated $\mathfrak{A}$ cycle convolutions was employed in the integral representations of Enriquez kernels in [58].

4.3 Expressing $\mathbf{D}_\delta$ in terms of multiple $\mathfrak{A}$ periods of C_δ

With the integration contour prescription of section 4.2 in place, we are now equipped to recast the multiplets $D_\delta^{I_1\cdots I_r}(1,\cdots,n)$ with $n \geq 1$ in terms of the multiple $\mathfrak{A}$ periods of cyclic products C_δ defined in (4.8). To do so, we multiply both sides of (4.3) by B_L and sum the result over all L,[11]

$$B_L \mathbf{D}^L_\delta(\mathbf{z};B) = \sum_L \beta_L \oint_{\mathfrak{A}^L} dt\, \mathbf{D}_\delta(t,\mathbf{z};B) \tag{4.10}$$

where β_L is given by the following series in B_L,

$$\beta_L = \frac{e^{-2\pi i B_L}-1}{-2\pi i} = B_L + \sum_{k=2}^{\infty} \frac{(-2\pi i)^{k-1}}{k!}(B_L)^k \tag{4.11}$$

Next, we use equation (3.10) to express the left side of (4.10) in terms of $\mathbf{D}_\delta(\mathbf{z};B)$ and $C_\delta(\mathbf{z})$. Doing so, we obtain an integral equation that relates generating functions $\mathbf{D}_\delta$ with different numbers of points to one another and $C_\delta(\mathbf{z})$,

$$\mathbf{D}_\delta(\mathbf{z};B) = C_\delta(\mathbf{z}) + \sum_L \beta_L \oint_{\mathfrak{A}^L} dt\, \mathbf{D}_\delta(t,\mathbf{z};B) \tag{4.12}$$

The first term on the right side is independent of the generators B while the Taylor series expansion of β starts at first order B, as shown in the second equality in (4.11). Therefore, by iterating equation (4.12) repeatedly, we obtain the Taylor expansion of $\mathbf{D}_\delta(\mathbf{z};B)$ in powers of β with coefficients given by the multiple $\mathfrak{A}$ period integrals $\mathfrak{D}^{I_1\cdots I_r}_\delta(\mathbf{z})$ defined in (4.8) with the contour prescription (4.9). For example, a single iteration of (4.12) gives,

$$\begin{aligned}\mathbf{D}_\delta(\mathbf{z};B) &= C_\delta(\mathbf{z}) + \beta_{I_1}\mathfrak{D}^{I_1}_\delta(\mathbf{z}) + \sum_{I_1,I_2} \beta_{I_2}\beta_{I_1} \oint_{\mathfrak{A}^{I_2}} dt_2 \oint_{\mathfrak{A}^{I_1}} dt_1\, \mathbf{D}_\delta(t_1,t_2,\mathbf{z};B) \\ &= C_\delta(\mathbf{z}) + \beta_{I_1}\mathfrak{D}^{I_1}_\delta(\mathbf{z}) + \beta_{I_2}\beta_{I_1}\mathfrak{D}^{I_1I_2}_\delta(\mathbf{z}) + \mathcal{O}(\beta^3) \end{aligned} \tag{4.13}$$

while the all-order expression for an arbitrary number $n \geq 1$ of points in $\mathbf{z}$ takes the form,

$$\mathbf{D}_\delta(\mathbf{z};B) = C_\delta(\mathbf{z}) + \sum_{r=1}^{\infty} \beta_{I_r}\cdots\beta_{I_1}\, \mathfrak{D}^{I_1\cdots I_r}_\delta(\mathbf{z}) \tag{4.14}$$

By expanding each β_{I_ℓ} in powers of B_{I_ℓ} according to (4.11) and extracting the coefficient of $B_{I_r}\cdots B_{I_1}$, we obtain the components $D^{I_1\cdots I_r}_\delta(\mathbf{z})$, which at low rank are given by,

$$\begin{aligned} D^{I_1}_\delta(\mathbf{z}) &= \mathfrak{D}^{I_1}_\delta(\mathbf{z}) \\ D^{I_1I_2}_\delta(\mathbf{z}) &= \mathfrak{D}^{I_1I_2}_\delta(\mathbf{z}) - i\pi\, \delta^{I_1}_{I_2}\, \mathfrak{D}^{I_2}_\delta(\mathbf{z}) \\ D^{I_1I_2I_3}_\delta(\mathbf{z}) &= \mathfrak{D}^{I_1I_2I_3}_\delta(\mathbf{z}) - i\pi \Big[\delta^{I_1}_{I_2}\, \mathfrak{D}^{I_2I_3}_\delta(\mathbf{z}) + \delta^{I_2}_{I_3}\, \mathfrak{D}^{I_1I_3}_\delta(\mathbf{z})\Big] - \frac{2\pi^2}{3}\delta^{I_2I_1}_{I_3}\, \mathfrak{D}^{I_3}_\delta(\mathbf{z}) \end{aligned} \tag{4.15}$$

with no summation over any of the repeated indices.

[11] Whenever the summation over repeated indices does not appear in the customary presentation, we shall explicitly include the corresponding summation sign in order to avoid any possible confusion. When convenient, we shall use the shorthand notation $\mathbf{z} = (z_1, \cdots, z_n)$ for the ordered set of $n \geq 1$ points.

4.4 Construction of the constant multiplets $D_\delta^{I_1\cdots I_r}$

We now return to completing the construction of the system of descent equations of Theorem 2.3 and to proving the remaining items (c) and (d), both of which involve constant multiplets denoted by $D_\delta^{I_1\cdots I_r}$. In this subsection, we will connect the definition (2.36) of these constant multiplets with the $\mathfrak{A}$ periods of the previous subsections. This will pave the way for proving their cyclic symmetry property (2.37) of item (c) in section 4.5 and then deriving the representation (2.38) of the one-point function $D_\delta^{I_1\cdots I_r}(1)$ of item (d) in section 4.6.

To motivate the particular construction of the constant multiplets $D_\delta^{I_1\cdots I_r}$ to be given below, we begin by recasting the monodromy relation (3.22) for the one-point function of Lemma 3.4 in the following form,

$$\mathbf{D}_\delta(\mathfrak{B}_L \cdot 1; B) = e^{-2\pi i \, \mathrm{ad}_{B_L}} \mathbf{D}_\delta(1; B) \tag{4.16}$$

To arrive at a relation of the form (2.38) in item (d) of Theorem 2.3 we need a combination of constant multiplets and Enriquez kernels whose monodromy matches that of $\mathbf{D}_\delta(1; B)$ in (4.16). Inspection of the monodromy relation for the generating function $\mathbf{K}_J(x, y; B)$ of Enriquez kernels, given in (3.5), shows that $\mathbf{K}_J(1, p; \mathrm{ad}_B)$, whose last argument is ad_B instead of B, fulfills this requirement. Although the dependence of $\mathbf{K}_J(1, p; \mathrm{ad}_B)$ on an extra point p and the presence of a pole in z_1 at p, neither of which were present in $\mathbf{D}_\delta(1; B)$, may at first appear as a drawback, we shall soon show that consistency of the system of descent equations implies the absence of both.

To proceed, we define the following combination,

$$\tilde{\mathbf{D}}_\delta(1, p; B) = \mathbf{D}_\delta(1; B) - \mathbf{K}_J(1, p; \mathrm{ad}_B) \mathbf{D}_\delta^J(B) \tag{4.17}$$

in terms of a generating function $\mathbf{D}_\delta^J(B)$ for as yet unknown constant multiplets $D_\delta^{I_1\cdots I_r}$,

$$\mathbf{D}_\delta^J(B) = \sum_{r=1}^{\infty} D_\delta^{I_1\cdots I_r J} B_{I_r} \cdots B_{I_1} \tag{4.18}$$

To determine $\mathbf{D}_\delta^J(B)$, we impose the vanishing of the $\mathfrak{A}$ periods in t of $\tilde{\mathbf{D}}_\delta(t, p; B)$,

$$\oint_{\mathfrak{A}^L} dt \, \tilde{\mathbf{D}}_\delta(t, p; B) = 0 \tag{4.19}$$

To extract the condition this relation imposes on the constant multiplets, we use the fact that the $\mathfrak{A}$ periods of $\mathbf{K}_J(1, p; \mathrm{ad}_B)$ are independent of p and may be read off from (3.8),

$$\oint_{\mathfrak{A}^L} dt \, \mathbf{K}_J(t, p; \mathrm{ad}_B) = \delta_J^L \, \frac{-2\pi i \, \mathrm{ad}_{B_L}}{e^{-2\pi i \, \mathrm{ad}_{B_L}} - 1} = \delta_J^L \sum_{k=0}^{\infty} (-2\pi i)^k \frac{\mathrm{Ber}_k}{k!} (\mathrm{ad}_{B_L})^k \tag{4.20}$$

The corresponding $h \times h$ matrix in the indices J, L is diagonal and invertible. Therefore, the vanishing of the $\mathfrak{A}$ periods of $\tilde{\mathbf{D}}_\delta(1,p;B)$ implies the following expression for $\mathbf{D}^J_\delta(B)$,

$$\mathbf{D}^J_\delta(B) = \frac{e^{-2\pi i \, \mathrm{ad}_{B_J}} - 1}{-2\pi i \, \mathrm{ad}_{B_J}} \oint_{\mathfrak{A}^J} dt \, \mathbf{D}_\delta(t;B) \tag{4.21}$$

whose coefficients of $B_{I_1} \cdots B_{I_r}$ feature the definition (2.36) for the constant multiplets in item (c) of Theorem 2.3. This justifies the vanishing $\mathfrak{A}$ periods of $\tilde{\mathbf{D}}_\delta(t,p;B)$ which were imposed in (4.19).

The pole of $\tilde{\mathbf{D}}_\delta(1,p;B)$ in the variable z_1 at the point p is given solely by the contribution of $g^I{}_J(1,p) \, \mathrm{ad}_{B_I}$ to $\mathbf{K}_J(1,p;\mathrm{ad}_B)$ and its residue is,

$$\operatorname*{Res}_{z_1=p} \tilde{\mathbf{D}}_\delta(1,p;B) = -\sum_J \mathrm{ad}_{B_J} \mathbf{D}^J_\delta(B) = -[B_J, \mathbf{D}^J_\delta(B)] \tag{4.22}$$

The commutator may be expressed in terms of the constant multiplets $D^{I_1 I_2 \cdots I_r}_\delta$ in the expansion (4.18) and their cyclically permuted counterparts $D^{I_2 \cdots I_r I_1}_\delta$ as follows,

$$[B_J, \mathbf{D}^J_\delta(B)] = \sum_{r=2}^\infty \Big(D^{I_1 I_2 \cdots I_r}_\delta - D^{I_2 \cdots I_r I_1}_\delta \Big) B_{I_r} \cdots B_{I_1} \tag{4.23}$$

Independence of $\tilde{\mathbf{D}}_\delta(1,p;B)$ on the point p will require, at the very least, that its residue in z_1 at p cancels, namely that the commutator $[B_J, \mathbf{D}^J_\delta(B)]$ vanishes. We therefore proceed to showing the cyclicity of $D^{I_1 \cdots I_r}_\delta$ in item (c) of Theorem 2.3 which implies the vanishing of the commutator $[B_J, \mathbf{D}^J_\delta(B)]$ by (4.23).

4.5 Invariance of constant multiplets under cyclic permutations

In this subsection, we shall prove item (c) of Theorem 2.3, namely the invariance of the constant multiplets $D^{I_1 \cdots I_r}_\delta$ under cyclic permutations of its indices.

The starting point is the expression (4.21) for the generating function $\mathbf{D}^L_\delta(B)$ of the constant multiplets $D^{I_1 \cdots I_r}_\delta$. Combining this formula with the integral of the expression (4.14) for the one-point function, expanded in powers of the composite letters β_I in (4.11), we obtain,

$$\mathbf{D}^L_\delta(B) = \frac{e^{-2\pi i \, \mathrm{ad}_{B_L}} - 1}{-2\pi i \, \mathrm{ad}_{B_L}} \sum_{r=1}^\infty \beta_{I_r} \cdots \beta_{I_1} \mathfrak{D}^{I_1 \cdots I_r L}_\delta \tag{4.24}$$

Expanding the left side and the exponentials on the right side in Taylor series in powers of B gives the constant multiplets $D^{I_1 \cdots I_r}_\delta$ in terms of the multiple $\mathfrak{A}$ periods $\mathfrak{D}^{J_1 \cdots J_s}_\delta$ with

$s \leq r$ in (4.8) where all points t_i of the cyclic product C_δ are integrated over. To low orders we obtain, for example,

$$\begin{aligned}
D_\delta^{I_1 I_2} &= \mathfrak{D}_\delta^{I_1 I_2} \\
D_\delta^{I_1 I_2 I_3} &= \mathfrak{D}_\delta^{I_1 I_2 I_3} - i\pi \left[\delta_{I_2}^{I_1} \mathfrak{D}_\delta^{I_2 I_3} + \delta_{I_3}^{I_2} \mathfrak{D}_\delta^{I_1 I_3} - \delta_{I_3}^{I_1} \mathfrak{D}_\delta^{I_2 I_3} \right] \\
D_\delta^{I_1 I_2 I_3 I_4} &= \mathfrak{D}_\delta^{I_1 I_2 I_3 I_4} - i\pi \left[\delta_{I_2}^{I_1} \mathfrak{D}_\delta^{I_2 I_3 I_4} + \delta_{I_3}^{I_2} \mathfrak{D}_\delta^{I_1 I_3 I_4} + \delta_{I_4}^{I_3} \mathfrak{D}_\delta^{I_1 I_2 I_4} - \delta_{I_4}^{I_1} \mathfrak{D}_\delta^{I_2 I_3 I_4} \right] \\
&\quad - \pi^2 \bigg[\delta_{I_2}^{I_1} \delta_{I_4}^{I_3} \mathfrak{D}_\delta^{I_2 I_4} + \frac{2}{3} \delta_{I_3}^{I_1 I_2} \mathfrak{D}_\delta^{I_3 I_4} + \frac{2}{3} \delta_{I_4}^{I_2 I_3} \mathfrak{D}_\delta^{I_1 I_4} \\
&\qquad - \delta_{I_4}^{I_1} \delta_{I_3}^{I_2} \mathfrak{D}_\delta^{I_3 I_4} - \frac{4}{3} \delta_{I_4}^{I_1 I_3} \mathfrak{D}_\delta^{I_2 I_4} + \frac{2}{3} \delta_{I_4}^{I_1 I_2} \mathfrak{D}_\delta^{I_3 I_4} \bigg]
\end{aligned} \tag{4.25}$$

with no summation over repeated indices implied. The generating series identity (4.24) for constant multiplets lends itself particularly well to the evaluation of the commutator in (4.22) and (4.23) and we obtain,

$$[B_J, \mathbf{D}_\delta^J(B)] = \frac{i}{2\pi} \sum_L \left(e^{-2\pi i \, \mathrm{ad}_{B_L}} - 1 \right) \sum_{r=1}^{\infty} \beta_{I_r} \cdots \beta_{I_1} \mathfrak{D}_\delta^{I_1 \cdots I_r L} \tag{4.26}$$

This formula expresses the commutator $[B_J, \mathbf{D}_\delta^J(B)]$ and therefore the transformation of the constant multiplets under cyclic permutations in its components in terms of multiple $\mathfrak{A}$ periods $\mathfrak{D}_\delta^{I_1 \cdots I_r}$. As we will see soon, the vanishing of the right side of (4.26) through interrelations of the summands for different values of r hinges on the transformation properties of $\mathfrak{D}_\delta^{I_1 \cdots I_r}$ under cyclic permutations that we shall derive in the next subsection.

4.5.1 Transformation of $\mathfrak{D}_\delta^{I_1 \cdots I_r}$ under cyclic permutations

A cyclic permutation of the indices $\mathfrak{D}_\delta^{I_1 I_2 \cdots I_r} \to \mathfrak{D}_\delta^{I_2 \cdots I_r I_1}$ modifies the order in which the curves $\mathfrak{A}^{I_1}, \cdots, \mathfrak{A}^{I_r}$ are extended into the interior D^o of the fundamental domain D by the prescription for multiple $\mathfrak{A}$ periods given in section 4.2. In particular, the outermost curve $\mathfrak{A}_\varepsilon^{I_1}$ of $\mathfrak{D}_\delta^{I_1 I_2 \cdots I_r}$ is moved to the innermost position $\mathfrak{A}_{r\varepsilon}^{I_1}$ under the cyclic permutation, as illustrated in figure 3. By doing so, the curve $\mathfrak{A}_\varepsilon^{I_1}$ crosses all of those curves $\mathfrak{A}_{2\varepsilon}^{I_2}, \cdots, \mathfrak{A}_{r\varepsilon}^{I_r}$ with $I_j = I_1$ when comparing the prescriptions for their relative ordering in $\mathfrak{D}_\delta^{I_1 I_2 \cdots I_r}$ and $\mathfrak{D}_\delta^{I_2 \cdots I_r I_1}$. The quantitative relation is given by the following lemma.

Lemma 4.1. *The multiple $\mathfrak{A}$ period $\mathfrak{D}_\delta^{I_1 \cdots I_r}$ of the cyclic product of Szegö kernels defined in (4.8) transforms as follows under cyclic permutations of its indices,*

$$\mathfrak{D}_\delta^{I_2 \cdots I_r I_1} = \mathfrak{D}_\delta^{I_1 I_2 \cdots I_r} + 2\pi i \big(\delta_{I_r}^{I_1} - \delta_{I_2}^{I_1} \big) \mathfrak{D}_\delta^{I_2 \cdots I_r} \tag{4.27}$$

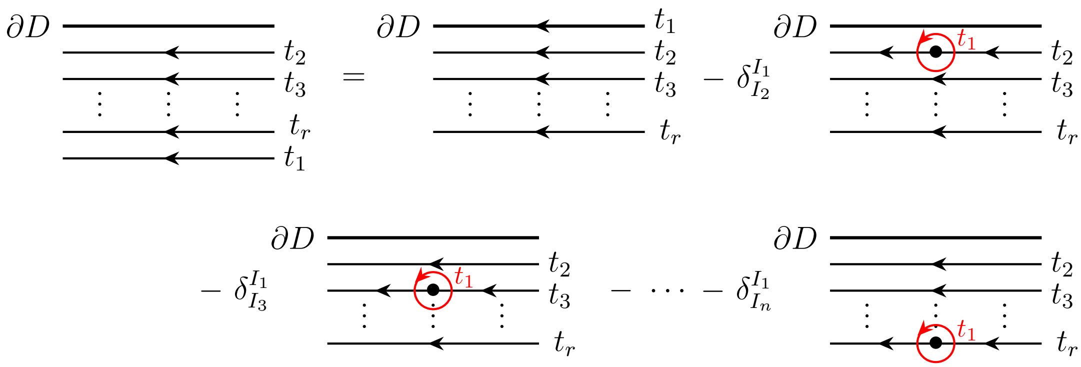

Figure 3: Contour deformation that brings the cyclically permuted ordering of contour displacements with t_1 in the innermost position back to the original order of (4.8) with t_1 in the outermost position. The crossing of the t_1 contour with those of $t_2, \cdots, t_r$ through the contour deformation is homotopic to infinitesimal circles around t_j drawn in red which only arise if $I_1 = I_j$. Moreover, the residue structure (4.28) of the integrand implies that the circles around $t_3, \cdots, t_{r-1}$ in the second line of the figure do not contribute to (4.27).

To prove the lemma, we trace the effects of $\mathfrak{A}^{I_1}_{\varepsilon}$ crossing $\mathfrak{A}^{I_2}_{2\varepsilon}, \cdots, \mathfrak{A}^{I_r}_{r\varepsilon}$. As shown in figure 3, each time $\mathfrak{A}^{I_1}_{\varepsilon}$ crosses a curve $\mathfrak{A}^{I_k}_{k\varepsilon}$ with $I_k = I_1$ the pole in t_1 at t_k produces an extra contribution given by the integral in t_1 around a small circle centered at t_k. These integrals may be evaluated using the residues,

$$\operatorname*{Res}_{t_k = t_{k\pm 1}} C_\delta(t_1, \cdots, t_r) = \pm C_\delta(t_1, \cdots, \hat{t}_k, \cdots, t_r) \tag{4.28}$$

of the cyclic products C_δ for nearest neighbor points. No contributions arise for $I_k \neq I_1$ or for non-nearest neighbor points. Thus, the only non-trivial residues arise for the circles in t_1 around the points t_2 and t_r provided $I_1 = I_2$ and/or $I_1 = I_r$, while the contributions around $t_3, \cdots, t_{r-1}$ vanish. Taking into account the residue structure of (4.28) proves formula (4.27) and thus Lemma 4.1. A similar relation for the exchange symmetry between $\mathfrak{A}$ periods due to simple poles in the integrand was pointed out in [58].

4.5.2 Lemma 4.1 implies item (c) of Theorem 2.3

With the cyclic transformation law in (4.27) established in Lemma 4.1, we shall now prove the invariance of the constant multiplets $D^{I_1 \cdots I_r}_{\delta}$ under cyclic permutations of its indices and thereby provide a proof of item (c) of Theorem 2.3.

The starting point is the expression in (4.26) for the commutator, whose vanishing is equivalent to the cyclic symmetry of all constant multiplets in view of (4.23). The expression (4.26) is rendered more explicit upon the use of the following relation,

$$\big(e^{-2\pi i \, \mathrm{ad}_{B_L}} - 1\big) X = e^{-2\pi i B_L} X e^{2\pi i B_L} - X = -2\pi i \big[\beta_L, X\big] \big(1 - 2\pi i \beta_L\big)^{-1} \tag{4.29}$$

for arbitrary combinations X of words in B_I. Here, we have used the definition of β_L given in (4.11) to express the exponential as follows $e^{-2\pi i B_L} = 1 - 2\pi i \beta_L$ and to pass from the first equality to the second. Applying this formula to the series for the commutator in (4.24) gives,

$$\big[B_L, \mathbf{D}^L_\delta(B)\big] = \sum_{r=2}^\infty \big[\beta_{I_r}, \beta_{I_{r-1}} \cdots \beta_{I_1}\big] \, (1 - 2\pi i \beta_{I_r})^{-1} \, \mathfrak{D}_\delta^{I_1 \cdots I_r} \tag{4.30}$$

Writing out the commutator of the β in the summand on the right and using a suitable relabelling of the indices, we obtain,

$$\begin{aligned} \big[B_L, \mathbf{D}^L_\delta(B)\big] = \sum_{r=2}^\infty \beta_{I_r} \cdots \beta_{I_2} \beta_{I_1} \Big\{ & (1 - 2\pi i \beta_{I_r})^{-1} \, \mathfrak{D}_\delta^{I_1 I_2 \cdots I_r} \\ & - (1 - 2\pi i \beta_{I_1})^{-1} \, \mathfrak{D}_\delta^{I_2 \cdots I_r I_1} \Big\} \end{aligned} \tag{4.31}$$

Substituting the expression for the cyclic permute $\mathfrak{D}_\delta^{I_2 \cdots I_r I_1}$ from (4.27) of Lemma 4.1, the above expression becomes,

$$\begin{aligned} \big[B_L, \mathbf{D}^L_\delta(B)\big] = \sum_{r=2}^\infty \beta_{I_r} \cdots \beta_{I_2} \beta_{I_1} \Big\{ & (1 - 2\pi i \beta_{I_r})^{-1} \, \mathfrak{D}_\delta^{I_1 I_2 \cdots I_r} \\ & - (1 - 2\pi i \beta_{I_1})^{-1} \, \Big[\mathfrak{D}_\delta^{I_1 I_2 \cdots I_r} + 2\pi i \big(\delta^{I_1}_{I_r} - \delta^{I_1}_{I_2}\big) \, \mathfrak{D}_\delta^{I_2 \cdots I_r} \Big] \Big\} \end{aligned} \tag{4.32}$$

The correction terms proportional to $2\pi i (\delta^{I_1}_{I_r} - \delta^{I_1}_{I_2})$ on the second line cancel for $r=2$ and only contribute starting at $r=3$. After shifting the summation variable r by one in the summation of these terms to restore the original range $r \geq 2$, and a suitable relabeling of the indices, we obtain,

$$\begin{aligned} \big[B_L, \mathbf{D}^L_\delta(B)\big] = \sum_{r=2}^\infty \beta_{I_r} \cdots \beta_{I_2} \beta_{I_1} \, \mathfrak{D}_\delta^{I_1 I_2 \cdots I_r} \Big\{ & (1 - 2\pi i \beta_{I_r})^{-1} - (1 - 2\pi i \beta_{I_1})^{-1} \\ & - 2\pi i \beta_{I_r} \, (1 - 2\pi i \beta_{I_r})^{-1} + 2\pi i \beta_{I_1} \, (1 - 2\pi i \beta_{I_1})^{-1} \Big] \Big\} \end{aligned} \tag{4.33}$$

The first and third terms inside the braces add up to 1 while the second and fourth terms add up to -1 so that the above sum cancels term by term and we conclude that,

$$\big[B_L, \mathbf{D}^L_\delta(B)\big] = 0 \tag{4.34}$$

which by (4.23) proves the invariance of the constant multiplets $D_\delta^{I_1 \cdots I_r}$ under cyclic permutations of its indices, and thereby proves item (c) of Theorem 2.3.

4.6 Completing the proof of Theorem 2.3: (c) implies (d)

Having established cyclic symmetry of the constant multiplets $D_\delta^{I_1\cdots I_r}$ in the previous subsection, it follows that the combination $\tilde{\mathbf{D}}_\delta(1,p;B)$ in (4.17) simplifies to

$$\begin{aligned}\tilde{\mathbf{D}}_\delta(1,p;B) &= \mathbf{D}_\delta(1;B) - \mathbf{W}_J(1;\mathrm{ad}_B)\mathbf{D}_\delta^J(B) + \mathbf{X}(1,p;\mathrm{ad}_B)[B_J,\mathbf{D}_\delta^J(B)] \\ &= \mathbf{D}_\delta(1;B) - \mathbf{W}_J(1;\mathrm{ad}_B)\mathbf{D}_\delta^J(B) \end{aligned} \tag{4.35}$$

which exposes holomorphicity in z_1 and independence on p. Moreover, the discussion around (4.19) to (4.21) revealed that $\tilde{\mathbf{D}}_\delta(1,p;B)$ has vanishing $\mathfrak{A}$ periods in z_1 (as an equivalent of the definition (2.36) of $D_\delta^{I_1\cdots I_r}$), and (4.17) together with (3.22) imply that $\tilde{\mathbf{D}}_\delta(1,p;B)$ has the following monodromy,

$$\tilde{\mathbf{D}}_\delta(\mathfrak{B}_L\cdot 1,p;B) = e^{-2\pi i B_L}\tilde{\mathbf{D}}_\delta(1,p;B)e^{2\pi i B_L} \tag{4.36}$$

Taylor expanding in powers of B, the components $\tilde{D}_\delta^{I_1\cdots I_r}(1,p)$ defined by,[12]

$$\tilde{\mathbf{D}}_\delta(1,p;B) = \sum_{r=1}^\infty \tilde{D}_\delta^{I_1\cdots I_r}(1,p)B_{I_r}\cdots B_{I_1} \tag{4.37}$$

have the following monodromy in z_1,

$$\tilde{D}_\delta^{I_1\cdots I_r}(\mathfrak{B}_L\cdot 1,p) = \sum_{k=0}^r\sum_{\ell=0}^{r-k}\frac{(-)^k(2\pi i)^{k+\ell}}{k!\,\ell!}\,\delta_L^{I_1\cdots I_\ell}\,\tilde{D}_\delta^{I_{\ell+1}\cdots I_{r-k}}(1,p)\,\delta_L^{I_{r-k+1}\cdots I_r} \tag{4.38}$$

The above properties are essential to prove the following proposition that relates the one-point function to the generating series of the constant multiplet.

Proposition 4.2. *The combination $\tilde{\mathbf{D}}_\delta(1,p;B)$ defined by (4.17) vanishes identically,*

$$\tilde{\mathbf{D}}_\delta(1,p;B) = 0 \tag{4.39}$$

which implies the following expression for the one-point function $\mathbf{D}_\delta(1;B)$ in terms of the constant multiplets in $\mathbf{D}_\delta^J(B)$,

$$\mathbf{D}_\delta(1;B) = \mathbf{K}_J(1,p;\mathrm{ad}_B)\mathbf{D}_\delta^J(B) = \mathbf{W}_J(1;\mathrm{ad}_B)\mathbf{D}_\delta^J(B) \tag{4.40}$$

[12] Given that the series expansion of both ingredients $\mathbf{D}_\delta(1;B)$ and $\mathbf{D}_\delta^J(B)$ of $\tilde{\mathbf{D}}_\delta(1,p;B)$ starts at the first order in B_I, terms at the zeroth order in B_I are guaranteed to be absent from (4.37).

The first statement (4.39) of the proposition is proven by contradiction. Let us assume that the lowest order term in the Taylor expansion (4.37) of $\tilde{\mathbf{D}}_\delta(1,p;B)$ in B is non-zero. In view of the monodromy relation (4.36), this lowest order term must have vanishing $\mathfrak{B}$ monodromy. Since it is a $(1,0)$ form holomorphic in z_1, it must be a linear combination of the Abelian differentials $\omega_J(1)$. But since its $\mathfrak{A}$ periods vanish in view of (4.19), this lowest order term must vanish, in contradiction to our initial hypothesis that the lowest order term is non-vanishing. This implies the vanishing of $\tilde{\mathbf{D}}_\delta(1,p;B)$ in (4.39).

The expression (4.40) for the one-point function in terms of $\mathbf{K}_J(1,p;\mathrm{ad}_B)$ is an immediate consequence of (4.39) and the expression (4.17) for $\tilde{\mathbf{D}}_\delta(1,p;B)$. The second equality in (4.40) follows from (4.34), concluding the proof of Proposition 4.2.

