

# Boundary layers, transport and universal distribution in boundary driven active systems

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## Abstract

We discuss analytical results for a run-and-tumble particle (RTP) in one dimension in presence of boundary reservoirs. It exhibits ‘kinetic boundary layers’, nonmonotonous distribution, **current without density gradient**, diffusion facilitated current reversal and optimisation on tuning dynamical parameters, and **a new transport effect** in the steady state. The spatial and internal degrees of freedom together possess a symmetry, using which we find the eigenspectrum for large systems. The eigenvalues are arranged in two bands which can mix in certain conditions resulting in a crossover in the relaxation. The late time distribution for large systems is obtained analytically; it retains a strong and often dominant ‘active’ contribution in the bulk rendering an effective passive-like description inadequate. A nontrivial ‘Milne length’ also emerges in the dynamics. Finally, a novel universality is proposed in the absorbing boundary problem for dynamics with short-ranged colored noise. Active processes driven by active reservoirs may thus provide a common physical ground for diverse and new nonequilibrium phenomena.

## Contents

|          |                                                                           |           |
|----------|---------------------------------------------------------------------------|-----------|
| <b>1</b> | <b>Introduction</b>                                                       | <b>2</b>  |
| <b>2</b> | <b>Model: boundary driven RTPs</b>                                        | <b>4</b>  |
| <b>3</b> | <b>Steady state distribution</b>                                          | <b>5</b>  |
| 3.1      | Steady state result for zero boundary magnetisation:                      | 9         |
| <b>4</b> | <b>Eigenspectrum and the time-dependent solution</b>                      | <b>10</b> |
| 4.1      | Formalism                                                                 | 10        |
| 4.2      | Symmetry of the model and solution for the spectrum                       | 12        |
| 4.2.1    | Spectrum in the large- $L$ limit                                          | 13        |
| 4.3      | Crossover behaviour of the relaxation time                                | 15        |
| 4.4      | Eigenfunctions and the large-time distribution                            | 16        |
| 4.5      | Semi-infinite line with absorbing barrier at $x = 0$ :                    | 17        |
| 4.5.1    | Discussion: thermal noise and the late time behaviour of active particles | 19        |
| <b>5</b> | <b>Proposed universality in the large time distribution</b>               | <b>21</b> |

|                                                                                                                                                                                                                                   |           |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------|
| <b>6 Conclusion</b>                                                                                                                                                                                                               | <b>22</b> |
| <b>A Comments on the boundary magnetisation control</b>                                                                                                                                                                           | <b>24</b> |
| <b>B Exit probability of RTP at the left boundary</b>                                                                                                                                                                             | <b>25</b> |
| <b>C Subleading correction to the band near <math>\lambda = 0</math></b>                                                                                                                                                          | <b>25</b> |
| <b>D Calculation of the coefficients <math>\beta^{(i)}</math>, expression for the left eigenvectors of <math>\mathcal{L}</math> in the <math>\lambda = 0</math> band and the distribution for large but finite <math>L</math></b> | <b>26</b> |
| <b>E Large time distribution of a passive underdamped Brownian motion in presence of an absorbing barrier</b>                                                                                                                     | <b>28</b> |
| <b>F Distribution of ABPs on a semi-infinite line with absorbing condition at <math>x = 0</math></b>                                                                                                                              | <b>29</b> |
| <b>References</b>                                                                                                                                                                                                                 | <b>30</b> |

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## 1 Introduction

Systems with self-propelled or active particles became a paradigm of nonequilibrium processes. Active agents consume energy from environment and convert it into mechanical work to propel themselves. Constant energy consumption drives active systems far from equilibrium [1–3]. There are numerous real life examples of active systems across scales including bacterial colony, school of fish, flock of birds, Janus particles [4, 5], vibrated granular rods and disks [6, 7] and many more. Such systems exhibit a plethora of nontrivial collective phenomena, e.g. pattern formation, motility-induced phase separation and clustering [2, 3, 8–10], giant number fluctuations [11], Casimir effect [12]. They reveal rich nonequilibrium properties even at the single particle level: examples include non-Boltzmann steady-state, accumulation near boundary, shape transition in probability distribution in a trapping potential, a condensation-like transition in the free motion with random activity [13–18]. Active motion is modelled with a Langevin-like equation with an ‘active noise’, which is typically coloured and violates the fluctuation-response relation, leading to such novel behaviour. In literature there are three widely studied scalar active particle models: run-and-tumble particles (RTP), active Brownian particles (ABP) and active-Ornstein-Uhlenbeck particles (AOUP).

Active particles show interesting behaviour in the presence of drives, thermal noise, obstacles and different boundary conditions. For example, RTPs subject to spatially periodic drive go through a transition from non-ergodic trapped states to moving states [19]. In presence of shear forces in two dimension, underdamped ABPs develop a boundary layer and flow reversal occurs near the wall [20]. Confining boundaries are known to have strong effects on active systems, e.g. boundary accumulation and boundary layer formation [13, 21], long-range effects and system size dependent behaviour [22–27], etc. Interestingly, the nature of the boundary layer and the pressure exerted by the particles on a wall depend on the shape of the wall [21] as well as on the interactions [22]. Thermal fluctuations, ubiquitous but usually neglected, can have nontrivial effects on active motion e.g. formation of bound states [28], current modulation on a ratchet [29], and triggering phase separation [30, 31].

In this work we focus on boundary-driven processes. Systems connected to particle or energy reservoirs have been studied for centuries, but there is a renewed interest in these pro-

cesses particularly in the context of low dimensional transport. The canonical picture is given by Brownian motion in a finite system connected to particle baths: the steady state density profile is a linear interpolation of the densities at the boundaries [32] while the current is proportional the density gradient as given by the Fick's law. However in the presence of interaction, bulk drive, or more than one conserved quantities etc. richer and often nonintuitive behaviour is observed. Models of boundary driven interacting diffusive systems exhibit long ranged correlation [33–35]; introduction of a bulk drive gives rise to different phases [36,37]. The thermal transport become anomalous and often violate Fourier's law in one dimensional interacting anharmonic chains [38–41]. Nonlinear bulk density profile and nongradient steady state current emerges in simple noninteracting cases where the particle dynamics is governed by strongly correlated noise, for instance Levy walk connected to particle reservoirs [40,42]. In contrast to the above cases where density or temperature gradient drives the current, a finite thermal conduction without temperature gradient was demonstrated for nonlinearly coupled passive systems driven by non-Gaussian noises [43]. New effects induced by 'active heat baths' in a passive harmonic (linearly coupled) chain, e.g. current reversal in the energy transport and boundary features in the steady state, have also been reported recently [44,45]. In contrast, results for active systems in presence of boundary reservoirs are rare [46], and to our knowledge the effects of active reservoirs are not yet discussed. It is natural to ask what are the steady state, transport properties, dynamical fluctuations and relaxation behaviour of different boundary driven active systems and how do they differ across models and from their passive counterpart. To explore the nonequilibrium physics in this class of processes, we consider an elementary model amenable to simple analysis and exact results.

In this paper we study non-interacting free RTPs in presence of thermal noise in a one-dimensional channel connected to particle reservoirs at both ends maintaining fixed particle densities and magnetisation<sup>1</sup>. The results are surprisingly rich and intriguing. The solution for steady state and large time distribution show the existence of 'kinetic boundary layers' [47] characterised by exponentially decaying components in the density profile near the reservoirs. These are distinct from the usual boundary layers reported earlier for confining walls or obstacles in that, here these occur in the presence of particle fluxes accross the boundaries and can correspond to both accumulation and depletion. For zero boundary magnetisation, the steady state current and bulk density profile show a passive-like behaviour while the effect of activity, which is notable for large persistence, becomes prominent at the boundary layers. In the bulk, the signature of activity appears through a constant magnetisation. Remarkably, in the presence of nonzero boundary magnetisations, the density profile can become nonmonotonic, there is current without density gradient, and in particular a diffusion-facilitated current reversal and current maximisation upon tuning the particle parameters can take place. [In this connection a new transport behaviour and related coefficient are identified.](#)

The time-dependent solution to the boundary value problem is identical to that with absorbing boundaries, which is relatively less discussed for RTPs [48–51]. We observe that the boundary condition endows the system with a 'reflection symmetry'. Using this, the eigenspectrum and time-dependent solutions are obtained analytically for large system-sizes. The eigenvalues of the time evolution operator is arranged in two distinct bands separated by the tumble rate. The relaxation is diffusive for large system size  $L$ ; as  $L$  is reduced the bands start overlapping, and for small enough systems the relaxation rate crosses over to an  $L$ -independent value. Such crossovers had been found earlier for one and two particle systems on a ring [28,52]. We observe that in each of the eigenstate the 'active' contribution appears only as  $O(L^{-1})$  correction to the leading 'passive-like' term for large  $L$ ; but, contrary to expectations, at large times both the active and passive contributions occur at the same order in the bulk. For higher

<sup>1</sup>In 1-dimension each RTP tumbles between two possible internal orientations; the magnetisation is defined as the difference of densities of the differently oriented particles.

persistence, the large distance late time distribution and related properties are in fact dominated by the active contribution. Interestingly, although the distribution satisfies the absorbing condition at the boundaries in the presence of thermal noise, a ‘Milne length’ [53–55] is still identified. [This length quantifies the apparent shift in the position of an absorbing boundary, when seen from afar, from its true position.](#) A nonzero Milne length has also been found recently in the context of survival probability of random flights [56] and athermal RTPs [50]. In the present case it depends nontrivially on the diffusivity and particle parameters and plays a significant role both in the steady state and dynamics. A kinetic boundary layer appears in the time-dependent solution as well and this is argued to be a consequence of thermal diffusion. We further argue that this is related to a net imbalance of the differently oriented particles in the system, which goes away as soon as the diffusivity vanishes. As a whole, the interplay of thermal noise and active motion brings important ramifications in the dynamics in the presence of boundary driving.

