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Something's got to give
English proverb

BCOV on the Large Hilbert Space

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Abstract

We formulate the BCOV theory of deformations of complex structures as a pull-back to the super moduli space of the worldline of a spinning particle. In this approach the appearance of a non-local kinetic term in the target space action has the same origin as the mismatch of pictures in the Ramond sector of super string field theory and is resolved by the same type of auxiliary fields in shifted pictures. The BV-extension is manifest in this description. A compensator for the holomorphic 3-form can be included by resorting to a description in the large Hilbert space.

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1 Introduction

The idea of constructing a field theory that could give rise to string amplitudes dates back to Siegel [1]. For various string theories it is known how to associate a perturbative BV-action in the form of an L_∞ structure for off-shell strings [2], [3]. Likewise for the string B-model, whose target space field content are the deformations of complex structures, there is a well established history: [4, 5, 6], to cite just a few of the many important contributions. The field theory resulting from the seminal work of Bershadsky–Cecotti–Ooguri–Vafa [4] is a celebrated result and it has been named BCOV after the authors.

Moreover the complex modulus is related by mirror symmetry [7] to the Kähler modulus, which also admits a field theory capturing its deformations [8]. Coincidentally, a powerful framework to deal with mirror symmetry happens to be what is commonly referred to as topological string theory, that is a sigma model on maps(Σ, M) with a global $\mathcal{N} = (2, 2)$ supersymmetry on the source space [9], twisted to decouple local super reparametrizations of the worldsheet [10]. Depending on the twist of the supercharges (A or B), this sigma model is then sensitive either to the complex structure, or to the Kähler modulus for the target space. In particular, in the B-twist, (part of) the BRST cohomology is isomorphic to the space of equivalence classes of linear deformation of complex structures of a Calabi–Yau manifold. Linear deformations are parametrized by the Beltrami differentials as antiholomorphic 1-forms, with values in the holomorphic tangent bundle. While the original Kodaira–Spencer theory of deformations of complex structures (e.g. [11]) is well defined on any complex manifold, the more restrictive CY-structure is required to formulate a topological string theory with such a target space.

Regarding the off-shell formulation, the tricky business is usually in the identification of a bilinear form that builds up the kinetic term in the action functional in Lagrangian formalism. In BCOV, an extra complication comes from the fact that there is no degree-preserving pairing for holomorphic vector fields. Instead the minimal model to the cohomology (quasi-isomorphic L_∞ algebra) was recently worked out in [12].

Combining these two approaches (on-shell vs off-shell) suggests that the Kodaira–Spencer theory of complex structure deformations could be a natural playground to explore string field theory in this setting and this was indeed discussed in the original BCOV description of Kodaira–Spencer gravity [4]. On the other hand, one could reverse the question and ask if string field theory could help us sharpen our understanding of BCOV and, in particular, the complication with the presence of a non-local kinetic term in the aforementioned reference. In this note we point out a close relation with the problem of pictures in the Ramond sector of string theory. We will therefore mimic the solution from string field theory by adding auxiliary fields of picture shifted by -1.

In fact, as we will point out, a worldline (rather than a worldsheet) sigma-model is sufficient to construct such an action[4]. Indeed, since in the B-model Maps(Σ, M) localize on constant maps in the target space M, one should also be able to capture the field theory from the first quantized worldline. This leads to some technical simplifications for the super ghost sector which has a different representation in worldsheet conformal field theory. Below, in section 3, we will construct a BV-action for KS-gravity theory starting from the spinning worldline with $\mathcal{N} = (2; 2)$ supersymmetry[5] on the source space, where, in contrast to the standard construction, one of the supersymmetries is gauged. This results in a picture changing operator that is equivalent to the divergence operator on polyvector fields whose inverse enters in the kinetic term of BCOV. Then, introducing an auxiliary field of picture shifted by -1, this results in BV-theory for complex structure deformation with auxiliary fields and a local kinetic term where, however, only the auxiliary fields enjoy a non-linear gauge redundancy. Upon elimination of the auxiliary fields this reproduces the BV-extension of the Kontsevich–Barannikov action [5]. The construction is fairly closely related to that in the Ramond sector of string field theory although the gauge sector looks different. An additional feature, common in string field theory, is that the existence of such an action functional implies a natural pairing of odd degree together with the complete BV spectrum of fields and anti-fields.

There exists an extension of BCOV with the inclusion of an extra scalar field (function) g that plays the role of a compensator to ensure the holomorphicity of the Calabi–Yau 3-form [6]. This function is not accounted for in the discussion just described. We will then consider an alternative formulation in section 4, which is reminiscent of the "large" Hilbert space in string theory [13, 14], where the superghost sector is represented by a Laurent series rather than a polynomial (or δ-function representations) with a novel degenerate inner product on the super ghost sector. This allows us to include the compensator g in the multiplet (with an even differential!) at the price of redefining the relation between fields and antifields. Concretely, the inner product entering in the definition of the kinetic terms differs from the symplectic form that defines the pairing of fields with anti-fields. The result complements the BV-formulations of [6] with a local kinetic term. However, as in [4, 6] the BV-bracket involves a constrained variation of the anti fields.

Another feature of the worldline, in contrast to the string, is that it can be quantized on any complex manifold. So one may wonder if we can formulate a target space field theory on a generic Kähler manifold that is not necessarily Calabi–Yau. In section 5 we will argue that this can indeed be done, however in a different formulation which is often referred to as a "theory of background fields" (e.g. [15]). Here, the integrability conditions for complex structure deformations are implied by the nilpotency of the BRST differential. This is then a background independent formulation of BCOV. However, the operator state correspondence between deformation of the BRST differential and perturbative states in BCOV still requires a holomorphic 3-form defined at least locally.

Two appendices are attached to this manuscript. In the first one, some fundamentals on the deformation theory of complex structures are recalled, while appendix B discusses, for the curious reader, a generalization of the worldline model of section 3.2.

2 Set-up

Kodaira–Spencer or BCOV is the field theory on the target space of closed topological strings (for the open string case in the A-model, the target space field theory is Chern–Simons theory instead). The worldsheet theory is known as B-model, see [10] and [16]. However, since Maps(Σ, M) localize on points in the target manifold [17], it is feasible to directly construct a particle model with superdiffeomorphism invariance (though strictly speaking invariance under diffeomorphisms is trivialised/removed) on the $\mathcal{N} = (2; 2)$ superline and still obtain the very same field theory [18]. This is the perspective that we adopt in this article, and we illustrate several variants of it in section 3. For the moment being, we would like to give a non-exhaustive review on Kodaira–Spencer theory (while Appendix A refreshes the reader on the topic of complex structure deformations and at the same time collects useful formulas and identities for multivector fields).

2.1 Review

We begin with a brief review of BCOV theory [4] and its extension in [5] and [19], [6]. The objective is an action functional whose linearized field equations modulo gauge symmetries capture

[4]Note that we will not address mirror symmetry here.

[5]We use the semicolon in order to epmphasize that these are not left- and right moving SUSY's

the cohomology

$$H^{\bullet}_{\bar\partial}(\ker \mathrm{div} \subset PV^{\bullet,\bullet}),$$

where the differential operators $\bar\partial$ and div, act on $PV^{\bullet,\bullet} := \Omega^{(0,\bullet)}(M, \Lambda^{\bullet} TM_{(1,0)})$ (see Appendix A). The interactions arise from a 2-product given by the Schouten–Nijenhuis bracket

$$[-,-]_{SN} : PV^{i,j} \wedge PV^{k,m} \to PV^{i+k-1,j+m}.$$

Tian [20] showed that the Schouten-Nijenhuis bracket is a derived bracket with the divergence operator (A.4). Therefore, if $\alpha, \beta \in \ker \mathrm{div} \subset PV^{\bullet,\bullet}$, this boils down to

$$[\alpha, \beta] = \mathrm{div}(\alpha \wedge \beta).$$

In addition we require a trace on polyvectors defined by

$$\int_{CY^3} \alpha := \int_{CY^3} \Upsilon \wedge \iota_\alpha \Upsilon, \quad \alpha \in PV^{3,3}, \quad \Upsilon \in \Omega^{3,0}_{\mathrm{hol}}(CY^3),$$

with the properties (if the CY manifold has no boundary): $\int \alpha \bar\partial \beta = -(-1)^{|\alpha|} \int \bar\partial \alpha \wedge \beta$, and $\int \alpha \wedge \mathrm{div}\beta = -(-1)^{|\alpha|} \int \mathrm{div}\alpha \wedge \beta$. The latter is due to the linearity and graded symmetry of the inner derivation $\iota_{\alpha\wedge\beta} = \iota_\alpha \iota_\beta = \pm \iota_\beta \iota_\alpha$, and the defining property that relates div to ∂ (see (A.3) in appendix A).