Expanding the last statement $\mathbf{D}_\delta(1;B) = \mathbf{W}_J(1;\mathrm{ad}_B)\mathbf{D}^J_\delta(B)$ of Proposition 4.2 in components according to (3.2) and (3.9) results in (2.38) and thereby concludes the proof of item (d) of Theorem 2.3.

5 Decomposing cyclic products into Enriquez kernels

In this section, we describe the simplified representations of cyclic products C_δ in terms of Enriquez kernels and constant multiplets $D_\delta^{I_1\cdots I_r}$ that result from the meromorphic descent in Theorem 2.3 and list several properties of the main constituents.

5.1 Meromorphic decomposition of cycles of Szegö kernels

The meromorphic descent described in the previous sections leads to the decomposition of the cyclic product $C_\delta(1,\cdots,n)$ of Szegö kernels in the following theorem, which was already announced in (1.3) and will be referred to as the *meromorphic decomposition*.[13]

Theorem 5.1. *The spin-structure dependence of C_δ can be fully separated from the dependence on the points $z_1,\cdots,z_n$ for $n\geq 2$ through the meromorphic decomposition,*

$$C_\delta(1,\cdots,n) = \mathcal{W}(1,\cdots,n) + \sum_{r=2}^{n} \mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)\, D_\delta^{I_1\cdots I_r} \tag{5.1}$$

The multiplets $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ are independent of the spin structure δ, meromorphic and single-valued $(1,0)$ forms in $z_1,\cdots,z_n\in\Sigma$, cyclically symmetric in the indices $I_1\cdots I_r$ and expressible in terms of (products, index contractions and derivatives of) Enriquez kernels. In particular, the $r=n$ term is given by a cyclically symmetrized product of holomorphic Abelian differentials,

$$\mathcal{W}_{I_1\cdots I_n}(1,\cdots,n) = \frac{1}{n}\,\omega_{I_1}(1)\,\cdots\,\omega_{I_n}(n) + \mathrm{cycl}(I_1,\cdots,I_n) \tag{5.2}$$

The coefficients $D_\delta^{I_1\cdots I_r}$ in (5.1) are those produced in the descent procedure of Theorem 2.3 for $r=2,\cdots,n$; they are independent of the points $z_1,\cdots,z_n$; cyclically symmetric; locally holomorphic in the moduli of Σ and carry all the dependence of C_δ on δ.

Note that, while the multiplets $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ at fixed rank $r\leq n$ are distinct for different values of n, their coefficients $D_\delta^{I_1\cdots I_r}$ are independent of n and thus universal.

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[13] Similar decomposition formulas for $C_\delta(1,\cdots,n)$ were proposed in [41, 69], but it remains unclear how to reconcile those claims with the expressions in this work beyond genus one.

5.1.1 Proof of Theorem 5.1

We present a constructive proof of Theorem 5.1 which proceeds by rearranging the descent equations (2.34) as follows with $n \geq 3$ and $r \geq 0$,

$$\begin{aligned} D_\delta^{I_1\cdots I_r}(1,\cdots,n) &= \omega_J(1) D_\delta^{I_1\cdots I_r J}(2,\cdots,n) + \sum_{i=0}^{r-1} g^{I_r\cdots I_{i+1}}{}_J(1,2) D_\delta^{I_1\cdots I_i J}(2,\cdots,n) \\ &\quad + \Big(\chi^{I_r\cdots I_1}(1,n) - \chi^{I_r\cdots I_1}(1,2)\Big) C_\delta(2,\cdots,n) \end{aligned} \tag{5.3}$$

where $D_\delta^{\emptyset}(1,\cdots,n) = C_\delta(1,\cdots,n)$. We also rearrange the descent equations (2.35) and (2.38), for $r \geq 0$ in the first line below and $r \geq 1$ in the second line below, as follows,

$$\begin{aligned} D_\delta^{I_1\cdots I_r}(1,2) &= \omega_J(1) D_\delta^{I_1\cdots I_r J}(2) + \sum_{i=0}^{r-1} g^{I_r\cdots I_{i+1}}{}_J(1,2) D_\delta^{I_1\cdots I_i J}(2) + \partial_2 \chi^{I_r\cdots I_1}(1,2) \\ D_\delta^{I_1\cdots I_r}(1) &= \omega_J(1) D_\delta^{I_1\cdots I_r J} + \sum_{\substack{0\leq i<j\leq r \\ (i,j)\neq(0,r)}} (-1)^i \varpi^{I_1\cdots I_i \sqcup I_r\cdots I_{j+1}}{}_J(1) D_\delta^{I_{i+1}\cdots I_j J} \end{aligned} \tag{5.4}$$

The meromorphic decomposition (5.1) along with the explicit form of the multiplets $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ may now be obtained by eliminating from this system of n equations (and their cyclic permutations in the points) all the intermediate multiplets $D_\delta^{I_1\cdots I_s}(r,\cdots)$, that depend on at least one point and at most $n-1$ points, in favor of the constant coefficients $D_\delta^{I_1\cdots I_s}$. This may be done recursively by increasing the rank r or, equivalently, by decreasing the number of points via iterative use of (5.3) and (5.4). The last step of this recursive procedure gives rise to the holomorphic $r=n$ term $\omega_{I_1}(1)\cdots\omega_{I_n}(n) D_\delta^{I_1\cdots I_n}$ in (5.1) and (5.2) by the first term on the right side of (5.4). The Enriquez-kernel representation of $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ with $r \leq n-1$ follows by tracking the appearance of the constant $D_\delta^{I_1\cdots I_r}$ in each of the n steps of the meromorphic descent, where $\mathcal{W}(1,\cdots,n)$ without any indices is obtained from the contributions without any accompanying factor of $D_\delta^{I_1\cdots I_r}$. This procedure leads to the n terms in the meromorphic decomposition (5.1).

The cyclic symmetry of $D_\delta^{I_1\cdots I_r}$ in its indices $I_1,\cdots,I_r$, guaranteed by item (c) of Theorem 2.3, implies that we may choose $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ to be cyclically invariant in its indices,

$$\mathcal{W}_{I_1 I_2\cdots I_r}(1,\cdots,n) = \mathcal{W}_{I_2\cdots I_r I_1}(1,\cdots,n) \tag{5.5}$$

and we shall do so throughout. The single-valuedness of $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ emerges through the particular combinations of Enriquez kernels that arise in the descent procedure, but this property is not manifest term by term. To prove it, one may use the fact that Theorem 2.3 involves only the cyclic invariance of $D_\delta^{I_1\cdots I_r}$ with $2 \leq r \leq n$ without relying on any relation between $D_\delta^{I_1\cdots I_r}$ of different rank. Since $C_\delta(1,\cdots,n)$ is single-valued it follows that the individual terms $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)D_\delta^{I_1\cdots I_r}$ must also be single-valued. Our choice (5.5) then ensures that already the individual coefficients $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ are single-valued, irrespective on their contraction with the cyclic $D_\delta^{I_1\cdots I_r}$. This completes the proof of the single-valuedness of $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ in all of $z_1,\cdots,z_n$ and of Theorem 5.1.

5.1.2 Examples of single-valued combinations of Enriquez kernels

We shall now spell out the explicit forms of the meromorphic and single-valued multiplets $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ for $n=2,3$ points that follow from the constructive procedure outlined in the above proof. Their $n=4$ point counterparts can be found in appendix E.1. There is no need to spell out the cases with $r=n$ in view of (5.2).

For $n=2$ points, matching the expression (2.10) for the two-cycle of Szegö kernels with the general form (5.1) of the decomposition readily allows us to identify,

$$\mathcal{W}(1,2)=\partial_2\chi(1,2) \tag{5.6}$$

For $n=3$ points, eliminating all of $C_\delta(2,3), D_\delta^I(2,3), D_\delta^K(3), D_\delta^{IJ}(3)$ from the descent equations (2.22) with the help of the two-point identities (2.10) and (2.11) casts $C_\delta(1,2,3)$ into the form of (5.1). By isolating the coefficient of D_δ^{JK} and the terms without any accompanying $D_\delta^{JK}, D_\delta^{IJK}$ in all steps of the three-point descent, we are led to the following expressions for the forms $\mathcal{W}_{JK}(1,2,3)$ and $\mathcal{W}(1,2,3)$, respectively,

$$\begin{aligned}\mathcal{W}_{JK}(1,2,3)&=\frac{1}{2}\Big[\big(\chi(1,3)-\chi(1,2)\big)\omega_J(2)\omega_K(3)+\omega_I(1)g^I{}_J(2,3)\omega_K(3)\\&\qquad+\big(\omega_J(1)\omega_I(2)-\omega_I(1)\omega_J(2)\big)\varpi^I{}_K(3)+(J\leftrightarrow K)\Big]\\\mathcal{W}(1,2,3)&=\big(\chi(1,3)-\chi(1,2)\big)\partial_3\chi(2,3)+\omega_I(1)\partial_3\chi^I(2,3)\end{aligned} \tag{5.7}$$

The prescription to add the image under $J\leftrightarrow K$ applies to both lines of the expression for $\mathcal{W}_{JK}(1,2,3)$ and implements the cyclic symmetry according to our choice (5.5).

5.2 Further properties of $D_\delta^{I_1\cdots I_r}$ and $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$

The propositions below collect further properties of the multiplets in the meromorphic decomposition of cyclic products of Szegö kernels in Theorem 5.1, including a reflection symmetry of $D_\delta^{I_1\cdots I_r}$ and various further properties of $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$.

Proposition 5.2. *The multiplets $D_\delta^{I_1\cdots I_r}$ are invariant under alternating reflection symmetry for any $r\geq 2$,*

$$D_\delta^{I_1I_2\cdots I_{r-1}I_r}=(-1)^rD_\delta^{I_rI_{r-1}\cdots I_2I_1} \tag{5.8}$$

The proof is given in appendix D.3.

Proposition 5.3. *The multiplets* $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ *exhibit the following further properties.*

(a) *For* $n\geq 3$, $r\leq n-1$ *and* $2\leq k\leq n-1$ *they have simple poles at adjacent points,*

$$\begin{aligned}
\bar\partial_1\mathcal{W}_{I_1\cdots I_r}(1,2,\cdots,n) &= \pi\big(\delta(1,2)-\delta(1,n)\big)\,\mathcal{W}_{I_1\cdots I_r}(2,\cdots,n)\\
\bar\partial_k\mathcal{W}_{I_1\cdots I_r}(1,2,\cdots,n) &= \pi\big(\delta(k,k+1)-\delta(k,k-1)\big)\,\mathcal{W}_{I_1\cdots I_r}(1,\cdots,\hat k,\cdots,n)\\
\bar\partial_n\mathcal{W}_{I_1\cdots I_r}(1,2,\cdots,n) &= \pi\big(\delta(n,1)-\delta(n,n-1)\big)\,\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n-1)
\end{aligned}\tag{5.9}$$

and are holomorphic in all points if $r=n$.

(b) *For* $n\geq 2$ *and* $0\leq r\leq n$ *and for a fixed choice of their indices* $I_1,\cdots,I_r$*, they are invariant under cyclic permutations of the points,*

$$\mathcal{W}_{I_1\cdots I_r}(1,2,\cdots,n)=\mathcal{W}_{I_1\cdots I_r}(2,\cdots,n,1)\tag{5.10}$$

(c) *They are invariant under the simultaneous reflection of the indices and the points,*

$$\mathcal{W}_{I_1I_2\cdots I_r}(1,2,\cdots,n)=(-1)^{n+r}\mathcal{W}_{I_r\cdots I_2I_1}(n,\cdots,2,1)\tag{5.11}$$

A direct proof of item (a) uses the fact that the proof of Theorem 2.3 does not involve any properties of $D_\delta^{I_1\cdots I_r}$ with $2\leq r\leq n$ other than their cyclic invariance, so that they may be treated effectively as linearly independent of one another. Substituting the relation (5.1) into (2.39) and identifying the coefficients of the various multiplets $D_\delta^{I_1\cdots I_r}$ then proves (5.9). In the process, one makes use of the fact that the multiplet $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ for $r=n$ in (5.2) is holomorphic in all points.

The proofs of items (b), (c) are relegated to appendix D.2. They rely on the modular counterpart of the meromorphic decomposition (5.1) to be described in section 6.4.

Note that the expressions for the multiplets $\mathcal{W}_{I_1I_2\cdots I_r}(1,\cdots,n)$ with $r\geq 3$ obtained from the meromorphic descent equations as described below (5.4) do not match the images $(-1)^r\mathcal{W}_{I_r\cdots I_2I_1}(1,\cdots,n)$ of reflecting only the indices and not the points. More precisely, the multiplets $D_\delta^{I_1\cdots I_r}$ in the second line of the descent equation (5.4) and the decomposition (5.1) are understood as cyclically invariant but otherwise unspecified symbols for the purpose of defining the $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ as coefficients, without any reference to their reflection property (5.8). This approach is possible since the meromorphic descent does not rely on (5.8) and allows us to construct a larger class of single-valued and meromorphic $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ (expected to be relevant to future work) than the combinations realized in (5.1) upon contraction with $D_\delta^{I_1\cdots I_r}$.

6 Descent in terms of DHS kernels

The descent relations of the cyclic product of Szegö kernels $C_\delta(1, \cdots, n)$ in terms of the single-valued modular invariant but non-meromorphic DHS kernels introduced in [60] were presented in [61], where a proof was given for low rank only. Here, we shall offer a convenient reformulation of these descent relations and provide a full proof.

6.1 Definition and properties of DHS kernels

The DHS kernels $f^{I_1 \cdots I_r}{}_J(x, y)$ for $r \geq 0$ and $I_1, \cdots, I_r, J \in \{1, \cdots, h\}$ were defined in [60] in terms of the Arakelov Green function $\mathcal{G}(x, y)$ (see appendix A.3 for its definition and properties), holomorphic Abelian differentials $\omega_I(x)$ and their complex conjugates. The DHS kernel for $r = 1$ is defined by the integral,

$$f^I{}_J(x, y) = \int_\Sigma d^2t \, \partial_x \mathcal{G}(x, t) \Big(\bar{\omega}^I(t) \omega_J(t) - \delta^I_J \, \delta(t, y) \Big) \qquad d^2t = \frac{i}{2} dt \wedge d\bar{t} \tag{6.1}$$

while for $r \geq 2$ the kernels are defined recursively in the rank by,

$$f^{I_1 \cdots I_r}{}_J(x, y) = \int_\Sigma d^2t \, \partial_x \mathcal{G}(x, t) \, \bar{\omega}^{I_1}(t) f^{I_2 \cdots I_r}{}_J(t, y) \tag{6.2}$$

The DHS kernels satisfy a Massey system of differential relations, as may be established using the differential equations (A.6) and (A.12) for the Arakelov Green function,

$$\begin{aligned} \bar{\partial}_x f^{I_1}{}_J(x, y) &= -\pi \, \bar{\omega}^{I_1}(x) \, \omega_J(x) + \pi \, \delta^{I_1}_J \, \delta(x, y) \\ \bar{\partial}_y f^{I_1}{}_J(x, y) &= \pi \, \delta^{I_1}_J \, \omega_K(x) \, \bar{\omega}^K(y) - \pi \, \delta^{I_1}_J \, \delta(x, y) \end{aligned} \tag{6.3}$$

and for $r \geq 2$,

$$\begin{aligned} \bar{\partial}_x f^{I_1 \cdots I_r}{}_J(x, y) &= -\pi \, \bar{\omega}^{I_1}(x) \, f^{I_2 \cdots I_r}{}_J(x, y) \\ \bar{\partial}_y f^{I_1 \cdots I_r}{}_J(x, y) &= \pi \, \delta^{I_r}_J \, f^{I_1 \cdots I_{r-1}}{}_K(x, y) \, \bar{\omega}^K(y) \end{aligned} \tag{6.4}$$

Finally, $f^{I_1 \cdots I_r}{}_J(x, y)$ admits a decomposition into a traceless and trace part, given by,

$$f^{I_1 \cdots I_r}{}_J(x, y) = \partial_x \Phi^{I_1 \cdots I_r}{}_J(x) - \delta^{I_r}_J \, \partial_x \mathcal{G}^{I_1 \cdots I_{r-1}}(x, y) \tag{6.5}$$

where $\Phi^{I_1 \cdots I_{r-1} J}{}_J(x) = 0$ and, similar to (2.8), the dependence on y is concentrated in the trace $\delta^{I_r}_J$ with respect to the rightmost indices. Using this relation for $r = 1$, one may recast the recursion relation (6.2) entirely in terms of DHS kernels f,

$$f^{I_1 \cdots I_r}{}_J(x, y) = -\int_\Sigma d^2t \, f^{I_1}{}_K(x, t) \, \bar{\omega}^K(t) f^{I_2 \cdots I_r}{}_J(t, y) \tag{6.6}$$

The DHS kernel $f^{I_1\cdots I_r}{}_J(x,y)$ is a single-valued $(1,0)$ form in $x\in\Sigma$ and a single-valued $(0,0)$ form in $y\in\Sigma$ and transforms as a modular tensor under $\mathrm{Sp}(2h,\mathbb{Z})$ [70, 71, 25]. The form $f^{I_1\cdots I_r}{}_J(x,y)$ is real analytic for $x,y\in\Sigma$ with $x\neq y$ for all $r\geq 0$, just as $\mathcal{G}(x,y)$ is. In view of (A.11), the $r=1$ DHS kernel exhibits a single simple pole at $x=y$,

$$f^I{}_J(x,y)=\frac{\delta^I_J}{x-y}+\text{regular} \tag{6.7}$$

While the form $f^{I_1I_2}{}_J(x,y)$ for $r=2$ does not have a limit as $y\to x$, the combination $f^{I_1I_2}{}_J(x,y)+\delta^{I_2}_J\frac{\pi}{x-y}\int_y^x\bar\omega^{I_1}$ does, see Lemma 8.1 of [62]. For $r\geq 3$ the form $f^{I_1\cdots I_r}{}_J(x,y)$ has a limit as $y\to x$.

6.2 Formulation of the descent in terms of DHS kernels

The descent of the cyclic product of Szegö kernels in terms of DHS kernels was presented in equations (33-35) of [61] and may be restated as follows.

The functions $C^{I_1\cdots I_r}_\delta(1,\cdots,n)$ are defined for $r\geq 0$ to be differential $(1,0)$ forms in the variables $z_1,\cdots,z_n$ for $n\geq 1$ and constants on Σ for $n=0$. For $r=0$ and $n\geq 2$ they are defined by the cyclic product of n Szegö kernels introduced in (1.1), namely,

$$C^{\emptyset}_\delta(1,\cdots,n)=C_\delta(1,\cdots,n) \tag{6.8}$$

and to vanish for $n=0,1$. For $r\geq 1$ and $n\geq 0$, they are defined recursively by,

$$C^{I_1\cdots I_r}_\delta(1,\cdots,n)=\int_\Sigma d^2t_r\,\bar\omega^{I_r}(t_r)\,C^{I_1\cdots I_{r-1}}_\delta(t_r,1,\cdots,n) \tag{6.9}$$

and vanish for $(r,n)=(1,0)$. These integrals are absolutely convergent for $n+r\geq 3$ and conditionally convergent for $n+r=2$. In particular, the constant modular tensors are obtained from (6.9) for $n=0$. For $r\geq 3$ they are given by,

$$C^{I_1\cdots I_r}_\delta=\int_\Sigma d^2t_1\,\bar\omega^{I_1}(t_1)\cdots\int_\Sigma d^2t_r\,\bar\omega^{I_r}(t_r)\,C_\delta(t_1,\cdots,t_r) \tag{6.10}$$

for $r=2$ by $C^{IJ}_\delta=D^{IJ}_\delta-\pi Y^{IJ}$ with D^{IJ}_δ given by (2.11) and vanish for $r=0,1$.

Proposition 6.1. *The functions $C^{I_1\cdots I_r}_\delta(1,\cdots,n)$ satisfy the following properties [61].*

(a) They are modular tensors under the congruence subgroup $\Gamma_h(2)$ of $\mathrm{Sp}(2h,\mathbb{Z})$,

$$\Gamma_h(2)=\big\{M\in\mathrm{Sp}(2h,\mathbb{Z})\,|\,M=I \bmod 2\big\} \tag{6.11}$$

that leaves each spin structure δ and associated Szegö kernel $S_\delta(x,y)$ invariant.

(b) The constant tensors $C_\delta^{I_1\cdots I_r}$ are invariant under cyclic permutations of its indices,

$$C_\delta^{I_1 I_2\cdots I_r} = C_\delta^{I_2\cdots I_r I_1} \tag{6.12}$$

(c) and transform as follows under reflection of its indices,

$$C_\delta^{I_1 I_2\cdots I_{r-1} I_r} = (-)^r C_\delta^{I_r I_{r-1}\cdots I_2 I_1} \tag{6.13}$$

The proof of the proposition is readily obtained by inspection of the definition of $C_\delta^{I_1\cdots I_r}(1,\cdots,n)$. In particular item (a) follows from the invariance of the cyclic product C_δ under $\Gamma_h(2)$ (see (2.12) for the $\mathrm{Sp}(2h,\mathbb{Z})$ transformation of the spin structures) together with the modular transformation of the differentials $\bar\omega^{I_r}(t_r)$ in the recursion (6.9), while items (b) and (c) follow from the cyclic and reflection properties of $C_\delta(t_1,\cdots,t_r)$ in the integrand of (6.10).

The theorem below presents a constructive proof of the *modular descent* in terms of DHS kernels, already presented without proof in equations (33-35) of [61].

Theorem 6.2. *The family of modular tensors $C_\delta^{I_1\cdots I_r}(1,\cdots,n)$ solves the following system of descent equations. For $n\geq 3$ and $r\geq 0$, we have,*

$$\begin{aligned} C_\delta^{I_1\cdots I_r}(1,\cdots,n) &= \omega_J(1) C_\delta^{I_1\cdots I_r J}(2,\cdots,n) + \sum_{i=0}^{r-1} f^{I_r\cdots I_{i+1}}{}_J(1,2) C_\delta^{I_1\cdots I_i J}(2,\cdots,n) \\ &\quad + \Big(\partial_1 \mathcal{G}^{I_r\cdots I_1}(1,n) - \partial_1 \mathcal{G}^{I_r\cdots I_1}(1,2)\Big) C_\delta(2,\cdots,n) \end{aligned} \tag{6.14}$$

For $n=2$ with $r\geq 0$ and for $n=1$ with $r\geq 1$, we have,

$$\begin{aligned} C_\delta^{I_1\cdots I_r}(1,2) &= \omega_J(1) C_\delta^{I_1\cdots I_r J}(2) + \sum_{i=0}^{r-1} f^{I_r\cdots I_{i+1}}{}_J(1,2) C_\delta^{I_1\cdots I_i J}(2) + \partial_1\partial_2 \mathcal{G}^{I_r\cdots I_1}(1,2) \\ C_\delta^{I_1\cdots I_r}(1) &= \omega_J(1) C_\delta^{I_1\cdots I_r J} + \sum_{\substack{0\leq i<j\leq r \\ (i,j)\neq(0,r)}} (-1)^i \partial_1 \Phi^{I_1\cdots I_i \sqcup I_r \cdots I_{j+1}}{}_J(1) C_\delta^{I_{i+1}\cdots I_j J} \end{aligned} \tag{6.15}$$

where we use the notation $f^{\emptyset}{}_J(1,2)=\omega_J(1)$ while $\partial\mathcal{G}$ and $\partial\Phi$ are introduced in (6.5).

6.3 Proof of Theorem 6.2

To prove Theorem 6.2, we begin by proving the following lemma.

Lemma 6.3. *The functions* $C_\delta^{I_1\cdots I_r}(1,\cdots,n)$*, defined in (6.9), satisfy the following system of differential equations. For* $r=0$ *and* $n\geq 3$ *with* $1\leq k\leq n$*,*

$$\bar\partial_k C_\delta(1,\cdots,n) = \pi\left(\delta(k,k+1)-\delta(k,k-1)\right) C_\delta(1,\cdots,\hat k,\cdots,n) \tag{6.16}$$

For $r\geq 1$ *and* $n\geq 2$ *with* $2\leq k\leq n-1$*, we have,*

$$\begin{aligned}
\bar\partial_1 C_\delta^{I_1\cdots I_r}(1,\cdots,n) &= \pi\,\delta(1,2) C_\delta^{I_1\cdots I_r}(2,\cdots,n) - \pi\,\bar\omega^{I_r}(1)\, C_\delta^{I_1\cdots I_{r-1}}(1,\cdots,n) \\
\bar\partial_k C_\delta^{I_1\cdots I_r}(1,\cdots,n) &= \pi\left(\delta(k,k+1)-\delta(k,k-1)\right) C_\delta^{I_1\cdots I_r}(1,\cdots,\hat k,\cdots,n) \\
\bar\partial_n C_\delta^{I_1\cdots I_r}(1,\cdots,n) &= \pi\bar\omega^{I_1}(n)\, C_\delta^{I_2\cdots I_r}(1,\cdots,n) - \pi\,\delta(n,n-1)\, C_\delta^{I_1\cdots I_r}(1,\cdots,n-1)
\end{aligned} \tag{6.17}$$

For $n=1$ *and* $r\geq 2$*, we have,*

$$\bar\partial_1 C_\delta^{I_1\cdots I_r}(1) = \pi\bar\omega^{I_1}(1) C_\delta^{I_2\cdots I_r}(1) - \pi\bar\omega^{I_r}(1) C_\delta^{I_1\cdots I_{r-1}}(1) \tag{6.18}$$

The proof of the lemma proceeds by induction in r for all values of n. For $r=0$ and all values of $n\geq 3$, equation (6.16) is an immediate consequence of the $\bar\partial$ derivative of the Szegö kernel. For $r=1$ and $n\geq 2$ equation (6.9) reduces to,

$$C_\delta^I(1,\cdots,n) = \int_\Sigma d^2t\,\bar\omega^I(t)\, C_\delta(t,1,\cdots,n) \tag{6.19}$$

Its $\bar\partial_1$ and $\bar\partial_n$ derivatives may be evaluated using (6.16) on $C_\delta(t,1,\cdots,n)$ and readily produce the first and last equations in (6.17) while its $\bar\partial_k$ derivative gives the middle equations for $2\leq k\leq n-1$. Assuming now that the system of equations (6.17) holds for all $r\leq s-1$ we shall show that it also holds for $r=s$. Indeed, from the definition of $C_\delta^{I_1\cdots I_s}(1,\cdots,n)$ in (6.9), we have for all $1\leq j\leq n$,

$$\partial_j C_\delta^{I_1\cdots I_s}(1,\cdots,n) = \int_\Sigma d^2t\,\bar\omega^{I_s}(t)\,\bar\partial_j C_\delta^{I_1\cdots I_{s-1}}(t,1,\cdots,n) \tag{6.20}$$

For each value of j we use the corresponding equation in (6.17) for $r=s-1$ to evaluate the $\bar\partial_j$ derivative and readily verify that the result is the corresponding equality of (6.17) for $r=s$, which completes the proof of the lemma.

6.3.1 Completing the proof of Theorem 6.2

To prove Theorem 6.2 we introduce the following combination for $n\geq 3$,

$$\begin{aligned}
\tilde C_\delta^{I_1\cdots I_r}(1,\cdots,n) = C_\delta^{I_1\cdots I_r}(1,\cdots,n) &- \sum_{i=0}^{r} f^{I_r\cdots I_{i+1}}{}_J(1,2) C_\delta^{I_1\cdots I_i J}(2,\cdots,n) \\
&+\Big(\partial_1\mathcal{G}^{I_r\cdots I_1}(1,2)-\partial_1\mathcal{G}^{I_r\cdots I_1}(1,n)\Big) C_\delta(2,\cdots,n)
\end{aligned} \tag{6.21}$$

while for $n=2$ and $n=1$, we define,

$$\tilde{C}_\delta^{I_1\cdots I_r}(1,2) = C_\delta^{I_1\cdots I_r}(1,2) - \sum_{i=0}^{r} f^{I_r\cdots I_{i+1}}{}_J(1,2) C_\delta^{I_1\cdots I_i J}(2) - \partial_1\partial_2 \mathcal{G}^{I_r\cdots I_1}(1,2) \tag{6.22}$$

$$\tilde{C}_\delta^{I_1\cdots I_r}(1) = C_\delta^{I_1\cdots I_r}(1) - \omega_J(1) C_\delta^{I_1\cdots I_r J} - \sum_{\substack{0\leq i<j\leq r\\(i,j)\neq(0,r)}} (-1)^i \partial_1 \Phi^{I_1\cdots I_i \sqcup I_r\cdots I_{j+1}}{}_J(1) C_\delta^{I_{i+1}\cdots I_j J}$$

Manifestly, the vanishing of these quantities is equivalent to equation (6.14) for $n\geq 3$ and equation (6.15) for $n=2,1$. By Lemma 6.3 and the Massey system (6.3), (6.4) of DHS kernels, their $\bar\partial_1$ derivatives for $n\geq 2$ and $r\geq 1$ evaluate as follows,

$$\begin{aligned}
\bar\partial_1 \tilde{C}_\delta(1,\cdots,n) &= 0 \\
\bar\partial_1 \tilde{C}_\delta^{I_1\cdots I_r}(1,\cdots,n) &= -\pi \bar\omega^{I_r}(1) \tilde{C}_\delta^{I_1\cdots I_{r-1}}(1,\cdots,n)
\end{aligned} \tag{6.23}$$

while for $n=1$ we have,

$$\bar\partial_1 \tilde{C}_\delta^{I_1\cdots I_r}(1) = \pi \bar\omega^{I_1}(1) \tilde{C}_\delta^{I_2\cdots I_r}(1) - \pi \tilde{C}_\delta^{I_1\cdots I_{r-1}}(1) \bar\omega^{I_r}(1) \tag{6.24}$$

The derivation of (6.24) makes use of the cyclic symmetry of the constants of (6.12). Furthermore, the integral against $\bar\omega^{I_{r+1}}(1)$ over $z_1\in\Sigma$ may be evaluated with the help of (6.9) and the fact that the integral of $\bar\omega^{I_r}(1)$ against $f^{I_r\cdots I_{i+1}}{}_J(1,2)$ and $\partial_1\mathcal{G}^{I_r\cdots I_1}(1,k)$ vanishes for all $i\neq r$ and all k and we find,

$$\int_\Sigma d^2t\, \bar\omega^{I_{r+1}}(t)\, \tilde{C}_\delta^{I_1\cdots I_r}(t,2,\cdots,n) = 0 \tag{6.25}$$

We now proceed to a proof by induction in r. For $r=0$ the first equation in (6.23) tells us that $\tilde{C}_\delta(1,\cdots,n)$ is holomorphic in z_1, so that it must be a linear combination of the holomorphic Abelian differentials $\omega_J(1)$ with z_1-independent coefficients. Then (6.25) implies that $\tilde{C}_\delta(1,\cdots,n)=0$. Let us now assume that $\tilde{C}_\delta^{I_1\cdots I_r}(1,\cdots,n)=0$ for all $r\leq s-1$. Therefore, the second equation of (6.23) and (6.24) imply that $\tilde{C}_\delta^{I_1\cdots I_s}(1,\cdots,n)$ is holomorphic in z_1 for all $n\geq 1$ so that it must be a linear combination of the holomorphic Abelian differentials $\omega_J(1)$ with z_1-independent coefficients. Then (6.25) implies that $\tilde{C}_\delta^{I_1\cdots I_s}(1,\cdots,n)=0$. Thus, we conclude that $\tilde{C}_\delta^{I_1\cdots I_r}(1,\cdots,n)=0$ for all n and r, which completes the proof of Theorem 6.2.