It turns out that many of the said features obtained for RTPs also appear in underdamped passive particle, athermal AOUP, and ABP, and are possibly generic to dynamics with coloured noise. In the absorbing boundary problem the emergence of kinetic boundary layers and Milne length were known long back for the underdamped passive motion [55, 57–59]. The large time density profile in that case [60] is in fact quantitatively similar to that obtained here for RTPs. It prompts us to propose that the late time distribution away from the absorbing wall is independent of the realisation of the noise, provided that the noise correlation is short ranged.

Rest of the paper is organized as follows. In Section 2, we describe the model system of a boundary-driven run-and-tumble particle. The steady-state and the time-dependent properties are discussed in Sections 3 and 4, respectively. A universality in the large time distribution is proposed in Section 5. In Section 6 we summarise our perspective and conclude.

## 2 Model: boundary driven RTPs

We consider a run-and-tumble particle (RTP) moving in a finite one-dimensional box of length  $L$  subject to a translational noise. The box is connected to particle reservoirs at the two ends. We assume that the microscopic details of the interaction of the reservoir with the system at the boundary is not important except that it provides a fixed boundary condition. The equation of motion of the particle is,

$$\dot{x} = v\sigma(t) + \eta(t), \quad (1)$$

where  $v$  is the self-propulsion speed,  $\sigma(t)$  is the intrinsic orientation (spin) at time  $t$  characterized by a telegraphic noise that takes values  $\pm 1$  and it flips between the two states with rate  $\omega$ , and  $\eta(t)$  is the Gaussian white noise with  $\langle \eta(t) \rangle = 0$ ,  $\langle \eta(t)\eta(t') \rangle = 2D\delta(t-t')$ ,  $D$  being the diffusion constant. The run and tumble motion described in terms of  $(x(t), \sigma(t))$  is a Markov process. The state of the particle at a time  $t$  is given by,  $|P(x, t)\rangle = \begin{bmatrix} P_+(x, t) \\ P_-(x, t) \end{bmatrix}$ , where  $P_+(x, t)$  and  $P_-(x, t)$  are the probabilities to find a particle at  $x$  at time  $t$  with instantaneous spin  $+1$  and  $-1$  respectively.  $P_+$  and  $P_-$  satisfy the Master equations,

$$\partial_t P_+ = D\partial_x^2 P_+ - v\partial_x P_+ - \omega P_+ + \omega P_-, \quad (2a)$$

$$\partial_t P_- = D\partial_x^2 P_- + v\partial_x P_- + \omega P_+ - \omega P_-, \quad (2b)$$

with the boundary conditions,

$$\begin{aligned} \text{At } x = 0 : P_+(0, t) &= P_0^+, P_-(0, t) = P_0^-; \\ \text{At } x = L : P_+(L, t) &= P_1^+, P_-(L, t) = P_1^-. \end{aligned} \quad (3)$$

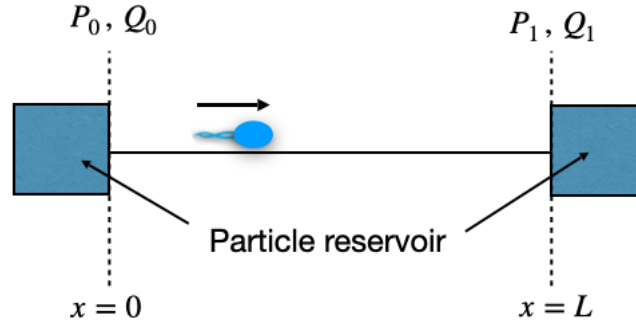


Figure 1: Schematic of the boundary driven active system. An RTP moves on a one-dimensional line bounded between  $x = 0$  and  $L$ . The system is connected to particle reservoirs at both the ends maintaining fixed particle densities  $P_0$  and  $P_1$  and magnetisation  $Q_0$  and  $Q_1$  at  $x = 0, L$  respectively.

$P_0^\pm$  and  $P_1^\pm$  are the probability densities of particles with spin  $+$  and  $-$  at the two boundaries  $x = 0$  and  $x = L$  respectively. The total probability density is  $P(x, t) = P_+(x, t) + P_-(x, t)$ . We also define a quantity  $Q(x, t) := \langle \sigma \rangle(x, t) = P_+(x, t) - P_-(x, t)$  which is analogous to magnetization. In terms of  $P$  and  $Q$  the boundary conditions are rewritten as,

$$\begin{aligned} \text{At } x = 0 : P(0) &= P_0, Q(0) = Q_0, \\ \text{At } x = L : P(L) &= P_1, Q(L) = Q_1. \end{aligned} \quad (4)$$

The schematic of the setup is shown in Figure 1.

### 3 Steady state distribution

In the steady state  $\partial_t P_+ = \partial_t P_- = 0$ . Using the definitions of  $P$  and  $Q$ , the equations for the steady state can be written as,

$$D \partial_x^2 P - v \partial_x Q = 0 \quad (5)$$

$$D \partial_x^2 Q - v \partial_x P - 2\omega Q = 0. \quad (6)$$

The solution to Eqs. (5)-(6) for the density and magnetisation profiles and the current  $J_0 = -D \partial_x P + v Q$  subject to the boundary conditions Eq. (4) are,

$$P(x) = P_0 + M \left[ B_0 - \Delta Q \frac{e^{-\mu L} (1 + e^{-\mu L})}{(1 - e^{-\mu L})^2} \right] + \frac{\Delta P - M(Q_0 + Q_1)}{M \frac{v}{\omega} + L} x - \frac{v}{\mu D} \frac{B_0 (e^{-\mu x} - e^{-\mu(L-x)}) - \Delta Q e^{-\mu(L-x)}}{1 + e^{-\mu L}}, \quad (7)$$

$$Q(x) = -\frac{\Delta P - M(Q_0 + Q_1)}{2(M + L \frac{\omega}{v})} + \frac{B_0 (e^{-\mu x} + e^{-\mu(L-x)}) + \Delta Q e^{-\mu(L-x)}}{1 + e^{-\mu L}}, \quad (8)$$

$$J_0 = -\frac{D_e}{M \frac{v}{\omega} + L} (\Delta P - M(Q_0 + Q_1)), \quad (9)$$

where,  $\Delta P = P_1 - P_0$ ,  $\Delta Q = Q_1 - Q_0$ ,  $D_e = D + \frac{v^2}{2\omega}$ ,  $\mu = \frac{\sqrt{2\omega D_e}}{D}$ ,  $M = \frac{v}{\mu D} \frac{1 - e^{-\mu L}}{1 + e^{-\mu L}}$ , and  $B_0 = \frac{\Delta P - M(Q_0 + Q_1)}{2(M + L \frac{\omega}{v})} + Q_0 - \Delta Q \frac{e^{-\mu L}}{1 - e^{-\mu L}}$ . The terms proportional to  $e^{\pm \mu x}$  represent the ‘kinetic boundary layers’ of width  $\mu^{-1}$ . Here  $D_e$  is the effective diffusivity for free RTP motion;  $M$  is

related to a Péclet number  $Pe = \frac{v}{\sqrt{2\omega D}}$  as  $M = \frac{Pe}{\sqrt{Pe^2 + 1}}$  for  $L \gg \mu^{-1}$ ; later we shall see that  $M$  is the ratio of Milne length and persistence length  $l_p = \frac{v}{2\omega}$ ; and  $B_0$  is an integration constant related to the boundary layer amplitude. In the  $D = 0$  case  $\mu$  diverges and the exponential terms vanish. This gives rise to discontinuities in the density and magnetisation profiles at the boundaries which is smoothened to boundary layers for  $D > 0$ . Below we discuss a few implications of the Eqs. (7)-(9).

**I. New features in the distribution:** The steady state density and magnetisation profiles are shown in Figure 2<sup>2</sup>; we can identify the boundary layers as given by the  $e^{\pm\mu x}$  terms in Eqs. (7)-(8). For  $L \gg \mu^{-1}$ , the magnetisation in the bulk (i.e.  $\mu^{-1} \ll x \ll L$ ) assumes a constant value  $Q_b = \frac{v}{2\omega D_e} J_0$ . In this region the density is linear in  $x$  and the density gradient which equals  $\frac{-J_0}{D_e}$  is proportional to  $(L + 2l_M)^{-1}$ , where  $l_M = \lim_{L \gg \mu^{-1}} M \frac{v}{2\omega} = \frac{v^2}{2\omega \sqrt{2\omega D_e}}$  is the Milne length as shown later. This increased effective system length suggests that the behaviour in the bulk pertains to boundary conditions at a distance  $l_M$  beyond the actual boundaries. This naive picture based on the usual notion of the Milne length however holds only for  $Q_0 = Q_1 = 0$  (section 3.1): in this case the linear bulk profile can be extrapolated to  $x = -l_M$  to the left and  $x = L + l_M$  to the right to satisfy the boundary values, implying that the density far from the boundaries is mimicked by that of a passive dynamics in a slightly larger system. For nonzero boundary magnetisations this picture no longer works.

A connection of the Milne length to the boundary layers can be observed. The distribution in the bulk, when extrapolated to the boundaries, deviates from the boundary condition, as the boundary discontinuities for  $D = 0$ . For  $D > 0$  this deviation is precisely compensated by the kinetic boundary layers with strength proportional to  $l_M$  representing particle accumulation at one end and depletion at the other.

Finally, the density profile may become nonmonotonic near the boundaries. In fact, for given boundary conditions the distribution can occur in two distinct categories for different parameter values, one with nonmonotoncities (Figure 2 left panel) and another being monotonous (Figure 2 right panel). Crucially, with parameters for which the current is co-directed with the boundary magnetisation(s) give rise to the nonmonotonous profile<sup>3</sup>. Below we discuss the properties of the steady state current in the large  $L$  limit.