Consider now $\mu \in \ker \mathrm{div} \subset PV^{1,1}$ a Beltrami differential and $\hat A \in \mathcal{H}$ its harmonic component. Kodaira–Spencer theory, as described in the seminal work [4], is then described by the action functional

$$S_{BCOV}[\mu] = \frac{1}{2} \int_{CY^3} \mu \wedge \frac{1}{\mathrm{div}} \bar\partial \mu + \frac{1}{3} (\mu + \hat A)^{\wedge^3}. \tag{2.1}$$

Because of the $\partial\bar\partial$-lemma, $\iota_{\bar\partial\mu} \Upsilon = \partial \iota_{\bar\partial v} \Upsilon = \iota_{\mathrm{div}\bar\partial v} \Upsilon$. So

$$\frac{1}{\mathrm{div}} \bar\partial \mu = \bar\partial v + \mathrm{div}\rho + z \in \mathrm{im}\, \bar\partial \oplus \mathrm{im}\, \mathrm{div} \oplus \mathcal{H}, \tag{2.2}$$

which implies that the action is well-defined despite its manifestly non-local kinetic term. Another feature to stress is that BCOV requires a holomorphic volume form so that Kähler geometry is not sufficient (see, however, section 5 for considerations on general Kähler geometries). S_{BCOV} has gauge symmetries given by

$$\delta\mu = \bar\partial \epsilon + [\epsilon, \mu]_{SN}, \quad \epsilon \in \ker \mathrm{div} \subset PV^{1,0},$$

which can be verified using the derivation property of $\bar\partial$ and div and Jacobi identity for $[-,-]_{SN}$. In [4], the BV formulation of the theory was also presented. We comment only on the classical part of it —the antifields and the odd BV bracket. One way to achieve a BV formulation on a CY 3-fold is by considering $PV^{0\le i\le 3, 0\le j\le 3}$ and determine that

$$\begin{cases} \oplus_{i+k\le 2} PV^{i,k} & \text{are fields,} \\ \oplus_{m+n>2} PV^{m,n} \ \cap m \neq 3 & \text{are antifields.} \end{cases} \tag{2.3}$$

Note that this assignment returns precisely an even dimensional manifold (cotangent manifold to the space of fields). Then the BV bracket on functionals F, L (whose arguments are elements of $PV^{3,3}$ and appear on the RHS of the expression below denoted with the same symbol) is

$$\{F, L\} = \sum_{\Phi} \int_{CY^3} \Upsilon \wedge \Big(\langle\langle \mathrm{div} \frac{\delta_R F}{\delta \Phi}, \Upsilon \rangle, \frac{\delta_L L}{\delta \Phi^*} \rangle - \langle\langle \mathrm{div} \frac{\delta_R F}{\delta \Phi^*}, \Upsilon \rangle, \frac{\delta_L L}{\delta \Phi} \rangle \Big). \tag{2.4}$$

Here pointed brackets refer to the canonical contraction of polyvectors with complex forms.

Let us now take the Beltrami differential $\mu = \mathrm{div} A \in \mathrm{im}\, \mathrm{div} \subset PV^{1,1}$. Then, Kodaira–Spencer theory as found in Barannikov–Kontsevich [5] is given by

$$S_{BCOV}[A] = \frac{1}{2} \int \bar\partial A \wedge \mathrm{div} A + \frac{1}{3} (\mathrm{div} A + \hat A)^{\wedge^3}. \tag{2.5}$$

Both of the functionals for the classical fields (2.5) and the BV functional (obtained by considering the set of polyvectors in (2.3)) capture only deformations of complex structures. Instead, there is no room for a function that could play the role of the scalar compensator required to ensure that the new volume form is again holomorphic, see eq. (A.6).

An observation due to Costello and Li [6] fixes this gap, at least on-shell and in an elegant, unified framework. The clue is to consider a parameter u of ghost degree 2, and to shift the polyvectors in degree 2. This means we should look at the module $\mathbb{C}[[u]] \otimes PV^{\bullet,\bullet}[2]$ where the total ghost number of a generic element $u^n \omega^{(i,j)}$ is:

$$\mathrm{gh}(u^n \omega^{(i,j)}) = 2 - (i + j + 2n).$$

Therefore it is possible to combine the differentials into an equivariant operator of ghost degree -1

$$\bar{\partial} + u\mathrm{div} =: Q,$$

for which, at order u^0, (2.4) becomes the derived bracket of Q with exterior multiplication of polyvectors. Then the following Maurer-Cartan equation encompasses both the equation for the deformation of complex structures and that for the preservation of the holomorphic volume form

$$Qa + \frac{1}{2}[a, a] = 0, \quad \mathrm{gh}(a) = 0. \tag{2.6}$$

Indeed, when a has ghost degree zero and $a = \mu + ug$, the equation splits into

$$\bar{\partial}\mu + \frac{1}{2}[\mu, \mu] = 0, \tag{2.7}$$

$$u(\bar{\partial} g + \mathrm{div}\mu + [\mu, g]) = 0. \tag{2.8}$$

The form (2.6) is particularly nice because the gauge symmetries can be found right away in ghost degree $+1$. However it was not possible to come up with an action functional that has the above MC equations as e.o.m.'s. Instead Costello and Li suggested a cubic interaction term S_{int} that satisfies the BV master equation with Q and the bracket given in (2.4)

$$QS_{int} + \frac{1}{2}\{S_{int}, S_{int}\} = 0.$$

The expression for S_{int} is ($\alpha_i \in PV^{\bullet,\bullet}$)

$$S_{int} = \sum_{n\geq 3} \frac{1}{n!} \int_M^{PV} \langle u^{k_1}\alpha_1 \otimes \cdots \otimes u^{k_n}\alpha_n \rangle_0, \quad \langle u^{k_1}\alpha_1 \otimes \cdots \otimes u^{k_n}\alpha_n \rangle_0 := \binom{n-3}{k_1 \cdots k_n} \alpha_1 \wedge \cdots \wedge \alpha_n \,.$$

Thus the cubic term agrees with BCOV's cubic term. Costello–Li also pointed out that BCOV theory can be formulated in any Calabi–Yau n-fold. Unless explicitly mentioned, for simplicity we will always specialize our description to $n = 3$ in the remainder of this article.

3 The Model

Mimicking the realization of Kodaira–Spencer theory as a target space theory for the topological string, the worldline description is based on the isomorphism

$$\mathcal{T} : C^\infty(U)[\bar{\theta}^1, \ldots, \bar{\theta}^n | \bar{\psi}^1, \ldots, \bar{\psi}^n] \xrightarrow{\sim} PV^{\bullet,\bullet}(U)\,, \quad U \subset M, \tag{3.1}$$

with the former represented as a Fock module of the operator formulation of the spinning relativistic point particle with global $\mathcal{N} = (2; 2)$ supersymmetry, described by the canonical pairs

$$\{\theta, \bar{\theta}\} = \{\psi, \bar{\psi}\} = [x, p] = 1\,. \tag{3.2}$$

where x is a local coordinate on $U \subset M$.[6] On this module, the supercharges $\bar{q} = \bar{\psi}\bar{p}$ and $q = \theta p$ represent the Dolbeault differential and divergence on $PV^{\bullet,\bullet}$ with

$$\{q, \bar{q}\} = 0 = \{q, q\} = \{\bar{q}, \bar{q}\}, \qquad q \sim \mathrm{div}, \quad \bar{q} \sim \bar{\partial}. \tag{3.3}$$

Following the procedure of the topological string one may then declare $\bar{q}$ to be the linear BV-differential (implementing holomorphic reparametrizations in the target space) and construct an appropriate complex for $\bar{q}$. The unconventional feature of this construction is the well-known non-local kinetic term in the action[7],

$$S = (\mu, \frac{\bar{q}}{\partial}\mu) + (\mu, \mu \cdot \mu)\,, \tag{3.4}$$

[6] This realization is quasi-isomorphic to the traditional description of the B-model.

[7] Recalling that μ takes values in TM the presence of $\frac{1}{\partial}$ can be understood form the fact that in 3 dimensions the natural paring is between 1- and 2-vector fields, rather than two 1-vector fields.

as described in [4]. This is in contrast with the general expectation [3] for any string field action where the action is of the form

$$S = (\psi, Q\psi) + (\psi, [\psi, \psi]) + \cdots , \tag{3.5}$$

where Q implements some linear gauge transformation in target space and $[\cdot, \cdot]$ is a bilinear map on the space of string fields. The origin of this discrepancy is easily traced back to the fact that the worldline/worldsheet model (3.3) is not a string theory because the worldsheet reparametrizations are not gauged. Therefore, the structure of the target space action is not implied by the BV-structure of the underlying super-moduli space of Riemann surfaces/worldliness. The key motivation of the present paper is to provide such a connection.

However, let us begin by presenting a way to "localize" the kinetic term in (3.4) with the help of auxiliary fields.