6.4 Modular decomposition of cyclic products of Szegö kernels

The modular descent described in the previous subsections leads to the decomposition of the cyclic product $C_\delta(1,\cdots,n)$ of Szegö kernels in the following theorem, which was already announced in (1.4) and will be referred to as the *modular decomposition*,

Theorem 6.4. *The spin-structure dependence of C_δ can be fully separated from the dependence on the points $z_1, \cdots, z_n$ for $n \geq 2$ through the modular decomposition,*

$$C_\delta(1, \cdots, n) = \mathcal{V}(1, \cdots, n) + \sum_{r=2}^{n} \mathcal{V}_{I_1 \cdots I_r}(1, \cdots, n)\, C_\delta^{I_1 \cdots I_r} \tag{6.26}$$

The multiplets $\mathcal{V}_{I_1 \cdots I_r}(1, \cdots, n)$ are independent of the spin structure, meromorphic and single-valued $(1,0)$ forms in $z_1, \cdots, z_n \in \Sigma$, modular tensors under the full modular group $\mathrm{Sp}(2h, \mathbb{Z})$*, cyclically symmetric in the indices $I_1 \cdots I_r$ and expressible in terms of (products, index contractions and derivatives of) DHS kernels. In particular, the term with $r = n$ is given by cyclically symmetrized products of holomorphic Abelian differentials,*

$$\mathcal{V}_{I_1 \cdots I_n}(1, \cdots, n) = \frac{1}{n}\, \omega_{I_1}(1) \cdots \omega_{I_n}(n) + \mathrm{cycl}(I_1, \cdots, I_n) \tag{6.27}$$

The proof of the theorem largely follows the ideas in the proof of Theorem 5.1. The modular decomposition (6.26) and the explicit form of $\mathcal{V}_{I_1 \cdots I_r}(1, \cdots, n)$ in terms of DHS kernels follow iteratively from the modular descent in (6.14) and (6.15) in the same way as the meromorphic decomposition (5.1) is obtained from the meromorphic descent in (5.3) and (5.4). The resulting $\mathcal{V}_{I_1 \cdots I_r}(1, \cdots, n)$ are term by term modular tensors under the full modular group $\mathrm{Sp}(2h, \mathbb{Z})$ since this is already the case for the composing DHS kernels [60] and Abelian differentials. Meromorphicity of the $\mathcal{V}_{I_1 \cdots I_r}(1, \cdots, n)$ with $r < n$ in $z_1, \cdots, z_n$ is not manifest term by term in their DHS-kernel representation but guaranteed by the meromorphicity (6.16) of the cyclic product C_δ and the analysis of antiholomorphic derivatives in the modular descent. Finally, by the cyclic symmetry of $C_\delta^{I_1 \cdots I_r}$, we may choose to define the $\mathcal{V}_{I_1 \cdots I_r}(1, \cdots, n)$ to be cyclically symmetric in their indices,

$$\mathcal{V}_{I_1 I_2 \cdots I_r}(1, \cdots, n) = \mathcal{V}_{I_2 \cdots I_r I_1}(1, \cdots, n) \tag{6.28}$$

and we shall do so throughout. This completes the proof of Theorem 6.4.

Note that, while the modular tensors $\mathcal{V}_{I_1 \cdots I_r}(1, \cdots, n)$ at fixed rank $r \leq n$ are distinct for different values of n, their coefficients $C_\delta^{I_1 \cdots I_r}$ are independent of n and thus universal.

Following our choice for their meromorphic counterparts explained at the end of section 5.2, the modular tensors $\mathcal{V}_{I_1 I_2 \cdots I_r}(1, \cdots, n)$ with $r \geq 3$ are not taken to match their reflection images $(-1)^r \mathcal{V}_{I_r \cdots I_2 I_1}(1, \cdots, n)$. This is again possible by considering the descent equations (6.15) and decomposition (6.26) for cyclically symmetric but otherwise unspecified $C_\delta^{I_1 \cdots I_r}$ and has the advantage of introducing a larger class of single-valued and meromorphic modular tensors $\mathcal{V}_{I_1 \cdots I_r}(1, \cdots, n)$ to be used in future work.

6.4.1 Examples for $n = 2, 3$ points

For $n = 2, 3$ points, the modular descent in (6.14) and (6.15) leads to the following examples of $\mathcal{V}_{I_1 \cdots I_r}(1, 2, \cdots, n)$ at non-maximal rank $r < n$ (see (6.27) for the $n = r$ case),

$$
\begin{aligned}
\mathcal{V}(1,2) &= \partial_1 \partial_2 \mathcal{G}(1,2) \\
\mathcal{V}_{JK}(1,2,3) &= \frac{1}{2} \Big[\big(\partial_1 \mathcal{G}(1,3) - \partial_1 \mathcal{G}(1,2) \big) \omega_J(2) \omega_K(3) + \omega_I(1) f^I{}_J(2,3) \omega_K(3) \\
&\quad + \big(\omega_J(1) \omega_I(2) - \omega_I(1) \omega_J(2) \big) \partial_3 \Phi^I{}_K(3) + (J \leftrightarrow K) \Big] \\
\mathcal{V}(1,2,3) &= \big(\partial_1 \mathcal{G}(1,3) - \partial_1 \mathcal{G}(1,2) \big) \partial_2 \partial_3 \mathcal{G}(2,3) + \omega_I(1) \partial_2 \partial_3 \mathcal{G}^I(2,3)
\end{aligned}
\tag{6.29}
$$

The symmetrization $J \leftrightarrow K$ applies to both lines of the expression for $\mathcal{V}_{JK}(1,2,3)$ and implements our choice (6.28).

Note that the relation $\partial_1 \partial_2 \mathcal{G}(1,2) = \partial_2 \chi(1,2) + \pi Y^{IJ} \omega_I(1) \omega_J(2)$ between the z_i-dependent parts $\mathcal{W}(1,2)$ and $\mathcal{V}(1,2)$ in the decompositions of $C_\delta(1,2)$ implies the relation,

$$
C_\delta^{IJ} = D_\delta^{IJ} - \pi Y^{IJ} \tag{6.30}
$$

with D_δ^{IJ} given by (2.11). The term involving the inverse imaginary part Y^{IJ} of the period matrix exemplifies the non-meromorphicity of the constants $C_\delta^{I_1 \cdots I_r}$ in the moduli of Σ and is needed for C_δ^{IJ} to transform as a modular tensor of $\Gamma_h(2)$.

6.4.2 Further properties of $\mathcal{V}_{I_1 \cdots I_r}(1, \cdots, n)$

Similar to the meromorphic case in section 5.2, we gather several properties of the constituents of the modular decomposition (6.26) in the following proposition.

Proposition 6.5. *The modular tensors $\mathcal{V}_{I_1 \cdots I_r}(1, \cdots, n)$ in the modular decomposition (6.26) of cyclic products exhibit the following properties.*

(*a*) *For $n \geq 3$, $r \leq n-1$ and $2 \leq k \leq n-1$ they have simple poles in adjacent points,*

$$
\begin{aligned}
\bar\partial_1 \mathcal{V}_{I_1 \cdots I_r}(1, 2, \cdots, n) &= \pi \big(\delta(1,2) - \delta(1,n) \big) \mathcal{V}_{I_1 \cdots I_r}(2, \cdots, n) \\
\bar\partial_k \mathcal{V}_{I_1 \cdots I_r}(1, 2, \cdots, n) &= \pi \big(\delta(k,k+1) - \delta(k,k-1) \big) \mathcal{V}_{I_1 \cdots I_r}(1, \cdots, \hat{k}, \cdots, n) \\
\bar\partial_n \mathcal{V}_{I_1 \cdots I_r}(1, 2, \cdots, n) &= \pi \big(\delta(n,1) - \delta(n,n-1) \big) \mathcal{V}_{I_1 \cdots I_r}(1, \cdots, n-1)
\end{aligned}
\tag{6.31}
$$

and are holomorphic in all points at $r = n$.

(*b*) *For $n \geq 2$ and $0 \leq r \leq n$ and for a fixed choice of their indices $I_1, \cdots, I_r$, they are invariant under cyclic permutations of the points,*

$$
\mathcal{V}_{I_1 \cdots I_r}(1, 2, \cdots, n) = \mathcal{V}_{I_1 \cdots I_r}(2, \cdots, n, 1) \tag{6.32}
$$

(c) *They are invariant under simultaneous reflection of the indices and the points,*

$$\mathcal{V}_{I_1 I_2 \cdots I_r}(1,2,\cdots,n) = (-1)^{n+r} \mathcal{V}_{I_r \cdots I_2 I_1}(n,\cdots,2,1) \tag{6.33}$$

The proof of item (a) follows the same logic used to prove item (a) of Proposition 5.3 where the tensor $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ replaces the multiplet $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ and the tensor $C_\delta^{I_1\cdots I_r}$ replaces the multiplet $D_\delta^{I_1\cdots I_r}$.

Items (b) and (c) are equivalent to the vanishing of the following combinations,

$$\begin{aligned} \mathcal{P}_{I_1\cdots I_r}(1,2,\cdots,n) &= \mathcal{V}_{I_1\cdots I_r}(1,2,\cdots,n) - \mathcal{V}_{I_1\cdots I_r}(2,\cdots,n,1) \\ \mathcal{Q}_{I_1\cdots I_r}(1,2,\cdots,n) &= \mathcal{V}_{I_1 I_2\cdots I_r}(1,2,\cdots,n) - (-1)^{n+r}\mathcal{V}_{I_r\cdots I_2 I_1}(n,\cdots,2,1) \end{aligned} \tag{6.34}$$

which we shall prove by induction in $n-r$. For $r=n$, the vanishing of $\mathcal{P}_{I_1\cdots I_n}(1,\cdots,n)$ and $\mathcal{Q}_{I_1\cdots I_n}(1,\cdots,n)$ immediately follows from the expression (6.27) for $\mathcal{V}_{I_1\cdots I_n}(1,\cdots,n)$ in terms of cyclically symmetrized Abelian differentials. As an inductive step, let us assume that $\mathcal{P}_{I_1\cdots I_r}(1,\cdots,n)$ and $\mathcal{Q}_{I_1\cdots I_r}(1,\cdots,n)$ in (6.34) vanish for $n-r=s$, then we will show that this implies their vanishing at $n-r=s+1$. For this purpose, we note that the antiholomorphic derivatives (6.31) of $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ established as item (a) imply

$$\bar\partial_k \mathcal{P}_{I_1\cdots I_r}(1,2,\cdots,n) = \pi\big(\delta(k,k{+}1) - \delta(k,k{-}1)\big)\,\mathcal{P}_{I_1\cdots I_r}(1,\cdots,\hat{k},\cdots,n) \tag{6.35}$$

for $k=2,\cdots,n$ and $\bar\partial_1 \mathcal{P}_{I_1\cdots I_r}(1,\cdots,n)=0$. As a result, $\mathcal{P}_{I_1\cdots I_r}(1,\cdots,n)$ at $n-r=s+1$ is holomorphic since its counterparts at $n-r=s$ on the right side of (6.35) vanish by the inductive assumption. So it can at best be a combination of Abelian differentials

$$\mathcal{P}_{I_1\cdots I_r}(1,2,\cdots,n)\,\Big|_{n-r=s+1} = \omega_{J_1}(1)\cdots\omega_{J_n}(n)\,\mathfrak{P}^{J_1\cdots J_n}_{I_1\cdots I_r} \tag{6.36}$$

with modular tensors $\mathfrak{P}^{J_1\cdots J_n}_{I_1\cdots I_r}$ independent on the points. The constant tensors $\mathfrak{P}^{J_1\cdots J_n}_{I_1\cdots I_r}$ are obtained by integrating $\mathcal{P}_{I_1\cdots I_r}(1,\cdots,n)$ over n copies of Σ against $\bar\omega^{J_1}(1)\cdots\bar\omega^{J_n}(n)$. These surface integrals vanish since each $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ in the modular decomposition (6.26) with $r\leq n-1$ integrates to zero. This follows from the fact that each contribution from the modular descent equations (6.14), (6.15) is a total derivative of a single-valued function in at least one of the points. The same reasoning applies to $\mathcal{Q}_{I_1\cdots I_r}(1,\cdots,n)$ in (6.34) where the $\bar\partial_k$ derivatives take the form of (6.35) with $\mathcal{P}\to\mathcal{Q}$ on both sides for all of $k=1,\cdots,n$. This completes our inductive proof of items (b) and (c).

We also note that the modular tensors $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ and $C_\delta^{I_1\cdots I_r}$ are generally not locally holomorphic in the complex-structure moduli of Σ, as will be detailed in a forthcoming paper [72]. Still, the antiholomorphic moduli variations of $C_\delta^{I_1\cdots I_r}$ in (38-41) of [61] identify the totally symmetrized components $C_\delta^{(I_1\cdots I_r)}$ as locally holomorphic.

6.5 Meromorphic versus modular descent and decomposition

As was already advertised in the Introduction, there exists a remarkable correspondence between the modular decomposition (6.26) and the meromorphic decomposition (5.1) of the cyclic product of Szegö kernels, which follows from the same correspondence between the respective systems of descent equations. Indeed, the meromorphic descent equations in (5.3) and (5.4) are mapped, term by term, to the modular descent equations in (6.14) and (6.15) by simultaneously swapping their elements as follows,

$$\begin{aligned} D_\delta^{I_1\cdots I_r}(1,\cdots,n) \quad &\longleftrightarrow \quad C_\delta^{I_1\cdots I_r}(1,\cdots,n) \\ g^{I_1\cdots I_r}{}_J(x,y) \quad &\longleftrightarrow \quad f^{I_1\cdots I_r}{}_J(x,y) \\ \chi^{I_1\cdots I_s}(x,y) \quad &\longleftrightarrow \quad \partial_x \mathcal{G}^{I_1\cdots I_s}(x,y) \\ \varpi^{I_1\cdots I_r}{}_J(x) \quad &\longleftrightarrow \quad \partial_x \Phi^{I_1\cdots I_r}{}_J(x) \end{aligned} \tag{6.37}$$

This correspondence of the modular and meromorphic descent equations under the map (6.37) implies that the constituents $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ and $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ are mapped into one another,

$$\mathcal{V}_{I_1\cdots I_r}(1,2,\cdots,n) = \mathcal{W}_{I_1\cdots I_r}(1,2,\cdots,n)\,\Big|_{(6.37)} \tag{6.38}$$

Concretely, $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ is obtained from $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ by simultaneously mapping their constituents term-by-term according to (6.37). This reproduces the modular $n=2,3$ examples in section 6.4.1 from the meromorphic ones in section 5.1.2, and the expressions for $\mathcal{V}_{I_1\cdots I_r}(1,2,3,4)$ with $r=0,2,3$ may be read off term-by-term from those of $\mathcal{W}_{I_1\cdots I_r}(1,2,3,4)$ in (E.1).

6.5.1 Correspondence of $\mathfrak{A}$ periods and surface integrals

At a more formal level, the correspondence (6.37) extends to swapping the $\mathfrak{A}$ periods for the meromorphic descent with surface integrals for the modular descent,

$$\oint_{\mathfrak{A}^L} dt \quad \longleftrightarrow \quad \int_\Sigma d^2t\,\bar\omega^L(t) \tag{6.39}$$

and the first line of (6.37). However, the $\mathfrak{A}$ integral representations (4.25) of the constant multiplets $D_\delta^{I_1\cdots I_r}$ with $r\geq 3$ exhibit a tail of lower-rank convolutions with (rational multiples of) powers of $2\pi i$ as coefficients which do not have any modular counterpart in the surface-integral representation (6.10) of $C_\delta^{I_1\cdots I_r}$. Similar tails of lower-rank convolutions on $\mathfrak{A}$ cycles also occur for the multiplets $D_\delta^{I_1\cdots I_r}(1,\cdots,n)$ at $n\geq 1$ in (4.15) which are absent from the modular $C_\delta^{I_1\cdots I_r}(1,\cdots,n)$ in (6.9). Still, the leading order

$\beta_I = B_I + \mathcal{O}(2\pi i B^2)$ in the generating series (4.14) implies that the terms in the $\mathfrak{A}$-integral representation of $D_\delta^{I_1\cdots I_r}(1,\cdots,n)$ without reference to $2\pi i$ correspond to the expressions for $C_\delta^{I_1\cdots I_r}(1,\cdots,n)$ as surface integrals (6.9) under (6.39) for all $n,r \geq 1$ (also see (4.24) for the cases with $n=0$ and $r \geq 2$). An analogous correspondence via (6.39) was observed in [58] between the surface integral representation (6.1) and (6.2) of DHS kernels and the $\mathfrak{A}$-integral representation of Enriquez kernels, including a tail of simpler $\mathfrak{A}$ convolutions with powers of $2\pi i$ as prefactors in the meromorphic case.

6.5.2 An informal argument: derivatives versus monodromies

The combinations $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ in (6.26) and $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ in (5.1) are both single-valued and meromorphic in the points $z_1,\cdots,z_n \in \Sigma$, even if these properties are realized in different ways on $\mathcal{V}$ and $\mathcal{W}$. While $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ is manifestly single-valued, its meromorphicity relies on the cancellation of the $\bar\omega^I(z_i)$ from their derivatives $\bar\partial_k$ that result from applying (6.3) and (6.4) term-by-term and eventually conspire to (6.31). Conversely, while $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ is manifestly meromorphic, its single-valuedness relies on cancellations between individual $\mathfrak{B}$-monodromies (2.3) of the Enriquez kernels.

The combinatorial mechanisms for the cancellations of $\bar\partial_k$ derivatives in $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ and $\mathfrak{B}_L$ monodromies in $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ turn out to be closely related. A direct link may be established by *formally truncating* the $\mathfrak{B}_L$ monodromies $\Delta_L^{(k)}$ of the Enriquez kernels in (2.3) to the first order in $2\pi i$, and denoting the corresponding operation by $\delta_L^{(k)}$,

$$\begin{aligned}
\delta_L^{(x)} g^{I_1\cdots I_r}{}_J(x,y) &= -2\pi i\, \delta_L^{I_1}\, g^{I_2\cdots I_r}{}_J(x,y) \\
\delta_L^{(y)} g^{I_1\cdots I_r}{}_J(x,y) &= 2\pi i\, \delta_J^{I_r}\, g^{I_1\cdots I_{r-1}}{}_L(x,y)
\end{aligned} \tag{6.40}$$

These *differential monodromies* $\delta_L^{(k)}$ are in one-to-one correspondence with the coefficients of $\bar\omega^L(z_k)$ in the $\bar\partial_k$ derivatives of the DHS kernels. More specifically, the Massey system (6.3) and (6.4) is in formal correspondence with (6.40) through the substitution rules,

$$\begin{aligned}
g^{I_1\cdots I_r}{}_J(x,y) \quad &\longleftrightarrow \quad f^{I_1\cdots I_r}{}_J(x,y) \\
2\pi i\, \delta_L^I \text{ in } \delta_L^{(k)} \quad &\longleftrightarrow \quad \pi\bar\omega^I(k) \text{ in } \bar\partial_k
\end{aligned} \tag{6.41}$$

By this link between antiholomorphic derivatives of DHS kernels and differential monodromies of Enriquez kernels, the meromorphicity (6.31) of $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ can be viewed as a consequence of the vanishing $\mathfrak{B}_L$ monodromies of $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ (as established in Theorem 5.1), truncated to the first order in $2\pi i$. More generally, the differential $\mathfrak{B}$ monodromies of the statements in Theorem 2.3 on the meromorphic descent offer an alternative, if informal, proof of the modular descent.

It would be interesting to investigate whether the contributions with ≥ 2 powers of $2\pi i$ in the $\mathfrak{B}_L$ monodromies (2.3) of Enriquez kernels have an echo in terms of DHS kernels.

7 Descent of linear chain products of Szegö kernels

Cyclic products of Szegö kernels for even spin structure δ at generic moduli constitute an important ingredient in the evaluation of superstring amplitudes in the RNS formulation. However, other ingredients are required as well, such as the correlators involving two worldsheet gravitino fields or the insertion of a fermion stress tensor into a cyclic product of Szegö kernels [73, 39]. They may all be reduced to linear chain products of Szegö kernels for even spin structure δ at generic moduli which are defined as follows,

$$L_\delta(x; z_1, \cdots, z_n; y) = S_\delta(x, z_1) S_\delta(z_1, z_2) \cdots S_\delta(z_{n-1}, z_n) S_\delta(z_n, y) \tag{7.1}$$

For example, the insertion of the stress tensor T_{ww} into a cyclic product of Szegö kernels is achieved by taking the following limit,

$$\lim_{x,y \to w} \left(\frac{1}{2} \partial_x L_\delta(x; z_1, \cdots, z_n; y) - \frac{1}{2} \partial_y L_\delta(x; z_1, \cdots, z_n; y) \right) \tag{7.2}$$

The cyclic product $C_\delta(z_1, \cdots, z_{n+1})$ itself may be obtained trivially from the linear chain product $L_\delta(x; z_1, \cdots, z_n; y)$ by *closing the chain*,

$$C_\delta(z_1, \cdots, z_{n+1}) = \lim_{x,y \to z_{n+1}} L_\delta(x; z_1, \cdots, z_n; y) \tag{7.3}$$

The functions $L_\delta(x; z_1, \cdots, z_n; y)$ are meromorphic in all their arguments; single-valued $(1, 0)$ forms in the *internal points* $z_1, \cdots, z_n$; and $(\frac{1}{2}, 0)$ forms in the *end points* x, y in which they inherit the monodromies associated with the spin structure δ. Following earlier notation, $L_\delta(x; z_1, \cdots, z_n; y)$ will often be abbreviated $L_\delta(x; 1, \cdots, n; y)$.

In this section, we shall extend the descent procedures and resulting decomposition formulae of sections 2, 5 and 6 to the case of linear chain products. The goal of the descent procedure here, as it was in the case of the cyclic product of Szegö kernels, is to reduce the spin structure dependence to elements that are as simple as possible, and in particular independent of the internal points $z_1, \cdots, z_n$. Since $L_\delta(x; z_1, \cdots, z_n; y)$ is a $(\frac{1}{2}, 0)$ form in the end points x, y, the simple spin structure dependent elements we seek for linear chain products must inevitably retain the dependence on x, y.

7.1 The case $n = 1$

A simple example is provided by the case $n = 1$, where the Fay trisecant identity (see footnote 6) may be used to obtain the decomposition of $L_\delta(x; z; y) = S_\delta(x, z) S_\delta(z, y)$,

$$L_\delta(x; z; y) = \partial_z \ln \frac{E(z, y)}{E(z, x)} \, S_\delta(x, y) - \omega_I(z) \, \frac{\partial^I \vartheta[\delta](x - y)}{\vartheta[\delta](0) \, E(x, y)} \tag{7.4}$$

We refer to (A.2) and (A.5) of appendix A for the definition and properties of Riemann ϑ-functions and their appearance in the Szegö kernel. Throughout, the argument $x-y$ of the ϑ function stands for the Abel map so that $\vartheta[\delta](x-y)$ is a shorthand for $\vartheta[\delta](\int_y^x \omega)$.

In each term on the right side of (7.4), the left factor is δ-independent and contains all the dependence of the term on the internal point z; the right factor is z-independent and contains all the δ-dependence of the term; and both factors may depend on the end points x, y. Using the second relation in (2.9), the z-dependence of both terms may be expressed in terms of Enriquez kernels,

$$L_\delta(x;z;y) = \big(\chi(z,x) - \chi(z,y)\big) M_\delta(x,y) + \omega_I(z) M_\delta^I(x,y) \tag{7.5}$$

where the entire spin-structure dependence is carried by,

$$\begin{aligned} M_\delta(x,y) &= S_\delta(x,y) \\ M_\delta^I(x,y) &= -\frac{\partial^I \vartheta[\delta](x-y)}{\vartheta[\delta](0)\, E(x,y)} = -\frac{\partial^I \vartheta[\delta](x-y)}{\vartheta[\delta](x-y)} S_\delta(x,y) \end{aligned} \tag{7.6}$$

Alternatively, the decomposition may be formulated in terms of DHS kernels,

$$L_\delta(x;z;y) = \big(\partial_z \mathcal{G}(z,x) - \partial_z \mathcal{G}(z,y)\big) L_\delta(x,y) + \omega_I(z) L_\delta^I(x,y) \tag{7.7}$$

where the spin-structure dependence is now carried by,

$$\begin{aligned} L_\delta(x,y) &= S_\delta(x,y) \\ L_\delta^I(x,y) &= \left(-\frac{\partial^I \vartheta[\delta](x-y)}{\vartheta[\delta](x-y)} + 2\pi i \,\mathrm{Im} \int_x^y \omega^I\right) S_\delta(x,y) \end{aligned} \tag{7.8}$$

The advantage of the decomposition (7.7) is that its terms are individually modular tensors under the congruence subgroup $\Gamma_h(2)$ of (6.11), at the cost of introducing non-meromorphic dependence on the end points x, y into $L_\delta^I(x,y)$ which is compensated by the Arakelov Green functions in (7.7).

In the remainder of this section, we shall generalize both the manifestly meromorphic decomposition (7.5) of the linear chain $L_\delta(x;z;y)$ and its modular counterpart (7.7) to linear chains (7.1) with an arbitrary number $n \geq 2$ of internal points. The modular descent of Theorem 6.2 and the decomposition formula in (6.26) of the cyclic products $C_\delta(1,\cdots,n)$ admit integral representations (6.10) that are somewhat simpler than those of the meromorphic formulation in Theorem 5.1, see section 6.5.1. Therefore, we shall start by presenting a modular decomposition of the linear chains (7.1) in section 7.2, 7.3 and then proceed to its meromorphic counterpart in section 7.4.

7.2 Descent for chains of arbitrary length via DHS kernels

The descent procedure for linear chains $L_\delta(x;1,\cdots,n;y)$ with an arbitrary number n of internal points in terms of DHS kernels f may be formulated analogously to the descent procedure for the case of cyclic products of Szegö kernels in Theorem 6.2.

The functions $L_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y)$ are differential $(1,0)$ forms in the variables $z_1,\cdots,z_n$ and $(\frac{1}{2},0)$ forms in x and y. For $r=0$ they are defined by,

$$L_\delta^{\emptyset}(x;1,\cdots,n;y)=L_\delta(x;1,\cdots,n;y) \tag{7.9}$$

while for $r\geq 1$ they are defined recursively as follows,

$$L_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y)=\int_\Sigma d^2t_r\,\bar\omega^{I_r}(t_r)\,L_\delta^{I_1\cdots I_{r-1}}(x;t_r,1,\cdots,n;y) \tag{7.10}$$

In particular, for $n=0$ the repeated iteration of (7.10) gives,

$$L_\delta^{I_1\cdots I_r}(x,y)=\int_\Sigma d^2t_1\,\bar\omega^{I_1}(t_1)\cdots\int_\Sigma d^2t_r\,\bar\omega^{I_r}(t_r)L_\delta(x;t_1,\cdots,t_r;y) \tag{7.11}$$

Proposition 7.1. *The functions $L_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y)$ satisfy the following properties.*

(a) For arbitrary n, they are tensors under the congruence subgroup $\Gamma_h(2)$ of (6.11).

(b) For $n=0$, the spinors $L_\delta^{I_1\cdots I_r}(x,y)$ transform as follows under reflections,

$$L_\delta^{I_1I_2\cdots I_r}(y,x)=(-1)^{r+1}L_\delta^{I_r\cdots I_2I_1}(x,y) \tag{7.12}$$

(c) their antiholomorphic derivatives for $r\geq 1$ are given by,

$$\begin{aligned}\bar\partial_x L_\delta^{I_1\cdots I_r}(x,y)&=\pi\bar\omega^{I_1}(x)L_\delta^{I_2\cdots I_r}(x,y)\\ \bar\partial_y L_\delta^{I_1\cdots I_r}(x,y)&=-\pi\bar\omega^{I_r}(y)L_\delta^{I_1\cdots I_{r-1}}(x,y)\end{aligned} \tag{7.13}$$

(d) and their monodromies are given by,

$$\begin{aligned}L_\delta^{I_1\cdots I_r}(\mathfrak{A}^K\cdot x,y)&=e^{2\pi i\delta'_K}L_\delta^{I_1\cdots I_r}(x,y)\\ L_\delta^{I_1\cdots I_r}(\mathfrak{B}_K\cdot x,y)&=e^{2\pi i\delta''_K}L_\delta^{I_1\cdots I_r}(x,y)\\ L_\delta^{I_1\cdots I_r}(x,\mathfrak{A}^K\cdot y)&=e^{2\pi i\delta'_K}L_\delta^{I_1\cdots I_r}(x,y)\\ L_\delta^{I_1\cdots I_r}(x,\mathfrak{B}_K\cdot y)&=e^{2\pi i\delta''_K}L_\delta^{I_1\cdots I_r}(x,y)\end{aligned} \tag{7.14}$$

where we use the standard decomposition $\delta=\delta''+\Omega\delta'$ with $2\delta',2\delta''\in\mathbb{Z}_2^h$.