**II. Magnetisation induced current and a new transport coefficient:** The current in Eq. (9) has two components, one is due to the global density gradient consistent with the Fick's law, and the other is a new contribution due to the boundary magnetisations. Consider  $\Delta P = 0$ : The current is still finite,  $J_0 = \frac{D_e}{L + 2l_M} M(Q_0 + Q_1)$ . This *prima facie* suggests that the boundary magnetisations act as drives giving rise to a current in the system. This way of understanding obviously works for  $\omega = 0$  (i.e. the driven passive case) for which both  $P_+$  and  $P_-$  are locally conserved. In that case the density is uniform except at the boundary layers, and the values of the current and bulk magnetisation are simply  $v(Q_0 + Q_1)/2$  and  $(Q_0 + Q_1)/2$  respectively. The active ( $\omega > 0$ ) scenario on the other hand is very different. We might expect that the effect of boundary magnetisation, that induces a biased motion  $u = v \frac{Q_0}{P_0}$  locally, would disappear beyond a length because of the tumble dynamics and overdamped situation. And indeed the influence is much reduced, but it does not vanish. Instead a residual bulk magnetisation and current both diminished to  $O(L^{-1})$  compared to those for  $\omega = 0$  persist. This is nonobvious

<sup>2</sup>The probability of finding the particle in the system is,  $\int_0^L P(x) dx = \frac{P_0 + P_1}{2} L$ , which is identical to that of a passive particle. This constrains  $P_0$  and  $P_1$  to  $O(L^{-1})$  or less since the total probability cannot exceed 1. We choose the boundary values accordingly.

<sup>3</sup>It can also be shown that the extrema occurs  $O(\ln L)$  away from the boundaries.

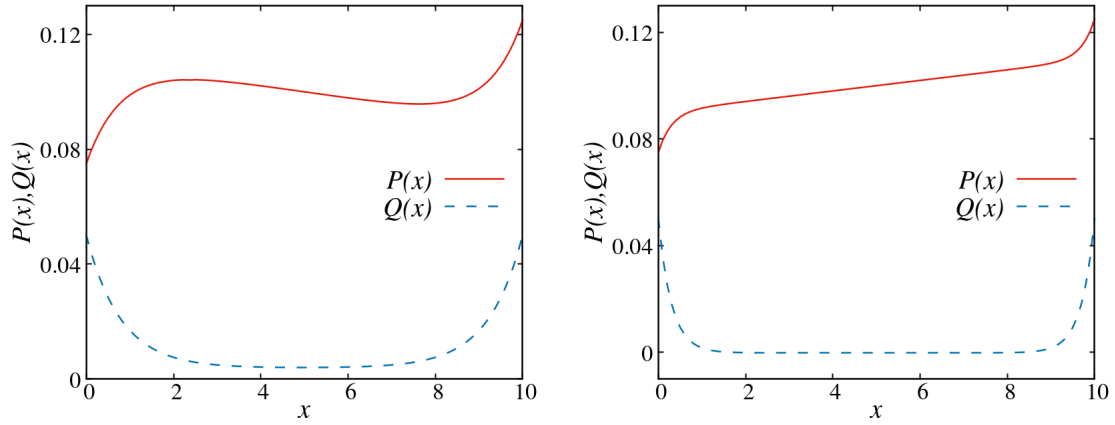


Figure 2: Left panel: Density (solid line) and magnetisation (dashed line) profiles in the steady state of the boundary driven RTP. Here  $v = 1.0$ ,  $D = 1.0$ ,  $w = 0.3$ ,  $L = 10.0$  and  $P_0 = 0.075$ ,  $P_1 = 0.125$ ,  $Q_0 = Q_1 = 0.05$ . The distribution is nonmonotonous and the slope in the bulk is negative, that corresponds to a current from the left to the right. Right panel: Density and magnetisation plots with the same boundary conditions and system size, but now with particle parameters  $v = 1.0$ ,  $D = 1.0$ ,  $w = 5.0$ . The density profile is monotonous in this case and the slope in the bulk is positive corresponding to a current in the reverse direction. The boundary layers can be identified in both the cases.

from a microscopic point of view<sup>4</sup>.

The competing components give rise to an interesting transport property. With suitable boundary magnetisations such that  $Q_0 + Q_1 = \Delta P/M$ , the current vanishes. Putting it in reverse, in a setting with zero current, e.g. in a closed 1D box such that particles cannot escape, maintaining a nonzero net boundary magnetisation induces a global density difference,  $\Delta P = M(Q_0 + Q_1)$ . This phenomenon to our knowledge classifies a new transport effect with coefficient  $M$  and does not have a thermodynamic analogue<sup>5</sup>. In particular, in the zero-current condition both magnetisation and local density gradient vanish in the bulk, and the effect is maintained by forces confined near the boundaries only. This is a global effect, entirely different from usual near equilibrium current balance phenomena. In the latter the thermodynamic forces (gradients) remain nonzero while the respective mass currents balance each other at all points.

To put this further in context, some previous works on heat transport in interacting active systems, e.g. an active chain connected to thermal baths [46] and a passive chain connected to active heat reservoirs [44], also reported competing contributions in the heat flux, which allow nontrivial current balance on tuning certain parameters. Here we argue that this is not only generic for active systems at a more elementary level but can also be leveraged to extract additional transport behaviour<sup>6</sup>.

<sup>4</sup>Consider diffusion on a channel of length  $L$  connected at both ends to reservoirs of density  $P_0$ . If the particle is driven with a bias velocity  $u$  within a length  $\ell \ll L$ ,  $D/u$  near the boundaries, a current  $j_\ell = u P_0 \ell / L$  is induced which vanishes as  $\ell \rightarrow 0$  for finite  $u$ . Only at  $u \rightarrow \infty$  a finite current can be sustained, e.g. RTP in the white noise limit:  $\lim_{v \rightarrow \infty, \omega \rightarrow \infty} \frac{v^2}{2\omega} = D_a$ . In this limit,  $u = v Q_0 / P_0 \rightarrow \infty$ , and  $J_0$  is finite, although  $Q_b \sim \omega^{-1/2} \rightarrow 0$ .

<sup>5</sup>A loose resemblance to a Seebeck-like effect can be noted with boundary magnetisations in place of temperatures. If the particles are charged, a voltage difference would also emerge.

<sup>6</sup>In the spirit of a thermo-couple, one can make an ‘active-couple’ by configuring a closed circuit of two channels and generate particle current with nonzero net magnetisation at the junctions.

**III. Current reversal and optimisation:** In the athermal ( $D = 0$ ) case,  $M$  equals 1 irrespective of the dynamical parameters. Thus  $J_0 \propto (\Delta P - Q_0 - Q_1) = P_1^- - P_0^+$ , implying that the direction of the current is determined entirely by the boundary conditions. The presence of thermal noise drastically alters the picture. For any  $D > 0$ ,  $M = \frac{v}{\sqrt{2\omega D_e}} \leq 1$  in the large  $L$  limit, and either of the activity parameters  $v, \omega$  can be tuned to access all values of  $M \in (0, 1)$ . Consequently, for a fixed boundary condition such that  $P_1^- < P_0^+$  i.e.  $0 < \frac{\Delta P}{Q_0+Q_1} < 1$ , we can always tune  $v, \omega$  in order to tune  $M$  across  $M^* = \frac{\Delta P}{Q_0+Q_1}$  leading to a reversal of the direction of the current. The two plots in Figure 2 correspond to different  $\omega$  values such that the respective currents are oppositely directed, but with same boundary conditions. The same is possible by tuning  $D$ , as shown in the left panel of Figure 3.

For  $D > 0$  interesting nonmonotonicities are present in the current. A *maximal* current occurs in two distinct conditions in the large  $L$  limit. Defining  $D_a = \frac{v^2}{2\omega}$ , the expression of the current in the thermodynamic limit becomes,  $J_0 \approx -\frac{D+D_a}{L} [\Delta P - \sqrt{\frac{D_a}{D+D_a}} (Q_0 + Q_1)]$  with  $M = \sqrt{\frac{D_a}{D+D_a}}$ . We can see that the competing terms in the expression for  $J_0$  are both enhanced but in a different manner as  $D$  and  $D_a$  are increased. The extremal current for fixed activity parameters and changing thermal noise occurs at  $\frac{\partial J_0}{\partial D} = 0$ , which translates to  $M = M_p \equiv 2 \frac{\Delta P}{Q_0+Q_1}$  under the condition  $0 < \frac{\Delta P}{Q_0+Q_1} < \frac{1}{2}$ . This condition also allows for the current reversal, implying that an optimal current can be achieved and the direction can be controlled by tuning  $D$  only (Figure 3 left panel).

On the other hand, the extremal current condition for changing activity parameters is given by  $\frac{\partial J_0}{\partial D_a} = 0$ , which implies,  $M = M_a \equiv \frac{\Delta P}{Q_0+Q_1} - \sqrt{(\frac{\Delta P}{Q_0+Q_1})^2 - 1}$ . This occurs only when  $\frac{\Delta P}{Q_0+Q_1} > 1$ , which does not allow for a current reversal (Figure 3 right panel). Thus the conditions for optimal current clearly distinguishes the role of thermal diffusivity and the effective diffusivity due to activity. For  $\frac{1}{2} < \frac{\Delta P}{Q_0+Q_1} < 1$  the current is monotonous for all parameter values and only a current reversal occurs at  $M^*$ .

It is useful to recall that the nature of the density profile depends crucially on the direction of the current – the latter is not determined solely by the boundary conditions, but can be controlled independently on tuning the dynamical parameters.