3.1 A local action with global SUSY

We choose the ghost degree of an element of $PV^{i,j}$, locally isomorphic to $C^\infty(U)[\bar\theta^1, \dots, \bar\theta^n|\bar\psi^1, \dots, \bar\psi^n]$, as

$$gh(v^{(i,j)}) = j - i \,. \tag{3.6}$$

Furthermore, both $\bar q \sim \bar\partial$ and $q \sim \text{div}$ increase this degree. We consider a Calabi–Yau ($n = 3$)-fold as our underlying manifold M and perform a decomposition in subspaces of definite ghost degree. Take

$$\tilde V \ni \tilde a = (\mu^{(1,0)} + \mu^{(2,1)} + \mu^{(3,2)}), \tag{3.7}$$

which forms a multiplet for the non-local and even-parity differential $d = \bar q q^{-1}$ entering in the kinetic term of (3.4). Upon acting with q on (3.7) this produces a second "string field"[8]

$$V \ni a = g^{(0,0)} + \mu^{(1,1)} + \mu^{(2,2)} \,, \tag{3.8}$$

which, in addition to the Beltrami differential, also contains the compensator $g^{(0,0)}$. Now we would like to write down a classical action functional S with a kinetic term given by $\bar q$. For this we first choose a pairing

$$(a, b) = \int_{CY^3} \mathrm{d}vol_{CY^3} \int \langle\!\langle ab, \Upsilon(\theta^{\wedge 3})\Upsilon(\psi^{\wedge 3})\rangle\!\rangle, \quad a, b \in V, \tilde V \,. \tag{3.9}$$

In the above, the double pointed brackets means "normal ordering", i.e. complete contraction of the $\bar\theta$'s and $\bar\psi$'s. This pairing has ghost number 0. Moreover, $q = \theta \cdot p$ and $\bar q = \bar\psi \cdot \bar p$ are self-adjoint w.r.t. the pairing. We then consider the bilinear term

$$S_1 = (\tilde a, \bar q a) \,. \tag{3.10}$$

To display this in component fields, we switch to the isomorphic representation in the polyvectors, denoted as $\langle -, - \rangle$:

$$\langle \mu, \nu \rangle := \int_{CY^3} \Upsilon \wedge \iota_{\mu\wedge\nu} \Upsilon = \int_{CY^3} \Upsilon \wedge \iota_\mu(\iota_\nu \Upsilon) \,.$$

Then S_1 becomes[9]

$$S_1 = \langle \mu^{(2,1)}, \bar\partial \mu^{(1,1)} \rangle + \langle \mu^{(3,2)}, \bar\partial g^{(0,0)} \rangle + \langle \mu^{(1,0)}, \bar\partial \mu^{(2,2)} \rangle \tag{3.11}$$

We then add the term

$$S_2 = \frac{1}{2}(\tilde a, \bar q q \tilde a) = \frac{1}{2}\langle \mu^{(2,1)}, \bar\partial \text{div} \mu^{(2,1)} \rangle + \langle \mu^{(2,1)}, \bar\partial \mu^{(1,1)} \rangle \,, \tag{3.12}$$

together with the cubic interaction

$$\frac{1}{3!}(a, a \cdot a) = \frac{1}{6}\langle \mu^{(1,1)}, \mu^{(1,1)} \wedge \mu^{(1,1)} \rangle + \langle \mu^{(2,2)}, \mu^{(1,1)} \wedge g^{(0,0)} \rangle \,. \tag{3.13}$$

Summing up (3.11), (3.12) and (3.13) yields Kodaira–Spencer theory. Since this will be extensively analyzed in a related context in section 4.2, we will keep the discussion concise here: Half of the

[8] Note that this does not form a multiplet for d but we won't need such a structure below.

[9] Due to the isomorphism in (B.3) we use the same symbol to denote the component polyvector fields.

E-L equations imply the divergence-less condition for the mutiplet a. Acting with the div operator on the other half and substituting the solution of the first half of the set yields

$$\bar{\partial} g^{(0,0)} + \mathrm{div}(\mu^{(1,1)} g^{(0,0)}) = 0, \tag{3.14}$$

$$\bar{\partial}\mu^{(1,1)} + \mathrm{div}(\frac{1}{2}\mu^{(1,1)}\mu^{(1,1)} + \mu^{(2,2)} g^{(0,0)}) = 0, \tag{3.15}$$

$$\bar{\partial}\mu^{(2,2)} + \mathrm{div}(\mu^{(2,2)}\mu^{(1,1)}) = 0\,. \tag{3.16}$$

These are integrability conditions for the Beltrami differential $\mu^{(1,1)}$, for the compensator $g^{(0,0)}$ and for a higher polyvector $\mu^{(2,2)}$.

3.2 A model with local SUSY

In order to obtain a string field theory understanding of the non-locality of the action in the last section we will now consider a worldline model with $\mathcal{N} = (2;2)$ symmetry where, however, one of the supersymmetries is gauged. A way to describe it is to begin with a supersymmetric version of a Baulieu–Singer type topological σ-model [21]. That is, we start with the topological action for parametrized holomorphic curves

$$I = \int \bar{\pi}(z) \cdot \mathrm{d}\bar{z}\,, \tag{3.17}$$

where $\bar{\pi}(z)\mathrm{d}\bar{z}$ is a closed one form on the curve. I has a local invariance $\delta\bar{z}(t) = \bar{\epsilon}(t)$ so there is a corresponding BRST-differential $\bar{q} = \bar{\psi}\cdot\bar{p}$, with $s\bar{z} = \bar{\psi}$, $s\bar{\psi} = 0$. If we then add the trivial pair $\psi, \bar{p}$, with $s\psi = \bar{p}$ together with the gauge fixing fermion

$$\Psi = \int \psi \cdot \mathrm{d}\bar{z}, \tag{3.18}$$

one sees that

$$I \sim I + s\Psi = I + \int \bar{p}\cdot\mathrm{d}\bar{z} + \psi\cdot\mathrm{d}\bar{\psi}\,. \tag{3.19}$$

To continue we set $\bar{\pi} \equiv 0$. We then extend this model to one with a *local* odd symmetry on the worldline. That is, we replace I by

$$s\Psi + \int p\cdot(\mathrm{d}z + \chi\cdot\theta) + \bar{\theta}\cdot\dot{\theta} + \beta\dot{\gamma}\,. \tag{3.20}$$

Here χ is a worldline gravitino, geometrically a super Beltrami differential on the worldline. The canonical pair $[\beta, \gamma] = 1$ represent the Faddeev-Popov ghosts arising form the gauge fixing of χ. We should note that the additional term in (3.20) is not topological. After gauge fixing, the BRST algebra for this system is summarized in the commuting differentials

$$\bar{q} \quad \text{and} \quad \gamma q = \gamma p\cdot\theta\,, \tag{3.21}$$

and the module is thus

$$C^\infty(M)[\bar{\theta}^1, \ldots, \bar{\theta}^n|\bar{\psi}^1, \ldots, \bar{\psi}^n] \otimes \mathbb{C}[[\gamma]]\,, \tag{3.22}$$

on which we define a pairing by

$$(a, b) = \int_{CY^3} \mathrm{d}vol_{CY^3} \int \mathrm{d}\gamma\, \langle\!\langle ab, \Upsilon(\theta^{\wedge 3})\Upsilon(\psi^{\wedge 3})\rangle\!\rangle\,. \tag{3.23}$$

The integral over γ means that our pairing has "picture 1". This means that alongside with polynomials in γ, we have to include a dual multiplet which is distributional (Dirac delta) in γ. Furthermore, the path-integral evaluation with the locally symmetric action (3.20) involves integration over an odd subbundle of super moduli space $\mathcal{M} = \mathcal{J}/D$ of χ's modulo globally defined odd reparamerizations (see e.g. [22, 23] for details). On distributions, this is realized as a nilpotent, even-parity picture changing operator $X = \delta(\beta)q$ with $\delta(\beta)\delta(\gamma) \sim 1$. This picture changing operator is generally not unique but this choice arises naturally in the worldline path integral [23] and furthermore happens to match precisely $\partial \sim \mathrm{div}$. So, the observation here is then that integration over the odd moduli space of the locally symmetric action (3.20) naturally provides the extra nilpotent operator entering in (3.4). In fact, for a worldline with just two punctures, relevant for the kinetic term, the moduli space is just a point [23]. Then γ and $\delta(\gamma)$ are the Faddeev–Popov ghosts for setting to zero the global odd transformations in (3.20).

A minimal set of fields[10] containing the Beltrami differential, is

$$a^{(0)} = \mu^a \bar{\theta}_a + \mu^b_a \bar{\psi}^a \bar{\theta}_b \equiv \mu^{(1,0)} + \mu^{(1,1)} \,, \tag{3.24}$$

where $\mu^{(1,0)}$ is a gauge symmetry for the Beltrami differential $\mu^{(1,1)}$ with the differential $\bar{q}$ of degree -1 for our definition of the ghost degree,

$$\mathrm{gh} := -\#\bar{\psi} + (\#\bar{\theta} + \#\gamma) \,. \tag{3.25}$$

It is clear that (3.24) in itself does not complete a multiplet for $\bar{q}$. However the remaining fields will be accounted for by antifields below, as is common in BV quantization.

We will see that it is possible to obtain the $\bar{q}$ cohomology from the field equations without restricting the fields to the image of the divergence operator, $\mathrm{im}\, q \sim \mathrm{im}\, \mathrm{div}$. Furthermore, our odd symplectic pairing (3.23) already determines the anti-field of $a^{(0)}$ as (gh $\delta(\gamma) = -1$)

$$a^{(0)*} = \mu^{*(2,2)} \delta(\gamma) + \mu^{*(2,3)} \delta(\gamma) \,. \tag{3.26}$$

Another option would be to have the Beltrami differential in a picture -1 multiplet, based on the $\delta(\gamma)$. In that case the interaction term would make up an integral form on $\mathcal{M}$ instead, e.g. [23].