The following theorem shows how the functions $L_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y)$ solve the descent equations for the case of linear chain products of Szegö kernels.

Theorem 7.2. *For $n\geq 2$ the functions $L_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y)$ satisfy the following system of descent equations,*

$$
\begin{aligned}
L_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y) = & \sum_{i=0}^{r} f^{I_r\cdots I_{i+1}}{}_J(1,2)\, L_\delta^{I_1\cdots I_i J}(x;2,\cdots,n;y) \\
& -\Big(\partial_1 \mathcal{G}^{I_r\cdots I_1}(1,2) - \partial_1 \mathcal{G}^{I_r\cdots I_1}(1,x)\Big) L_\delta(x;2,\cdots,n;y)
\end{aligned}
\tag{7.15}
$$

while for $n=1$ they obey,

$$
\begin{aligned}
L_\delta^{I_1\cdots I_r}(x;z;y) = & \sum_{i=0}^{r} f^{I_r\cdots I_{i+1}}{}_J(z,y)\, L_\delta^{I_1\cdots I_i J}(x,y) \\
& -\Big(\partial_z \mathcal{G}^{I_r\cdots I_1}(z,y) - \partial_z \mathcal{G}^{I_r\cdots I_1}(z,x)\Big) L_\delta(x,y)
\end{aligned}
\tag{7.16}
$$

To prove the theorem, we need the result analogous to Lemma 6.3 giving the differential equations. For $r=0$ with $1\leq k\leq n$ and setting $z_0=x$ and $z_{n+1}=y$ we have,

$$
\bar\partial_k L_\delta(x;1,\cdots,n;y) = \pi\big(\delta(k,k+1)-\delta(k,k-1)\big) L_\delta(x;1,\cdots,\hat{k},\cdots,n;y) \tag{7.17}
$$

For $r\geq 1$ with $k=1$, $2\leq k\leq n-1$ and $k=n$ we have respectively,

$$
\begin{aligned}
\bar\partial_1 L_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y) &= \pi\delta(1,2)\, L_\delta^{I_1\cdots I_r}(x;2,\cdots,n;y) \\
& \quad -\pi\bar\omega^{I_r}(z_1) L_\delta^{I_1\cdots I_{r-1}}(x;2,\cdots,n;y) \\
\bar\partial_k L_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y) &= \pi\big(\delta(k,k+1)-\delta(k,k-1)\big) \\
& \quad \times L_\delta^{I_1\cdots I_r}(x;1,\cdots,\hat{k},\cdots,n;y) \\
\bar\partial_n L_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y) &= \pi\big(\delta(n,y)-\delta(n,n-1)\big) L_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y)
\end{aligned}
\tag{7.18}
$$

The proof of the theorem proceeds along the same lines as the proof of Theorem 6.2.

7.3 Modular decomposition of linear chain products

As a result of iterating the modular descent in (7.15) and (7.16), linear chains (7.1) of Szegö kernels with an arbitrary number n of internal points can be decomposed as,

$$
L_\delta(x;1,\cdots,n;y) = \sum_{r=0}^{n} \mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y) L_\delta^{I_1\cdots I_r}(x,y) \tag{7.19}
$$

Similar to the modular decomposition (6.26) of the cyclic products C_δ of Szegö kernels, all the dependence on the internal points $z_1, \cdots, z_n$ is carried by DHS kernels, namely by the tensors f and $\partial\mathcal{G}$ in (7.15) and (7.16). The DHS kernels are grouped into combinations $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ meromorphic in the internal points and single-valued in all points according to the indices of the accompanying spinors $L_\delta^{I_1\cdots I_r}(x,y)$ in (7.11) that capture the entire spin-structure dependence of (7.19). By the composition of $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ from DHS kernels and the integral representation (7.11) of $L_\delta^{I_1\cdots I_r}(x,y)$, they are modular tensors of $\text{Sp}(2h,\mathbb{Z})$ and $\Gamma_h(2)$, respectively, and we refer to (7.19) as the *modular decomposition* of linear chain products. In particular, for $r=n$ one readily establishes,

$$\mathcal{V}_{I_1\cdots I_n}(x;1,\cdots,n;y) = \omega_{I_1}(1)\cdots\omega_{I_n}(n) \tag{7.20}$$

A significant difference between the decompositions of cyclic products and linear chain products of Szegö kernels is that the extra dependence on the end points in the latter case enters both the $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ and the $L_\delta^{I_1\cdots I_r}(x,y)$ on the right side of (7.19).

7.3.1 Examples of the modular tensors $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$

The simplest non-trivial example of the modular tensors $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ in (7.19) involving one internal point can be read off from (7.7),

$$\mathcal{V}(x;1;y) = \partial_1\mathcal{G}(1,x) - \partial_1\mathcal{G}(1,y) \tag{7.21}$$

With two internal points, the non-trivial $\mathcal{V}$ tensors in (7.19) are given by,

$$\begin{aligned}
\mathcal{V}_J(x;1,2;y) &= \partial_1\big(\mathcal{G}(1,x)-\mathcal{G}(1,2)\big)\omega_J(2) + \omega_I(1)f^I{}_J(2,y) \\
\mathcal{V}(x;1,2;y) &= \partial_1\big(\mathcal{G}(1,x)-\mathcal{G}(1,2)\big)\partial_2\big(\mathcal{G}(2,x)-\mathcal{G}(2,y)\big) \\
&\quad + \omega_I(1)\partial_2\big(\mathcal{G}^I(2,x)-\mathcal{G}^I(2,y)\big)
\end{aligned} \tag{7.22}$$

Their analogues with three internal points and rank ≤ 2 can be found in appendix E.2. The cases at highest rank $r=n$ follow the simple formula in (7.20).

7.3.2 Properties of the modular tensors $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$

We gather several properties of the modular tensors $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ entering the modular decomposition (7.19) of linear chains in the following proposition.

Proposition 7.3. *The modular tensors $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ satisfy the properties.*

(a) *The antiholomorphic derivatives of* $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ *with* $r \leq n-1$ *and* $k = 1,\cdots,n$ *are given by (setting* $z_0 = x$ *and* $z_{n+1} = y$*),*

$$\begin{aligned}
\bar\partial_k \mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y) &= \pi\big(\delta(k,k+1)-\delta(k,k-1)\big)\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,\hat{k},\cdots,n;y) \\
\bar\partial_x \mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y) &= -\pi\bar\omega^K(x)\mathcal{V}_{KI_1\cdots I_r}(x;1,\cdots,n;y) \\
&\quad +\pi\delta(1,x)\mathcal{V}_{I_1\cdots I_r}(x;2,\cdots,n;y) \\
\bar\partial_y \mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y) &= \pi\mathcal{V}_{I_1\cdots I_rK}(x;1,\cdots,n;y)\bar\omega^K(y) \\
&\quad -\pi\delta(n,y)\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n{-}1;y)
\end{aligned} \tag{7.23}$$

while the cases with $r = n$ *in (7.20) are holomorphic.*

(b) *The modular tensors* $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ *exhibit the following alternating parity under simultaneous reflection of the indices and the points,*

$$\mathcal{V}_{I_1I_2\cdots I_r}(y;n,\cdots,2,1;x) = (-1)^{n-r}\mathcal{V}_{I_r\cdots I_2I_1}(x;1,2,\cdots,n;y) \tag{7.24}$$

We note that the cyclic symmetry of $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$, established in (6.28), has no counterpart for $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$.

To prove item (a) of Proposition 7.3, we proceed as follows. In the antiholomorphic derivatives (7.23) of item (a), the δ-functions follow from the pole structure (7.17) of the linear chain product. The modular descent is designed to preserve this pole structure without referring to any relations between spinors $L_\delta^{I_1\cdots I_r}(x,y)$ of different rank, so it must hold for each term in the decomposition (7.19), except for the non-singular term of rank $r = n$ since $L_\delta^{I_1\cdots I_n}(x,y)$ does not enter linear chains with $n-1$ internal points. The contributions of $\bar\omega^K(x), \bar\omega^K(y)$ to the antiholomorphic derivatives (7.23) of item (a) can be inferred by imposing meromorphicity of the modular decomposition. They follow by evaluating $\bar\partial_x, \bar\partial_y$ of (7.19), using the antiholomorphic derivatives of $L_\delta^{I_1\cdots I_r}(x,y)$ in item (a) and imposing the coefficients of the resulting $L_\delta^{J_1\cdots J_s}(x,y)$ to be free of $\bar\omega^K(x), \bar\omega^K(y)$. This can be imposed separately for each value of $s = 0,1,\cdots,n-1$ since the modular descent is designed to reflect the meromorphicity of linear chain products without relying on any relations between spinors $L_\delta^{I_1\cdots I_r}(x,y)$ of different rank.

Item (b) is equivalent to the vanishing of,

$$\mathcal{R}_{I_1\cdots I_r}(x;1,\cdots,n;y) = \mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y) - (-1)^{n-r}\mathcal{V}_{I_r\cdots I_1}(y;n,\cdots,1;x) \tag{7.25}$$

and proven by induction in $n-r$ similar to the proof of items (b) and (c) of Proposition 6.5. The base case of $r = n$ in (7.20) evidently gives rise to a vanishing $\mathcal{R}$. The inductive step consists of relating (7.25) with $n-r = s$ and $n-r = s+1$ by antiholomorphic derivatives in $z_1,\cdots,z_n$ and noting that all instances of $\mathcal{R}$ with $n-r \geq 1$ vanish upon integrating $z_1,\cdots,z_n$ over n copies of the surface against $\bar\omega^{J_1}(1)\cdots\bar\omega^{J_n}(n)$.

Note that both of $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ at $r \leq n-1$ and $L_\delta^{I_1\cdots I_r}(x,y)$ at $r \geq 1$ individually depend non-meromorphically on the complex-structure moduli of Σ. Still, their combination in (7.19) resulting in a linear chain of Szegö kernels is guaranteed to yield a meromorphic function of the moduli.

7.4 Descent for chains of arbitrary length via Enriquez kernels

The meromorphic descent procedure given in Theorem 2.3 and the modular descent procedure given in Theorem 6.2 for cyclic products of Szeö kernels were found to be related by the correspondence of (6.37) converting Enriquez kernels into DHS kernels and vice-versa. The same type of correspondence relates the modular descent for linear chain products of Szegö kernels described in the previous sections to their meromorphic counterpart.

The multiplets $M_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y)$ are differential $(1,0)$ forms in the internal points $z_1,\cdots,z_n$ and $(\frac{1}{2},0)$ forms in the end points x and y. For $r=0$, they are defined by,

$$M_\delta^{\emptyset}(x;1,\cdots,n;y) = L_\delta(x;1,\cdots,n;y) \tag{7.26}$$

and for $r \geq 1$ they are defined recursively in the rank r as follows,

$$\begin{aligned} M_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y) = & \oint_{\mathfrak{A}^{I_r}} dt\, M_\delta^{I_1\cdots I_{r-1}}(x;t,1,\cdots,n;y) \\ & - \sum_{\ell=1}^{r-1} (-2\pi i)^{r-\ell} \frac{\mathrm{Ber}_{r-\ell}}{(r-\ell)!} \, \delta_J^{I_r\cdots I_\ell} \, M_\delta^{I_1\cdots I_{\ell-1}J}(x;1,\cdots,n;y) \end{aligned} \tag{7.27}$$

The following theorem shows how the functions $M_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y)$ solve the system of descent equations for linear chain products of Szegö kernels in terms of Enriquez kernels.

Theorem 7.4. *For $n \geq 2$ the functions $M_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y)$ satisfy the following system of descent equations,*

$$\begin{aligned} M_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y) = & \sum_{i=0}^{r} g^{I_r\cdots I_{i+1}}{}_J(1,2)\, M_\delta^{I_1\cdots I_i J}(x;2,\cdots,n;y) \\ & - \Big(\chi^{I_r\cdots I_1}(1,2) - \chi^{I_r\cdots I_1}(1,x)\Big) L_\delta(x;2,\cdots,n;y) \end{aligned} \tag{7.28}$$

while for $n=1$ we have,

$$\begin{aligned} M_\delta^{I_1\cdots I_r}(x;z;y) = & \sum_{i=0}^{r} g^{I_r\cdots I_{i+1}}{}_J(z,y)\, M_\delta^{I_1\cdots I_i J}(x,y) \\ & - \Big(\chi^{I_r\cdots I_1}(z,y) - \chi^{I_r\cdots I_1}(z,x)\Big) S_\delta(x,y) \end{aligned} \tag{7.29}$$

Iterating the recursion relations reduces all the spin structure dependence of a general function $M_\delta^{I_1\cdots I_s}(x;1,\cdots,n;y)$ to a linear combination of the basic multiplets $M_\delta^{I_1\cdots I_r}(x,y)$ with δ-independent coefficients. These basic multiplets are expressible through multiple $\mathfrak{A}$ convolutions of linear chain products,

$$\mathfrak{M}_\delta^{I_1\cdots I_r}(x,y) = \oint_{\mathfrak{A}^{I_r}} dt_r \cdots \oint_{\mathfrak{A}^{I_1}} dt_1 \, L_\delta(x;t_1,\cdots,t_r;y) \tag{7.30}$$

according to the generating series,

$$\sum_{r=1}^{\infty} B_{I_r}\cdots B_{I_1} M_\delta^{I_1\cdots I_r}(x,y) = \sum_{r=1}^{\infty} \beta_{I_r}\cdots\beta_{I_1}\mathfrak{M}_\delta^{I_1\cdots I_r}(x,y) \tag{7.31}$$

see (4.11) for the series expansion of β_I in terms of B_I and section 4.2 for the integration contour prescription that defines the right side of (7.30).

The proof of the theorem proceeds as for Theorem 2.3 and is left to the reader. A few remarks are as follows. First, the recursion (7.28) in the rank implies the recursion (7.27) in the number of points by integrating in z_1 over $\mathfrak{A}^{I_r}$, using the $\mathfrak{A}$ period (2.7) and the vanishing $\mathfrak{A}$ periods of $\chi^{I_r\cdots I_1}(1,2) - \chi^{I_r\cdots I_1}(1,x)$ in z_1. Second, the relation of (7.31) follows from iterations of (7.27), which are conveniently collected in a generating series akin to (3.9) by closely following the derivation in section 4.3 for cyclic products. Note that for linear chain products, the edge case $n=0$ does not require a separate treatment in contrast to the case for cyclic products.

7.4.1 Meromorphic decomposition of linear chains

By iterating the meromorphic descent (7.28) and (7.29), we find the following alternative decomposition of the linear chain product (7.1):

$$L_\delta(x;1,\cdots,n;y) = \sum_{r=0}^{n} \mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y) M_\delta^{I_1\cdots I_r}(x,y) \tag{7.32}$$

Just like in the term-by-term modular decomposition (7.19), the dependence on the internal points $z_1,\cdots,z_n$ of the linear chain via $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ is fully separated from that on the even spin structure δ via $M_\delta^{I_1\cdots I_r}(x,y)$. In contrast to (7.19), the decomposition (7.32) is meromorphic term by term in both the points $x,y,z_1,\cdots,z_n$ and the moduli of Σ. However, the single-valuedness of the composing Szegö kernels in the internal points is realized through cancellations of $\mathfrak{B}$ monodromies between individual contributions to $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ with $r \leq n-1$. Their counterparts at $r=n$ in turn are holomorphic and single-valued,

$$\mathcal{W}_{I_1\cdots I_n}(x;1,\cdots,n;y) = \omega_{I_1}(1)\cdots\omega_{I_n}(n) \tag{7.33}$$

As will be detailed in Proposition 7.5 below, the monodromies of (7.32) in the end points x, y according to the Szegö kernels in (7.1) additionally rely on the interplay between the multiplets $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ and $M_\delta^{I_1\cdots I_r}(x,y)$.

7.4.2 Correspondence with the modular descent

The meromorphic descent (7.28) and (7.29) may be obtained from the modular one in (7.15) and (7.16) by converting DHS kernels to Enriquez kernels according to,

$$
\begin{aligned}
L_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y) \quad &\longleftrightarrow \quad M_\delta^{I_1\cdots I_r}(x;1,\cdots,n;y) \\
f^{I_1\cdots I_r}{}_J(x,y) \quad &\longleftrightarrow \quad g^{I_1\cdots I_r}{}_J(x,y) \\
\partial_x \mathcal{G}^{I_1\cdots I_s}(x,y) \quad &\longleftrightarrow \quad \chi^{I_1\cdots I_s}(x,y) \\
\partial_x \Phi^{I_1\cdots I_r}{}_J(x) \quad &\longleftrightarrow \quad \varpi^{I_1\cdots I_r}{}_J(x)
\end{aligned}
\tag{7.34}
$$

which adapt the analogous substitutions (6.37) for the cyclic products to linear chain products. Hence, the combinations of Enriquez kernels $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ that carry the entire dependence on the internal points in (7.32) are simply obtained by applying the substitution (7.34) to the DHS kernels within the $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ in the modular decomposition,

$$
\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y) = \mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)\Big|_{(7.34)}
\tag{7.35}
$$

The same correspondence $g^{J_1\cdots J_s}{}_K(x,y) \leftrightarrow f^{J_1\cdots J_s}{}_K(x,y)$ was found in (6.38) to relate $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n) \leftrightarrow \mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ in the decompositions of cyclic products.

7.4.3 Examples of the multiplets $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ and $M_\delta^{I_1\cdots I_r}(x,y)$

The simplest examples of the multiplets $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ with $r \leq n-1$ can be obtained by applying the correspondence (7.35) to the expressions for the modular tensors $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ with $n \leq 3$ in (7.21), (7.22) and (E.2).

The simplest examples of $M_\delta^{I_1\cdots I_r}(x,y)$, written as multiple $\mathfrak{A}$ convolution integrals (7.30), are obtained by expanding the letters β_I on the right side of (7.31) via (4.11) and isolating the coefficients of $B_{I_r}\cdots B_{I_1}$,

$$
\begin{aligned}
M_\delta^{I_1}(x,y) &= \mathfrak{M}_\delta^{I_1}(x,y) \\
M_\delta^{I_1 I_2}(x,y) &= \mathfrak{M}_\delta^{I_1 I_2}(x,y) - i\pi\, \delta^{I_1}_{I_2}\, \mathfrak{M}_\delta^{I_2}(x,y) \\
M_\delta^{I_1 I_2 I_3}(x,y) &= \mathfrak{M}_\delta^{I_1 I_2 I_3}(x,y) - i\pi \left[\delta^{I_1}_{I_2}\, \mathfrak{M}_\delta^{I_2 I_3}(x,y) + \delta^{I_2}_{I_3}\, \mathfrak{M}_\delta^{I_1 I_3}(x,y)\right] - \frac{2\pi^2}{3}\delta^{I_2 I_1}_{I_3}\, \mathfrak{M}_\delta^{I_3}(x,y)
\end{aligned}
\tag{7.36}
$$

The composition of indices of the terms of lower rank is in one-to-one correspondence with the analogous representations of the multiplets $D_\delta^{I_1\cdots I_r}(1,\cdots,n)$ for cyclic products in (4.15) since the respective generating series match. However, the expressions for the constant multiplets $D_\delta^{I_1\cdots I_r}$ in terms of similar convolutions in (4.25) feature extra terms whose index structure does not have any counterparts in (7.36).

7.4.4 Properties of the multiplets $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ and $M_\delta^{I_1\cdots I_r}(x,y)$

We gather several properties of the multiplets $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ and $M_\delta^{I_1\cdots I_r}(x,y)$ entering the meromorphic decomposition (7.32) of linear chains in the following proposition:

Proposition 7.5. *The multiplets $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ and $M_\delta^{I_1\cdots I_r}(x,y)$ exhibit the following properties:*

(a) *The simple poles of the multiplets $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ with $r\leq n-1$ as well as $k=1,\cdots,n$ are determined by (again setting $z_0=x$ and $z_{n+1}=y$),*

$$\begin{aligned}
&\bar\partial_k \mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y) = \pi\big(\delta(k,k+1)-\delta(k,k-1)\big)\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,\hat{k},\cdots,n;y)\\
&\bar\partial_x \mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y) = \pi\delta(1,x)\mathcal{W}_{I_1\cdots I_r}(x;2,\cdots,n;y)\\
&\bar\partial_y \mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y) = -\pi\delta(n,y)\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n{-}1;y)
\end{aligned}\tag{7.37}$$

while the cases with $r=n$ in (7.33) are holomorphic.

(b) *The multiplets $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ are single-valued in the internal points. Their monodromies in the end points x,y are trivial for $\mathfrak{A}$ cycles and take the following form for $\mathfrak{B}$ cycles,*

$$\begin{aligned}
\Delta_L^{(x)}\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y) &= \sum_{k=1}^{n-r}\frac{(-2\pi i)^k}{k!}\mathcal{W}_{\vec{L}_k I_1\cdots I_r}(x;1,\cdots,n;y)\\
\Delta_L^{(y)}\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y) &= \sum_{k=1}^{n-r}\frac{(2\pi i)^k}{k!}\mathcal{W}_{I_1\cdots I_r\vec{L}_k}(x;1,\cdots,n;y)
\end{aligned}\tag{7.38}$$

with the shorthand $\vec{L}_k$ for k consecutive indices $LL\cdots L$.

(c) *The $\mathfrak{A}$ monodromies of the spinors $M_\delta^{I_1\cdots I_r}(x,y)$ are identical to those of the individual Szegö kernels and their modular counterparts $L_\delta^{I_1\cdots I_r}(x,y)$ in (7.14). Their $\mathfrak{B}$ monodromies in turn are given by*

$$\begin{aligned}
M_\delta^{I_1\cdots I_r}(\mathfrak{B}_L\cdot x,y) &= e^{2\pi i\delta''_L}\sum_{k=0}^{r}\frac{(2\pi i)^k}{k!}\delta_L^{I_1\cdots I_k}M_\delta^{I_{k+1}\cdots I_r}(x,y)\\
M_\delta^{I_1\cdots I_r}(x,\mathfrak{B}_L\cdot y) &= e^{2\pi i\delta''_L}\sum_{k=0}^{r}\frac{(-2\pi i)^k}{k!}\delta_L^{I_{r-k+1}\cdots I_r}M_\delta^{I_1\cdots I_{r-k}}(x,y)
\end{aligned}\tag{7.39}$$

(*d*) *The multiplets* $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ *exhibit the following alternating parity property under simultaneous reflection of the indices and the points,*

$$\mathcal{W}_{I_1 I_2\cdots I_r}(y;n,\cdots,2,1;x) = (-1)^{n-r}\mathcal{W}_{I_r\cdots I_2 I_1}(x;1,2,\cdots,n;y) \tag{7.40}$$

(*e*) *The analogous reflection properties of the spinors* $M_\delta^{I_1\cdots I_r}(x,y)$ *are,*

$$M_\delta^{I_1 I_2\cdots I_r}(y,x) = (-1)^{r+1} M_\delta^{I_r\cdots I_2 I_1}(x,y) \tag{7.41}$$

Item (*a*) is proven through the same arguments that give rise to the poles in item (*b*) of Proposition 7.3, i.e. the delta distrubutions in (7.23) while ignoring the additional terms involving $\bar\omega^K$.

Items (*b*) and (*c*) can be understood from the notion of *differential* $\mathfrak{B}$ *monodromies* introduced in section 6.5.2. The first order in $2\pi i$ on the right side of (7.38) follows from (7.35) and the terms $\bar\omega^K(x), \bar\omega^K(y)$ in (7.23) through the correspondence between the differential monodromies of Enriquez kernels and antiholomorphic derivatives of DHS kernels in (6.41). The differential monodromies in the $k=1$ terms of (7.39) are then a consequence of imposing single-valuedness on the meromorphic decomposition (7.32) of linear chain products to first order in $2\pi i$ in the term-by-term monodromies using linear independence of $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ at different rank.

The full proof of items (*b*) and (*c*) including higher orders in $2\pi i$ is most conveniently carried out by organizing the Enriquez kernels and spinors $M_\delta^{I_1\cdots I_r}(x,y)$ into generating functions similar to those in section 3. The differential monodromies derived in the previous paragraphs imply the exact statements (7.38) and (7.39) of items (*b*) and (*c*) once the powers of $2\pi i$ in the monodromies of these generating functions are shown to exponentiate as it is the case in (3.5).

The proof of items (*d*) and (*e*) again exploits the correspondence (7.34) between the constituents of the meromorphic and modular descents for linear chain products, following the logic of the proof of Proposition 5.2 and items (*b*), (*c*) of Proposition 5.3.

- We prove (7.40) in item (*d*) by adapting the arguments in appendix D.2: checking the reflection properties (7.24) of individual $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ solely requires the interchange lemma and Fay identities of DHS kernels which hold in identical form for Enriquez kernels [62] and thereby reduce the verifications of (7.40) to those of (7.24) which are proven in section 7.3.2. Note that the decompositions of linear chain products in this section do not involve any derivatives of DHS or Enriquez kernels, so the derivation of (7.40) does not require the relations (D.2) and (D.4) among derivatives.

- The statement (7.41) of item (e) can be inferred from the reflection parity $(-1)^{n+1}$ of linear chain products $L_\delta(x;1,\cdots,n;y)$, imposed at the level of the meromorphic decomposition (7.32). The proof is most conveniently carried out by induction in the rank r of $M_\delta^{I_1\cdots I_r}(x,y)$ as done in appendix D.3 in the context of cyclic products, using the reflection properties (7.40) of the multiplets $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ in item (d). At rank $r\leq 3$, a direct proof of the reflection properties based on the representations (7.36) and contour deformation techniques can be found in appendix D.4.

7.5 Coincident limits of $M_\delta^{I_1\cdots I_r}(x,y)$ and $L_\delta^{I_1\cdots I_r}(x,y)$

In this subsection, we relate the coincident limit $y\to x$ of the multiplets $M_\delta^{I_1\cdots I_r}(x,y)$ and the tensors $L_\delta^{I_1\cdots I_r}(x,y)$ to the constant multiplets $D_\delta^{I_1\cdots I_r}$ and the constant tensors $C_\delta^{I_1\cdots I_r}$, respectively. These relations derive from the fact that the cyclic case is given by the closure $y\to x$ of the linear chain case,

$$\lim_{y\to x} L_\delta(x;1,\cdots,n;y)=C_\delta(1,\cdots,n,x) \tag{7.42}$$

We shall treat the meromorphic and modular cases separately below as the structure of their limit differs to some degree.

7.5.1 The limit of $M_\delta^{I_1\cdots I_r}(x,y)$

Imposing the meromorphic decompositions of (5.1) and (7.32) on the relation (7.42) leads to recursion relations for the coincident limits $\lim_{y\to x} M_\delta^{I_1\cdots I_r}(x,y)$. These limits exist and are finite for $r\geq 1$. For example, to the lowest few ranks we obtain,

$$\begin{aligned}
M_\delta^{I}(x,x)&=D_\delta^{IJ}\omega_J(x) \\
M_\delta^{IJ}(x,x)&=D_\delta^{IJK}\omega_K(x)+D_\delta^{IK}\varpi^J{}_K(x)-D_\delta^{JK}\varpi^I{}_K(x) \\
M_\delta^{IJK}(x,x)&=D_\delta^{IJKL}\omega_L(x)+D_\delta^{IJL}\varpi^K{}_L(x)-D_\delta^{JKL}\varpi^I{}_L(x)+D_\delta^{IL}\varpi^{KJ}{}_L(x) \\
&\quad+D_\delta^{KL}\varpi^{IJ}{}_L(x)-D_\delta^{JL}\big(\varpi^{IK}{}_L(x)+\varpi^{KI}{}_L(x)\big)
\end{aligned} \tag{7.43}$$

The combinatorial structure of these expressions for $M_\delta^{I_1\cdots I_r}(x,x)$ at $r=1,2,3$ suggests the following generalization to arbitrary rank.

Proposition 7.6. *The coincident limit $y\to x$ of the spinors $M_\delta^{I_1\cdots I_r}(x,y)$ in the meromorphic decomposition (7.32) of linear chain products is given as follows in terms of Enriquez kernels and constant multiplets $D_\delta^{J_1\cdots J_s}$,*

$$M_\delta^{I_1\cdots I_r}(x,x)=D_\delta^{I_1\cdots I_rK}\omega_K(x)+\sum_{\substack{0\leq i<j\\(i,j)\neq(0,r)}}^{r}(-1)^i D_\delta^{I_{i+1}\cdots I_jK}\varpi^{I_1\cdots I_i\sqcup I_r\cdots I_{j+1}}{}_K(x) \tag{7.44}$$

The proof relates $M_\delta^{I_1\cdots I_r}(x,x)$ to the one-point function $D_\delta^{I_1\cdots I_r}(x)$, which may be expressed in terms of the constant multiplets using item (d) of Theorem 2.3. We note the formal similarity of (7.43) with the formulae for the coincident limits of $\chi^I(x,y), \chi^{IJ}(x,y)$ and $\chi^{IJK}(x,y)$ in section 9.4 of [62]. More specifically, Conjecture 9.8 of [62] relates $\chi^{I_1\cdots I_r}(x,x)$ to combinations of $\varpi^{J_1\cdots J_s}{}_K(x)$ and certain constants $\mathfrak{N}^{I_p\cdots I_q K}$ in the place of $D_\delta^{I_p\cdots I_q K}$ which mirror the double sums and shuffle products of (7.44).