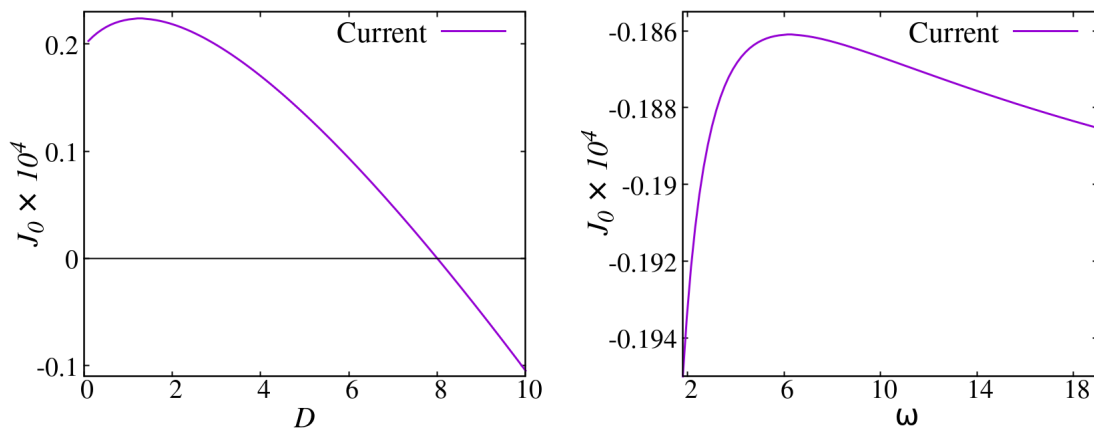


Figure 3: Nonmonotonicity of the steady state current. Left panel: Diffusivity  $D$  is tuned. Here  $v = 1.0, w = 0.5, L = 200.0, P_0 = Q_0 = 0.002, P_1 = Q_1 = 0.004$ . A current reversal also occurs at higher values of  $D$ . Right panel: Tumble rate  $\omega$  is tuned. Here  $v = 1.0, D = 1.0, L = 200.0, P_0 = 0.002, P_1 = 0.006, Q_0 = Q_1 = 0.001$ . (In both panels the y-axis is amplified  $10^4$  times for better visibility.)

It is surprising that this simple model accommodates such remarkable out of equilibrium features, some of which are new according to our knowledge. These features might be general in the boundary driven active processes. [The crucial role of the boundary magnetisation calls for a discussion of its possible realisations in experiments and simulations. This is nontrivial, in particular in the presence of boundary fluxes and translational noise, and is beyond the scope of this paper. We only outline a few possible avenues in the Appendix A.](#)

### 3.1 Steady state result for zero boundary magnetisation:

For the special case  $P_0^+ = P_0^- = \frac{P_0}{2}$  and  $P_1^+ = P_1^- = \frac{P_1}{2}$ , i.e.  $Q_0 = Q_1 = 0$ , the solution reduces to a simpler form,

$$P(x) = P_0 + \frac{\Delta P}{L + M \frac{v}{\omega}} \left[ x + l_M \frac{(1 - e^{-\mu L}) - (e^{-\mu x} - e^{-\mu(L-x)})}{1 + e^{-\mu L}} \right], \quad (10)$$

$$Q(x) = \frac{\Delta P}{2(L \frac{\omega}{v} + M)} \left[ \frac{e^{-\mu x} + e^{-\mu(L-x)}}{1 + e^{-\mu L}} - 1 \right], \text{ and} \quad (11)$$

$$J_0 = -\frac{D_e}{M \frac{v}{\omega} + L} \Delta P. \quad (12)$$

The equations for the steady state and the boundary conditions in this case are related to the exit probability  $E_{\pm}(x)$  of the particle at the left boundary starting at an initial position  $x$  with initial spin  $\pm 1$ . More details are in Appendix B. It gives a simulation scheme for the steady state. [We first simulate  \$E\_{\pm}\(x\)\$  for an RTP as follows: Consider an RTP with initial position  \$x \in \[0, L\]\$  and initial spin  \$m = +1\$ ; the dynamics stops as soon as the particle hits one of the boundaries. Counting the fraction of times it hits the left boundary gives  \$E\_+\(x\)\$ . We similarly find  \$E\_-\(x\)\$  with  \$m = -1\$ . The distribution is obtained thereafter using Eq. \(B.1\).](#) The data with zero boundary magnetisation is shown in Figure 4, which agrees with the theoretical expressions in Eqs. (10)-(11). We however couldn't simulate the steady state with nonzero boundary magnetisations.

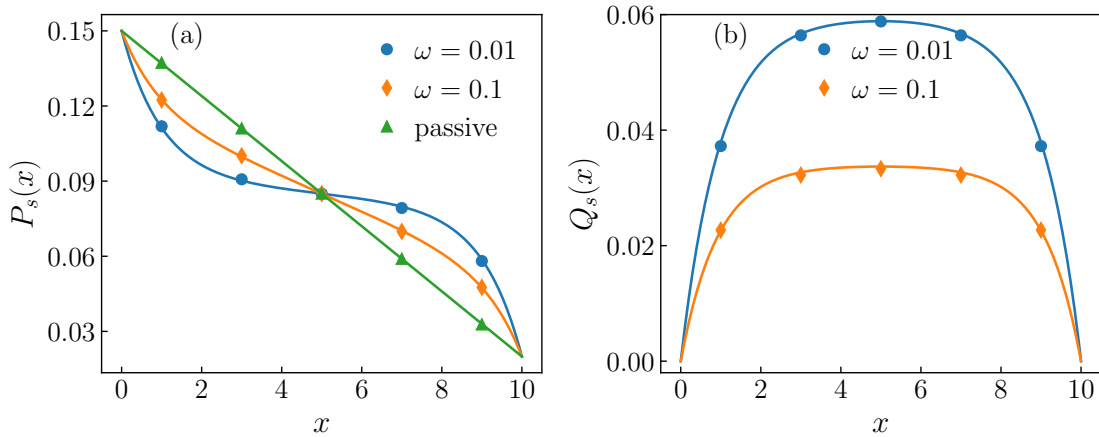


Figure 4: Steady state for  $Q_0 = Q_1 = 0$  : (a) the probability density; (b) magnetisation for different tumble rate  $\omega$ . Here  $L = 10, v = 1.0, D = 1.0$  and the boundary densities are  $P_0 = 0.15, P_1 = 0.02$ . Points are obtained by simulating the exit probabilities and using Eq. (B.1).

**Large- $L$  limit:** For  $L \gg \frac{v}{\omega}, \mu^{-1}$ , the particle undergoes many tumble events and we expect the behaviour to be largely diffusive with effective diffusion constant  $D_e$ . In this limit the

steady state results can be quite accurately approximated as <sup>7</sup>,

$$J_0 \simeq -D_e \frac{\Delta P}{L + 2l_M}, \quad (13)$$

$$P(x) \simeq P_0 + \frac{\Delta P}{L + 2l_M} (x + l_M) - \frac{\Delta P}{L + 2l_M} l_M (e^{-\mu x} - e^{-\mu(L-x)}), \quad (14)$$

$$Q(x) \simeq -\frac{\Delta P}{L + 2l_M} \frac{v}{2\omega} (1 - e^{-\mu x} - e^{-\mu(L-x)}). \quad (15)$$

The expressions suggest that the current and density in the bulk are passive-like with a slightly larger effective system size, whereas the signature of activity shows up at the boundary layer. Nontrivial effect of activity appears in the bulk through  $Q(x)$ , for which there is no passive analogue and which assumes a constant value for  $\mu^{-1} \ll x \ll L$ .

## 4 Eigenspectrum and the time-dependent solution

### 4.1 Formalism

The evolution of the distribution of the RTP, given by the master equation (2a)-(2b), can be rewritten in terms of the state vector  $|P(x, t)\rangle = \begin{bmatrix} P_+(x, t) \\ P_-(x, t) \end{bmatrix}$  as,

$$\partial_t |P(x, t)\rangle = \mathcal{L} |P(x, t)\rangle, \quad (16)$$

$\mathcal{L}$  being a linear operator,

$$\mathcal{L} = \begin{bmatrix} D\partial_x^2 - v\partial_x - \omega & \omega \\ \omega & D\partial_x^2 + v\partial_x - \omega \end{bmatrix}, \quad (17)$$

along with the boundary conditions Eq. (3). The solution to Eq. (16) subject to the initial condition gives the full state vector  $|P(x, t)\rangle$ , which can be written as,

$$|P(x, t)\rangle = |P_{ss}(x)\rangle + \sum_{n=1}^{\infty} c_n e^{\lambda_n t} |\phi_n(x)\rangle, \quad (18)$$

where  $|P_{ss}(x)\rangle$  is the steady state, and  $|\phi_n(x)\rangle \equiv [\phi_n^+(x), \phi_n^-(x)]^T$  are the eigenfunctions of the time evolution operator  $\mathcal{L}$ . Substituting Eq. (18) in Eq. (16) and using the linear independence of  $e^{\lambda_n t}$ 's for different  $\lambda_n$ 's, we find,

$$\mathcal{L} |\phi_n(x)\rangle = \lambda_n |\phi_n(x)\rangle. \quad (19)$$

Here  $\lambda_n$ 's are the eigenvalues of  $\mathcal{L}$  with corresponding eigenfunctions  $|\phi_n(x)\rangle$ , both yet unknown, satisfying the boundary conditions  $|\phi_n(0)\rangle = |\phi_n(L)\rangle = 0$  as we shall argue soon. The coefficients  $c_n$  are determined from the initial condition. Let us consider the eigenfunctions  $|\phi_n(x)\rangle$  to be of the form,

$$|\phi_n(x)\rangle = \beta_n e^{k_n x} |A_n\rangle \quad (20)$$

where  $|A_n\rangle$  is a constant vector. Using the above, Eq. (19) can be written as,

$$M(k_n) |A_n\rangle = \lambda_n |A_n\rangle \quad (21)$$

<sup>7</sup>The approximation carries a tiny error  $\sim O(e^{-\mu L})$ .

with

$$M(k_n) = \begin{bmatrix} Dk_n^2 - vk_n - \omega & \omega \\ \omega & Dk_n^2 + vk_n - \omega \end{bmatrix}. \quad (22)$$

The eigenvalues of  $M(k_n)$  gives the ‘dispersion relation’:

$$\lambda_n = -(\omega - Dk_n^2) \pm \sqrt{\omega^2 + v^2 k_n^2}. \quad (23)$$

Inverting the above relation we express  $k_n$  in terms of  $\lambda_n$ ,

$$k_n^2 = \frac{\lambda_n}{D} + \frac{\omega D_e}{D^2} \pm \sqrt{\left(\frac{\lambda_n}{D} + \frac{\omega D_e}{D^2}\right)^2 - \frac{\lambda_n^2 + 2\omega\lambda_n}{D^2}}. \quad (24)$$