Let us now explain how the problem of the non-local quadratic term in the BCOV action is naturally mapped to the problem of pictures in the Ramond sector of open super string theory [24]. The natural kinetic term of the BV-action

$$S_2 = (a^{(0)}, \bar{q} a^{(0)}) \tag{3.27}$$

is not admissible since the picture does not add up to 0, i.e. the integration over γ is not well defined[11]. This is precisely what happens in the Ramond sector of string theory. In that theory there is a solution to this problem by introducing an auxiliary field with picture shifted by -1 [25]. This then suggests a solution in our case as well. We add to $a^{(0)}$ a set of auxiliary fields in picture -1,

$$a^{(-1)} = \mu^{(2,0)} \delta(\gamma) + \mu^{(2,1)} \delta(\gamma) \,, \tag{3.28}$$

together with its anti-field,

$$a^{(-1)*} = \mu^{*(1,2)} + \mu^{*(1,3)} \,. \tag{3.29}$$

The BV-multiplets ((3.28) $\oplus$ (3.26)) and ((3.24) $\oplus$ (3.29)) are related to each other by the picture raising operator

$$X = \delta(\beta) q \,, \qquad \mathrm{gh} X = 0 \,, \tag{3.30}$$

which commutes with $\bar{q}$. We thus have two complexes related by the non-invertible cochain map X

$$\begin{array}{ccccccc} 1 & & 0 & & -1 & & -2 \\ \mu^{(1,0)} & \xrightarrow{\bar{q}} & \mu^{(1,1)} & \xrightarrow{\bar{q}} & \mu^{*(1,2)} & \xrightarrow{\bar{q}} & \mu^{*(1,3)} \\ X\uparrow & & X\uparrow & & X\uparrow & & X\uparrow \\ \delta(\gamma)\mu^{(2,0)} & \xrightarrow{\bar{q}} & \delta(\gamma)\mu^{(2,1)} & \xrightarrow{\bar{q}} & \delta(\gamma)\mu^{*(2,2)} & \xrightarrow{\bar{q}} & \delta(\gamma)\mu^{*(2,3)} \,, \end{array} \tag{3.31}$$

on which we have a well defined kinetic term

$$\begin{aligned} S_2 &= \frac{1}{2}(a^{(-1)}, \bar{q} X a^{(-1)}) + (a^{(-1)}, \bar{q} a^{(0)}) \\ &= \frac{1}{2}\langle \mu^{(2,1)}, \bar{\partial} \mathrm{div} \mu^{(2,1)} \rangle + \langle \mu^{(2,1)}, \bar{\partial} \mu^{(1,1)} \rangle \,, \end{aligned} \tag{3.32}$$

where, in the second line we used the isomorphic representation in the polyvectors as before. We thus recover the local kinetic term from the last subsection, however without the field g and

[10] We opt for coloring the fields so to track and assemble them more easily.

[11] If we were to insert a $\delta(\gamma)$ in the pairing (B.5), then (3.27) would be well defined and would instead result in the appropriate kinetic term for Chern–Simons theory.

without the coupling to $\mu^{(1,0)}$. The latter will naturally appear paired with anti-fields in the BV-extension given below. The fact that the compensator g is absent, while not an inconsistency, is a shortcoming that will be addressed in the large Hilbert space description below. On the other hand, the div operator appears naturally in this description in the form of picture changing as a consequence of working on super moduli space. An alternative way to match the picture without introducing auxiliary fields would be to insert the inverse X^{-1} of the picture raising operator into (3.27). However, the inverse is not well defined since X has a non-vanishing (co)-kernel. This is just the same problem of the $\frac{1}{\partial}$ operator in the BCOV-kinetic term, which we propose to address in this paper.

Furthermore, the BV-extension is already at hand —as usually happens in string theory—, by simply including the antifield with respect to the natural pairing on super moduli space $\mathcal{M}$. Indeed, the BV-extension of the quadratic action is simply

$$\begin{aligned} S_{2,BV} &= \frac{1}{2}(a^{(-1)}, \bar{q}Xa^{(-1)}) + (a^{(-1)}, \bar{q}a^{(0)}) + (a^{(0)*}, \bar{q}a^{(0)}) + (a^{(-1)*}, \bar{q}a^{(-1)}) \\ &= \frac{1}{2}\langle \mu^{(2,1)}, \bar{\partial}\mathrm{div}\mu^{(2,1)}\rangle + \langle \mu^{(2,1)}, \bar{\partial}\mu^{(1,1)}\rangle + \langle \mu^{*(2,2)}, \bar{\partial}\mu^{(1,0)}\rangle + \langle \mu^{*(1,2)}, \bar{\partial}\mu^{(2,0)}\rangle\,. \end{aligned} \tag{3.33}$$

This makes the role of $\mu^{(1,0)}$ and $\mu^{(2,0)}$ as gauge parameters for $\mu^{(1,1)}$ and $\mu^{(2,1)}$ manifest. Geometrically, $\mu^{(1,0)}$ represents holomorphic reparametrizations.

Before moving on to the next part, we should perhaps emphasize that the algebra generated by q does not close into a translation and is thus not a supersymmetry but rather a nilpotent symmetry on $\mathbb{R}^{0|1}$. We can promote this to a $\mathcal{N}=2$ SUSY by adding the terms $\bar{p}\cdot\bar{\chi}(\bar{\theta}) - \frac{e}{2}\bar{p}\cdot p$ to the worldline action. We will comment on this point in appendix B. In that case, one needs to go to a reduced module V_{red} and refer to a different presymplectic pairing.

3.3 Interacting theory and its BV formulation

Let us now turn to the cubic interaction. For the 3-punctured line the moduli space $\mathcal{M}\simeq\mathbb{R}^{0|1}$ is (0|1)-dimensional [23]. To eliminate any dependence of this contribution on $\mathcal{M}$ one inserts the Poincaré dual $Y=\eta\delta(\mathrm{d}\eta)$ in the path integral, thus eliminating all dependence on the odd coordinates. Then the 3-point correlator representing the cubic term should produce a function, rather than a pseudoform on $\mathcal{M}$. This means that the insertions must have picture zero. In addition, we insert a $\delta(\gamma)$ (anywhere on the line) to provide the Jacobian for isolating the global odd transformation[12], as described below Eqn. (3.23). Adding this to $S_{2,BV}$ we end up with

$$\begin{aligned} S_{BV} =& \frac{1}{2}(a^{(-1)}, \bar{q}Xa^{(-1)}) + (a^{(-1)}, \bar{q}a^{(0)}) + \frac{1}{3!}(a^{(0)}, \delta(\gamma)a^{(0)}\wedge a^{(0)}) + (a^{(-1)*}, \bar{q}a^{(-1)}) \\ =& \frac{1}{2}\langle \mu^{(2,1)}, \bar{\partial}\mathrm{div}\mu^{(2,1)}\rangle + \langle \mu^{(2,1)}, \bar{\partial}\mu^{(1,1)}\rangle + \frac{1}{3!}\langle \mu^{(1,1)}, \mu^{(1,1)}\wedge\mu^{(1,1)}\rangle + \langle \mu^{*(1,2)}, \bar{\partial}\mu^{(2,0)}\rangle\,. \end{aligned} \tag{3.34}$$

Note that we have removed the term $(a^{(0)*}, \bar{q}a^{(0)}) = \langle \mu^{*(2,2)}, \bar{\partial}\mu^{(1,0)}\rangle$ compared to (3.33), because it does not relate to a symmetry of the cubic term. Equation (3.34) is then the action for BCOV with a local kinetic term. Indeed the E-L equations for the fields are

$$\bar{q}(q\mu^{(2,1)} + \mu^{(1,1)}) = 0\,, \tag{3.35}$$

$$\bar{q}\mu^{(2,1)} + \frac{1}{2}\mu^{(1,1)}\wedge\mu^{(1,1)} = 0\,, \tag{3.36}$$

If $\mu^{(1,1)}$ has no projection on the cohomology of q, the first equation can be solved to express $\mu^{(1,1)} = -q\mu^{(2,1)}$, implying, in particular, that $\mu^{(1,1)}$ is divergence free. If, on the other hand, $\mu^{(1,1)}\in H_{\bar{\partial}}(PV)$ then it follows from the $\partial\bar{\partial}$-lemma that $\mu^{(1,1)}$ can be assumed to be divergence-free (there is a quasi-iso between $H_{\bar{\partial}}(PV)$ and $(\ker \mathrm{div}, \bar{\partial})$, see [5]). Upon left action by q on the second equation and by using the first equation, we get

$$\bar{q}\mu^{(1,1)} + \frac{1}{2}q(\mu^{(1,1)}\wedge\mu^{(1,1)}) = 0\,, \tag{3.37}$$

which is equivalent to the KS-equation for $\mu^{(1,1)}$ or, in other words to the integrability condition of the Beltrami differential, where Tian's lemma [20] is used (reviewed also in the Appendix (A.5)).

[12] Alternatively we could have one insertion in picture -1 and two insertion in picture 0. However, consulting (3.31) we can see that there is a non-vanishing contraction of that kind.