7.5.2 The limit of $L_\delta^{I_1\cdots I_r}(x,y)$

The correspondence of the meromorphic and modular decompositions of linear chain products in (7.34) does not always extend on a term-by-term basis to coincident limits: the offset between the rank-one spinors in (7.8),

$$L_\delta^I(x,y) = M_\delta^I(x,y) + 2\pi i\, S_\delta(x,y)\, \mathrm{Im} \int_x^y \omega^I \tag{7.45}$$

does not have a well-defined limit $y \to x$ in view of the direction dependence of the limit of the last term above. Instead, the proper limit should be taken as follows,

$$\lim_{y\to x} \left(L_\delta^I(x,y) + \frac{\pi}{x-y} \int_x^y \bar\omega^I \right) = M_\delta^I(x,x) - \pi\omega^I(x) = C_\delta^{IJ}\omega_J(x) \tag{7.46}$$

The case for arbitrary rank $r \geq 2$ follows its meromorphic counterpart (7.44) under the correspondence $(M_\delta, D_\delta, \varpi) \leftrightarrow (L_\delta, C_\delta, \partial\Phi)$ and is given by the following proposition.

Proposition 7.7. *The coincident limit $y \to x$ of the spinors $L_\delta^{I_1\cdots I_r}(x,y)$ for rank $r \geq 2$ exists and is given in terms of DHS kernels and the constant tensors $C_\delta^{J_1\cdots J_s}$ as follows,*

$$L_\delta^{I_1\cdots I_r}(x,x) = C_\delta^{I_1\cdots I_r K}\omega_K(x) + \sum_{\substack{0\leq i<j \\ (i,j)\neq(0,r)}}^{r} (-1)^i C_\delta^{I_{i+1}\cdots I_j K}\partial_x \Phi^{I_1\cdots I_i \sqcup I_r \cdots I_{j+1}}{}_K(x) \tag{7.47}$$

The proof proceeds by relating $L_\delta^{I_1\cdots I_r}(x,x)$ to the one-point tensor $C_\delta^{I_1\cdots I_r}(x)$ and using the last item in Theorem 6.2 to express the result in terms of the constant tensors $C_\delta^{I_1\cdots I_r}$.

8 Specializing to genus one

In this section, we shall examine the decompositions of cyclic products of Szegö kernels in (5.1), (6.26) and of linear chain products of Szegö kernels in (7.19), (7.32) for the special case of genus one, namely $h = 1$. The torus Σ will be represented as the quotient $\Sigma = \mathbb{C}/\Lambda$ where the lattice Λ is generated by the periods 1 and τ with $\text{Im}\,\tau > 0$. In the formulas below, the torus Σ will be considered fixed and the dependence on τ will generally not be exhibited. As will be shown below, the spin-structure dependence can be separated from the dependence on the points of the Szegö kernels in terms of Jacobi ϑ_κ functions for $\kappa = 1, 2, 3, 4$ and coefficients of the Kronecker-Eisenstein series that furnish the integration kernels of elliptic polylogarithms [42, 16, 40, 43, 63]. In particular, the integral representations of the constants $D_\delta^{I_1 \cdots I_r}$ and $C_\delta^{I_1 \cdots I_r}$ in section 4.4 and (6.10), respectively, can be made fully explicit.

8.1 Enriquez and DHS kernels

We begin by reviewing the restriction of the Enriquez and DHS kernels to genus one and expressing them in terms of expansion coefficients of Kronecker-Eisenstein series. More specifically, we will encounter two closely related variants of the Kronecker-Eisenstein series that are given in terms of the unique odd Jacobi theta function ϑ_1 by,

$$F(z, \eta) = \frac{\vartheta_1'(0)\vartheta_1(z+\eta)}{\vartheta_1(z)\vartheta_1(\eta)} \qquad \Omega(z, \eta) = \exp\left(2\pi i \eta \frac{\text{Im}\, z}{\text{Im}\,\tau}\right) F(z, \eta) \tag{8.1}$$

The function $F(z, \eta)$ is meromorphic and multiple-valued in $z \in \Sigma$, while $\Omega(z, \eta)$ is single-valued but non-meromorphic in $z \in \Sigma$. The $\mathfrak{A}$ monodromy $z \to z + 1$ of $F(z, \eta)$ is trivial while its $\mathfrak{B}$ monodromy $z \to z + \tau$ is given by,

$$F(z+\tau, \eta) = e^{-2\pi i \eta} F(z, \eta) \tag{8.2}$$

Both $F(z, \eta)$ and $\Omega(z, \eta)$ are meromorphic and multiple-valued in η and their monodromies may be obtained from those in z by using the relation $F(z, \eta) = F(\eta, z)$. The Kronecker-Eisenstein integration kernels $g^{(r)}$ and $f^{(r)}$ are obtained as the coefficients of the Laurent expansions of $F(z, \eta)$ and $\Omega(z, \eta)$ in powers of η, respectively,

$$F(z, \eta) = \frac{1}{\eta} + \sum_{r=1}^{\infty} \eta^{r-1} g^{(r)}(z) \qquad \Omega(z, \eta) = \frac{1}{\eta} + \sum_{r=1}^{\infty} \eta^{r-1} f^{(r)}(z) \tag{8.3}$$

The Enriquez and DHS kernels reduce to $g^{(r)}(z)$ and $f^{(r)}(z)$ for $r \geq 1$, respectively [57, 60],

$$g^{I_1 \cdots I_r}{}_J(x, y)\big|_{h=1} = g^{(r)}(x-y) \qquad f^{I_1 \cdots I_r}{}_J(x, y)\big|_{h=1} = f^{(r)}(x-y) \tag{8.4}$$

where all the indices on the left sides are restricted to take the value 1. In particular, the y-independent traceless parts in the decompositions (2.8) and (6.5) vanish, so that,

$$\begin{aligned} \varpi^{I_1\cdots I_r}{}_J(x)\,\Big|_{h=1} &= 0 & \chi^{I_1\cdots I_s}(x,y)\,\Big|_{h=1} &= -g^{(s+1)}(x-y) \\ \partial_x \Phi^{I_1\cdots I_r}{}_J(x)\,\Big|_{h=1} &= 0 & \partial_x \mathcal{G}^{I_1\cdots I_s}(x,y)\,\Big|_{h=1} &= -f^{(s+1)}(x-y) \end{aligned} \tag{8.5}$$

For $r \geq 3$, the coincident limit of $g^{(r)}(x-y)$ produces the holomorphic Eisenstein series G_r,

$$\lim_{y\to x} g^{(r)}(x-y) = -\mathrm{G}_r = -\sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^r} \tag{8.6}$$

which are modular forms of weight $(r,0)$ under $\mathrm{SL}(2,\mathbb{Z})$ and vanish for odd r. For $r=2$, the double sum in (8.6) is conditionally convergent and may be defined by using the Eisenstein summation prescription [74, 2] which results in the quasi-modular form G_2. A modular-covariant counterpart of G_2 is given by the almost holomorphic modular form,

$$\widehat{\mathrm{G}}_2 = \mathrm{G}_2 - \frac{\pi}{\mathrm{Im}\,\tau} \tag{8.7}$$

8.2 Szegö kernels for even spin structures

For genus one, the Szegö kernel $S_\delta(x,y)$ for an even spin structure δ may be expressed in terms of the even Jacobi theta functions ϑ_δ with $\delta = 2,3,4$,

$$S_\delta(z) = \frac{\vartheta_1'(0)\vartheta_\delta(z)}{\vartheta_1(z)\vartheta_\delta(0)} \tag{8.8}$$

where translation invariance on the torus implies that $S_\delta(x,y)|_{h=1} = S_\delta(z)$ only depends on the difference of the points $z = x-y$. The Szegö kernel $S_\delta(z)$ of (8.8) is closely related to the Kronecker-Eisenstein series $F(z,\eta)$ of (8.1) when η is set to the corresponding half-period ω_δ given as follows,

$$\omega_\delta = u_\delta\tau + v_\delta \qquad\qquad (u_2,v_2) = (0,\tfrac{1}{2}),\ (u_3,v_3) = (\tfrac{1}{2},\tfrac{1}{2}),\ (u_4,v_4) = (\tfrac{1}{2},0) \tag{8.9}$$

and we have [41],

$$S_\delta(z) = e^{2\pi i z u_\delta} F(z,\omega_\delta) = F(z,\omega_\delta) \times \begin{cases} 1 & : \ \delta = 2 \\ e^{i\pi z} & : \ \delta = 3,4 \end{cases} \tag{8.10}$$

Alternatively, we may express $S_\delta(z)$ in terms of $\Omega(z,\omega_\delta)$ of (8.1), which is conveniently done with the help of real *co-moving coordinates* $u,v \in \mathbb{R}/\mathbb{Z}$ related to z by $z = u\tau + v$,

$$S_\delta(z) = \Omega(\omega_\delta, z) = e^{2\pi i(v\,u_\delta - u\,v_\delta)}\,\Omega(z,\omega_\delta) \tag{8.11}$$

The Fourier expansion [75, 45],

$$\Omega(z,\eta) = \sum_{m,n\in\mathbb{Z}} \frac{e^{2\pi i(mv-nu)}}{m\tau+n+\eta} \tag{8.12}$$

of the doubly-periodic Kronecker-Eisenstein series $\Omega(z,\eta)$ combined with (8.11) gives the Fourier expansion of the Szegö kernel [27],

$$S_\delta(z) = e^{2\pi i(v\,u_\delta - u\,v_\delta)} \sum_{m,n\in\mathbb{Z}} \frac{e^{2\pi i(mv-nu)}}{m\tau+n+\omega_\delta} \tag{8.13}$$

which exposes the signs $e^{2\pi i u_\delta}$ and $e^{-2\pi i v_\delta}$ produced by the $\mathfrak{A}$ monodromies $v \to v+1$ and $\mathfrak{B}$ monodromies $u \to u+1$ of the Szegö kernel, respectively.

8.3 Cyclic products of Szegö kernels

The cyclic product of n Szegö kernels on the torus with even spin structure δ is given by,

$$C_\delta(1,2,\cdots,n) = S_\delta(z_{12})S_\delta(z_{23})\cdots S_\delta(z_{n1}) \tag{8.14}$$

where we use the notation $z_{ij} = z_i - z_j$. In view of the relations (8.10) and (8.11), the cyclic product may be expressed alternatively as follows,

$$\begin{aligned} C_\delta(1,2,\cdots,n) &= F(z_{12},\omega_\delta)F(z_{23},\omega_\delta)\cdots F(z_{n1},\omega_\delta) \\ &= \Omega(z_{12},\omega_\delta)\Omega(z_{23},\omega_\delta)\cdots \Omega(z_{n1},\omega_\delta) \end{aligned} \tag{8.15}$$

In these products, all the non-trivial monodromy of the individual F factors and all the non-meromorphicity of the individual Ω factors cancels so that $C_\delta(1,\cdots,n)$ is meromorphic and single-valued in $z_1,\cdots,z_n \in \Sigma$. The decomposition of $C_\delta(1,\cdots,n)$ into a sum of δ-dependent constants with δ independent coefficients is given by the following proposition, which summarizes and clarifies some of the results obtained earlier in [29, 41].

Proposition 8.1. *The cyclic product $C_\delta(1,\cdots,n)$ of Szegö kernels on the torus with spin structure δ may be decomposed as follows,*

$$C_\delta(1,\cdots,n) = V_n(1,\cdots,n) + \sum_{k=1}^{[n/2]} R_{2k}(e_\delta)V_{n-2k}(1,\cdots,n) \tag{8.16}$$

where $R_{2k}(e_\delta)$ is constant on Σ and a modular form of weight $(2k,0)$ under the congruence subgroup $\Gamma(2) \subset \mathrm{SL}(2,\mathbb{Z})$, while $V_r(1,\cdots,n)$ are δ-independent elliptic (i.e. meromorphic *doubly periodic) functions in the points z_i. The functions $V_r(1,\cdots,n)$ may be expressed either in terms of the meromorphic Kronecker-Eisenstein coefficients,*

$$V_r(1,\cdots,n) = \sum_{\substack{s_1,s_2,\cdots,s_n\geq 0 \\ s_1+s_2+\cdots+s_n=r}} g^{(s_1)}(z_{12})g^{(s_2)}(z_{23})\cdots g^{(s_n)}(z_{n1}) \tag{8.17}$$

or in terms of their single-valued counterparts,

$$V_r(1,\cdots,n) = \sum_{\substack{s_1,s_2,\cdots,s_n\geq 0 \\ s_1+s_2+\cdots+s_n=r}} f^{(s_1)}(z_{12})f^{(s_2)}(z_{23})\cdots f^{(s_n)}(z_{n1}) \tag{8.18}$$

with $g^{(0)}(z) = f^{(0)}(z) = 1$ and thus $V_0(1,\cdots,n) = 1$. The modular forms $R_{2k}(e_\delta)$ under $\Gamma(2)$ may be expressed in terms of the Weierstrass function $\wp(z)$ and its derivatives, evaluated at the half period ω_δ in (8.9) corresponding to the spin structure δ,

$$R_2(e_\delta) = \wp(\omega_\delta) = e_\delta \qquad\qquad R_{2k}(e_\delta) = \frac{\wp^{(2k-2)}(\omega_\delta)}{(2k-1)!} - \mathrm{G}_{2k} \quad \text{for } k \geq 2 \tag{8.19}$$

or may alternatively be written as degree-two polynomials in $e_\delta = \wp(\omega_\delta)$,

$$R_{2k}(e_\delta) = a_{2k-4}e_\delta^2 + b_{2k-2}e_\delta + c_{2k} \tag{8.20}$$

where a_{2k-4}, b_{2k-2} and c_{2k} are modular forms under $\mathrm{SL}(2,\mathbb{Z})$ of weight $2k-4$, $2k-2$ and $2k$, respectively, for $k \geq 2$, with $a_0 = b_0 = 1$ and $a_2 = b_2 = 0$.

8.3.1 Proof of Proposition 8.1

To prove the proposition we use the relation between $C_\delta(1,\cdots,n)$ and the generating functions F and Ω evaluated at the half periods, as spelled out in (8.15). Considering the product of F or Ω factors with arbitrary values of the parameter $\eta \in \mathbb{C}$ instead, the Laurent expansions of (8.3) imply the following Laurent expansion [44, 29],

$$F(z_{12},\eta)F(z_{23},\eta)\cdots F(z_{n1},\eta) = \sum_{r=0}^{\infty} \eta^{r-n} V_r(1,2,\cdots,n) \tag{8.21}$$

The expressions (8.17) and (8.18) for $V_r(1,\cdots,n)$ in terms of $g^{(s)}$ or $f^{(s)}$ kernels readily follow from inserting the expansion (8.3) of the individual Kronecker-Eisenstein series into (8.21). It remains to show that, when η is set to the half period ω_δ, then its dependence may be assembled into the above modular forms of $\Gamma(2)$.

To do so, we use the fact that $F(z,\eta)$ satisfies $F(z,\eta+1) = F(z,\eta)$ and $F(z,\eta+\tau) = e^{-2\pi i z}F(z,\eta)$ to verify that the combination of (8.21) is an elliptic (i.e. meromorphic and doubly periodic) function in η, all of whose poles in $\eta \in \Sigma$ are located at $\eta = 0$ and are of maximum order n. Therefore, the combination of (8.21) is a linear combination of a finite number of derivatives $\wp^{(\ell)}(\eta)$ of $\wp(\eta)$ with $\ell + 2 \leq n$. The precise form of the coefficients is obtained by using the following expansion,

$$\wp(\eta) = \frac{1}{\eta^2} + \mathcal{O}(\eta) \qquad\qquad \frac{\wp^{(\ell)}(\eta)}{(\ell+1)!} - \mathrm{G}_{\ell+2} = \frac{(-)^\ell}{\eta^{\ell+2}} + \mathcal{O}(\eta) \quad \text{for } \ell \geq 1 \tag{8.22}$$

Matching the poles gives,

$$\begin{aligned} F(z_{12},\eta) \cdots F(z_{n1},\eta) = V_n(1,\cdots,n) + \wp(\eta) V_{n-2}(1,\cdots,n) \\ + \sum_{\ell=1}^{n-2} (-)^\ell \left(\frac{\wp^{(\ell)}(\eta)}{(\ell+1)!} - \mathrm{G}_{\ell+2} \right) V_{n-\ell-2}(1,\cdots,n) + \mathcal{O}(\eta) \end{aligned} \tag{8.23}$$

Since the left side is an elliptic function in η whose poles, which are all at $\eta = 0$ and translates by $\mathbb{Z}+\tau\mathbb{Z}$, are matched by the poles on the right side, the terms $\mathcal{O}(\eta)$ on the right side vanish by Liouville's theorem. Furthermore, the derivatives $\wp^{(\ell)}(\eta)$ evaluated at the half periods $\eta = \omega_\delta$ vanish for all odd values of ℓ, so that we recover (8.16) with $R_{2k}(e_\delta)$ given by (8.19) upon setting $\ell+2 = 2k$. Finally, using the defining equation of $\wp(\eta)$,

$$\wp'(\eta)^2 = 4\wp(\eta)^3 - 60\mathrm{G}_4\wp(\eta) - 140\mathrm{G}_6 \tag{8.24}$$

one readily expresses the function $\wp^{(2k-2)}(\eta)$ in terms of a polynomial in $\wp(\eta)$ of degree k. Evaluating the derivatives at $\eta = \omega_\delta$ and using the cubic equation satisfied by e_δ,

$$e_\delta^3 - 15\mathrm{G}_4 e_\delta - 35\mathrm{G}_6 = 0 \tag{8.25}$$

one iteratively reduces $\wp^{(2k-2)}(\omega_\delta)$ to a degree-two polynomial in $\wp(\omega_\delta)$ and thereby demonstrates the validity of (8.20). For the lowest values of k the δ-independent coefficients in these polynomials are given as follows,

$$\begin{aligned} a_0 &= 1 & b_2 &= 0 & c_4 &= 6\mathrm{G}_4 \\ a_2 &= 0 & b_4 &= 6\mathrm{G}_4 & c_6 &= 20\mathrm{G}_6 \\ a_4 &= 3\mathrm{G}_4 & b_6 &= 15\mathrm{G}_6 & c_8 &= 14\mathrm{G}_8 \\ a_6 &= 10\mathrm{G}_6 & b_8 &= 70\mathrm{G}_8 & c_{10} &= 120\mathrm{G}_{10} \end{aligned} \tag{8.26}$$

Note that, similar to the equation (8.25) for the roots at genus one, cyclic products of Szegö kernels at genus two can be decomposed into degree-two polynomials in the entries $\mathfrak{L}_\delta^{11}, \mathfrak{L}_\delta^{12}, \mathfrak{L}_\delta^{22}$ thanks to the system of trilinear equations these objects satisfy [54], also see [61] for a representation of $\mathfrak{L}_\delta^{IJ}$ at $I, J \in \{1,2\}$ in terms of Riemann theta functions.

8.3.2 Extracting the constants $C_\delta^{I_1\cdots I_r}$ and $D_\delta^{I_1\cdots I_r}$

As a consequence of Proposition 8.1, we obtain simple expressions for the reduction of the constants $C_\delta^{I_1\cdots I_r}$ and $D_\delta^{I_1\cdots I_r}$ to the case of genus one, as spelled out in following proposition.

Proposition 8.2. *For genus one, the constants $C_\delta^{I_1\cdots I_r}$ and $D_\delta^{I_1\cdots I_r}$ in the modular and meromorphic descents are given by,*

$$C_\delta^{I_1\cdots I_r}\big|_{h=1} = \begin{cases} e_\delta + \widehat{\mathrm{G}}_2 & : \ r=2 \\ R_r(e_\delta) + \mathrm{G}_n & : \ r \geq 4 \text{ even} \\ 0 & : \ r \geq 3 \text{ odd} \end{cases} \tag{8.27}$$

and

$$D_\delta^{I_1\cdots I_r}\big|_{h=1} = \begin{cases} e_\delta + \mathrm{G}_2 & : \ r=2 \\ R_r(e_\delta) + \mathrm{G}_n & : \ r \geq 4 \text{ even} \\ 0 & : \ r \geq 3 \text{ odd} \end{cases} \tag{8.28}$$

respectively. The polynomials $R_r(e_\delta)$ in $e_\delta = \wp(\omega_\delta)$ are obtained from (8.19) and (8.24).

The proof of the proposition can be found in appendix F.1. We note the examples

$$\begin{aligned} C_\delta^{I_1\cdots I_4}\big|_{h=1} &= e_\delta^2 - 5\mathrm{G}_4 & \qquad C_\delta^{I_1\cdots I_8}\big|_{h=1} &= 3\mathrm{G}_4 e_\delta^2 + 15\mathrm{G}_6 e_\delta + 15\mathrm{G}_8 \\ C_\delta^{I_1\cdots I_6}\big|_{h=1} &= 6\mathrm{G}_4 e_\delta + 21\mathrm{G}_6 & \qquad C_\delta^{I_1\cdots I_{10}}\big|_{h=1} &= 10\mathrm{G}_6 e_\delta^2 + 70\mathrm{G}_8 e_\delta + 121\mathrm{G}_{10} \end{aligned} \tag{8.29}$$

and the following general relation between the constants $C_\delta^{I_1\cdots I_r}$ and $D_\delta^{I_1\cdots I_r}$ at genus one which can be read off from (8.27) and (8.28):

$$D_\delta^{I_1\cdots I_r}\big|_{h=1} = \begin{cases} C_\delta^{I_1 I_2}\big|_{h=1} + \frac{\pi}{\mathrm{Im}\,\tau} & : \ r=2 \\ C_\delta^{I_1\cdots I_r}\big|_{h=1} & : \ r \geq 3 \end{cases} \tag{8.30}$$

The genus one instances of $C_\delta^{I_1\cdots I_r}$ at $r \leq 8$ were reported without proof in [61], and the expressions for $C_\delta^{I_1\cdots I_6}\big|_{h=1}$ and $C_\delta^{I_1\cdots I_8}\big|_{h=1}$ experienced corrections in the most recent arXiv version of the reference.

8.4 Linear chain products at genus one

For linear chain products (7.1) at genus one, the dependence on both the marked points and on the even spin structure δ can be made fully explicit as done in Proposition 8.1 for cyclic products. The results have not appeared in the literature prior to this work and are summarized in the following theorem.

Theorem 8.3. *The genus one instances of linear chain products (7.1) of Szegö kernels can be decomposed in two different ways,*

$$
\begin{aligned}
L_\delta(x;1,\cdots,n;y) &= \sum_{r=0}^{n} M_\delta^{I_1\cdots I_r}(x,y)\,\big|_{h=1} W_{n-r}(x;1,\cdots,n;y) \\
&= \sum_{r=0}^{n} L_\delta^{I_1\cdots I_r}(x,y)\,\big|_{h=1} V_{n-r}(x;1,\cdots,n;y)
\end{aligned}
\tag{8.31}
$$

where the first line is term-by-term meromorphic in all variables and the second line exposes the modular properties. In both cases, the dependence on the marked points is carried by elliptic functions of the internal points $z_1,\cdots,z_n$

$$
\begin{aligned}
W_r(x;1,2,\cdots,n;y) &= \sum_{\substack{s_1,\cdots,s_{n+1}\geq 0\\ s_1+\cdots+s_{n+1}=r}} g^{(s_1)}(x-z_1)g^{(s_2)}(z_{12})\cdots g^{(s_n)}(z_{n-1,n})g^{(s_{n+1})}(z_n-y) \\
V_r(x;1,2,\cdots,n;y) &= \sum_{\substack{s_1,\cdots,s_{n+1}\geq 0\\ s_1+\cdots+s_{n+1}=r}} f^{(s_1)}(x-z_1)f^{(s_2)}(z_{12})\cdots f^{(s_n)}(z_{n-1,n})f^{(s_{n+1})}(z_n-y)
\end{aligned}
\tag{8.32}
$$

and the accompanying δ dependent spinors in x and y are expressible in terms of Kronecker-Eisenstein derivatives and lattice sums (see (8.9) for the co-moving coordinates u_δ, v_δ and recall that $z=x-y=u\tau+v$ with $u,v\in\mathbb{R}/\mathbb{Z}$)

$$
\begin{aligned}
M_\delta^{I_1\cdots I_r}(x,y)\,\big|_{h=1} &= \frac{(-1)^r}{r!}\, e^{2\pi i u_\delta(x-y)}\, \partial_\eta^r F(x-y,\eta)\,\big|_{\eta=\omega_\delta} \\
L_\delta^{I_1\cdots I_r}(x,y)\,\big|_{h=1} &= e^{2\pi i(v u_\delta - u v_\delta)} \sum_{m,n\in\mathbb{Z}} \frac{e^{2\pi i(mv-nu)}}{(m\tau+n+\omega_\delta)^{r+1}}
\end{aligned}
\tag{8.33}
$$

The proof of Theorem 8.3 may be found in appendix F.3. The expressions (8.33) make the genus one instances of the integral representations (7.11) and (7.31) of the quantities $L_\delta^{I_1\cdots I_r}(x,y)$ and $M_\delta^{I_1\cdots I_r}(x,y)$ in the decompositions (7.19) and (7.32) fully explicit. In combination with the relation (7.47) between the coincident limits of $L_\delta^{I_1\cdots I_r}(x,x)$ and the constant tensors $C_\delta^{I_1\cdots I_r}$ at $r\geq 3$, we are led to the following corollary.

Corollary 8.4. *The constants $C_\delta^{I_1\cdots I_r}$ and $D_\delta^{I_1\cdots I_r}$ at genus one in (8.27) can be alternatively written in terms of the following lattice sums*

$$
C_\delta^{I_1\cdots I_r}\,\big|_{h=1} = D_\delta^{I_1\cdots I_r}\,\big|_{h=1} = \sum_{m,n\in\mathbb{Z}} \frac{1}{(m\tau+n+\omega_\delta)^r}\,, \qquad r\geq 3 \tag{8.34}
$$

The analogous expressions of the $r=2$ cases can be read off from (8.27) and 8.28) using the lattice-sum representation of $e_\delta=\wp(\omega_\delta)$.

9 Outlook

In this work, we have established that the integration kernels for polylogarithms on a Riemann surface Σ of arbitrary genus provide a natural space of functions in terms of which the dependence of fermion correlators on points in Σ may be expressed. For both cyclic products and linear chain products of Szegö kernels, the descent procedures in Theorems 2.3 and 6.2 systematically give their Σ-dependence in terms of Enriquez or DHS kernels. In this way, their dependence on the spin structure is concentrated in constants on Σ in the case of cyclic products or spinors on Σ depending solely on the end points of linear chain products. The spin-structure dependence is expressed in terms of multiple convolution integrals, either over homology cycles or over the surface, which extend the analogous convolution representations of Enriquez kernels [58] and DHS kernels [60].

The decomposition formulae (5.1), (6.26), (7.19) and (7.32) for products of Szegö kernels obtained from our descent procedure are expected to substantially simplify the evaluation of superstring amplitudes: First, the spin structure sums for arbitrary chiral measures can be performed at the level of constants or spinors depending only on the end points of linear chain products instead of functions of the other points. Second, the link to higher-genus polylogarithms offers a growing arsenal of techniques for the integration over the points of the fermion correlators in a low-energy expansion of string amplitudes.

More generally, the integration kernels produced in our decompositions of the cyclic products and linear chains are believed to span the function space needed to express the complete moduli-space integrands of string amplitudes. Under this assumption, our results should feed into bootstrap constructions of higher-point and higher-genus amplitudes beyond today's reach of direct calculations. In particular, the meromorphic function space of Enriquez kernels [57] is compatible with the chiral-splitting formulation of string amplitudes [76, 38, 77] and expected to fruitfully combine with the methods of [78, 79] to incorporate the information about the supergravity amplitudes in the field-theory limit.

Our results raise several follow-up questions in a broader mathematical context and suggest concrete steps towards their string theory applications including,

- converting the integral representations of the constants $D_\delta^{I_1\cdots I_r}$ and $C_\delta^{I_1\cdots I_r}$ in the decompositions (5.1), (6.26) of cyclic products into expansion formulae around boundaries of moduli space;
- relating $D_\delta^{I_1\cdots I_r} \leftrightarrow C_\delta^{I_1\cdots I_r}$ and the forms $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n) \leftrightarrow \mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ that capture the dependence of cyclic products on the points, using the gauge transformation and Lie-algebra automorphism relating the Enriquez and DHS connections [56];
- expressing higher-genus correlation functions of the current algebra of heterotic strings in terms of Enriquez kernels and DHS kernels as done at genus one [44].

A Function theory on a Riemann surface

Throughout, Σ will be a compact Riemann surface of arbitrary genus $h \geq 1$ with simply connected covering space $\tilde{\Sigma}$ and associated projection by $\pi : \tilde{\Sigma} \to \Sigma$. The first homology group $H_1(\Sigma, \mathbb{Z})$ is endowed with an intersection pairing $\mathfrak{J}$. A canonical basis for $H_1(\Sigma, \mathbb{Z})$ is given by cycles $\mathfrak{A}^I$ and $\mathfrak{B}_I$ that obeys $\mathfrak{J}(\mathfrak{A}^I, \mathfrak{A}^J) = \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$ and $\mathfrak{J}(\mathfrak{A}^I, \mathfrak{B}_J) = \delta^I_J$ for $I, J \in \{1, \cdots, h\}$. Choosing the cycles $\mathfrak{A}^I$ and $\mathfrak{B}_J$ to share a common base point q promotes them into a set of generators of the first homotopy group $\pi_1(\Sigma, q)$ of Σ, as illustrated in figure 1 for a Riemann surface of genus two.

A basis for the Dolbeault cohomology group $H_1^{(1,0)}(\Sigma)$ is given by the holomorphic Abelian differentials $\omega_J = \omega_J(x)dx$, whose integrals on $\mathfrak{A}^I$ cycles are normalized and whose integrals on $\mathfrak{B}_I$ cycles give the entries of the period matrix Ω of the surface Σ,

$$\oint_{\mathfrak{A}^I} \omega_J = \delta^I_J \qquad\qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ} \tag{A.1}$$

The Riemann relations imply that the period matrix Ω is symmetric $\Omega^t = \Omega$ and that its imaginary part $Y = \mathrm{Im}\,(\Omega)$ is a positive definite matrix.