Denoting the positive roots for the  $k_n$ ’s corresponding to plus and minus sign by  $k_a$  and  $k_b$  respectively, Eq. (24) implies that there are four  $k_n$  values for each  $\lambda_n$ ,

$$k_n^{(1)} = k_a(\lambda_n), \quad k_n^{(2)} = -k_a(\lambda_n) \quad (25a)$$

$$k_n^{(3)} = k_b(\lambda_n), \quad k_n^{(4)} = -k_b(\lambda_n). \quad (25b)$$

The corresponding eigenvectors of  $M(k_n)$  are

$$|A_n^{(i)}\rangle = \begin{bmatrix} a(k_n^{(i)}) \\ 1 \end{bmatrix}, i = 1, \dots, 4 \text{ with} \quad (26)$$

$$a(k) = (\lambda + \omega - kv - Dk^2)/\omega. \quad (27)$$

Note that,  $a(k_n^{(1)})a(k_n^{(2)}) = a(k_n^{(3)})a(k_n^{(4)}) = 1$ . Therefore, each eigenvalue  $\lambda_n$  is four-fold degenerate and assuming that the  $k_n^{(i)}$ ’s are all different, the eigenfunction  $|\phi_n(x)\rangle$  can now be written as a linear combination:

$$|\phi_n(x)\rangle = \sum_{i=1}^4 \beta_n^i e^{k_n^{(i)}x} |A_n^i\rangle. \quad (28)$$

The full time-dependent solution for the probability distribution then becomes,

$$|P(x, t)\rangle = |P_{ss}(x)\rangle + \sum_{n=1}^{\infty} c_n e^{\lambda_n t} \sum_{i=1}^4 \beta_n^i e^{k_n^{(i)}x} |A_n^i\rangle. \quad (29)$$

The task is to determine  $\lambda_n$ ’s and  $\beta_n^i$ ’s in terms of the parameters governing the dynamics, viz  $v, \omega, D$  and the system size  $L$ . Recall the boundary conditions from Eq. (3),

$$\begin{aligned} |P(0, t)\rangle &= \begin{bmatrix} P_0^+/2 \\ P_0^-/2 \end{bmatrix} = |P_{ss}(0)\rangle, \\ |P(L, t)\rangle &= \begin{bmatrix} P_1^+/2 \\ P_1^-/2 \end{bmatrix} = |P_{ss}(L)\rangle. \end{aligned} \quad (30)$$

The above imply that, at  $x = 0$ ,  $|P(0, t)\rangle - |P_{ss}(0)\rangle = 0$ , and similar at  $x = L$ . In fact,  $|P_{tr}(x, t)\rangle \equiv |P(x, t)\rangle - |P_{ss}(x)\rangle$  satisfies the masters equation (2a)-(2b) but with the boundary conditions  $|P_{tr}(0, t)\rangle = |P_{tr}(L, t)\rangle = 0$ . Consequently,  $\sum_{n=1}^{\infty} c_n e^{\lambda_n t} |\phi_n(0)\rangle = \sum_{n=1}^{\infty} c_n e^{\lambda_n t} |\phi_n(L)\rangle = 0$  at all times. Using the linear independence of  $e^{\lambda_n t}$ ’s once again, we find, for each  $n$ ,

$$|\phi_n(0)\rangle = |\phi_n(L)\rangle = 0. \quad (31)$$

The time-dependent behaviour of the boundary driven case is therefore identical to that of the one-dimensional dynamics with two absorbing ends at  $x = 0$  and  $L$ . Henceforth, we shall omit

the steady state. Using the absorbing boundary conditions in Eq.(28), we get the following set of equations for the coefficients  $\beta_n^i$ :

$$x = 0 : \sum_{i=1}^4 \beta_n^i = 0, \quad \sum_{i=1}^4 \beta_n^i a_n^i = 0, \quad (32a)$$

$$x = L : \sum_{i=1}^4 \beta_n^i e^{k_n^{(i)}L} = 0, \quad \sum_{i=1}^4 \beta_n^i e^{k_n^{(i)}L} a_n^i = 0, \quad (32b)$$

where  $a_n^i \equiv a(k_n^{(i)})$ . Above equations can be re-written as a matrix equation,  $S(k_n^{(i)})|\beta\rangle = 0$ , where  $|\beta\rangle = [\beta_n^{(1)}, \beta_n^{(2)}, \beta_n^{(3)}, \beta_n^{(4)}]^T$  and

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a_n^1 & a_n^2 & a_n^3 & a_n^4 \\ e^{k_n^{(1)}L} & e^{k_n^{(2)}L} & e^{k_n^{(3)}L} & e^{k_n^{(4)}L} \\ a_n^1 e^{k_n^{(1)}L} & a_n^2 e^{k_n^{(2)}L} & a_n^3 e^{k_n^{(3)}L} & a_n^4 e^{k_n^{(4)}L} \end{bmatrix} \quad (33)$$

To have a nontrivial solution for  $\beta$ 's we must have

$$\det[S] = 0, \quad (34)$$

which, upon solving, determines  $\lambda_n$ 's as a function of system parameters. We observe that,  $\lambda = 0, -2\omega$  satisfies the above determinant equation.  $\lambda = 0$  stands for the steady state which is just zero for the absorbing boundaries and is discarded.  $\lambda = -2\omega$  is the relaxation rate of the tumble dynamics; it governs the relaxation of the initial magnetisation.

## 4.2 Symmetry of the model and solution for the spectrum

The determinant equation (34) is a complicated transcendental equation and we cannot find explicit solutions in general. However, the problem can be greatly simplified. Note that, the master equation (16) satisfies a *reflection symmetry*, that is, whenever the spin is reversed ( $\sigma \rightarrow -\sigma$ ) along with  $x \rightarrow L - x$ , the equation and the boundary conditions remain invariant. To express it mathematically, let us define an operation  $\mathcal{O}_x$ :

$$\mathcal{O}_x[P^+(x), P^-(x)]^T = [P^-(L - x), P^+(L - x)]^T, \quad (35)$$

as shown in Figure 5. We find that,  $\mathcal{O}_x$  commutes with  $\mathcal{L}$ , i.e. it is a symmetry operation. It also satisfies  $\mathcal{O}_x^2 = I$ , implying that its eigenvalues are  $\sigma_x = \pm 1$ . Consequently, the eigenstates  $|\phi_n(x)\rangle$  of the operator  $\mathcal{L}$  will also be an eigenstate of  $\mathcal{O}_x$  corresponding to an eigenvalue  $\pm 1$ . In the following, we segregate all the eigenstates  $|\phi_n(x)\rangle$  into two symmetry sectors labelled by the values of  $\sigma_x$ , and determine the spectrum for each of the sectors.

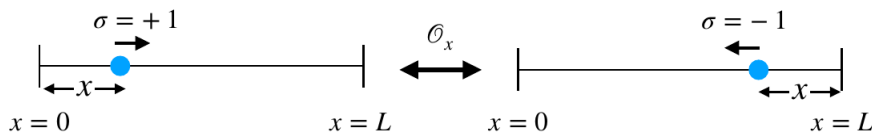


Figure 5: The symmetry of the dynamics. The master equations and the boundary conditions remain invariant under  $\sigma \rightarrow -\sigma$  along with  $x \rightarrow L - x$ .

**spectrum for  $o_x = 1$  (even sector):**

Here,  $\phi_n^+(x) = \phi_n^-(L-x)$ . Using Eq. (28) and the values of  $k_n$ 's from Eq. (25), we find,

$$\beta^{(1)} a^{(1)} e^{k_a x} + \beta^{(3)} a^{(3)} e^{k_b x} + \beta^{(2)} a^{(2)} e^{-k_a x} + \beta^{(4)} a^{(4)} e^{-k_b x} \quad (36)$$

$$= \beta^{(1)} e^{k_a (L-x)} + \beta^{(3)} e^{k_b (L-x)} + \beta^{(2)} e^{-k_a (L-x)} + \beta^{(4)} e^{-k_b (L-x)}. \quad (37)$$

We dropped the suffix  $n$  for convenience. Comparing the coefficients of each independent spatial modes  $e^{k^{(i)}x}$ , we obtain,

$$\beta^{(1)} a^{(1)} = \beta^{(2)} e^{-k_a L} \quad (38)$$

$$\beta^{(3)} a^{(3)} = \beta^{(4)} e^{-k_b L} \quad (39)$$

Using the above expressions in Eqs. (32) for the coefficients  $\beta^{(i)}$ , we obtain,

$$\frac{\beta^{(1)}}{\beta^{(3)}} = -\frac{1 + a^{(3)} e^{k_b L}}{1 + a^{(1)} e^{k_a L}} = -\frac{a^{(3)} + e^{k_b L}}{a^{(1)} + e^{k_a L}}. \quad (40)$$

Rearranging the last equality we get,

$$\frac{e^{-k_b L} - e^{-k_a L}}{1 - e^{-k_a L} e^{-k_b L}} = \frac{a^{(3)} - a^{(1)}}{a^{(3)} a^{(1)} - 1}. \quad (41)$$

Here  $k_a, k_b, a^{(1)}, a^{(3)}$  are all functions of  $\lambda$ . By solving the above expression, we can in principle find the  $\lambda$ 's in this sector.

**spectrum for  $o_x = -1$  (odd sector):**

Here,  $\phi_n^+(x) = -\phi_n^-(L-x)$ . Following the same procedure, we obtain the equation satisfied by the spectrum in this sector as,

$$\frac{e^{-k_b L} - e^{-k_a L}}{1 - e^{-k_a L} e^{-k_b L}} = -\frac{a^{(3)} - a^{(1)}}{a^{(3)} a^{(1)} - 1}. \quad (42)$$

Although the equations (41)-(42) are exact and much simpler compared to Eq. (34), these are not yet exactly solvable for  $\lambda$  in general. However, we earlier mentioned that  $\lambda = 0$  and  $\lambda = -2\omega$  are eigenvalues of the evolution operator since these satisfy Eq. (34). Here we explicitly find that  $\lambda = 0$  satisfies Eq. (42) and therefor belongs to the odd sector (which, as argued earlier, should be discarded for these set of solutions). Whereas  $\lambda = -2\omega$  satisfies Eq. (41) and belongs to the even sector. We find analytical expressions for the 'band' of eigenvalues close to 0 and  $-2\omega$  in the large  $L$  limit and the corresponding eigenfunctions in the two symmetry sectors. In the following, we demonstrate the results and discuss the relaxation behaviour.