Focusing on polyvectors for ease,[13] let us follow what happens when, using (3.35), we express the *divergence-free* $\mu^{(1,1)}$ in terms of $\mu^{(2,1)}$ and the harmonic part h, taken to be compactly supported so that $\bar{\partial}h = 0$. Then

$$\begin{aligned} S_{BV}[\mu] = &-\frac{1}{2}\langle\mu^{(2,1)}, \bar{\partial}\mathrm{div}\mu^{(2,1)}\rangle - \frac{1}{3!}\langle\mathrm{div}\mu^{(2,1)} + h, (\mathrm{div}\mu^{(2,1)} + h) \wedge (\mathrm{div}\mu^{(2,1)} + h)\rangle \\ &+ \langle\mu^{*(1,2)}, \bar{\partial}\mu^{(2,0)} + \mathrm{div}\mu^{(2,0)} \wedge (\mathrm{div}\mu^{(2,1)} + h)\rangle + \frac{1}{2}\langle\mu^{*(1,3)}, \mathrm{div}\mu^{(2,0)} \wedge \mathrm{div}\mu^{(2,0)}\rangle \end{aligned} \tag{3.38}$$

provides the BV formulation of Barannikov–Kontsevich action for BCOV. If the antifields $\mu^{*(1,2)}$ and $\mu^{*(1,3)}$ are div-exact, the action functional is invariant under the BV-transformations

$$\begin{aligned} \delta\mu^{(2,1)} &= \bar{\partial}\mu^{(2,0)} + \mathrm{div}\mu^{(2,0)} \wedge (\mathrm{div}\mu^{(2,1)} + h), \\ \delta\mu^{*(1,2)} &= -\bar{\partial}\mathrm{div}\mu^{(2,1)} - \frac{1}{2}\mathrm{div}((\mathrm{div}\mu^{(2,1)} + h) \wedge (\mathrm{div}\mu^{(2,1)} + h)) + \mathrm{div}(\mu^{*(1,2)} \wedge \mathrm{div}\mu^{(2,0)}), \\ \delta\mu^{(2,0)} &= \frac{1}{2}\mathrm{div}\mu^{(2,0)} \wedge \mathrm{div}\mu^{(2,0)}, \\ \delta\mu^{*(1,3)} &= \bar{\partial}\mu^{*(1,2)} + \mathrm{div}(\mu^{*(1,2)} \wedge (\mathrm{div}\mu^{(2,1)} + h)) + \mathrm{div}(\mu^{*(1,3)} \wedge \mathrm{div}\mu^{(2,0)}), \end{aligned} \tag{3.39}$$

as can be checked using integration by parts, $\bar{\partial}^2 = 0$, the derivation property of $\bar{\partial}$ w.r.t. the alternating product $\wedge$, the relation $\int \mathrm{div}a \wedge b = \mathrm{bdry} - (-1)^{|a|}\int a \wedge \mathrm{div}b$ and that div yields the Schouten bracket on a product of *div-free polyvectors*, which satisfies Jacobi identity.

For the same reasons, (3.38) satisfies the classical BV-master equation. Until now, we have overlooked the degree of ghost field, antifield and antighost field but at this point it becomes important to assess it. Given the ghost degree of the "vacuum" $\delta(\gamma)$,

$$\mathrm{gh}(\mu^{(2,0)}) = 1, \quad \mathrm{gh}(\mu^{*(1,2)}) = -1, \quad \mathrm{gh}(\mu^{*(1,3)}) = -2. \tag{3.40}$$

As customary, the parity is the ghost degree mod 2. Then, letting $F, L \in \Lambda^3(TM_{(1,0)} \oplus T^*M_{(0,1)})$, our odd BV bracket induced by the pairing is:

$$\{F, L\} = \sum_\Phi \int_{CY^3} \Upsilon \wedge \left(\iota_{\frac{\delta_R L}{\delta\Phi}}\iota_{\frac{\delta_L F}{\delta\Phi^*}} - \iota_{\frac{\delta_R L}{\delta\Phi^*}}\iota_{\frac{\delta_L F}{\delta\Phi}}\right)\Upsilon = \sum_\Phi \left\langle\frac{\delta_R L}{\delta\Phi}, \frac{\delta_L F}{\delta\Phi^*}\right\rangle - \left\langle\frac{\delta_R L}{\delta\Phi^*}, \frac{\delta_L F}{\delta\Phi}\right\rangle. \tag{3.41}$$

Then

$$\begin{aligned} \{S_{BV}, S_{BV}\} = &-\left\langle\bar{\partial}\mathrm{div}\mu^{(2,1)} + \frac{1}{2}\mathrm{div}((\mathrm{div}\mu^{(2,1)} + h) \wedge (\mathrm{div}\mu^{(2,1)} + h)), \bar{\partial}\mu^{(2,0)} + \mathrm{div}\mu^{(2,0)} \wedge (\mathrm{div}\mu^{(2,1)} + h)\right\rangle \\ &+ \langle\mathrm{div}(\mu^{*(1,2)} \wedge \mathrm{div}\mu^{(2,0)}), \bar{\partial}\mu^{(2,0)} + \mathrm{div}\mu^{(2,0)} \wedge (\mathrm{div}\mu^{(2,1)} + h)\rangle \\ &+ \frac{1}{2}\langle\mathrm{div}(\mu^{*(1,3)} \wedge \mathrm{div}\mu^{(2,0)}), \mathrm{div}\mu^{(2,0)} \wedge \mathrm{div}\mu^{(2,0)}\rangle \\ &+ \frac{1}{2}\langle\bar{\partial}\mu^{*(1,2)} + \mathrm{div}((\mathrm{div}\mu^{(2,1)} + h) \wedge \mu^{*(1,2)}), \mathrm{div}\mu^{(2,0)} \wedge \mathrm{div}\mu^{(2,0)}\rangle \end{aligned} \tag{3.42}$$

is zero after the aforementioned algebraic massaging. Note that although S_{BV} is a functional, we are effectively taking derivatives of the arguments of the BV action and shoving them in.

4 Large Hilbert space

A shortcoming of the formulation in the last section is that g is not included in the BV-multiplet[14]. In this section we attempt to get around this by formulating the theory in the large Hilbert space for the γ-ghost, which is achieved by allowing for negative powers of γ in the multiplet instead of pictures. Expanding the setup to the large Hilbert space has proven fruitful in string field theory literature, see for instance [26]. In this formulation one loses the connection to super moduli space seen before. On the other hand, it allows to combine multiplets that were of different pictures into a single multiplet.

[13] Of course all our considerations also hold for graded functions of picture 0 and $-1$and canonical operators entering in the first line of (3.34).

[14] In fact it could be included but simply drops out in the action.

4.1 Long multiplet

For instance, we can combine $a^{(0)}$ and $a^{(-1)}$ in section 3.2 into one multiplet of fields as

$$a = \frac{1}{\gamma^2} g^{(3,0)} + \frac{1}{\gamma^2} g^{(3,1)} + \frac{1}{\gamma} \mu^{(2,0)} + \frac{1}{\gamma} \mu^{(2,1)} + \mu^{(1,0)} + \mu^{(1,1)} + \gamma g^{(0,0)} + \gamma g^{(0,1)} \,, \tag{4.1}$$

where $g^{(3,0)}$, $\mu^{(1,0)}$, $\mu^{(2,0)}$ and $g^{(0,0)}$ are ghosts. The differential $\bar{q} = \bar{\psi}^a \bar{p}_a$, of ghost number -1, is represented as before, while the "picture changing operator" is represented as

$$X = \gamma \theta^a p_a \,, \tag{4.2}$$

which has a closed action on (4.1) and is now odd as well but of ghost number 0. We note in passing that $g^{(\bullet,\bullet)}$ could have been included also in the small Hilbert space in Section 3.2. However, such a field is not generated by X as indicated in (3.31).

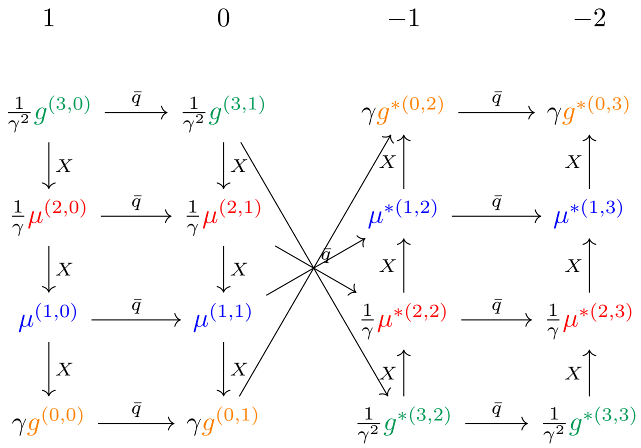
Since we do not have pictures (i.e. δ-function for the ghosts) we need to reconsider the definition of the symplectic form in the large Hilbert space. For this we will represent the delta function as a residue:

$$(a, b) = \frac{1}{2\pi i} \int_{CY^3} \mathrm{d}vol_{CY^3} \oint \mathrm{d}\gamma \, \langle\!\langle a, b \, \Upsilon(\psi^{\wedge 3}) \Upsilon(\theta^{\wedge 3}) \rangle\!\rangle. \tag{4.3}$$

Notice that, unlike in the previous section, this pairing is even and pairs (4.3) with

$$a^* = \frac{1}{\gamma^2} g^{*(3,2)} + \frac{1}{\gamma^2} g^{*(3,3)} + \frac{1}{\gamma} \mu^{*(2,2)} + \frac{1}{\gamma} \mu^{*(2,3)} + \mu^{*(1,2)} + \mu^{*(1,3)} + \gamma g^{*(0,2)} + \gamma g^{*(0,3)} \tag{4.4}$$

where now $\mu^{*(2,2)}$ is of even parity. We can display the complex built out of a and a^* schematically as



, (4.5)

where it is transparent which fields are gauge fields and all the roles are assigned accordingly to the prescription that the antighost degree of a is equal to $-\mathrm{gh}(a) - 1$. We have chosen a representation in which the fields and their respective duals are on the same line.