A.1 The prime form

A key ingredient in the theory of functions and differential forms on Σ is provided by the prime form $E(x, y)$, defined for $x, y \in \tilde{\Sigma}$ to be a differential form of conformal weight $(-\frac{1}{2}, 0)$ in both x and y, anti-symmetric $E(y, x) = -E(x, y)$ with a single zero for x, y in a given fundamental domain D, normalized by $E(x, y) = (x - y)dx^{-\frac{1}{2}}dy^{-\frac{1}{2}} + \mathcal{O}(x - y)^3$ in a system of local coordinates. The Riemann ϑ-function of rank h with characteristics $\kappa = (\kappa', \kappa'')$ with $\kappa', \kappa'' \in \mathbb{C}^h$ for period matrix Ω and $\zeta \in \mathbb{C}^h$ is defined by,[14]

$$\vartheta[\kappa](\zeta) = \sum_{n \in \mathbb{Z}^h} \exp\left\{\pi i(n + \kappa')^t \Omega (n + \kappa') + 2\pi i (n + \kappa')^t (\zeta + \kappa'')\right\} \tag{A.2}$$

An explicit formula for the prime form is given in [68] by,

$$E(x, y) = \frac{\vartheta[\nu](\int_y^x \omega)}{h_\nu(x)\, h_\nu(y)} \qquad\qquad h_\nu(x)^2 = \omega_I(x) \frac{\partial}{\partial \zeta_I} \vartheta[\nu](\zeta)\Big|_{\zeta=0} \tag{A.3}$$

where ν is an odd half-integer characteristic, or *spin structure on* Σ (for which $2\nu', 2\nu'' \in \mathbb{Z}_2^h$ and the integer $4(\nu')^t\nu''$ is odd-valued) and $h_\nu(x)$ is the holomorphic $(\frac{1}{2}, 0)$ form whose square is given by the second equation above. The above ratio defining the prime form is actually independent of the choice of odd half-integer characteristic ν.

[14] The dependence on Ω will be suppressed throughout.

A.2 The Szegö kernel

The Szegö kernel $S_\delta(x,y)$ for even spin structure δ and a *generic* period matrix Ω (namely, for which $\vartheta[\delta](0) \neq 0$) is a meromorphic $(\frac{1}{2},0)$ form in $x,y \in \Sigma$ defined by,

$$S_\delta(x,y) = \frac{\vartheta[\delta](\int_y^x \omega)}{\vartheta[\delta](0)\, E(x,y)} \tag{A.4}$$

It follows that $S_\delta(x,y) = -S_\delta(y,x)$ has a single pole in x at y with unit residue so that,

$$\bar\partial_x S_\delta(x,y) = \pi \delta(x,y) \tag{A.5}$$

For even spin structures and non-generic moduli for which $\vartheta[\delta](0) = 0$ and for all odd spin structures, the Cauchy-Riemann operator $\bar\partial_x$ acting on $(\frac{1}{2},0)$ forms has zero modes so that the definition of the Szegö kernel requires making choices. Here, we shall only consider the generic case.

A.3 The Arakelov Green function

The *Arakelov Green function* $\mathcal{G}(x,y) = \mathcal{G}(y,x)$ [80] (see also [81, 24] for its use in physics) is a real single-valued scalar in $x,y \in \Sigma$ uniquely defined by the following equations,

$$\bar\partial_x \partial_x \mathcal{G}(x,y) = -\pi\delta(x,y) + \pi\kappa(x) \qquad\qquad \int_\Sigma d^2t\, \kappa(t)\mathcal{G}(t,y) = 0 \tag{A.6}$$

where κ is the canonical volume form on Σ, defined by,

$$\kappa(x) = \frac{1}{h}\omega_I(x)\,\bar\omega^I(x) \tag{A.7}$$

An explicit formula for the Arakelov Green function may be obtained in terms of the *string Green function* $G(x,y)$ which is given by,

$$G(x,y) = -\ln|E(x,y)|^2 + 2\pi \left(\mathrm{Im}\int_y^x \omega_I\right)\left(\mathrm{Im}\int_y^x \omega^I\right) \tag{A.8}$$

as follows,

$$\mathcal{G}(x,y) = G(x,y) - \gamma(x) - \gamma(y) + \int_\Sigma d^2t\,\kappa(t)\gamma(t) \tag{A.9}$$

where

$$\gamma(x) = \int_\Sigma d^2t\,\kappa(t) G(x,t) \tag{A.10}$$

The defining equations (A.6) and the canonical volume form κ being conformal and modular invariant, it follows that the Arakelov Green function $\mathcal{G}(x,y)$ is a modular invariant conformal scalar (while the string Green function is not conformally invariant). The function $\mathcal{G}(x,y)$ is real-analytic for $x,y\in\Sigma$ away from $x=y$, where it has the following asymptotic behavior as $x\to y$,

$$\mathcal{G}(x,y) = -\ln|x-y|^2 + \text{regular} \tag{A.11}$$

It will also be useful to have the following mixed differential equation,

$$\bar\partial_y \partial_x \mathcal{G}(x,y) = \pi\delta(x,y) - \pi\omega_I(x)\bar\omega^I(y) \tag{A.12}$$

A.4 Enriquez kernels

The basic definition of the Enriquez kernel $g^{I_1\cdots I_r}{}_J(x,y)$, for $r\geq 0$ and $I_1,\cdots,I_r,J\in\{1,\cdots,h\}$ was presented in section 2.1 as a meromorphic $(1,0)$ form in $x\in\tilde\Sigma$ and $(0,0)$ form in $y\in\tilde\Sigma$. The Enriquez kernel $g^{I_1\cdots I_r}{}_J(x,y)$ is holomorphic in the interior $x,y\in D^o$ of a preferred fundamental domain for Σ for $r\geq 2$, has a single simple pole in x at y with residue $\delta^{I_1}_J$ for $r=1$, and is given by $g^{\emptyset}{}_J(x,y)=\omega_J(x)$ for $r=0$. Its monodromies in x and y around $\mathfrak{A}$ cycles are trivial, while its monodromies around $\mathfrak{B}$ cycles are given by (2.3) using the notations of (2.5) and (2.4). The forms $g^{I_1\cdots I_r}{}_J(x,y)$ may have poles in x at $\pi^{-1}(y)$ for all $r\geq 1$, as mandated by the monodromy relations.

In the sequel, it will be useful to have the monodromies around $\mathfrak{B}$ cycles of the traceless and trace parts defined by (2.8) separately at our disposal. They may be obtained by decomposing the monodromy relations of g given in (2.3) into their traceless and trace parts, and we find (see (2.5) for the $\Delta^{(x)}_L$ notation),

$$\Delta^{(x)}_L \varpi^{I_1\cdots I_r}{}_J(x) = \sum_{k=1}^{r}\frac{(-2\pi i)^k}{k!}\delta^{I_1\cdots I_k}_L \varpi^{I_{k+1}\cdots I_r}{}_J(x) - \frac{(-2\pi i)^r}{r!\,h}\delta^{I_r}_J\delta^{I_1\cdots I_{r-1}}_L\omega_L(x) \tag{A.13}$$

whereas the $\mathfrak{B}$ monodromies of $\chi^{I_1\ldots I_s}(x,y)$ with $s\geq 0$ are given by,

$$\begin{aligned}
\Delta^{(x)}_L\chi^{I_1\cdots I_s}(x,y) &= \sum_{k=1}^{s}\frac{(-2\pi i)^k}{k!}\delta^{I_1\cdots I_k}_L\chi^{I_{k+1}\cdots I_s}(x,y) - \frac{(-2\pi i)^{s+1}}{(s{+}1)!h}\delta^{I_1\cdots I_s}_L\omega_L(x)\\
\Delta^{(y)}_L\chi^{I_1\cdots I_s}(x,y) &= -\sum_{k=0}^{s}\frac{(2\pi i)^{s-k+1}}{(s{-}k{+}1)!}\, g^{I_1\cdots I_k}{}_L(x,y)\delta^{I_{k+1}\cdots I_s}_L
\end{aligned} \tag{A.14}$$

The periods around $\mathfrak{A}^L$ cycles on the boundary of the fundamental domain D in figure 1 are given in terms of Bernoulli numbers Ber_r by (2.7) and its decomposition into traceless and trace parts was given by (2.8). Thus, we have,

$$\oint_{\mathfrak{A}^L} dt\, \chi^{I_1\cdots I_r}(t,y) = -\frac{1}{h}(-2\pi i)^{r+1}\frac{\mathrm{Ber}_{r+1}}{(r+1)!}\,\delta_L^{I_1\cdots I_r}$$
$$\oint_{\mathfrak{A}^L} dt\, \varpi^{I_1\cdots I_r}{}_J(t) = (-2\pi i)^r\frac{\mathrm{Ber}_r}{r!}\left(\delta_J^{I_1\cdots I_r L} - \frac{1}{h}\delta_J^{I_r}\delta_L^{I_1\cdots I_{r-1}}\right) \tag{A.15}$$

A.4.1 Low order monodromy formulas

It will be useful to dispose of some low order formulas for the $\mathfrak{B}$ monodromies of g, ϖ and χ in all their arguments. The $\mathfrak{B}$ monodromies of g are given by,

$$\begin{aligned}
\Delta_L^{(x)} g^I{}_J(x,y) &= -2\pi i\,\delta_L^I\,\omega_J(x)\\
\Delta_L^{(y)} g^I{}_J(x,y) &= 2\pi i\,\delta_J^I\,\omega_L(x)\\
\Delta_L^{(x)} g^{I_1I_2}{}_J(x,y) &= -2\pi i\,\delta_L^{I_1}\,g^{I_2}{}_J(x,y) - 2\pi^2\,\delta_L^{I_1I_2}\,\omega_J(x)\\
\Delta_L^{(y)} g^{I_1I_2}{}_J(x,y) &= 2\pi i\,\delta_J^{I_2}\,g^{I_1}{}_L(x,y) - 2\pi^2\,\delta_J^{I_2}\,\delta_L^{I_1}\,\omega_L(x)
\end{aligned} \tag{A.16}$$

those of χ are given by,

$$\begin{aligned}
\Delta_L^{(x)} \chi(x,y) &= 2\pi i\,\omega_L(x)/h\\
\Delta_L^{(y)} \chi(x,y) &= -2\pi i\,\omega_L(x)\\
\Delta_L^{(x)} \chi^I(x,y) &= -2\pi i\,\delta_L^I\,\chi(x,y) + 2\pi^2\,\delta_L^I\,\omega_L(x)/h\\
\Delta_L^{(y)} \chi^I(x,y) &= -2\pi i\,g^I{}_L(x,y) + 2\pi^2\,\delta_L^I\,\omega_L(x)
\end{aligned} \tag{A.17}$$

and those of ϖ are given by,

$$\begin{aligned}
\Delta_L^{(x)} \varpi^I{}_J(x) &= -2\pi i\delta_L^I\,\omega_J(x) + 2\pi i\delta_J^I\,\omega_L(x)/h\\
\Delta_L^{(x)} \varpi^{I_1I_2}{}_J(x) &= -2\pi i\,\delta_L^{I_1}\,\varpi^{I_2}{}_J(x) - 2\pi^2\,\delta_L^{I_1I_2}\omega_J(x) + 2\pi^2\delta_L^{I_1}\delta_J^{I_2}\,\omega_L(x)/h
\end{aligned} \tag{A.18}$$

B Proof of Lemma 2.4

To prove Lemma 2.4, we note that (2.39) follows from (2.33) and the known poles of the Szegö kernels in the cyclic product $C_\delta(1,\cdots,n)$, while the relations (2.40) may be verified directly for the case $r=1$ using the first line in (2.32). To establish the relations (2.40) for $r \geq 2$, we proceed by induction in r. To do so we reformulate the system of equations in (2.40) in terms of the following combinations for $n \geq 2$ and $r \geq 1$,

$$\begin{aligned} S_{\delta\, k}^{I_1 \cdots I_r}(1,\cdots,n) &= \bar\partial_k D_\delta^{I_1\cdots I_r}(1,\cdots,n) \\ &\quad -\pi\Big(a_k\,\delta(k,k+1) - b_k\,\delta(k,k-1)\Big) D_\delta^{I_1\cdots I_r}(1,\cdots,\hat k,\cdots,n) \end{aligned} \tag{B.1}$$

where the constants a_k and b_k are given as follows,

$$\begin{cases} a_1 = 1 \\ b_1 = 0 \end{cases} \qquad \begin{cases} a_n = 0 \\ b_n = 1 \end{cases} \qquad \begin{cases} a_k = 1 \\ b_k = 1 \end{cases} \quad \text{for} \quad 2 \leq k \leq n-1 \tag{B.2}$$

With this notation, the system of equations in (2.40) is equivalent to $S_{\delta\, k}^{I_1\cdots I_r}(1,\cdots,n)=0$ for all values of k in the range $1 \leq k \leq n$. Next, by differentiating (2.34) with respect to z_k for $2 \leq k \leq n$, eliminating the terms proportional to $\chi^{I_r\cdots I_1}(1,2)-\chi^{I_r\cdots I_1}(1,n)$ in terms of D_δ functions whose argument z_k is missing, we find that the entire remaining relation may be recast in terms of the combinations $S_{\delta\, k}$ in (B.1) for $1 \leq k \leq n$ as follows,

$$\begin{aligned} \omega_J(1) S_{\delta\, k}^{I_1\cdots I_r J}(2,\cdots,n) &= S_{\delta\, k}^{I_1\cdots I_r}(1,\cdots,n) \\ &\quad - \sum_{i=0}^{r-1} g^{I_r\cdots I_{i+1}}{}_J(1,2)\, S_{\delta\, k}^{I_1\cdots I_i J}(2,\cdots,n) \end{aligned} \tag{B.3}$$

Recall that these relations are derived for $1 \leq r \leq s-1$ in view of the assumptions made in the formulation of Lemma 2.4. Using the fact that the rank on the left is $r+1$ but the rank on the right is at most r, and that $S_{\delta\, k}^{I_1\cdots I_r}(1,\cdots,n)=0$ for $r=1$, it follows by induction in r that $S_{\delta\, k}^{I_1\cdots I_r}(1,\cdots,n)=0$ for all $r \leq s$. This completes the proof of the system of equations (2.40).

To prove (2.41), it suffices to define $S_{\delta\, k}^{I_1\cdots I_r}(1,\cdots,n)$ for the case $n=1$ by setting $a_k=b_k=0$ in (B.1). The equations of (B.3) are then satisfied for $n=2$. Using (B.3) for $n=2$ and the fact that the case $r=1$ holds true by Proposition 2.1 then readily leads to a proof of (2.41) by induction in r and completes the proof of Lemma 2.4.

C Proof of Lemma 2.5

In this appendix, we shall prove the monodromy relations in (2.42) of Lemma 2.5 for the multiplets $D_\delta^{I_1\cdots I_r}(1,\cdots,n)$ up to rank $r \leq s$ under the assumption of the lemma that the descent equations (2.34) hold true for all $r \leq s-1$ and all $n \geq 3$. Recall that all $\mathfrak{A}$ cycle monodromies manifestly vanish term by term. The $\mathfrak{B}$ cycle monodromies in the points z_k of $D_\delta^{I_1\cdots I_r}(2,\cdots,n)$ will be evaluated separately for the three cases distinguished in (2.42), starting with the middle line for which $3 \leq k \leq n-1$, which turns out to be the simplest of the three and, in turn, will be used to prove the case $k=2$.

C.1 Monodromy of $D_\delta^{I_1\cdots I_r}(2,\cdots,n)$ in z_k for $3 \leq k \leq n-1$

Applying the monodromy operator $\Delta_L^{(k)}$ for $3 \leq k \leq n-1$ to both sides of (2.34) gives,

$$\begin{aligned}\omega_J(1)\Delta_L^{(k)} D_\delta^{I_1\cdots I_r J}(2,\cdots,n) &= \Delta_L^{(k)} D_\delta^{I_1\cdots I_r}(1,\cdots,n) \\ &\quad - \sum_{i=0}^{r-1} g^{I_r\cdots I_{i+1}}{}_J(1,2)\Delta_L^{(k)} D_\delta^{I_1\cdots I_i J}(2,\cdots,n)\end{aligned} \tag{C.1}$$

Evaluating the monodromy of the first equation in (2.32), we readily establish that $\Delta_L^{(k)} D_\delta^J(1,\cdots,n) = 0$ for all $n \geq 1$, which confirms equation (C.1) for the case $r=0$. For $r \geq 1$ we proceed by using (C.1) by induction in r. Since the rank on the left side is one higher than the maximum rank on the right side, the induction is straightforward and establishes the middle equation of (2.42).

C.2 Monodromy of $D_\delta^{I_1\cdots I_r}(2,\cdots,n)$ in z_2 for $3 \leq n$

In terms of the following combination,

$$\begin{aligned}T_\delta^{I_1\cdots I_r}(1,\cdots,n) &= \Delta_L^{(1)} D_\delta^{I_1\cdots I_r}(1,\cdots,n) \\ &\quad - \sum_{\ell=1}^{r} \frac{(-2\pi i)^\ell}{\ell!} \delta_L^{I_r\cdots I_{r+1-\ell}} D_\delta^{I_1\cdots I_{r-\ell}}(1,\cdots,n)\end{aligned} \tag{C.2}$$

the first equation of (2.42) is equivalent to $T_\delta^{I_1\cdots I_r}(1,\cdots,n) = 0$. Next, we apply the operator $\Delta_L^{(2)}$ to (2.34), and take care of including the double monodromy given by the sum on the last line below,

$$\begin{aligned}\omega_J(1)\,\Delta_L^{(2)} D_\delta^{I_1\cdots I_r J}(2,\cdots,n) = {}& \Delta_L^{(2)}\, D_\delta^{I_1\cdots I_r}(1,\cdots,n) + \Delta_L^{(2)} \chi^{I_r\cdots I_1}(1,2)\, C_\delta(2,\cdots,n) \\ & - \sum_{i=0}^{r-1} \Delta_L^{(2)} g^{I_r\cdots I_{i+1}}{}_J(1,2) D_\delta^{I_1\cdots I_i J}(2,\cdots,n) \\ & - \sum_{i=0}^{r-1} g^{I_r\cdots I_{i+1}}{}_J(1,2)\, \Delta_L^{(2)} D_\delta^{I_1\cdots I_i J}(2,\cdots,n) \\ & - \sum_{i=0}^{r-1} \Delta_L^{(2)} g^{I_r\cdots I_{i+1}}{}_J(1,2)\, \Delta_L^{(2)} D_\delta^{I_1\cdots I_i J}(2,\cdots,n) \end{aligned} \tag{C.3}$$

Using the results established in section C.1, the first term on the first line on the right side of (C.3) vanishes, $\Delta_L^{(2)} D_\delta^{I_1\cdots I_r}(1,\cdots,n)=0$. To compute the monodromies involving the Enriquez kernels, we use the second equation of (2.3) for g and the second equation of (A.14) for χ, and suitably rearrange the indices as follows,

$$\begin{aligned}\Delta_L^{(2)} g^{I_r\cdots I_i}{}_J(1,2) &= \delta_J^{I_i} \sum_{k=1}^{r+1-i} \frac{(2\pi i)^k}{k!} g^{I_r\cdots I_{k+i}}{}_L(1,2)\, \delta_L^{I_{k+i-1}\cdots I_{i+1}} \\ \Delta_L^{(2)} \chi^{I_r\cdots I_1}(1,2) &= -\sum_{k=1}^{r+1} \frac{(2\pi i)^k}{k!} g^{I_r\cdots I_k}{}_L(1,2)\, \delta_L^{I_{k-1}\cdots I_1} \end{aligned} \tag{C.4}$$

After some regrouping of sums, the result may be written as $L_1+L_2+L_3=0$ where,

$$\begin{aligned} L_1 &= \sum_{i=0}^{r} g^{I_r\cdots I_{i+1}}{}_J(1,2)\, \Delta_L^{(2)} D_\delta^{I_1\cdots I_i J}(2,\cdots,n) \\ L_2 &= \sum_{i=0}^{r} \sum_{k=1}^{r+1-i} \frac{(2\pi i)^k}{k!} g^{I_r\cdots I_{I+k}}{}_L(1,2)\, \delta_L^{I_{i+k-1}\cdots I_{i+1}} D_\delta^{I_1\cdots I_i}(2,\cdots,n) \\ L_3 &= \sum_{i=1}^{r} \sum_{k=1}^{r+1-i} \frac{(2\pi i)^k}{k!} g^{I_r\cdots I_{I+k}}{}_L(1,2)\, \delta_L^{I_{i+k-1}\cdots I_{i+1}} \Delta_L^{(2)} D_\delta^{I_1\cdots I_i}(2,\cdots,n) \end{aligned} \tag{C.5}$$

Next, we eliminate the $\Delta_L^{(2)} D_\delta$ terms in favor of the combination T_δ that was introduced in (C.2). The contributions involving T_δ are given by,

$$\begin{aligned} L_1^T &= \sum_{i=0}^{r} g^{I_r\cdots I_{i+1}}{}_J(1,2)\, T_\delta^{I_1\cdots I_i J}(2,\cdots,n) \\ L_3^T &= \sum_{i=1}^{r} \sum_{k=1}^{r+1-i} \frac{(2\pi i)^k}{k!} g^{I_r\cdots I_{i+k}}{}_L(1,2)\, \delta_L^{I_{i+k-1}\cdots I_{i+1}} T_\delta^{I_1\cdots I_i}(2,\cdots,n) \end{aligned} \tag{C.6}$$

In terms of L_1^T and L_3^T, the contributions L_1 and L_3 take the following form,

$$L_1 = L_1^T - \sum_{i=0}^{r} \sum_{\ell=1}^{i+1} \frac{(-2\pi i)^\ell}{\ell!} g^{I_r \cdots I_{i+1}}{}_L(1,2)\, \delta_L^{I_i \cdots I_{i+2-\ell}} D_\delta^{I_1 \cdots I_{i+1-\ell}}(2, \cdots, n) \tag{C.7}$$

$$L_3 = L_3^T + \sum_{i=1}^{r} \sum_{k=1}^{r+1-i} \sum_{\ell=1}^{i} (-)^\ell \frac{(2\pi i)^{k+\ell}}{k!\,\ell!} g^{I_r \cdots I_{i+k}}{}_L(1,2)\, \delta_L^{I_{i+k-1} \cdots I_{i-\ell+1}} D_\delta^{I_1 \cdots I_{i-\ell}}(2, \cdots, n)$$

where we have used the identity,

$$\delta_L^{I_{k+i-1} \cdots I_{i+1}}\, \delta_L^{I_i \cdots I_{i+1-\ell}} = \delta_L^{I_{k+i-1} \cdots I_{i+1-\ell}} \tag{C.8}$$

to simplify the summand of the triple sum in L_3. As a result of using this identity, the factors g and D_δ in the summand depend only on the combinations $k+i$ and $i-\ell$. Hence, it is possible to carry out one of the three summations explicitly. To do so, we perform a double change of variables in the triple sum for L_3, with the following ranges,

$$\begin{array}{lll} (k,i) \to (p,i) & p = k+i & 1 \leq i \leq p \leq r+1 \\ (\ell,i) \to (q,i) & q = i-\ell & 0 \leq q \leq i-1 \leq p-1 \end{array} \tag{C.9}$$

so that the three summations in L_3 may be rearranged as follows,

$$\sum_{i=1}^{r} \sum_{k=1}^{r+1-i} \sum_{\ell=1}^{i} = \sum_{p=2}^{r+1} \sum_{q=0}^{p-2} \sum_{i=q+1}^{p-1} \tag{C.10}$$

As a result, L_3 may be expressed as follows,

$$L_3 = L_3^T + \sum_{p=2}^{r+1} \sum_{q=0}^{p-2} \sum_{i=q+1}^{p-1} \frac{(-)^{i+q}(2\pi i)^{p-q}}{(p-i)!\,(i-q)!} g^{I_r \cdots I_p}{}_L(1,2)\, \delta_L^{I_{p-1} \cdots I_{q+1}} D_\delta^{I_1 \cdots I_q}(2, \cdots, n) \tag{C.11}$$

The sum over i may be carried out exactly,

$$\sum_{i=q+1}^{p-1} \frac{(-)^{i+q}}{(p-i)!\,(i-q)!} = -\frac{1+(-)^{p-q}}{(p-q)!} \tag{C.12}$$

leaving the following double sum,

$$L_3 = L_3^T - \sum_{p=2}^{r+1} \sum_{q=0}^{p-2} \big(1+(-)^{p-q}\big) \frac{(2\pi i)^{p-q}}{(p-q)!} g^{I_r \cdots I_p}{}_L(1,2)\, \delta_L^{I_{p-1} \cdots I_{q+1}} D_\delta^{I_1 \cdots I_q}(2, \cdots, n) \tag{C.13}$$

In view of the factor $(1+(-)^{p-q})$ in the summand, we are free to extend the summation over q to include $q=p-1$ and, once that has been done, we are also free to extend the summation range of p to include $p=1$ since the second sum then forces $q=0$ which again vanishes in view of the factor $(1+(-)^{p-q})$. Separating now the contributions from the two terms in the factor $(1+(-)^{p-q})$, we have,

$$\begin{aligned} L_3 = L_3^T \; & - \sum_{p=1}^{r+1} \sum_{q=0}^{p-1} \frac{(2\pi i)^{p-q}}{(p-q)!} g^{I_r \cdots I_p}{}_L(1,2)\, \delta_L^{I_{p-1}\cdots I_{q+1}} D_\delta^{I_1 \cdots I_q}(2,\cdots,n) \\ & - \sum_{p=1}^{r+1} \sum_{q=0}^{p-1} \frac{(-2\pi i)^{p-q}}{(p-q)!} g^{I_r \cdots I_p}{}_L(1,2)\, \delta_L^{I_{p-1}\cdots I_{q+1}} D_\delta^{I_1 \cdots I_q}(2,\cdots,n) \end{aligned} \tag{C.14}$$

Changing summation variables in $L_1 - L_1^T$ from i to $p=i+1$ and ℓ to $q=p-\ell$, and in L_2 from k to $p=i+k$ and from i to q, we obtain,

$$\begin{aligned} L_1 &= L_1^T + \sum_{p=1}^{r+1} \sum_{q=0}^{p-1} \frac{(-2\pi i)^{p-q}}{(p-q)!} g^{I_r \cdots I_p}{}_L(1,2) \delta_L^{I_{p-1}\cdots I_{q+1}} D_\delta^{I_1 \cdots I_q}(2,\cdots,n) \\ L_2 &= \sum_{p=1}^{r+1} \sum_{q=0}^{p-1} \frac{(2\pi i)^{p-q}}{(p-q)!} g^{I_r \cdots I_p}{}_L(1,2)\, \delta_L^{I_{p-1}\cdots I_{q+1}} D_\delta^{I_1 \cdots I_q}(2,\cdots,n) \end{aligned} \tag{C.15}$$

We see that, in the sum $L_1+L_2+L_3$, the sum of $L_1-L_1^T$ cancels the sum of the second line of $L_3-L_3^T$ while L_2 cancels the sum on the first line of $L_3-L_3^T$. Therefore, the relation $L_1+L_2+L_3=0$ reduces to $L_1^T+L_3^T=0$ or more explicitly,

$$\begin{aligned} \omega_J(1) T_\delta^{I_1 \cdots I_r J}(2,\cdots,n) = & - \sum_{i=0}^{r-1} g^{I_r \cdots I_{i+1}}{}_J(1,2)\, T_\delta^{I_1 \cdots I_i J}(2,\cdots,n) \\ & - \sum_{i=1}^{r} \sum_{k=1}^{r+1-i} \frac{(2\pi i)^k}{k!} g^{I_r \cdots I_{i+k}}{}_L(1,2)\, \delta_L^{I_{i+k-1}\cdots I_{i+1}} T_\delta^{I_1 \cdots I_i}(2,\cdots,n) \end{aligned} \tag{C.16}$$

where the term on the left corresponds to the $i=r$ contribution to L_1^T.

We are now ready to complete the proof by induction. For $r=1$ we have,

$$T_\delta^I(2,\cdots,n) = \Delta_L^{(1)} D_\delta^I(2,\cdots,n) + 2\pi i\, \delta_L^I C_\delta(2,\cdots,n) \tag{C.17}$$

which vanishes in view of the expression for $D_\delta^{I_1}(2,\cdots,n)$ in the first line of (2.32) and the $\mathfrak{B}$ monodromy transformation of $\chi(1,2)$ in the point z_2, given in (A.17). Since the left side of equation (C.16) is of rank $r+1$ while the maximum rank on the right side is r, it follows by induction in r that we have $T_\delta^{I_1\cdots I_r}(2,\cdots,n)=0$ for all r and all $n\geq 2$. This completes the proof of the monodromy formula of (2.42) for $k=2$.