#### 4.2.1 Spectrum in the large- $L$ limit

**Eigenvalues near  $\lambda = 0$  and relaxation:** In the limit of large system size, we expect that the distribution takes very long to relax to the steady state and the particle dynamics is close to diffusion. Consequently the low lying  $\lambda$  values are close to zero and from Eq. (24), we get,

$$k_a \approx \frac{\sqrt{2\omega D_e}}{D} (= \mu), \quad k_b \approx \sqrt{\lambda/D_e}, \quad (43)$$

The finite value of  $k_a$  renders  $e^{-k_a L} \approx 0$  in Eqs. (41)-(42). Further, from Eq. (27) one can find that,  $a^{(3)} \approx 1$ ,  $a^{(1)} \approx -\frac{1+\nu/\sqrt{2\omega D_e}}{1-\nu/\sqrt{2\omega D_e}}$ , and the ratio in the rhs of the Eqs. (41)-(42) takes the value,  $\frac{a^{(3)}-a^{(1)}}{a^{(3)}a^{(1)}-1} \approx -1$ . Consequently, we have, for  $o_x = \pm 1$ ,

$$e^{k_b L} \approx \mp 1 \quad \Rightarrow \quad e^{\sqrt{\frac{\lambda}{D_e}} L} \approx \mp 1 \quad (44)$$

The solutions are,

$$\lambda_n \approx -\frac{(2n-1)^2 \pi^2 D_e}{L^2}, \quad n > 0 \text{ (for } o_x = 1), \quad (45a)$$

$$\lambda_{n'} \approx -\frac{(2n')^2 \pi^2 D_e}{L^2}, \quad n' > 0 \text{ (for } o_x = -1). \quad (45b)$$

These are the alternate symmetric and antisymmetric bands near  $\lambda = 0$ . We find that the relaxation rate, that is given by the eigenvalue with real part closest to zero, lies in the symmetric sector and in the large  $L$  limit it is given by,

$$|\lambda_1| = \frac{\pi^2 D_e}{L^2}. \quad (46)$$

Alternatively, the relaxation time is,  $\tau_L = |\lambda_1|^{-1} = \frac{L^2}{\pi^2 D_e}$ . This suggests that, for large systems the relaxation behaviour resembles that of a Brownian particle with an effective diffusion rate  $D_e$ . Note that, the solution given in Eq. (45) for the band of eigenvalues are valid as long as  $|\lambda_n| \ll 2\omega$ , or,  $n \ll \sqrt{\tau_L/\tau_t}$ , where  $\tau_t = (2\omega)^{-1}$  is the relaxation rate of the tumble dynamics.

**Correction to the leading behaviour:** To evaluate the subleading behaviour we need to keep the first correction terms in  $\lambda$  in Eqs. (41)-(42). Since  $k_a$  is finite, in the large  $L$  limit we shall neglect  $e^{-k_a L}$  and therefore the l.h.s. in both the equations is  $e^{-k_b L} \approx e^{-L\sqrt{\lambda/D_e}}$ . On the other hand, for small  $\lambda$ ,  $a^{(3)} \approx 1 - \frac{v}{\omega\sqrt{D_e}}\sqrt{\lambda}$ , whereas  $a^{(1)} \approx -\frac{1+v/\sqrt{2\omega D_e}}{1-v/\sqrt{2\omega D_e}} + O(\lambda)$ . This implies that the corrections in the rhs of Eqs. (41)-(42) are of  $O(\sqrt{\lambda})$ , and the equations to the first subleading order reads,

$$e^{-L\sqrt{\lambda/D_e}} \approx \mp \left( 1 + \frac{v^2}{\omega D_e} \sqrt{\frac{\lambda}{2\omega}} \right), \text{ for } o_x = \pm 1 \text{ respectively.} \quad (47)$$

One can solve the above equation perturbatively by considering  $\lambda_n = \lambda_n^{(0)} + \lambda_n^{(1)}$ , where  $\lambda_n^{(0)} = -\frac{n^2 \pi^2 D_e}{L^2}$  are the leading order solution (Eq. (45)) and  $\lambda_n^{(1)} \ll \lambda_n^{(0)}$  are the corrections:

$$\lambda_n^{(1)} \approx -\frac{2v^2}{\omega\sqrt{2\omega D_e}} \frac{\lambda_n^{(0)}}{L}. \quad (48)$$

The steps are shown in Appendix C. Consequently, the band near  $\lambda = 0$  takes the form,

$$\lambda_n = \lambda_n^{(0)} + \lambda_n^{(1)} \approx \lambda_n^{(0)} \left( 1 - \frac{2v^2}{\omega\sqrt{2\omega D_e}} \frac{1}{L} \right). \quad (49)$$

**Eigenvalues near  $\lambda = -2\omega$ :** We have already mentioned that  $-2\omega$  is an eigenvalue of the evolution operator that belongs to the symmetric sector. Let us consider  $\lambda = -2\omega + \varepsilon$ , where  $\varepsilon$  is presumably small in the large  $L$  limit. In this limit,

$$k_a \approx \sqrt{\frac{v^2 - 2\omega D}{D^2}} + O\left(\frac{\varepsilon}{D}\right), \quad k_b \approx \sqrt{\varepsilon/D}$$

$$a^{(1)} = 1 - \frac{v^2}{\omega D} - \frac{k_a v}{\omega}, \quad a^{(3)} = -1 - \frac{k_b v}{\omega}.$$

Since  $\varepsilon \rightarrow 0$ ,  $a^{(3)} \approx -1$ . For large  $L$ ,  $e^{-k_a L} \rightarrow 0$ <sup>8</sup>. Therefore, Eqs (42)-(43) reduce to

$$e^{-k_b L} = \pm 1 \text{ for } o_x = \pm 1. \quad (50)$$

<sup>8</sup>Here we disregard the particular case of  $v^2 = 2\omega D$ .

The solutions are,

$$\varepsilon_n \approx -\frac{(2n)^2 \pi^2 D}{L^2}, \quad n \geq 0 \quad (\text{for } o_x = 1), \quad (51a)$$

$$\varepsilon_{n'} \approx -\frac{(2n' - 1)^2 \pi^2 D}{L^2}, \quad n' > 0 \quad (\text{for } o_x = -1), \quad (51b)$$

and therefore  $\lambda_n = -2\omega + \varepsilon_n$ . The corrections to the above expressions can be calculated, which is not shown in the present paper.

### 4.3 Crossover behaviour of the relaxation time

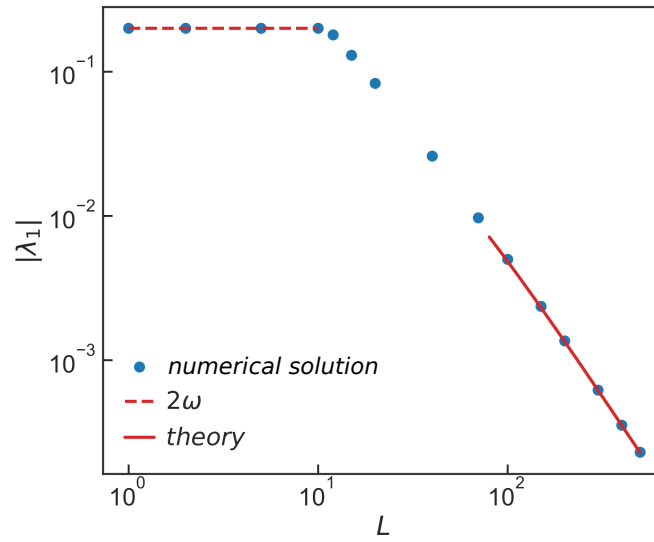


Figure 6: Crossover behaviour of the relaxation rate for different system sizes. Here,  $D = 1.0, \nu = 1.0, \omega = 0.1$ . Points are obtained by numerically solving Eq. (41) (symmetric sector) and identifying the solution closest to zero using Mathematica. The crossover to  $L$ -dependent relaxation is seen to occur for system sizes very close to  $L_p = \frac{\nu}{\omega} = 10.0$ . The large  $L$  behaviour is consistent with Eq. (49) for  $n = 1$ .

For passive particles the relaxation time because of diffusion is proportional to  $L^2/D$ . RTPs on the other hand also carries a tumble dynamics which relaxes in a timescale  $\tau_t \sim (2\omega)^{-1}$ . When the system size is much larger than  $L_c \sim \nu/\omega$ , several tumble events occur before reaching the steady state and the relaxation dynamics becomes effectively passive-like. In this regime the bands are also well-separated. As  $L$  is reduced, the corrections due to activity (Eq. (49)) become significant and a deviation from diffusive behaviour shows up. When  $L \sim L_c$ , the two bands would mix and for  $L < L_c$  they cross, implying that the diffusive dynamics now relaxes faster compared to the tumble dynamics and  $\lambda = -2\omega$  determines the relaxation of the system. This behaviour is shown in Figure 6. We expect this to be a generic feature of finite-sized active systems, and more generally of processes with coloured noise.