A free action for the multiplet a is easily found to be:

$$\frac{1}{2}(a, \bar{q} X a) + \frac{1}{2}(a, \bar{q} a) + (a^*, \bar{q} a) = \langle g^{(3,1)}, \bar{\partial} \mathrm{div} \mu^{(1,1)} \rangle + \langle g^{(3,1)}, \bar{\partial} g^{(0,1)} \rangle + \langle \mu^{(2,1)}, \bar{\partial} \mathrm{div} \mu^{(2,1)} \rangle$$
$$+ \langle \mu^{(2,1)}, \bar{\partial} \mu^{(1,1)} \rangle + \langle g^{*(3,2)}, \bar{\partial} g^{(0,0)} \rangle + \langle \mu^{*(1,2)}, \bar{\partial} \mu^{(2,0)} \rangle + \langle \mu^{*(2,2)}, \bar{\partial} \mu^{(1,0)} \rangle + \langle g^{*(0,2)}, \bar{\partial} g^{(3,0)} \rangle \,. \tag{4.6}$$

However, now we cannot add a cubic interaction term preserving any of the gauge symmetries. The reason for this is the rigidity introduced by unifying both the picture 1 and −1 multiplets in the "long" multiplet (4.1).

4.2 A non-local differential

Alternatively, we may consider a short multiplet starting with the ghost $\mu^{(1,0)}$ as

$$\tilde{A} = \frac{1}{\gamma^2} g^{(3,2)} + \frac{1}{\gamma} \mu^{(2,1)} + \mu^{(1,0)}. \tag{4.7}$$

This forms a complex[15] for the non-local and even-parity differential $\mathrm{d} = \bar{q}X^{(-1)}$, which is well defined only in im$X$. However, d will not appear in the action, so that the variation of the action functional will be free. Upon acting with X as in (3.31) we obtain a second multiplet

$$A = \frac{1}{\gamma}\mu^{(2,2)} + \mu^{(1,1)} + \gamma g^{(0,0)} \tag{4.8}$$

which combines $\mu^{(1,1)}$ with what previously was its anti-field $\mu^{(2,2)}$. These two multiplets are parity odd and even, and have homogeneous BV-degree 0 and 1 respectively, if we define the latter as

$$deg = 2 - (\#\psi + \#\theta + 2\#\gamma)\,, \tag{4.9}$$

in agreement with [6]. Given that $\tilde{A}$ contains the auxiliary field $\mu^{(2,1)}$ we then choose the action

$$\begin{aligned} S(\tilde{A}, A) &= \frac{1}{2}\left(\tilde{A}, \bar{q}\,X\tilde{A}\right) + \left(A, \bar{q}\tilde{A}\right) + \frac{1}{6}\left(A, \frac{1}{\gamma}A \wedge A\right) \\ &= \frac{1}{2}\langle \mu^{(2,1)}, \bar{q}\,q\mu^{(2,1)}\rangle + \langle g^{(3,2)}, \bar{q}\,q\mu^{(1,0)}\rangle + \langle \mu^{(2,1)}, \bar{q}\mu^{(1,1)}\rangle + \langle g^{(3,2)}, \bar{q}g^{(0,0)}\rangle \\ &\quad + \langle \mu^{(2,2)}, \bar{q}\mu^{(1,0)}\rangle + \frac{1}{6}\langle \mu^{(1,1)}, \mu^{(1,1)} \wedge \mu^{(1,1)}\rangle + \langle \mu^{(2,2)}, \mu^{(1,1)} \wedge g^{(0,0)}\rangle\,. \end{aligned} \tag{4.10}$$

The equations of motion following from (4.10) are then

$$\bar{q}q\mu^{(2,1)} + \bar{q}\mu^{(1,1)} = 0\,, \tag{4.11}$$

$$\bar{q}q\mu^{(1,0)} + \bar{q}g^{(0,0)} = 0\,, \tag{4.12}$$

$$\bar{q}qg^{(3,2)} + \bar{q}\mu^{(2,2)} = 0\,, \tag{4.13}$$

$$\bar{q}\mu^{(2,1)} + \frac{1}{2}\mu^{(1,1)} \wedge \mu^{(1,1)} + \mu^{(2,2)} \wedge g^{(0,0)} = 0\,, \tag{4.14}$$

$$\bar{q}\mu^{(1,0)} + \mu^{(1,1)} \wedge g^{(0,0)} = 0\,, \tag{4.15}$$

$$\bar{q}g^{(3,2)} + \mu^{(2,2)} \wedge \mu^{(1,1)} = 0\,. \tag{4.16}$$

We can now repeat the discussion below (3.35): If A has no projection on the cohomology of q, the equation for A can be solved to express $A = -X\tilde{A}$, implying, in particular, that A is divergence-free. If, on the other hand, $A \in H_{\bar{\partial}}(PV)$ then it follows (existence of a quasi-iso to $(\ker \mathrm{div}, \bar{\partial})$ guaranteed by the $\partial\bar{\partial}$-lemma) that A can be assumed to be divergence free. Upon left action by q on eq. (4.14) and using eq. (4.11) (and similarly for the last two equations), we find

$$\bar{q}\mu^{(1,1)} + \frac{1}{2}q(\mu^{(1,1)} \wedge \mu^{(1,1)}) + q(\mu^{(2,2)} \wedge g^{(0,0)}) = 0\,, \quad \bar{q}\mu^{(2,2)} + q(\mu^{(2,2)} \wedge \mu^{(1,1)}) = 0, \tag{4.17}$$

$$\bar{q}g^{(0,0)} + q(\mu^{(1,1)} \wedge g^{(0,0)}) = 0\,, \tag{4.18}$$

which agrees with (2.7), (2.8) obtained by Costello and Li [6] for divergence free μ.

Let us now consider the gauge symmetries: $S(A, \tilde{A})$ is invariant under the non-linear transformations of $\mu^{(1,1)}$ and $\mu^{(2,2)}$

$$\delta\mu^{(1,1)} = -\bar{q}\mu^{(1,0)} - \mu^{(1,1)} \wedge g^{(0,0)}, \tag{4.19}$$

$$\delta\mu^{(2,2)} = \bar{q}\mu^{(2,1)} + \frac{1}{2}\mu^{(1,1)} \wedge \mu^{(1,1)} + \mu^{(2,2)} \wedge g^{(0,0)}. \tag{4.20}$$

The remaining fields do not transform. These symmetries are rather strange, their structure almost suggesting that we should take $g^{(0,0)}$ and $\mu^{(1,1)}$ as gauge fields so to have linear/quadratic gauge transformations (or else there would be terms that do not depend on gauge fields). We will not delve more into that.

Now we would like to make contact with section 3.3 and [5, 19, 6]. Hence we use eq. (4.11), (4.12) and (4.13), that follow from variation w.r.t. to the auxiliary fields in $\tilde{A}$ and substitute A in $S(A, \tilde{A})$ as $A = -q\tilde{A} + h$, with h a compactly-supported harmonic term, to get:

$$\begin{aligned} S(\tilde{A}) = &-\frac{1}{2}\langle \mu^{(2,1)}, \bar{q}q\mu^{(2,1)}\rangle + \langle g^{(3,2)}, \bar{q}g^{(0,0)}\rangle + \langle qg^{(3,2)} + k^{(2,2)}, (q\mu^{(2,1)} + h^{(1,1)}) \wedge g^{(0,0)}\rangle \\ &- \frac{1}{6}\langle q\mu^{(2,1)} + h^{(1,1)}, (q\mu^{(2,1)} + h^{(1,1)}) \wedge (q\mu^{(2,1)} + h^{(1,1)})\rangle. \end{aligned} \tag{4.21}$$

[15] The anti fields for $\tilde{A}$ are contained in $\tilde{A}^* = \gamma g^{(0,1)} + \mu^{(1,2)} + \frac{1}{\gamma}\mu^{(2,3)}$ which will, however, not appear in the action functional below.

This is then the extension of the action in [5] to include the compensator field g. It is invariant under the gauge transformations

$$\begin{aligned}\delta\mu^{(2,1)} &= \bar{q}\mu^{(2,0)} + q\mu^{(2,0)} \wedge (q\mu^{(2,1)} + h^{(1,1)}),\\ \delta\mu^{(1,0)} &= g \wedge q\mu^{(2,0)},\\ \delta g^{(3,2)} &= (qg^{(3,2)} + k^{(2,2)}) \wedge q\mu^{(2,0)}\,.\end{aligned} \tag{4.22}$$

Upon left action by q and using eq. (4.11), (4.12) and (4.13) these take the more intuitive form

$$\begin{aligned}\delta\mu^{(1,1)} &= -\bar{q}\lambda^{(1,0)} - [\lambda^{(1,0)}, \mu^{(1,1)}],\\ \delta g^{(0,0)} &= -[g^{(0,0)}, \lambda^{(1,0)}],\\ \delta\mu^{(2,2)} &= -[\mu^{(2,2)}, \lambda^{(1,0)}]\,.\end{aligned} \tag{4.23}$$

where we defined $\lambda^{(1,0)} \equiv -q\mu^{(2,0)}$. Notably, there is just one gauge parameter: we cannot change $g^{(3,2)}$ by a $\bar{q}$-exact contribution. This fact is reminiscent of the discussion around (3.34) in the small Hilbert space: only one gauge parameter could remain a symmetry of the interacting theory (whereas there were two for the linear theory).