C.3 Monodromy of $D_\delta^{I_1\cdots I_r}(2,\cdots,n)$ in z_n for $3 \leq n$

The final element in the proof of Lemma 2.5 consists of proving the last equation in (2.42). To do this we introduce the following combination,

$$U_\delta^{I_1\cdots I_r}(1,\cdots,n) = \Delta_L^{(n)} D_\delta^{I_1\cdots I_r}(1,\cdots,n) - \sum_{\ell=1}^{r} \frac{(2\pi i)^\ell}{\ell!} \delta_L^{I_1\cdots I_\ell} D_\delta^{I_{\ell+1}\cdots I_r}(1,\cdots,n) \tag{C.18}$$

in terms of which the last equation in (2.42) is equivalent to $U_\delta^{I_1\cdots I_r}(1,\cdots,n) = 0$. Applying $\Delta_L^{(n)}$ to equation (2.34) gives,

$$\begin{aligned} 0 = \Delta_L^{(n)} D_\delta^{I_1\cdots I_r}(1,\cdots,n) - \Delta_L^{(n)} \chi^{I_r\cdots I_1}(1,n)\, C_\delta(2,\cdots,n) \\ - \sum_{i=0}^{r} g^{I_r\cdots I_{i+1}}{}_J(1,2) \Delta_L^{(n)} D_\delta^{I_1\cdots I_i J}(2,\cdots,n) \end{aligned} \tag{C.19}$$

Eliminating $\Delta_L^{(n)} D_\delta$ in favor of U_δ, we may organize the result as $L_4^T + L_4 = 0$ where,

$$L_4^T = U^{I_1\cdots I_r}(1,\cdots,n) - \sum_{i=0}^{r} g^{I_r\cdots I_{i+1}}{}_J(1,2) U_\delta^{I_1\cdots I_i J}(2,\cdots,n) \tag{C.20}$$

and L_4 simplifies to give,

$$\begin{aligned} L_4 = \sum_{\ell=1}^{r} \frac{(2\pi i)^\ell}{\ell!} \delta_L^{I_1\cdots I_\ell} \Big[& D_\delta^{I_{\ell+1}\cdots I_r}(1,\cdots,n) \\ & + \big(\chi^{I_r\cdots I_{\ell+1}}(1,2) - \chi^{I_r\cdots I_{\ell+1}}(1,n)\big) C_\delta(2,\cdots,n) \\ & - \sum_{k=\ell+1}^{r+1} g^{I_r\cdots I_k}{}_J(1,2) D_\delta^{I_{\ell+1}\cdots I_{k-1} J}(2,\cdots,n) \Big] \end{aligned} \tag{C.21}$$

The rank of the terms on the right is at most $r-1$. The expression coincides with equation (2.34) and therefore vanishes. The remaining expression for L_4^T in (C.20) involves only U_δ and may again be analyzed by induction in r, starting with the vanishing of $U_\delta^I(2,\cdots,n)$ which follows from the monodromy in z_n of the first equation in (2.34). This concludes the proof of the z_n monodromy and therefore of the entire Lemma 2.5.

D Proving properties of meromorphic multiplets

This appendix gathers proofs of several symmetry properties of the meromorphic multiplets $D_\delta^{I_1\cdots I_r}$, $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$, and $M_\delta^{I_1\cdots I_r}(x,y)$ in the decompositions (5.1) and (7.32) of cyclic products and linear chain products, respectively.

D.1 Reflection property of $D_\delta^{I_1\cdots I_r}$ from contour deformations

The contour deformation techniques of figure 3, which were the key to proving the cyclic invariance of the constant multiplets $D_\delta^{I_1\cdots I_r}$ in section 4.5, can also be used to establish their reflection property at fixed rank r stated in Proposition 5.2. Before presenting an indirect proof of the proposition for arbitrary rank in appendix D.3, we shall provide some explicit calculations to rank $r\leq 5$.

For this purpose, we shall evaluate $\mathfrak{D}_\delta^{I_1I_r\cdots I_2}$ in terms of $\mathfrak{D}_\delta^{I_1I_2\cdots I_r}$ which will give the departures from reflection symmetry of the convolution integrals in (4.8) for $n=0$ by contour deformations similar to those of figure 3. [15]

$$\begin{aligned}
\mathfrak{D}_\delta^{I_1I_3I_2} &= -\mathfrak{D}_\delta^{I_1I_2I_3} + 2\pi i\delta_{I_2}^{I_3}\mathfrak{D}_\delta^{I_1I_2}\\
\mathfrak{D}_\delta^{I_1I_4I_3I_2} &= \mathfrak{D}_\delta^{I_1I_2I_3I_4} + 2\pi i\delta_{I_3}^{I_4}\mathfrak{D}_\delta^{I_1I_3I_2} - 2\pi i\delta_{I_2}^{I_3}\mathfrak{D}_\delta^{I_1I_2I_4}\\
\mathfrak{D}_\delta^{I_1I_5I_4I_3I_2} &= -\mathfrak{D}_\delta^{I_1I_2I_3I_4I_5} + 2\pi i\delta_{I_4}^{I_5}\mathfrak{D}_\delta^{I_1I_4I_3I_2} + 2\pi i\delta_{I_2}^{I_3}\mathfrak{D}_\delta^{I_1I_2I_4I_5}\\
&\quad + 2\pi i\delta_{I_3}^{I_4}\big[\mathfrak{D}_\delta^{I_1I_2I_3I_5} - 2\pi i\delta_{I_3}^{I_2}\mathfrak{D}_\delta^{I_1I_2I_5}\big]
\end{aligned}\tag{D.1}$$

These expressions are derived by deforming the cascade of displaced integration contours for $t_1,\cdots,t_r$ associated with $\mathfrak{D}_\delta^{I_1I_r\cdots I_2}$ to the simpler arrangement of contours on the right side of figure 3 that corresponds to $\mathfrak{D}_\delta^{I_1I_2\cdots I_r}$, with t_1 at the outermost, t_2 at the next-to-outermost and t_r at the innermost placement in the interior D^o of the fundamental domain. The contours for $t_2,\cdots,t_r$ of the starting point $\mathfrak{D}_\delta^{I_1I_r\cdots I_2}$ are ordered in the opposite way, with t_2 at the innermost placement in D^o and t_r at the outermost one besides t_1. One proceeds by deforming the contour for t_2 past all of those for $t_3,\cdots,t_r$, followed by deforming all further contours for t_j past those for $t_{j+1},\cdots,t_r$ in the order of increasing $j=3,\cdots,r-1$. All crossings of contours for t_j with those of t_i contribute via integrals over infinitesimal circles of t_j around t_i if $I_i=I_j$ which we evaluate via residues as in the proof of Lemma 4.1. Again, the pole structure of the integrand $C_\delta(t_1,\cdots,t_r)$ implies that the only non-vanishing residues (4.28) arise for circles of t_k around $t_{k\pm 1}$.

[15] Note that the coefficient of $\delta_{I_3}^{I_4}$ in the last line can be rewritten as $\mathfrak{D}_\delta^{I_1I_2I_3I_5} - 2\pi i\delta_{I_3}^{I_2}\mathfrak{D}_\delta^{I_1I_2I_5} = \mathfrak{D}_\delta^{I_1I_5I_3I_2} - 2\pi i\delta_{I_3}^{I_5}\mathfrak{D}_\delta^{I_1I_5I_2}$ by the rank-four identity in (D.1) which establishes the symmetry of the combination $\mathfrak{D}_\delta^{I_1I_2I_3I_4I_5} + \mathfrak{D}_\delta^{I_1I_5I_4I_3I_2}$ under the simultaneous swap $I_2\leftrightarrow I_5$ and $I_3\leftrightarrow I_4$.

The alternating reflection parity (5.8) at rank $r \leq 5$ then follows by applying the properties (D.1) and (4.27) of the $\mathfrak{A}$ convolutions to the expressions for $D_\delta^{I_1\cdots I_r}$ in terms of $\mathfrak{D}_\delta^{J_1\cdots J_s}$. At rank $r \leq 4$, these expressions can be found in (4.25), and the $r=5$ case is obtained from the components of (4.21), where the integrand $\mathbf{D}_\delta(t;B)$ on the right side is expanded via (4.13).

D.2 Proof of items (b) and (c) of Proposition 5.3

We shall here prove the symmetry properties (5.10), (5.11) of the $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ in items (b), (c) of Proposition 5.3, using the fact that the same properties are already established for their modular counterparts $\mathcal{W} \leftrightarrow \mathcal{V}$ in items (b), (c) of Proposition 6.5.

At fixed n and r, the proven symmetry properties (6.32) and (6.33) of $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ can alternatively be checked by means of the Fay identities and interchange lemmas [62] of the DHS kernels. For instance, the weight-one interchange lemma $\omega_I(x)f^I{}_J(y,x) + \omega_I(y)f^I{}_J(x,y) = 0$ [25] together with the alternating symmetry of double derivatives

$$\partial_x\partial_y\mathcal{G}^{I_1\cdots I_r}(x,y) = (-1)^r\partial_x\partial_y\mathcal{G}^{I_r\cdots I_1}(y,x) \tag{D.2}$$

suffice to check that the simplest instances of $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ in (6.29) obey

$$\begin{aligned}\mathcal{V}(1,2) &= \mathcal{V}(2,1)\\ \mathcal{V}_{JK}(1,2,3) &= -\mathcal{V}_{KJ}(2,1,3)\\ \mathcal{V}(1,2,3) &= -\mathcal{V}(2,1,3)\end{aligned} \tag{D.3}$$

The key idea of this proof is to transfer these symmetry checks from the DHS kernels in $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ to the Enriquez kernels in $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$. This can be done since the explicit form of $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ is obtained from $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ by substituting $f^{J_1\cdots J_s}{}_K(x,y) \to g^{J_1\cdots J_s}{}_K(x,y)$ term by term, see (6.38).

As detailed in section 9 of [62], the interchange lemmas and Fay identities of Enriquez kernels take the same form as those of the DHS kernels (see [62] for a proof of the DHS-kernel identities and the meromorphic interchange lemma and [82] for a proof of meromorphic Fay identities). Similarly, the alternating symmetry (D.2) has a direct counterpart [62]

$$\partial_y\chi^{I_1\cdots I_r}(x,y) = (-1)^r\partial_x\chi^{I_r\cdots I_1}(y,x) \tag{D.4}$$

at the level of Enriquez kernels. Accordingly, the identities (D.3) among modular tensors at $n \leq 3$ points propagate to

$$\begin{aligned}\mathcal{W}(1,2) &= \mathcal{W}(2,1)\\ \mathcal{W}_{JK}(1,2,3) &= -\mathcal{W}_{KJ}(2,1,3)\\ \mathcal{W}(1,2,3) &= -\mathcal{W}(2,1,3)\end{aligned} \tag{D.5}$$

since the underlying manipulations of Enriquez kernels take the same form as the DHS-kernel identities required for the derivation of (D.3).

More generally, the one-to-one correspondence between linear and quadratic relations among Enriquez kernels and those of DHS kernels implies that the Fay identities and interchange lemmas needed to establish the properties (6.32) and (6.33) of $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ are preserved under $f^{J_1\cdots J_s}{}_K(x,y) \to g^{J_1\cdots J_s}{}_K(x,y)$. This relies on the uniqueness results on the relations among Enriquez kernels in Lemma 11 and Theorem 12 of [82]. Hence, the images $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ of $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ under (6.37) will obey the formal images of the properties (6.32) and (6.33) under $\mathcal{V} \to \mathcal{W}$. These images are the statements in items (b) and (c) of Proposition 5.3 which thereby completes their proof.

D.3 Proof of Proposition 5.2

We prove the alternating reflection parity $D_\delta^{I_1 I_2\cdots I_r} = (-1)^r D_\delta^{I_r\cdots I_2 I_1}$ in Proposition 5.2 by induction in the rank r. As a base case at $r=2$, the parity $D_\delta^{I_1 I_2} = D_\delta^{I_2 I_1}$ is clear both from the cyclicity of $D_\delta^{I_1\cdots I_r}$ in Theorem 2.3 and from the explicit form of $D_\delta^{I_1 I_2}$ in (2.11).

Assuming that $D_\delta^{I_1 I_2\cdots I_s} = (-1)^s D_\delta^{I_s\cdots I_2 I_1}$ hold for $s \leq r-1$, we will now infer the $s=r$ case from item (c) of Proposition 5.3 and the alternating parity

$$C_\delta(1,2,\cdots,r) = (-1)^r C_\delta(r,\cdots,2,1) \tag{D.6}$$

of cyclic products which follows from the antisymmetry $S_\delta(x,y) = -S_\delta(y,x)$ of Szegö kernels. For this purpose, the alternating parity (D.6) will be imposed at the level of the meromorphic decomposition (5.1) which we split into the form of

$$\begin{aligned} C_\delta(1,\cdots,r) &= \check{C}_\delta(1,\cdots,r) + \omega_{I_1}(1)\cdots\omega_{I_r}(r) D_\delta^{I_1\cdots I_r} \\ \check{C}_\delta(1,\cdots,r) &= \mathcal{W}(1,\cdots,r) + \sum_{s=2}^{r-1} \mathcal{W}_{I_1\cdots I_s}(1,\cdots,r)\, D_\delta^{I_1\cdots I_s} \end{aligned} \tag{D.7}$$

The alternating parity of $\mathcal{W}_{I_1\cdots I_s}(1,\cdots,r)$ established in item (c) of Proposition 5.3 together with the inductive assumption imply that (D.6) holds separately for the part $\check{C}_\delta(1,\cdots,r)$ in (D.7),

$$\begin{aligned} \check{C}_\delta(1,\cdots,r) &= (-1)^r \mathcal{W}(r,\cdots,1) + \sum_{s=2}^{r-1} (-1)^{r-s} \mathcal{W}_{I_s\cdots I_1}(r,\cdots,1)\, D_\delta^{I_1\cdots I_s} \\ &= (-1)^r \left\{ \mathcal{W}(r,\cdots,1) + \sum_{s=2}^{r-1} \mathcal{W}_{I_s\cdots I_1}(r,\cdots,1)\, D_\delta^{I_s\cdots I_1} \right\} \\ &= (-1)^r\, \check{C}_\delta(r,\cdots,1) \end{aligned} \tag{D.8}$$

where we emphasize that $D_\delta^{I_1\cdots I_s} = (-1)^s D_\delta^{I_s\cdots I_1}$ has only been used for $s \leq r-1$. As a consequence of (D.8), the contributions from $\check{C}_\delta(1,\cdots,r)$ drop out in the following rewriting of (D.6):

$$\begin{aligned}
0 &= C_\delta(1,2,\cdots,r) - (-1)^r C_\delta(r,\cdots,2,1) \\
&= \omega_{I_1}(1)\cdots\omega_{I_r}(r) D_\delta^{I_1\cdots I_r} - (-1)^r \omega_{I_1}(r)\cdots\omega_{I_r}(1) D_\delta^{I_1\cdots I_r} \\
&= \omega_{I_1}(1)\cdots\omega_{I_r}(r)\big(D_\delta^{I_1\cdots I_r} - (-1)^r D_\delta^{I_r\cdots I_1}\big)
\end{aligned} \tag{D.9}$$

which completes the inductive step and therefore the proof of Proposition 5.2 for arbitrary rank r (see appendix D.1 for an alternative proof at rank $r \leq 5$).

D.4 Reflection properties of $M_\delta^{I_1\cdots I_r}$ from contour deformations

The reflection property (7.12) of the spinors $L_\delta^{I_1\cdots I_r}(x,y)$ in the modular decomposition (7.19) of linear chain products is manifest from their surface-integral representation (7.11). However, this does not immediately carry over to the reflection properties (7.41) of their meromorphic counterparts $M_\delta^{I_1\cdots I_r}(x,y)$ when expressed in terms of the $\mathfrak{A}$ convolutions generated by (7.31). As an alternative to the indirect proof in section 7.4.4 that is valid for arbitrary rank, we shall here provide a direct proof of (7.41) at rank $r \leq 3$, using the contour deformations of section 4.5.1 which may also be extended to $r \geq 4$.

The salient point is that the $\mathfrak{A}$ convolutions $\mathfrak{M}_\delta^{I_1\cdots I_r}(x,y)$ at $r \geq 2$ fail to exhibit any reflection parity ± 1 since the transition to $\mathfrak{M}_\delta^{I_r\cdots I_1}(y,x)$ requires the crossing of certain integration contours $\mathfrak{A}^{I_k}$ in (7.30) which results in residue contributions similar to those in section 4.5.1. By following the integration contour deformation techniques illustrated in figure 3, one can derive examples such as,

$$\begin{aligned}
\mathfrak{M}_\delta^{I_1}(y,x) &= \mathfrak{M}_\delta^{I_1}(x,y) \\
\mathfrak{M}_\delta^{I_1 I_2}(y,x) &= -\mathfrak{M}_\delta^{I_2 I_1}(x,y) + 2\pi i \delta^{I_2}_{I_1} \mathfrak{M}_\delta^{I_1}(x,y) \\
\mathfrak{M}_\delta^{I_1 I_2 I_3}(y,x) &= \mathfrak{M}_\delta^{I_3 I_2 I_1}(x,y) - 2\pi i \big[\delta^{I_3}_{I_2} \mathfrak{M}_\delta^{I_2 I_1}(x,y) + \delta^{I_2}_{I_1} \mathfrak{M}_\delta^{I_3 I_1}(x,y)\big] \\
&\quad + (2\pi i)^2 \delta^{I_1 I_2}_{I_3} \mathfrak{M}_\delta^{I_3}(x,y)
\end{aligned} \tag{D.10}$$

see (D.1) for the analogous reflection formulae for $\mathfrak{A}$ convolutions of cyclic products. Just as it was the case for the constants (4.25) in the cyclic products, the correction terms in (7.36) by convolutions of shorter linear chain products compensate for the lack of simple reflection parities ± 1 in (D.10).

E Low rank examples

This appendix gathers further examples of the meromorphic and single-valued forms $\mathcal{W}$ and $\mathcal{V}$ in the reduction of products of Szegö kernels. We follow the presentation of sections 5.1, 7.3 and spell out the explicit form of $\mathcal{W}_{I_1\cdots I_r}(1,2,3,4)$ for cyclic products in terms of Enriquez kernels and $\mathcal{V}_{I_1\cdots I_r}(x;1,2,3;y)$ for linear chains in terms of DHS kernels. Their counterparts $\mathcal{V}_{I_1\cdots I_r}(1,2,3,4)$ and $\mathcal{W}_{I_1\cdots I_r}(x;1,2,3;y)$ involving the opposite type of higher-genus kernels can be straightforwardly obtained from the subsequent expressions via $g^{I_1\cdots I_r}{}_J(x,y) \leftrightarrow f^{I_1\cdots I_r}{}_J(x,y)$, i.e. through the correspondences (6.38) and (7.35).

E.1 Meromorphic $\mathcal{W}$ for cyclic products of four Szegö kernels

The single-valued combinations $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ of Enriquez kernels at $n=2,3$ points can be found in section 5.1.2. At $n=4$, the expressions below for $\mathcal{W}_{I_1\cdots I_r}(1,2,3,4)$ at $r=0,2,3$ are derived by matching the results of the descent (2.24) with the meromorphic decomposition (5.1). Similar to the $n=3$ case, one eliminates $C_\delta(2,3,4), C_\delta(3,4)$ and all the multiplets $D_\delta^{I_1\cdots I_s}(r,\cdots)$ that depend on at least one point $z_r,\cdots$ from (2.24). The resulting coefficients of D_δ^{JKL}, D_δ^{JK} and δ-independent terms are given by,

$$
\begin{aligned}
\mathcal{W}_{JKL}(1,2,3,4) = \frac{1}{3}\Big[&\big(\chi(1,4)-\chi(1,2)\big)\omega_J(2)\omega_K(3)\omega_L(4) \\
&+\omega_I(1)g^I{}_J(2,3)\omega_K(3)\omega_L(4)+\omega_J(1)\omega_I(2)g^I{}_K(3,4)\omega_L(4) \\
&+\big(\omega_J(1)\omega_K(2)\omega_I(3)-\omega_I(1)\omega_J(2)\omega_K(3)\big)\varpi^I{}_L(4)+\mathrm{cycl}(J,K,L)\Big] \qquad \text{(E.1)}\\
\mathcal{W}_{JK}(1,2,3,4) = \frac{1}{2}\Big[&\big(\chi(1,4)-\chi(1,2)\big)\big(\chi(2,4)-\chi(2,3)\big)\omega_J(3)\omega_K(4) \\
&+\big(\chi(1,4)-\chi(1,2)\big)\omega_I(2)g^I{}_J(3,4)\omega_K(4) \\
&+\big(\chi(1,4)-\chi(1,2)\big)\big(\omega_J(2)\omega_I(3)-\omega_I(2)\omega_J(3)\big)\varpi^I{}_K(4) \\
&+\omega_I(1)\big(\chi^I(2,4)-\chi^I(2,3)\big)\omega_J(3)\omega_K(4) \\
&+\omega_I(1)g^I{}_M(2,3)g^M{}_J(3,4)\omega_K(4)+\omega_I(1)\omega_M(2)g^{MI}{}_J(3,4)\omega_K(4) \\
&+\omega_I(1)\big(g^I{}_J(2,3)\omega_M(3)-g^I{}_M(2,3)\omega_J(3)\big)\varpi^M{}_K(4) \\
&+\omega_I(2)\big(\omega_J(1)g^I{}_M(3,4)-\omega_M(1)g^I{}_J(3,4)\big)\varpi^M{}_K(4) \\
&+\big(\omega_J(1)\omega_I(2)\omega_M(3)-\omega_I(1)\omega_J(2)\omega_M(3) \\
&\quad -\omega_M(1)\omega_J(2)\omega_I(3)+\omega_M(1)\omega_I(2)\omega_J(3)\big)\varpi^{MI}{}_K(4)+(J\leftrightarrow K)\Big]\\
\mathcal{W}(1,2,3,4) = {}&\big(\chi(1,4)-\chi(1,2)\big)\big(\chi(2,4)-\chi(2,3)\big)\partial_4\chi(3,4) \\
&+\big(\chi(1,4)-\chi(1,2)\big)\omega_I(2)\partial_4\chi^I(3,4)+\omega_I(1)g^I{}_J(2,3)\partial_4\chi^J(3,4) \\
&+\omega_I(1)\big(\chi^I(2,4)-\chi^I(2,3)\big)\partial_4\chi(3,4)+\omega_I(1)\omega_J(2)\partial_4\chi^{JI}(3,4)
\end{aligned}
$$

The prescriptions $+\text{cycl}(J,K,L)$ and $+(J \leftrightarrow K)$ refer to all lines of the expressions for $\mathcal{W}_{JKL}(1,2,3,4)$ and $\mathcal{W}_{JK}(1,2,3,4)$, respectively, and again implement the cyclic symmetrization in the indices according to the choice (5.5). The rank-four case of $\mathcal{W}_{IJKL}(1,2,3,4) = \frac{1}{4}\omega_I(1)\omega_J(2)\omega_K(3)\omega_L(4) + \text{cycl}(I,J,K,L)$ lines up with the general formula (5.2) for maximal rank $r = n$.

E.2 Modular $\mathcal{V}$ for linear chains with three internal points

The combinations $\mathcal{V}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ of DHS kernels that are meromorphic in the internal points $z_1,\cdots,z_n$ of linear chain products are spelt out for $n=1,2$ in section 7.3.1. Their three-point analogues following from the modular descent equations of Theorem 7.2 are given by

$$\begin{aligned}
\mathcal{V}_{JK}(x;1,2,3;y) &= \partial_1\big(\mathcal{G}(1,x)-\mathcal{G}(1,2)\big)\omega_J(2)\omega_K(3) \\
&\quad + \omega_I(1)f^I{}_J(2,3)\omega_K(3) + \omega_J(1)\omega_I(2)f^I{}_K(3,y) \\
\mathcal{V}_J(x;1,2,3;y) &= \partial_1\big(\mathcal{G}(1,x)-\mathcal{G}(1,2)\big)\partial_2\big(\mathcal{G}(2,x)-\mathcal{G}(2,3)\big)\omega_J(3) \\
&\quad + \partial_1\big(\mathcal{G}(1,x)-\mathcal{G}(1,2)\big)\omega_I(2)f^I{}_J(3,y) \\
&\quad + \omega_I(1)\partial_2\big(\mathcal{G}^I(2,x)-\mathcal{G}^I(2,3)\big)\omega_J(3) \\
&\quad + \omega_I(1)\omega_K(2)f^{KI}{}_J(3,y) + \omega_I(1)f^I{}_K(2,3)f^K{}_J(3,y) \\
\mathcal{V}(x;1,2,3;y) &= \partial_1\big(\mathcal{G}(1,x)-\mathcal{G}(1,2)\big)\partial_2\big(\mathcal{G}(2,x)-\mathcal{G}(2,3)\big)\partial_3\big(\mathcal{G}(3,x)-\mathcal{G}(3,y)\big) \\
&\quad + \partial_1\big(\mathcal{G}(1,x)-\mathcal{G}(1,2)\big)\omega_K(2)\partial_3\big(\mathcal{G}^K(3,x)-\mathcal{G}^K(3,y)\big) \\
&\quad + \omega_K(1)\partial_2\big(\mathcal{G}^K(2,x)-\mathcal{G}^K(2,3)\big)\partial_3\big(\mathcal{G}(3,x)-\mathcal{G}(3,y)\big) \\
&\quad + \omega_I(1)\omega_K(2)\partial_3\big(\mathcal{G}^{KI}(3,x)-\mathcal{G}^{KI}(3,y)\big) \\
&\quad + \omega_I(1)f^I{}_K(2,3)\partial_3\big(\mathcal{G}^K(3,x)-\mathcal{G}^K(3,y)\big)
\end{aligned} \tag{E.2}$$

The rank-three case of $\mathcal{V}_{IJK}(x;1,2,3;y) = \omega_I(1)\omega_J(2)\omega_K(3)$ is covered by the all-multiplicity formula in the second line of (7.19).

F Proofs at genus one

In this last appendix we gather the proofs of Proposition 8.2 and Theorem 8.3 on products of Szegö kernels at genus one.

F.1 Proof of Proposition 8.2

We shall derive the expressions (8.27) and (8.28) in Proposition 8.2 for the specializations of the constants $C_\delta^{I_1\cdots I_r}$ and $D_\delta^{I_1\cdots I_r}$ to genus one.

The first step is to derive the expressions (8.27) for $C_\delta^{I_1\cdots I_n}|_{h=1}$ at $n\geq 3$ by integrating all the n points $z_1,\cdots,z_n$ in (8.16) over the genus one surface against $\prod_{j=1}^n d^2z_j/(\operatorname{Im}\tau)$. The left side integrates to $C_\delta^{I_1\cdots I_n}|_{h=1}$ by the integral representation (6.10) of $C_\delta^{I_1\cdots I_n}$ at arbitrary genus. Integrating the right side of (8.16) over n copies of the torus produces G_n from the singular term $f^{(1)}(z_{12})f^{(1)}(z_{23})\cdots f^{(1)}(z_{n1})$ of $V_n(1,\cdots,n)$ whereas all $V_k(1,\cdots,n)$ at $1\leq k\leq n{-}1$ integrate to zero. This follows from the fact that $\int_\Sigma d^2z\, f^{(k)}(z{-}x)=0$ for any $k\geq 1$ and $x\in\mathbb{C}$ since the Fourier zero mode $m=n=0$ of (8.12) only features the singular term $1/\eta$. Finally, the summand in (8.16) at $k=\lfloor n/2\rfloor$ integrates to $R_n(e_\delta)$ if n is even since $V_0(1,\cdots,n)=1$ and to zero if n is odd.

The next step is to prove the $n=2$ instances of both (8.28) and (8.27) by specializing the explicit results (2.11) and (6.30) for D_δ^{IJ} and C_δ^{IJ} to genus $h=1$. The double-derivatives of theta functions in (2.11) reduce to those in the expressions,

$$e_\delta=-\partial_\eta^2\ln\vartheta_\delta(\eta)\Big|_{\eta=0}-\mathrm{G}_2 \tag{F.1}$$

as a consequence of $\wp(\eta)=-\partial_\eta^2\ln\vartheta_1(\eta)-\mathrm{G}_2$ at $\eta=\omega_\delta$. In this way, we define the regularization of the conditionally convergent integral of $C_\delta(1,2)$ over its two points.

The third and most involved part of the proof of Proposition 8.2 concerns the equality of $C_\delta^{I_1\cdots I_n}|_{h=1}$ and $D_\delta^{I_1\cdots I_n}|_{h=1}$ for $n\geq 3$. The key idea is to compare the expressions (8.18) and (8.17) for the elliptic functions $V_r(1,\cdots,n)$ of the points $z_1,\cdots,z_n$ with the genus one instances of the modular tensors $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ in the modular decomposition (6.26) and their counterparts $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ in the meromorphic decomposition (5.1).

By (8.4) and (8.5), all instances of $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ and $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ at non-zero rank $r\geq 2$ reduce to products of undifferentiated Kronecker-Eisenstein coefficients $g^{(s)}(z_{ij})$ and $f^{(s)}(z_{ij})$ akin to (8.17) and (8.18), respectively. The expressions are gathered in the following lemma to be proven in section F.2.1 below:

Lemma F.1. *The genus one instances of the meromorphic and single-valued functions* $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ *and* $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ *of* $z_1,\cdots,z_n$ *at rank* $r\geq 2$ *in the meromorphic and* *modular decompositions (5.1) and (6.26) can be expressed in terms of the elliptic functions* $V_s(1,\cdots,n)$ *in (8.18) via*

$$\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)\,\big|_{h=1} = V_{n-r}(1,\cdots,n)\,, \qquad 2\leq r\leq n \tag{F.2}$$

which coincides with the meromorphic counterparts at genus one:

$$\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)\,\big|_{h=1} = V_{n-r}(1,\cdots,n)\,, \qquad 2\leq r\leq n \tag{F.3}$$

The comparison of the scalars $\mathcal{V}(1,\cdots,n)$ and $\mathcal{W}(1,\cdots,n)$ at genus one with combinations of the elliptic $V_r(1,\cdots,n)$ requires a refined analysis since their construction from the respective descent procedures introduces derivatives of Enriquez or DHS kernels, see the examples in (5.7), (6.29) and (E.1). The appearance of these derivatives at genus one is captured by the following lemma to be proven in section F.2.2 below.

Lemma F.2. *The spin structure independent terms* $\mathcal{V}(1,\cdots,n)$ *and* $\mathcal{W}(1,\cdots,n)$ *in the meromorphic and modular decompositions (5.1) and (6.26) at genus one are given by the coefficients of* $1/\eta$ *in the generating functions,*

$$\mathcal{V}(1,\cdots,n)\,\big|_{h=1} = \operatorname*{Res}_{\eta=0} \Omega(z_{n1},\eta)\Omega(z_{12},\eta)\cdots\partial_{n-1}\Omega(z_{n-1,n},\eta) \tag{F.4}$$

as well as,

$$\mathcal{W}(1,\cdots,n)\,\big|_{h=1} = \operatorname*{Res}_{\eta=0} F(z_{n1},\eta)F(z_{12},\eta)\cdots\partial_{n-1}F(z_{n-1,n},\eta) \tag{F.5}$$

The expressions in (F.4) and (F.5) will be shown in section F.2.3 to imply the alternative representations in the following lemma which completely bypass derivatives of Kronecker-Eisenstein coefficients.