#### 4.4 Eigenfunctions and the large-time distribution

Once we have the spectrum, we can completely solve for the coefficients  $\beta_n^i, i = (1, \dots, 4)$ . For  $o_x = \pm 1$ , the coefficients are related as follows:

$$\beta^{(1)} = -\beta^{(3)} \frac{a^{(3)} \pm e^{k_b L}}{a^{(1)} \pm e^{k_a L}}, \quad (52)$$

$$\beta^{(2)} = \pm \beta^{(1)} a^{(1)} e^{k_a L}, \quad (53)$$

$$\beta^{(4)} = \pm \beta^{(3)} a^{(3)} e^{k_b L}, \quad (54)$$

$\beta^{(3)}$  is left arbitrary at this point. After incorporating the coefficients in Eq. (28), the expression of the eigenfunctions  $|\phi_n(x)\rangle$  becomes,

$$|\phi_n(x)\rangle = \beta_n^{(3)} \left[ \frac{a^{(3)} \pm e^{k_b L}}{a^{(1)} \pm e^{k_a L}} \left( e^{k_a x} \begin{bmatrix} a^{(1)} \\ 1 \end{bmatrix} \pm e^{k_a(L-x)} \begin{bmatrix} 1 \\ a^{(1)} \end{bmatrix} \right) - \left( e^{k_b x} \begin{bmatrix} a^{(3)} \\ 1 \end{bmatrix} \pm e^{k_b(L-x)} \begin{bmatrix} 1 \\ a^{(3)} \end{bmatrix} \right) \right]. \quad (55)$$

The set of equations (52)-(54) can be simplified and the coefficients can be determined order by order in the large  $L$  limit. We shall present results for the band near  $\lambda = 0$  and keep the terms which are  $\sim O(L^{-1})$ . The calculation steps are given in the Appendix D. Using the coefficients in Eq. (29), we find, at large times  $t \gg \omega^{-1}$ ,

$$\begin{aligned} |P_{\text{tr}}(x, t)\rangle &\approx \sum_{n=1}^{\infty} c_n e^{\lambda_n t} \beta_n^{(3)} \left( 2i \sin \frac{n\pi x}{L} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - i \frac{n\pi v}{\omega L} \cos \frac{n\pi x}{L} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right. \\ &\quad + \frac{n\pi v}{\omega L} \left\{ \left( 1 + \frac{v}{\sqrt{2\omega D_e}} \right) \sin \frac{n\pi x}{L} + i \frac{v}{\sqrt{2\omega D_e}} \left( 1 - \frac{2x}{L} \right) \cos \frac{n\pi x}{L} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &\quad \left. + i \frac{n\pi v}{\omega L} \left( 1 - \frac{v}{\sqrt{2\omega D_e}} \right) \left\{ (-1)^{n-1} e^{-k_a(L-x)} \begin{bmatrix} a_n^{(1)} \\ 1 \end{bmatrix} + e^{-k_a x} \begin{bmatrix} 1 \\ a_n^{(1)} \end{bmatrix} \right\} \right\}, \quad (56) \end{aligned}$$

with  $\lambda_n \approx -\frac{n^2 \pi^2 D_e}{L^2}$ ,  $n > 0$ ; here we have used  $a_n^{(1)} a_n^{(2)} = 1$ . Each term of the summation in Eq. (56) is an eigenfunction evaluated to  $O(L^{-1})$ . In the following we shall use  $\beta_n$  instead of  $\beta_n^{(3)}$ . To the *leading order*, each of the eigenfunctions is simply proportional to  $\sin \frac{n\pi x}{L}$  and the probability density  $P_{\text{tr}}(x, t) = [1 \ 1] |P_{\text{tr}}(x, t)\rangle$  takes a passive-like form with an effective diffusion constant  $D_e$ :

$$\lim_{t \rightarrow \infty} \lim_{L \rightarrow \infty} P_{\text{tr}}(x, t) \approx \sum_{n=1}^{\infty} c_n \beta_n e^{\lambda_n t} 4i \sin \frac{n\pi x}{L} \equiv P_{\text{passive}}(x, t)|_{D \rightarrow D_e}, \quad (57)$$

while the magnetisation vanishes. In particular, the relaxation mode is given by  $\phi_1(x) \propto \sin \frac{\pi x}{L}$ . Note that the *active* contributions in excess to the effective passivelike contributions, proportional to the active speed  $v$ , appear at  $O(L^{-1})$  in each term of Eq. (56). Thus at very large times when only very few  $n$ 's contribute, the active part in the distribution and the magnetisation  $Q_{\text{tr}}(x, t) = [1 \ -1] |P_{\text{tr}}(x, t)\rangle$  appears only at the subleading order. However, this is not true in the large but intermediate times when larger  $n$ 's also contribute.

To determine the full distribution we need to evaluate the  $n$ -dependent constants  $\beta_n$  and  $c_n$ . Let  $\langle \psi_n(x) |$  be the left eigenvector of the evolution operator  $\mathcal{L}$  corresponding to eigenvalue  $\lambda_n$ . Then the orthonormality of the  $\langle \psi_n(x) |$ 's and  $|\phi_n(x)\rangle$ 's give us  $\beta_n$ . The  $c_n$  is subsequently determined from the initial condition:  $c_n = \int dx \langle \psi_n(x) | P(x, 0) \rangle$ . We shall consider a fixed initial condition,  $|P(x, 0)\rangle = \delta(x - x_0) \begin{bmatrix} s_+ \\ s_- \end{bmatrix}$ , and find the propagator at large times. Here  $s_+, s_-$  are the probabilities of the initial spin to be plus and minus respectively:  $s_+ + s_- = 1$ ,

and  $m_0 = s_+ - s_-$  is the initial magnetisation. The detailed expressions for  $\langle \psi_n(x) \rangle$ ,  $\beta_n$  and  $c_n$ 's are given in the Appendix D. Incorporating these, we obtain the expression of the state to  $O(L^{-1})$  for times  $t \gg \omega^{-1}$ ,

$$\begin{aligned}
|P_{\text{tr}}(x, t)\rangle &= \frac{1}{2L} \sum_{n=1}^{\infty} e^{\lambda_n t} \left( 2 \left\{ 1 - \frac{v^2}{L\omega\sqrt{2\omega D_e}} \right\} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right. \\
&\quad + \frac{n\pi v^2}{L\omega\sqrt{2\omega D_e}} \left\{ \left(1 - \frac{2x_0}{L}\right) \cos \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} + \left(1 - \frac{2x}{L}\right) \cos \frac{n\pi x}{L} \sin \frac{n\pi x_0}{L} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&\quad - \frac{n\pi v}{L\omega} \left\{ \left(\frac{v}{\sqrt{2\omega D_e}} + m_0\right) e^{-k_a x_0} - (-1)^n \left(\frac{v}{\sqrt{2\omega D_e}} - m_0\right) e^{-k_a(L-x_0)} \right\} \sin \frac{n\pi x}{L} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&\quad + \frac{n\pi v}{L\omega} \left(1 - \frac{v}{\sqrt{2\omega D_e}}\right) \sin \frac{n\pi x_0}{L} \left\{ e^{-k_a x} \begin{bmatrix} 1 \\ a^{(1)} \end{bmatrix} - (-1)^n e^{-k_a(L-x)} \begin{bmatrix} a^{(1)} \\ 1 \end{bmatrix} \right\} \\
&\quad \left. + m_0 \frac{n\pi v}{L\omega} \cos \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{n\pi v}{L\omega} \sin \frac{n\pi x_0}{L} \cos \frac{n\pi x}{L} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right). \quad (58)
\end{aligned}$$

At relaxation time scales ( $t \gtrsim L^2/D_e$ ), only the small  $n$  terms ( $n = 1, 2$  etc.) survive. Here the dominating contribution comes from the first term of Eq. (58) which is just the effective passive expression, while the active contribution occurs as subleading corrections. However, at intermediate times  $\omega^{-1} \ll t \ll L^2/D_e$ , all the  $n$ 's contribute and it is not obvious whether the active contributions constitute only a correction to the effective passive behaviour. For this we aim to evaluate the summations in a closed form. Generally this is not doable for the summations at hand, but we can make progress for large values of  $L$  using the Poisson summation formula and the symmetries of the distribution. The details and a closed form expression of  $|P_{\text{tr}}(x, t)\rangle$  is given in Appendix D. For  $x, x_0 \ll L$  the system is well approximated by a semi-infinite line, for which we discuss the results below.

#### 4.5 Semi-infinite line with absorbing barrier at $x = 0$ :

On the semi-infinite geometry the summations in the Eq. (58) can be converted to integrals and the state vector at large times takes the form:

$$\begin{aligned}
|P_{\text{tr}}^{\infty}(x, t)\rangle &= \frac{1}{2} \left( \frac{1}{\sqrt{4\pi D_e t}} \left\{ e^{-\frac{(x-x_0)^2}{4D_e t}} - e^{-\frac{(x+x_0)^2}{4D_e t}} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right. \\
&\quad + \frac{m_0 v}{\omega} \frac{\pi}{(4\pi D_e t)^{3/2}} \left\{ (x-x_0) e^{-\frac{(x-x_0)^2}{4D_e t}} + (x+x_0) e^{-\frac{(x+x_0)^2}{4D_e t}} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&\quad + \frac{v}{\omega} \frac{2\pi}{(4\pi D_e t)^{3/2}} \left\{ \frac{v}{\sqrt{2\omega D_e}} (x+x_0) e^{-\frac{(x+x_0)^2}{4D_e t}} - \left(\frac{v}{\sqrt{2\omega D_e}} + m_0\right) x e^{-k_a x_0} e^{-\frac{x^2}{4D_e t}} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&\quad + \frac{v}{\omega} \left(1 - \frac{v}{\sqrt{2\omega D_e}}\right) \frac{2\pi}{(4\pi D_e t)^{3/2}} x_0 e^{-k_a x} e^{-\frac{x_0^2}{4D_e t}} \begin{bmatrix} 1 \\ a^{(1)} \end{bmatrix} \\
&\quad \left. + \frac{v}{\omega} \frac{\pi}{(4\pi D_e t)^{3/2}} \left\{ (x-x_0) e^{-\frac{(x-x_0)^2}{4D_e t}} - (x+x_0) e^{-\frac{(x+x_0)^2}{4D_e t}} \right\} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right). \quad (59)
\end{aligned}$$