The gauge invariant theory described by (4.21), (4.23) admits a minimal BV extension. Crucially for that, the (odd) ghost field $\frac{1}{\gamma}\mu^{(2,0)}$ must transform in the adjoint:

$$\delta\mu^{(2,0)} = -\frac{1}{2}q\mu^{(2,0)}q\mu^{(2,0)}. \tag{4.24}$$

We shall now take the following pairs of fields-antifields, where *the antifields are q-exact*:

$$\left(\frac{1}{\gamma}\mu^{(2,1)}, \mu^{*(1,2)}\right), \qquad \left(\frac{1}{\gamma^2}g^{(3,2)}, \gamma g^{*(0,1)}\right), \qquad \left(\mu^{(1,0)}, \frac{1}{\gamma}\mu^{*(2,3)}\right) \tag{4.25}$$

and for the ghost field $\frac{1}{\gamma}\mu^{(2,0)}$ we require the antighost field to be $\mu^{*(1,3)}$. Then we can define an odd BV bracket between two functionals as:

$$\{F, L\} = \sum_{\Phi} \oint \mathrm{d}\gamma \int_{CY^3} \Upsilon \wedge \left\langle\left\langle \frac{\delta_R F}{\delta\Phi}, \Upsilon\right\rangle, \frac{\delta_L L}{\delta\Phi^*}\right\rangle - \left\langle\left\langle \frac{\delta_R F}{\delta\Phi^*}, \Upsilon\right\rangle, \frac{\delta_L L}{\delta\Phi}\right\rangle.$$

Eventually

$$S = S(\tilde{A}) + \langle\mu^{*(1,2)}, \delta\mu^{(2,1)}\rangle + \langle g^{*(0,1)}, \delta g^{(3,2)}\rangle + \langle\mu^{*(2,3)}, \delta\mu^{(1,0)}\rangle - \langle\mu^{*(1,3)}, \delta\mu^{(2,0)}\rangle \tag{4.26}$$

satisfies the CME, $\frac{1}{2}\{S, S\} = 0$. The calculation is straightforward and the conclusion follows from q-exactness of the antifield, $\langle qa, b\rangle = -(-1)^{|a|}\langle a, qb\rangle$ and the usual compatibility conditions of the differential $\bar{q}$ with the Schouten bracket (the latter expressed in Tian's version), as well as its Jacobi identity.

5 Theory of background fields

In the previous section we derived a local and polynomial BV-action for the Kodaira–Spencer theory of complex structure deformations on a Calabi–Yau complex 3-fold. However, since Kodaira–Spencer theory is well posed for any complex manifold, one may wonder why the additional CY-structure is needed for the worldline. In this section we present an alternative, background independent formulation, where we absorb the Beltrami differential into a deformation of the differential Q. The obvious deformation is the "minimal coupling"

$$\bar{p}_a \to \bar{p}_a + \mu_a^b p_b \equiv (\bar{p}_\mu)_a \tag{5.1}$$

with $p_a, \bar{p}_a \sim \partial_{z^a}, \partial_{\bar{z}^a}$ on the representation space V. Now, what is the correct condition we should impose on the deformed $\bar{q}$ for the background to be Maurer–Cartan? We will say that a deformation of an almost complex structure is MC if the problem of infinitesimal deformations around that background is well-defined, i.e. it can be formulated as a cohomology problem to distinguish between fake (reparametrizations) and actual infinitesimal deformations. For this $\bar{q}$ needs to be nilpotent on the relevant vector space spanned by infinitesimal deformations, which is tantamount to

$$[(\bar{p}_\mu)_a, (\bar{p}_\mu)_b] = 0, \tag{5.2}$$

which is just the KS-equation for integrable deformations of complex structures on a complex manifold. See also [8] for an earlier discussion. Note that the Calabi–Yau condition is not required here. However, a locally invertible holomorphic 3-form is required to establish an "operator-state-correspondence" between states in V and infinitesimal deformations of $\bar{q}$ (see e.g. [27]). Indeed, we need to make use of the local isomorphism Υ in order to identify a state (3.24) with $\delta_\mu \bar{q}$.

6 Outlook and conclusions

In this paper we gave a construction of a BV action for BCOV starting from a spinning worldline with local supersymmetry instead of a topological sigma-model, mimicking familiar constructions from string field theory. Working in the small Hilbert space this gives a direct construction of the BV-extension of the action of Barannikov and Kontsevich [5]. In the large Hilbert space formulation there is a multiplet that includes the scalar compensator for the holomorphic 3-form. Compared to Costello–Li's on-shell formulation [28], the off-shell action constructed here includes a local kinetic term at the price of having a parity even inner product which does not pair with anti fields. Furthermore, instead of the equivariant differential $\bar{\partial} + u\mathrm{div}$ in [6] this multiplet has an even differential $\mathbf{d} = \bar{\partial} X^{-1}$ involving the inverse picture changing operator which represents the divergence operator dressed with the ghosts.

As for the question of the gauge symmetries, in the non-linear theory with the local kinetic term (either in the large or small Hilbert space) we observe only linear gauge symmetries ($\bar{q}$-coboundaries). Instead, after using the field equations to project the field onto its div-exact and harmonic part, more complicated symmetries (adjoint action of the vector field for holomorphic diffeos) emerge. We have also been able to find a short multiplet in the large Hilbert space comprising the Beltrami differential and its conjugated antifield with the presence of some additional BV-transformations.

We note that in the small Hilbert space formulation, with an arguably more geometric interpretation in terms of pictures, we are able to formulate Kodaira–Spencer theory just for complex structure deformations without inclusion of the compensator for the holomorphic 3-form. For that we needed to resort to the large Hilbert space with the drawbacks just described. However, it is not obvious that the compensator could be included in the small Hilbert space as well if one were to consider reducible representations (BV-multiplets).

While geometrizing the non-standard kinetic term of BCOV, we believe that our description also highlights Kodaira–Spencer gravity as a valuable toy model to explore and geometrize peculiar features of super string field theory such as picture changing and transition between the small and large Hilbert spaces. In particular, we gave a new interpretation of the pairing in the large Hilbert space. Also, it would be interesting to explore the relation between picture changing and the equivariant differential in [28].

The worldline description has the advantage that the superghost sector which plays a central role in our investigation, is readily included in the operator formulation while in a worldsheet sigma-model its implementation is less direct, in terms of bosonized ghost, which obscures its geometric interpretation. It may thus be instructive to revisit our construction for the worldsheet sigma-model with topological twist.

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A Deformation problem of complex structures

We present a short reminder about the deformation problem of complex structures following [11]. Given a complex structure J on a manifold M, a splitting between holomorphic and antiholomorphic vector fields is in place. If we are provided with a new (almost) complex structure J', sufficiently close to J, then the projection of TM' to its antiholomorphic component will induce an isomorphism and thus we end up with the chain of isomorphisms:

$$TM_{(1,0)} \xrightarrow{\pi^{-1}_{(0,1)}} TM'_{(1,0)} \xrightarrow{\pi_{(1,0)}} TM_{(0,1)}\,.$$

In the second slot, we must think of $TM'_{(1,0)}$ as $TM'_{(1,0)} \subset TM' \otimes \mathbb{C}$. Therefore, there exists a global antiholomorphic 1-form, with values in holomorphic tangent vectors, the *Beltrami differential*:

$$\mu \in \Omega^{(0,1)}(M, TM_{(1,0)})\,. \tag{A.1}$$

We then seek the conditions on μ so to promote the new almost complex structure to a complex structure. Newlander–Nirenberg theorem states that an almost complex structure is fully-fledged complex if it is integrable:

$$[X,Y] + J'([J'X,Y] + [X,J'Y]) - [J'X,J'Y] = 0, \quad \forall\, X,Y \in \mathfrak{X}(M).$$

Equivalently, $TM'_{(0,1)}$ is an involutive distribution. This reflects in the following Maurer–Cartan equation for μ:

$$\bar\partial\mu + \frac{1}{2}[\mu,\mu] = 0\,, \tag{A.2}$$

where $[-,-]$ is the Lie bracket of vector fields (the reader should be reminded that it extends on multivector fields as the Schouten–Nijenhuis bracket). There is another equivalent expression to (A.2) based on Tian's lemma [20] (see also [29] for a review), which we will display later. First let us recall that the holomorphic Dolbeault differential on forms induces a differential operator ∂_Υ on polyvector fields. The latter are forms that take values in multivector fields, i.e.

$$PV^{\bullet,\bullet}(M) := \Omega^{0,\bullet}(M, \Lambda^\bullet TM_{(1,0)})\,.$$

The differential operator ∂_Υ is then defined by the commutative diagram:

$$\begin{array}{ccc} PV^{\bullet,\bullet} & \xrightarrow{\partial_\Upsilon} & PV^{\bullet-1,\bullet} \\ \Upsilon\big\downarrow\wr & & \wr\big\downarrow\Upsilon \\ \Omega^{d-\bullet,\bullet} & \xrightarrow{\partial} & \Omega^{d+1-\bullet,\bullet} \end{array}\,. \tag{A.3}$$