Lemma F.3. *The spin structure independent terms* $\mathcal{V}(1,\cdots,n)$ *and* $\mathcal{W}(1,\cdots,n)$ *in the meromorphic and modular decompositions (5.1) and (6.26) at genus one can be expressed in terms of the elliptic functions* $V_r(1,\cdots,n)$ *in (8.18) and (almost) holomorphic Eisenstein series via*

$$\mathcal{V}(1,\cdots,n)\,\big|_{h=1} = V_n(1,\cdots,n) - \widehat{\mathrm{G}}_2 V_{n-2}(1,\cdots,n) - \sum_{k=2}^{\lfloor n/2\rfloor} \mathrm{G}_{2k} V_{n-2k}(1,\cdots,n) \tag{F.6}$$

as well as,

$$\mathcal{W}(1,\cdots,n)\,\big|_{h=1} = V_n(1,\cdots,n) - \mathrm{G}_2 V_{n-2}(1,\cdots,n) - \sum_{k=2}^{\lfloor n/2\rfloor} \mathrm{G}_{2k} V_{n-2k}(1,\cdots,n) \tag{F.7}$$

Note that the right side of (F.7) only differs from that of (F.6) by the appearance of the meromorphic G_2 in the place of the modular $\widehat{\mathrm{G}}_2$.

On the basis of Lemmas F.1 and F.3, we can complete the proof of Proposition 8.2: by equating the meromorphic and modular decompositions (5.1), (6.26) at genus $h=1$ and inserting all of (F.2), (F.3), (F.6), (F.7) for the dependence on the marked points, we can solve for $D_\delta^{I_1\cdots I_n}|_{h=1}$ in terms of $C_\delta^{I_1\cdots I_n}|_{h=1}$ and arrive at (8.30). This can be done separately for $n=2,3,4,\cdots$ which leads to a single undetermined instance of $D_\delta^{I_1\cdots I_n}|_{h=1}$ in each case. This concludes our proof of (8.30) which, in combination with (8.27) established in earlier steps, implies the second main statement (8.28) of Proposition 8.2.

F.2 Proof of Lemmas F.1 to F.3

In this section we shall prove Lemmas F.1, F.2 and F.3 which were used in the previous section to prove Proposition 8.2.

F.2.1 Proof of Lemma F.1

The first statement (F.2) of Lemma F.1 follows from the fact that the pole structure (6.31) of the left side matches that of the right side in all variables,

$$\bar{\partial}_k V_s(1,\cdots,n) = \pi\big(\delta(k,k+1)-\delta(k,k-1)\big)\, V_{s-1}(1,\cdots,\hat{k},\cdots,n) \tag{F.8}$$

with $s=n-r\geq 1$ and $k=1,\cdots,n$, so that the difference between the left and right sides must be independent of $z_1,\cdots,z_n$ and the fact that the multiple integral over all the points $z_1,\cdots,z_n$ over the torus vanishes on both sides. The two simple poles in z_k at $z_{k\pm1}$ follow from the generating function (8.21) of the elliptic functions $V_s(1,\cdots,n)$ and the singular behavior $F(z,\eta)=\frac{1}{z}+\mathcal{O}(z^0)$. In particular, one clearly reproduces $V_0(1,\cdots,n)=1$ from the simple formula (6.27) for $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ at rank $r=n$.

The meromorphic counterpart (F.3) of (F.2) follows from the facts that

- (a) the expressions for $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)$ and $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)$ are related by swapping Enriquez kernels and DHS kernels as in (6.38);
- (b) their $h=1$ instances are related by swapping $g^{(s)}(z_{ij})\leftrightarrow f^{(s)}(z_{ij})$ by virtue of (8.4);
- (c) undifferentiated $h=1$ kernels $g^{(s)}(z_{ij})$ and $f^{(s)}(z_{ij})$ obey the same Fay identities [40].

The outcome $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)|_{h=1}$ of the modular descent at genus one is related the manifestly cyclic expressions for $V_{n-r}(1,\cdots,n)$ in (8.18) by a sequence of Fay identities among $f^{(s)}(z_{ij})$. By item (a) and (b), the expressions for $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)|_{h=1}$ resulting from the meromorphic descent at genus one is obtained from $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)|_{h=1}$ via $f^{(s)}(z_{ij}) \to g^{(s)}(z_{ij})$. Given that $g^{(s)}(z_{ij})$ and $f^{(s)}(z_{ij})$ obey the same Fay identities, see item (c), one can attain manifestly cyclic rewritings of $\mathcal{W}_{I_1\cdots I_r}(1,\cdots,n)|_{h=1}$ from those of $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)|_{h=1}$ via $f^{(s)}(z_{ij}) \to g^{(s)}(z_{ij})$. Since the elliptic $V_{n-r}(1,\cdots,n)$ in the earlier result (F.2) for $\mathcal{V}_{I_1\cdots I_r}(1,\cdots,n)|_{h=1}$ are invariant under $g^{(s)}(z_{ij}) \leftrightarrow f^{(s)}(z_{ij})$, see (8.17) and (8.18), we are led to the second statement (F.3) of Lemma F.1.

F.2.2 Proof of Lemma F.2

The first statement (F.4) of Lemma F.2 is readily checked for the cases of $n = 2, 3$ through the Laurent expansion (8.3) of $\Omega(z, \eta)$ in η and the specializations of (6.29) to genus one,

$$\begin{aligned} \mathcal{V}(1,2)\,\Big|_{h=1} &= \partial_1 f^{(1)}_{12} \\ \mathcal{V}(1,2,3)\,\Big|_{h=1} &= (f^{(1)}_{12} - f^{(1)}_{13})\partial_2 f^{(1)}_{23} + \partial_2 f^{(2)}_{23} \end{aligned} \tag{F.9}$$

using the shorthands

$$f^{(s)}_{ij} = f^{(s)}(z_i - z_j) \tag{F.10}$$

Our proof of (F.4) at general n proceeds in three steps by showing that its right side

- (a) is cyclic in $z_1, z_2, \cdots, z_n$;
- (b) has the same recursive simple-pole structure in the number of points as the left side with $\mathrm{Res}_{z_n=z_1}\mathcal{V}(1,\cdots,n)|_{h=1} = \mathcal{V}(1,\cdots,n-1)|_{h=1}$ at $n \geq 3$;
- (c) vanishes upon integrating all of $z_1,\cdots,z_n$ over the torus.

Item (a) follows from the fact that the cyclic product $\Omega(z_{12},\eta)\Omega(z_{23},\eta)\cdots\Omega(z_{n1},\eta)$ without the z_{n-1} derivative in (F.4) is an elliptic function of η and therefore does not have a residue,

$$V_{n-1}(1,\cdots,n) = \operatorname*{Res}_{\eta=0}\Omega(z_{12},\eta)\Omega(z_{23},\eta)\cdots\Omega(z_{n1},\eta) = 0 \tag{F.11}$$

Taking derivatives of (F.11) in the z_i then leads to differences such as

$$\begin{aligned} 0 &= \partial_n \operatorname*{Res}_{\eta=0}\Omega(z_{12},\eta)\cdots\Omega(z_{n-1,n},\eta)\Omega(z_{n1},\eta) \\ &= \operatorname*{Res}_{\eta=0}\Omega(z_{12},\eta)\cdots\partial_n\big[\Omega(z_{n-1,n},\eta)\Omega(z_{n1},\eta)\big] \\ &= \operatorname*{Res}_{\eta=0}\big[\Omega(z_{12},\eta)\cdots\Omega(z_{n-1,n},\eta)\partial_n\Omega(z_{n1},\eta) - \Omega(z_{n1},\eta)\Omega(z_{12},\eta)\cdots\partial_{n-1}\Omega(z_{n-1,n},\eta)\big] \end{aligned} \tag{F.12}$$

using translation invariance $(\partial_i+\partial_j)\Omega(z_{ij},\eta) = 0$ in the last step. We conclude from (F.12) that the right side of (F.4) is cyclically invariant under $(z_1,z_2,\cdots,z_n) \to (z_2,\cdots,z_n,z_1)$.

Item (b) amounts to showing that the right side of (F.4) with $n \geq 3$ has a simple pole in z_{n1} with residue given by $\text{Res}_{\eta=0}\, \Omega(z_{12},\eta)\Omega(z_{23},\eta)\cdots\partial_{n-1}\Omega(z_{n-1,1},\eta)$. This immediately follows from $\Omega(z_{ij},\eta) = 1/z_{ij} + \mathcal{O}(z_{ij}^0)$ and by item (a) implies that the right side of (F.4) has simple poles in each pair $z_{i,i+1}$ of consecutive points, with an $(n-1)$-point instance of the same expression as a residue. Hence, the difference of the left and right side of (F.4) at n points is holomorphic if its $(n-1)$-point instance vanishes.

Item (c) follows from the fact that both $f_{ij}^{(s)}$ and $\partial_i f_{ij}^{(s)}$ with $s \geq 1$ vanish upon integrating either z_i or z_j over the torus. Accordingly, each term on the right side of

$$\operatorname*{Res}_{\eta=0} \Omega(z_{n1},\eta)\Omega(z_{12},\eta)\cdots\partial_{n-1}\Omega(z_{n-1,n},\eta) = \sum_{\substack{s_1,s_2,\cdots,s_n\geq 0\\ s_1+s_2+\cdots+s_n=n-1}} f_{n1}^{(s_1)} f_{12}^{(s_2)} \cdots \partial_{n-1} f_{n-1,n}^{(s_n)} \tag{F.13}$$

integrates to zero over one of $z_1,\cdots,z_n$ since $s_1+\cdots+s_n = n-1$ is incompatible with having all $s_j > 0$. So there are at most $n-1$ factors of $f_{ij}^{(s)}$ or $\partial_i f_{ij}^{(s)}$ with $s > 0$ per summand, and there is at least one point which only enters one of the factors. Since also the left side of (F.4) integrates to zero over z_i and in fact for $\mathcal{V}(1,\cdots,n)$ at arbitrary genus $h \geq 1$, the difference of the left and right sides of (F.4) integrates to zero at all $n \geq 2$.

We can now show the equality (F.4) by induction in n. The base cases at $n = 2, 3$ are already checked in (F.9). Item (b) implied that the difference of the left and right side of (F.4) is holomorphic at n points, assuming that it vanishes at $\leq n-1$ points and $n \geq 3$. As a holomorphic function of $z_1,\cdots,z_n$, the difference must be constant, and by the result of item (c), this constant vanishes.

Since $\mathcal{W}(1,\cdots,n)$ are obtained from $\mathcal{V}(1,\cdots,n)$ by the conversion (6.38) of DHS kernels into Enriquez kernels, their genus one instances are related by $f_{ij}^{(s)} \leftrightarrow g_{ij}^{(s)}$. Hence, (F.4) implies the second statement (F.5) of the lemma and thereby concludes its proof.

F.2.3 Proof of Lemma F.3

With the representations (F.4) and (F.5) of $\mathcal{V}(1,\cdots,n)|_{h=1}$ and $\mathcal{W}(1,\cdots,n)|_{h=1}$ at hand, we can now prove the statements (F.6) and (F.7) of Lemma F.3 by means of the identities

$$\begin{aligned}
\partial_z F(z,\eta) &= \partial_\eta F(z,\eta) + \Big(g^{(1)}(\eta) - g^{(1)}(z)\Big) F(z,\eta) \\
\partial_z \Omega(z,\eta) &= \partial_\eta \Omega(z,\eta) + \Big(g^{(1)}(\eta) + \frac{\pi\eta}{\operatorname{Im}\tau} - f^{(1)}(z)\Big)\Omega(z,\eta)
\end{aligned} \tag{F.14}$$

among Kronecker-Eisenstein series. The first line is a simple consequence of the theta function representations in (8.1) and the second line follows from the first one via $F(z,\eta) = e^{-2\pi i\eta \operatorname{Im} z/\operatorname{Im}\tau}\Omega(z,\eta)$.

By inserting the second line of (F.14) into (F.4), we find the alternative form

$$\mathcal{V}(1,\cdots,n)\,\big|_{h=1} = \operatorname*{Res}_{\eta=0} \Omega(z_{n1},\eta)\Omega(z_{12},\eta)\cdots\Omega(z_{n-2,n-1},\eta) \\ \times \left(g^{(1)}(\eta) + \frac{\pi\eta}{\operatorname{Im}\tau} - f^{(1)}(z) + \partial_\eta \right) \Omega(z_{n-1,n},\eta) \tag{F.15}$$

However, the last two terms of the second line give rise to an elliptic function in η whose residue at $\eta=0$ vanishes, see (F.11). Hence, only the first two terms in the parenthesis in the second line of (F.15) contribute to $\mathcal{V}(1,\cdots,n)|_{h=1}$. By inserting their Laurent expansion

$$g^{(1)}(\eta) + \frac{\pi\eta}{\operatorname{Im}\tau} = \frac{1}{\eta} - \widehat{\mathrm{G}}_2\,\eta - \sum_{k=4}^{\infty} \mathrm{G}_k\,\eta^{k-1} \tag{F.16}$$

into (F.15), we arrive at the generating series of the right side of (F.6),

$$\left(g^{(1)}(\eta) + \frac{\pi\eta}{\operatorname{Im}\tau} \right) \Omega(z_{12},\eta)\Omega(z_{23},\eta)\cdots\Omega(z_{n1},\eta) \\ = \left(\frac{1}{\eta} - \widehat{\mathrm{G}}_2\,\eta - \sum_{k=4}^{\infty} \mathrm{G}_k\,\eta^{k-1} \right) \frac{1}{\eta^n} \left(1 + \sum_{r=1}^{\infty} \eta^r V_r(1,\cdots,n) \right) \tag{F.17}$$

such that taking the residue of its simple pole at $\eta=0$ implies the first statement (F.6) of the lemma.

Similarly, after eliminating $\partial_{n-1}F(z_{n-1,n},\eta)$ from (F.5) through the first line of (F.14) and discarding the elliptic functions of η due to $\partial_\eta F(z_{n-1,n},\eta)$ and $g^{(1)}(z)F(z_{n-1,n},\eta)$, the residue in (F.5) can be rewritten as

$$\mathcal{W}(1,\cdots,n)\,\big|_{h=1} = \operatorname*{Res}_{\eta=0}\, g^{(1)}(\eta)F(z_{12},\eta)F(z_{23},\eta)\cdots F(z_{n1},\eta) \tag{F.18}$$

without the extra term $\frac{\pi\eta}{\operatorname{Im}\tau}$ which accompanied $g^{(1)}(\eta)$ in the doubly-periodic case (F.14). As a result, the Laurent expansion $g^{(1)}(\eta) = \frac{1}{\eta} - \mathrm{G}_2\eta - \sum_{k=4}^{\infty}\mathrm{G}_k\eta^{k-1}$ involves the holomorphic but quasi-modular Eisenstein series G_2 in the place of the modular but almost holomorphic $\widehat{\mathrm{G}}_2$ in (F.16). The generating series

$$g^{(1)}(\eta)F(z_{12},\eta)F(z_{23},\eta)\cdots F(z_{n1},\eta) \\ = \left(\frac{1}{\eta} - \mathrm{G}_2\,\eta - \sum_{k=4}^{\infty} \mathrm{G}_k\,\eta^{k-1} \right) \frac{1}{\eta^n} \left(1 + \sum_{r=1}^{\infty} \eta^r V_r(1,\cdots,n) \right) \tag{F.19}$$

is therefore identical to (F.17) up to $\mathrm{G}_2 \leftrightarrow \widehat{\mathrm{G}}_2$, and taking the residue at $\eta=0$ reproduces the second statement (F.7) of the lemma and thereby concludes its proof.

F.3 Proof of Theorem 8.3

The proof of Theorem 8.3 will be organized into multiple steps.

F.3.1 Matching elliptic functions of $z_1, \cdots, z_n$

The first step in proving Theorem 8.3 is to match the elliptic functions W_{n-r}, V_{n-r} of $z_1, \cdots, z_n$ in (8.32) with the $h = 1$ instance of the combinations $\mathcal{W}_{I_1\cdots I_r}(x; 1, \cdots, n; y)$ and $\mathcal{V}_{I_1\cdots I_r}(x; 1, \cdots, n; y)$ of Enriquez kernels and DHS kernels defined by the descents in sections 7.2 and 7.4. Similar to the elliptic functions (F.2) and (F.3) in the cyclic products, we will now derive the dictionary

$$\begin{aligned} \mathcal{V}_{I_1\cdots I_r}(x; 1, \cdots, n; y)\big|_{h=1} &= V_{n-r}(x; 1, \cdots, n; y) \\ \mathcal{W}_{I_1\cdots I_r}(x; 1, \cdots, n; y)\big|_{h=1} &= W_{n-r}(x; 1, \cdots, n; y) \end{aligned} \tag{F.20}$$

valid for $r = 0, 1, \cdots, n$.

For the modular case in the first line of (F.20), we observe that both sides are meromorphic in the internal points and by (7.23) have the same simple poles in adjacent points of the chain products with residues ± 1. Moreover, both sides vanish upon integrating $z_1, \cdots, z_n$ over the genus one surface since $\int_\Sigma d^2 z\, f^{(k)}(z-x) = 0$ for any $k \geq 1$ and each term in the second line of (8.32) at $r \leq n$ integrates to zero on these grounds.

The second line of (F.20) again follows from the identical Fay identities among the Kronecker-Eisenstein kernels $g^{(s)}(z_{ij})$ and $f^{(s)}(z_{ij})$: in the first place, the genus one expressions for $\mathcal{V}_{I_1\cdots I_r}(x; 1, \cdots, n; y)$ and $\mathcal{W}_{I_1\cdots I_r}(x; 1, \cdots, n; y)$ that follow from the modular and merormorphic descents of Theorems 7.2 and 7.4 via (8.4) do not line up with V_{n-r} and W_{n-r} in (8.32). At fixed n, the first line of (F.20) which was established on general grounds in the previous paragraph can be explicitly verified via repeated use of the Fay identities among the $f^{(s)}(z_{ij})$. The meromorphic counterparts $\mathcal{W}_{I_1\cdots I_r}(x; 1, \cdots, n; y)$ produced by the descent are obtained by converting the DHS kernels of $\mathcal{V}_{I_1\cdots I_r}(x; 1, \cdots, n; y)$ into Enriquez kernels, see (7.35), so the respective genus one instances are related by $f^{(s)}(z_{ij}) \leftrightarrow g^{(s)}(z_{ij})$. The Fay identities among $f^{(s)}(z_{ij})$ that produce the expressions (8.32) for V_{n-r} from the outcome of the modular descent apply in identical form to the $g^{(s)}(z_{ij})$ in the outcome of the meromorphic descent. Hence, the $\mathcal{W}_{I_1\cdots I_r}(x; 1, \cdots, n; y)$ at genus one admit alternative representations obtained from substituting $f^{(s)}(z_{ij}) \to g^{(s)}(z_{ij})$ in any expression for $\mathcal{V}_{I_1\cdots I_r}(x; 1, \cdots, n; y)|_{h=1}$. Applying this substitution $f^{(s)}(z_{ij}) \to g^{(s)}(z_{ij})$ to the expressions for V_{n-r} in the first line of (8.32) casts $\mathcal{W}_{I_1\cdots I_r}(x; 1, \cdots, n; y)|_{h=1}$ into the form of W_{n-r} in the second line of (8.32). This concludes the derivation of the second line in (F.20) from the first line.

Note that the reasoning of the previous paragraph relies on the absence of differentiated Enriquez kernels and DHS kernels in the $\mathcal{W}_{I_1\cdots I_r}(x; 1, \cdots, n; y)$ and $\mathcal{V}_{I_1\cdots I_r}(x; 1, \cdots, n; y)$

produced by the descents for linear chain products. As a consequence, we do not encounter any Kronecker-Eisenstein derivatives $\partial_{z_i} g^{(s)}(z_{ij})$ or $\partial_{z_i} f^{(s)}(z_{ij})$ in the genus one limit, and the Fay identities used to connect the two lines of (F.20) always involve three different points. In particular, we do not encounter the different coefficients G_2 vs. $\widehat{\mathrm{G}}_2$ in the η expansion of (F.14) when explicitly verifying (F.20) at fixed multiplicity.

F.3.2 Deriving the meromorphic decomposition at genus one

We shall next prove the first line of (8.31) using the Kronecker-Eisenstein representation of linear chain products

$$\begin{aligned} L_\delta(x;1,\cdots,n;y) &= S_\delta(x-z_1)S_\delta(z_{12})\cdots S_\delta(z_{n-1,n})S_\delta(z_n-y) \\ &= e^{2\pi i u_\delta(x-y)}F(x-z_1,\omega_\delta)F(z_{12},\omega_\delta)\cdots F(z_{n-1,n},\omega_\delta)F(z_n-y,\omega_\delta) \end{aligned} \tag{F.21}$$

The second line follows from the expression (8.10) for the genus one Szegö kernel and will be simplified by means of the auxiliary function

$$P_\delta(\eta|x;1,\cdots,n;y) = F(x-y,\omega_\delta-\eta)F(x-z_1,\eta)F(z_{12},\eta)\cdots F(z_{n-1,n},\eta)F(z_n-y,\eta) \tag{F.22}$$

As a meromorphic function of η at fixed $x,y,z_1,\cdots,z_n$, the right side of (F.22) is doubly periodic (the individual phases of $F(z,\eta+\tau)=e^{-2\pi iz}F(z,\eta)$ cancel from the product) and only has poles at the two points $\eta=0$ and $\eta=\omega_\delta$ in a fundamental domain of the torus. Hence, (F.22) as an elliptic function of η has vanishing total residue,

$$\operatorname*{Res}_{\eta=0} P_\delta(\eta|x;1,\cdots,n;y) + \operatorname*{Res}_{\eta=\omega_\delta} P_\delta(\eta|x;1,\cdots,n;y) = 0 \tag{F.23}$$

By the Laurent expansion $F(z,\eta)=\frac{1}{\eta}+\mathcal{O}(\eta^0)$, the second residue in (F.23) is given by

$$\begin{aligned} \operatorname*{Res}_{\eta=\omega_\delta} P_\delta(\eta|x;1,\cdots,n;y) &= -F(x-z_1,\omega_\delta)F(z_{12},\omega_\delta)\cdots F(z_{n-1,n},\omega_\delta)F(z_n-y,\omega_\delta) \\ &= -e^{2\pi i u_\delta(y-x)}L_\delta(x;1,\cdots,n;y) \end{aligned} \tag{F.24}$$

where the linear chain product $L_\delta(x;1,\cdots,n;y)$ has been identified using the second line of (F.21). The residue of the auxiliary function (F.22) at $\eta=0$ in turn follows from combining the Taylor expansion of $F(x-y,\omega_\delta-\eta)=F(x-y,\omega_\delta)-\eta\big(\partial_\eta F(x-y,\eta)|_{\eta=\omega_\delta}\big)+\mathcal{O}(\eta^2)$ around $\eta=\omega_\delta$ with the Laurent expansion

$$F(x-z_1,\eta)F(z_{12},\eta)\cdots F(z_{n-1,n},\eta)F(z_n-y,\eta) = \frac{1}{\eta^{n+1}}\bigg\{1+\sum_{r=1}^{\infty}\eta^r W_r(x;1,2,\cdots,n;y)\bigg\} \tag{F.25}$$

which generates the elliptic functions $W_r(x;1,\cdots,n;y)$ of the internal points $z_1,\cdots,z_n$ in the first line of (8.32) and leads to the representation

$$\operatorname*{Res}_{\eta=0} P_\delta(\eta|x;1,\cdots,n;y) = \sum_{r=0}^{n} \frac{(-1)^r}{r!}\, \partial_\eta^r F(x-y,\eta)\,\big|_{\eta=\omega_\delta} W_{n-r}(x;1,\cdots,n;y) \tag{F.26}$$

of the first residue in (F.23). By equating the right side of (F.26) with minus the expression (F.24) for the second residue of $P_\delta(\eta|x;1,\cdots,n;y)$, we arrive at the following equivalent of the meromorphic decomposition in the first lines of (8.31) to (8.33):

$$\begin{aligned} L_\delta(x;1,\cdots,n;y) &= -e^{2\pi i u_\delta(x-y)} \operatorname*{Res}_{\eta=\omega_\delta} P_\delta(\eta|x;1,\cdots,n;y) \\ &= e^{2\pi i u_\delta(x-y)} \sum_{r=0}^{n} \frac{(-1)^r}{r!}\, \partial_\eta^r F(x-y,\eta)\,\big|_{\eta=\omega_\delta} W_{n-r}(x;1,\cdots,n;y) \end{aligned} \tag{F.27}$$

In particular, the $W_{n-r}(x;1,\cdots,n;y)$ were shown in section F.3.1 to line up with the $h=1$ instance of the individual $\mathcal{W}_{I_1\cdots I_r}(x;1,\cdots,n;y)$ of the meromorphic decomposition (7.32). By the linear independence of $W_s(x;1,\cdots,n;y)$ at different values of $s=0,1,\cdots,n$, this implies that the expressions for $M_\delta^{I_1\cdots I_r}(x,y)|_{h=1}$ in the first line of (8.33) can indeed be read off by comparing (F.27) with (8.31).

Note that, by the ϑ-function representation (8.1) of the Kronecker-Eisenstein series, all instances of $M_\delta^{I_1\cdots I_r}(x,y)|_{h=1}$ in (8.33) can be represented via $\vartheta_1(x-y+\omega_\delta)$, $\vartheta_1(\omega_\delta)$ and their derivatives in the first argument. The rank $r=1$ case can be further simplified to $M_\delta^I(x,y)|_{h=1} = -S_\delta(x-y)\frac{\partial_x \vartheta_\delta(x-y)}{\vartheta_\delta(x-y)}$, consistently with (7.6) at genus $h=1$.

F.3.3 Matching with the modular decomposition

The meromorphic decomposition (F.27) of linear chain products at genus one can be reformulated in terms of the doubly-periodic generating series $\Omega(z,\eta)$ in (8.1). The dependence on the internal points then occurs through the coefficients V_r of

$$\Omega(x-z_1,\eta)\Omega(z_{12},\eta)\cdots\Omega(z_{n-1,n},\eta)\Omega(z_n-y,\eta) = \frac{1}{\eta^{n+1}}\left\{1+\sum_{r=1}^{\infty}\eta^r V_r(x;1,2,\cdots,n;y)\right\} \tag{F.28}$$

in the second line of (8.32). Comparing with the meromorphic generating function in (F.25) exposes that

$$W_r(x;1,\cdots,n;y) = \sum_{\ell=0}^{r} \frac{(-2\pi i u)^\ell}{\ell!}\, V_{r-\ell}(x;1,\cdots,n;y) \tag{F.29}$$

with the co-moving coordinates $u, v \in \mathbb{R}/\mathbb{Z}$ defined by $x - y = z = u\tau + v$. In order to deduce the second line of (8.31) from the first line (demonstrated in the previous section), it therefore remains to show that the spinors $L_\delta^{I_1\cdots I_r}(x,y)|_{h=1}$ in the modular decomposition (7.19) of linear chains are related by

$$M_\delta^{I_1\cdots I_r}(x,y)\,\big|_{h=1} = \sum_{\ell=0}^{r} \frac{(-2\pi i u)^{r-\ell}}{(r-\ell)!}\, L_\delta^{I_1\cdots I_\ell}(x,y)\,\big|_{h=1} \tag{F.30}$$

In the first place, their representation (7.11) as convolutions together with the Fourier expansion (8.13) of the Szegö kernel implies the alternative representation

$$L_\delta^{I_1\cdots I_r}(x,y)\,\big|_{h=1} = e^{2\pi i(vu_\delta - uv_\delta)} \sum_{m,n\in\mathbb{Z}} \frac{e^{2\pi i(mv-nu)}}{(m\tau+n+\omega_\delta)^{r+1}} \tag{F.31}$$

Consistency with the expression in (F.30) and $M_\delta^{I_1\cdots I_r}(x,y)|_{h=1}$ given by the first line of (8.33) can be seen from

$$\begin{aligned} \partial_\eta^r F(z,\eta) &= \partial_\eta^r e^{-2\pi i\eta u} \sum_{m,n\in\mathbb{Z}} \frac{e^{2\pi i(mv-nu)}}{(m\tau+n+\eta)} \\ &= e^{-2\pi i\eta u} \sum_{\ell=0}^{r} \binom{r}{\ell} (-2\pi i u)^{r-\ell} \sum_{m,n\in\mathbb{Z}} \frac{(-1)^\ell\, \ell!\, e^{2\pi i(mv-nu)}}{(m\tau+n+\eta)^{\ell+1}} \\ &= (-1)^r\, e^{-2\pi i\eta u} \sum_{\ell=0}^{r} \frac{r!}{(r-\ell)!} (2\pi i u)^{r-\ell} \sum_{m,n\in\mathbb{Z}} \frac{e^{2\pi i(mv-nu)}}{(m\tau+n+\eta)^{\ell+1}} \end{aligned} \tag{F.32}$$

where we have used the Fourier expansion (8.12) of $\Omega(z,\eta)$ in the first line. Setting $\eta = \omega_\delta$, inserting into the first line of (8.33) and identifying the Fourier expansion of $L_\delta^{I_1\cdots I_\ell}(x,y)|_{h=1}$ in the parenthesis of

$$M_\delta^{I_1\cdots I_r}(x,y)\,\big|_{h=1} = \sum_{\ell=0}^{r} \frac{(-2\pi i u)^{r-\ell}}{(r-\ell)!} \left(e^{2\pi i(vu_\delta - uv_\delta)} \sum_{m,n\in\mathbb{Z}} \frac{e^{2\pi i(mv-nu)}}{(m\tau+n+\omega_\delta)^{\ell+1}} \right) \tag{F.33}$$

then reproduces the form (F.30) of $M_\delta^{I_1\cdots I_r}(x,y)|_{h=1}$ mandated by (8.31). This concludes the proof of Theorem 8.3.

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