Here ∂ is the holomorphic Dolbeault differential and Υ is the non-degenerate holomorphic volume form thanks to which $PV^{k,m} \cong \Omega^{d-k,m}$. Such an element exists only for the case of Calabi–Yau manifolds. Note that it is possible to define a boundary operator on $PV^{\bullet,\bullet}$ without explicit dependence on Υ. Therefore we also refer to ∂_Υ as div because it extends the divergence operator to multivector fields. So [20]:

$$[\alpha,\beta]_{SN} \equiv \partial_\Upsilon(\alpha\wedge\beta) - (\partial_\Upsilon\alpha)\wedge\beta - (-1)^{|\alpha|}\alpha\wedge\partial_\Upsilon\beta, \tag{A.4}$$

where the wedge product is intended to be taken on both the form part and the multivector fields. Eventually, on div-free Beltrami differentials, (A.2) can be rewritten as:

$$\bar\partial\mu + \frac{1}{2}\mathrm{div}(\mu\wedge\mu) = 0, \quad \mu \in PV^{1,1}(M)\,. \tag{A.5}$$

Complex structure deformations of Calabi–Yau manifolds may or may not preserve the volume form. If $g \in PV^{(0,0)}$ parametrizes the freedom in choosing a global factor for Υ, then to preserve the holomorphic volume form one has to ask that $\iota_{\exp g \exp \mu}\Upsilon$ is closed. This yields:

$$\begin{aligned} 0 = \mathrm{d}(\iota_{\exp g \exp \mu}\Upsilon) =& \mathrm{d}(\iota_{\exp g}\iota_{\exp \mu}\Upsilon) = \mathrm{d}\iota_g\Upsilon + \mathrm{d}\iota_\mu\Upsilon - \iota_\mu\mathrm{d}(\iota_g\Upsilon) + \iota_g\mathrm{d}(\iota_\mu\Upsilon) + \mathrm{d}(\iota_g\iota_\mu\Upsilon) + \mathcal{O}(3) \\ =& \left(\iota_{\bar\partial g} + \iota_{\mathrm{div}\mu} + \iota_{[g,\mu]}\right)\Upsilon + \mathcal{O}(3). \end{aligned}$$

In conclusion, complex structure deformations that also preserve the holomorphic volume form on a Calabi–Yau manifold are ought to satisfy:

$$\bar\partial g + \mathrm{div}\mu + [g,\mu] = 0. \tag{A.6}$$

B A fully gauged $(2;2)$-worldline

Alternatively, we could motivate our choice for the operator algebra (3.3) for BCOV theory by starting with the spinning particle whose worldline has $\mathcal{N}=(2;2)$-supersymmetry. Then BRST quantization gives rise to the differential

$$Q = cH + \gamma q + \gamma^\dagger q^\dagger + \bar\gamma\bar q + \bar\gamma^\dagger \bar q^\dagger + b(\gamma\gamma^\dagger + \bar\gamma\bar\gamma^\dagger). \tag{B.1}$$

Written in Darboux coordinates, the $2+2$ supercharges in (B.1) take the form

$$\bar q^\dagger = \psi\cdot p, \quad \bar q = \bar\psi\cdot\bar p, \quad q = \theta\cdot p, \quad q^\dagger = \bar\theta\cdot\bar p\,.$$

with

$$\{\psi^a, \bar\psi^{\bar b}\} = h^{a\bar b} = \{\theta^a, \bar\theta^{\bar b}\} \implies \{\bar q, \bar q^\dagger\} = H = \{q, q^\dagger\}, \tag{B.2}$$

provided the target space is Kähler.[16] All other brackets are zero. H is the worldline Hamiltonian whose explicit form will not be relevant here, as it relates to diffeomorphism invariance, a non-topological feature that we are going to lift. Indeed, in order to make contact with the topological string, we shall project out all information about the punctures on super moduli space $\mathcal{M}$ of the $\mathcal{N}=(2;2)$ worldline. For this, we choose the path integral measure to produce a constant function (rather than a top form) on $\mathcal{M}$. Essentially, one does not want to know where the punctures are.

This is reasonable, because there is a canonical way to write interaction terms on the worldsheet (or worldline) whereas the kinetic terms involve some choices. Our choices will be as follows. We first eliminate the Hamiltonian constraint in Q by going from

$$V = C^\infty(U)[\bar\theta^1,\dots\bar\theta^n|\bar\psi^1,\dots\bar\psi^n]\otimes\mathbb{C}[[c,\gamma,\bar\gamma,\gamma^\dagger,\bar\gamma^\dagger]] \cong \Lambda^\bullet(TU_{(1,0)}[1]\oplus T^*U_{(0,1)})\otimes\mathbb{C}[[c,\gamma,\bar\gamma,\gamma^\dagger,\bar\gamma^\dagger]] \tag{B.3}$$

where $U\subset M$, to the reduced module

$$V_{red} := H_{b\gamma\gamma^\dagger}\cap H_{b\bar\gamma\bar\gamma^\dagger} = V/\{\gamma\gamma^\dagger, \bar\gamma\bar\gamma^\dagger\}\,, \tag{B.4}$$

where H is exact. This reduction does not yet delete all dependence on supermoduli but this can be achieved by a presymplectic formulation with a suitable degenerate symplectic form.

After this kick-starter, we shall now suggest a pairing. One possible presymplectic formulation amounts to simply insert δ-functions for all the ghosts which would be the canonical procedure for the interaction term. However, this completely trivalizes Q inside expectation values. Therefore we define the pairing by

$$(a,b) = \int_{CY^3}\mathrm{d}vol_{CY^3}\int \mathrm{d}\gamma\mathrm{d}\bar\gamma\mathrm{d}\gamma^\dagger\mathrm{d}\bar\gamma^\dagger\langle\!\langle ab, \Upsilon(\theta^{\wedge 3})\Upsilon(\psi^{\wedge 3})\rangle\!\rangle\ \delta(\bar\gamma^\dagger)\delta(\gamma^\dagger)\delta'(\bar\gamma), \qquad a,b\in V_{red} \tag{B.5}$$

in the path integral measure. The lack of $\delta(\gamma)$ implies that our pairing has "picture 1". Note how this pairing has total ghost number given by $gh(\delta(\gamma))$, i.e. the choice of the ghost number of the vacuum. Moreover, $q=\theta\cdot p$ and $\bar q = \bar\psi\cdot\bar p$ are self-adjoint w.r.t. the pairing (one way to see this is to use the existing isomorphisms with the module of $PV^{\bullet,\bullet}$ and integration thereof, and note that $\bar q\sim\bar\partial$ as well as $q\sim$ div, which are self-adjoint w.r.t. the integral.).

That done we can easily describe the structure states in V_{red}, modulo terms that will not contribute due to the degeneracy in the symplectic structure. We just need to choose a representation for the remaining ghost algebra $[\beta,\gamma]=1$. There are two inequivalent representations, given by polynomials in γ and derivatives of $\delta(\gamma)$. In the polynomial representation of V_{red} we have

$$\Phi^{(0)} = \oplus_{p,q}\left(\varphi_{0\ b_1\cdots b_q}^{a_1\cdots a_p}(z)\bar\theta_{a_1}\cdots\bar\theta_{a_p}\bar\psi^{b_1}\cdots\bar\psi^{b_q} + \bar\gamma\phi_{\bar 1^\dagger\ b_1\cdots b_q}^{a_1\cdots a_p}(z)\bar\theta_{a_1}\cdots\bar\theta_{a_p}\bar\psi^{b_1}\cdots\bar\psi^{b_q}\right) + O(\gamma). \tag{B.6}$$

Now, again due to the missing delta function $\delta(\gamma)$ in the pairing (B.5), this representation pairs with the "picture -1",

$$\Phi^{(-1)} = \oplus_{p,q}\left(\varphi_{\bar 1^\dagger\ b_1\cdots b_q}^{a_1\cdots a_p}(z)\bar\theta_{a_1}\cdots\bar\theta_{a_p}\bar\psi^{b_1}\cdots\bar\psi^{b_q}\,\bar\gamma\delta(\gamma) + \varphi_{0\ b_1\cdots b_q}^{a_1\cdots a_p}(z)\bar\theta_{a_1}\cdots\bar\theta_{a_p}\bar\psi^{b_1}\cdots\bar\psi^{b_q}\,\delta(\gamma)\right) + O(\partial_\gamma)\delta(\gamma). \tag{B.7}$$

This module and its dual are quite large and eventually we can focus only on (3.24), (3.26), (3.28) and (3.29) in the body of this article. An attentive eye has certainly already caught that the $\bar\gamma$

[16] As matter of fact, it would be appropriate to deploy covariant momenta and thus Christoffel symbols in the supercharges, but we will not need their explicit expressions in the following therefore we spare the effort.

dependence of $\Phi^{(-1)}$ has disappeared from $a^{*(0)}$ and $a^{*(-1)}$. The correspondence to that half-gauged sigma model is clear if one notes that there is an equivalent formulation to (B.5) where we absorb $\bar{\gamma}$ in the component fields, thus stripping away the $\bar{\gamma}$-dependence of the states (3.26) and (3.29), and work with the pairing

$$(a,b) = \int_{CY^3} \mathrm{d}vol_{CY^3} \int \mathrm{d}\gamma \mathrm{d}\bar{\gamma} \mathrm{d}\gamma^\dagger \mathrm{d}\bar{\gamma}^\dagger \langle\!\langle ab, \Upsilon(\theta^{\wedge 3})\Upsilon(\psi^{\wedge 3})\rangle\!\rangle \; \delta(\bar{\gamma}^\dagger)\delta(\gamma^\dagger)\delta(\bar{\gamma})) \tag{B.8}$$

with an un-differentiated $\delta(\bar{\gamma})$ and $\bar{\gamma}\bar{q} \to \bar{q} \sim \bar{\partial}$.

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