The same expression is obtained by taking the limit  $L \gg x, x_0$  in Eq. (D.13). All the terms in the above expression are  $\sim O(t^{-3/2})$ , and therefore 'passive' as well as the 'active' contributions are equally important in determining the large time behaviour in the semi infinite domain. From Eq. (59) we find the distribution and magnetisation profile by respectively adding and

subtracting the rows:

$$\begin{aligned}
 P_{\text{tr}}^{\infty}(x, t) = & \frac{1}{\sqrt{4\pi D_e t}} \left\{ e^{-\frac{(x-x_0)^2}{4D_e t}} - e^{-\frac{(x+x_0)^2}{4D_e t}} \right\} \\
 & + \frac{v^2}{\omega \sqrt{2\omega D_e}} \frac{2\pi}{(4\pi D_e t)^{3/2}} (x+x_0) e^{-\frac{(x+x_0)^2}{4D_e t}} \\
 & + \frac{m_0 v}{\omega} \frac{\pi}{(4\pi D_e t)^{3/2}} \left\{ (x-x_0) e^{-\frac{(x-x_0)^2}{4D_e t}} + (x+x_0) e^{-\frac{(x+x_0)^2}{4D_e t}} \right\} \\
 & - \frac{v}{\omega} \frac{2\pi}{(4\pi D_e t)^{3/2}} \left\{ \left( \frac{v}{\sqrt{2\omega D_e}} + m_0 \right) x e^{-k_a x_0} e^{-\frac{x^2}{4D_e t}} + \frac{v}{\sqrt{2\omega D_e}} x_0 e^{-k_a x} e^{-\frac{x_0^2}{4D_e t}} \right\}, \quad (60)
 \end{aligned}$$

$$\begin{aligned}
 Q_{\text{tr}}^{\infty}(x, t) = & \frac{v}{\omega} \frac{\pi}{(4\pi D_e t)^{3/2}} \left\{ (x-x_0) e^{-\frac{(x-x_0)^2}{4D_e t}} - (x+x_0) e^{-\frac{(x+x_0)^2}{4D_e t}} \right\} \\
 & + \frac{v}{\omega} \frac{2\pi}{(4\pi D_e t)^{3/2}} x_0 e^{-k_a x} e^{-\frac{x_0^2}{4D_e t}}. \quad (61)
 \end{aligned}$$

In Figure 7 we have shown the distribution and magnetisation obtained by simulating the RTP dynamics on the semi-infinite line and its comparison with the theoretical predictions. The

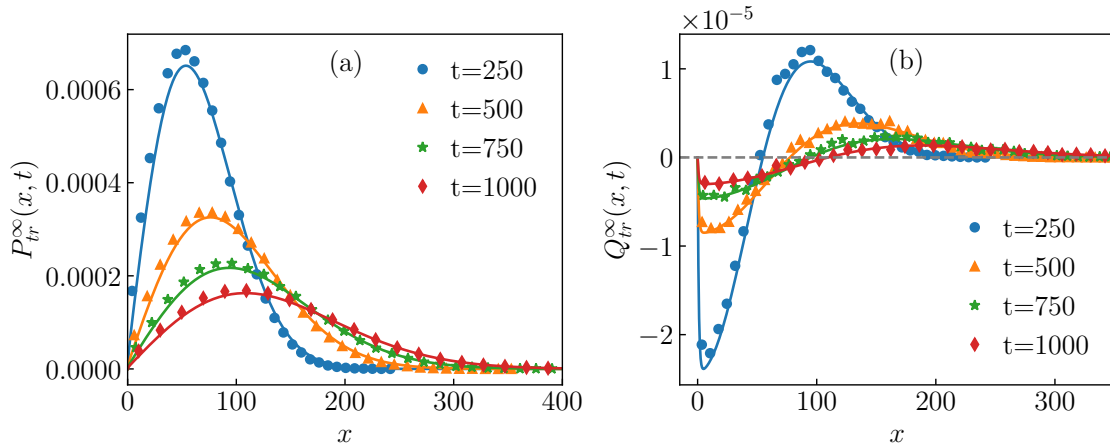


Figure 7: (a) Probability density at different times. Solid lines are the theoretical expression (Eq. (60)). (b) The magnetisation profile at different times. Solid lines are from the expression in Eq. (61). Here  $x_0 = 1$ ,  $m_0 = 0$ ,  $\omega = 0.1$ ,  $v = 1$ ,  $D = 1$ .

profiles carry an interesting structure. In each of Eqs. (60)-(61), the terms that fall exponentially in distance represents the kinetic boundary layer with a ‘skin depth’  $k_a^{-1}$ , identical to that in the steady state. The other terms form the ‘scaling’ part of the profile.

In Eq. (60) we can immediately identify the *passive* and *active* contributions to the distribution. It is useful to look at the relative magnitude of these two parts. We first note that, at space-time scales such that  $t \gg \{\frac{x_0^2}{D_e}, \frac{x^2}{D_e}\}$ ,  $P_{\text{passive}}^{\infty}(x, t) = \frac{1}{\sqrt{4\pi D_e t}} \left\{ e^{-\frac{(x-x_0)^2}{4D_e t}} - e^{-\frac{(x+x_0)^2}{4D_e t}} \right\} \approx \frac{x x_0}{2\sqrt{\pi}(D_e t)^{3/2}}$  and from Eq. (60),  $P_{\text{active}}^{\infty} = P_{\text{tr}}^{\infty} - P_{\text{passive}}^{\infty} \sim O(t^{-3/2})$ , i.e. both occur at the same order. The relative weight of active and passive parts is,

$$\tilde{R}(x_0, x) = \lim_{t \gg \frac{x_0^2}{D_e}, \frac{x^2}{D_e}} \frac{P_{\text{active}}^{\infty}}{P_{\text{passive}}^{\infty}} = \frac{v}{2\omega} \left[ \left( m_0 + \frac{v}{\sqrt{2\omega D_e}} \right) \frac{1 - e^{-k_a x_0}}{x_0} + \frac{v}{\sqrt{2\omega D_e}} \frac{1 - e^{-k_a x}}{x} \right], \quad (62)$$

which is finite throughout the system for any finite  $x_0$ , and can actually attain dominant role for highly persistent (small  $\omega$  or large  $v$ ) run and tumble motion. The effect of activity becomes more prominent when the particle is in the boundary layer  $x \lesssim k_a^{-1}$ . Beyond that this

part decays slowly as  $x^{-1}$ , and asymptotically the contribution from initial position  $x_0$  only remains. In fact, it can be shown from Eq. (60) that, when both  $x$  and  $t$  is very large such that  $y = x/\sqrt{D_e t}$  is finite, both the active and passive contributions are  $\sim (D_e t)^{-1}$ , and their ratio becomes,

$$R(x_0) = \lim_{\{x, t \rightarrow \infty, \frac{x}{\sqrt{D_e t}} = y\}} \frac{P_{\text{active}}^\infty}{P_{\text{passive}}^\infty} = \frac{v}{2\omega} \left( m_0 + \frac{v}{\sqrt{2\omega D_e}} \right) \frac{1 - e^{-k_a x_0}}{x_0}. \quad (63)$$

The nature of  $R(x_0)$  is shown in Figure 8. To see the impact of the active contribution, consider

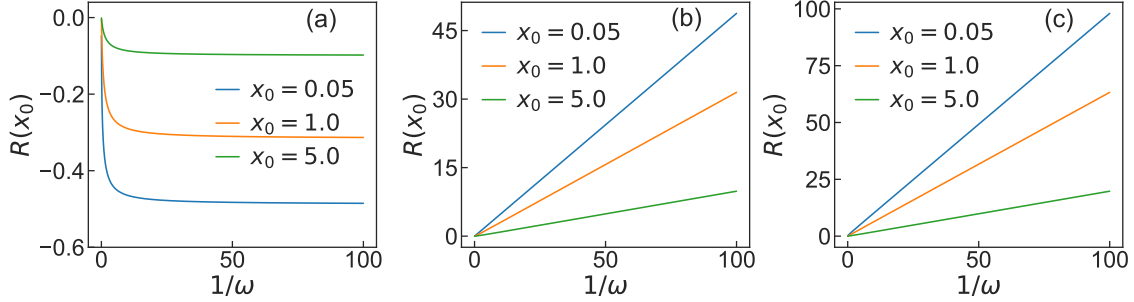


Figure 8:  $R(x_0)$  from Eq. (63) as a function of  $1/\omega$  for (a)  $m_0 = -1$ , (b)  $m_0 = 0$ , (c)  $m_0 = 1$  for different values of  $x_0$ . Here  $v=1, D=1$ .

a highly persistent (small  $\omega$ ) particle starting at some  $x_0 \lesssim k_a^{-1} = \frac{D}{\sqrt{2\omega D_e}} \approx \frac{D}{v}$ . In this case  $R(x_0) \approx \frac{v}{2\omega} \left( m_0 + \frac{v}{\sqrt{2\omega D_e}} \right) k_a \approx \frac{v^2}{\omega D} s_+ - \frac{1}{2}$ : if the probability  $s_+$  of the initial spin to be positive is finite, the active to passive ratio is very high and the distribution in the bulk is almost completely dominated by the active part. This includes the  $m_0 = 0$  case which is closest to a passive-like initial state. This also tells that a perturbation in the active case will spread very differently from its passive counterpart; moreover, an active ‘intrusion’ ( $x_0 \approx 0$ ) will in general stay longer and penetrate further into the system compared to a passive one of same effective diffusivity.

#### 4.5.1 Discussion: thermal noise and the late time behaviour of active particles

This everlasting and often dominant influence of activity to the distribution is partly related to the initial penetration of the particles before they start tumbling. But that effect is strongly modulated in the presence of thermal noise. The nontrivial effect of diffusion is reflected in the flux at  $x = 0$ :

$$\begin{aligned} J(0, t) &= \left[ -D \frac{\partial P(x, t)}{\partial x} + v Q(x, t) \right]_{x=0} \\ &= -\frac{1}{2\sqrt{\pi}(D_e t)^{\frac{3}{2}}} \left[ D_e x_0 + \frac{Dv}{2\omega} \left( m_0 + \frac{v}{\sqrt{2\omega D_e}} \right) (1 - e^{-k_a x_0}) \right]. \end{aligned} \quad (64)$$

For  $D = 0$ , the terms in the square bracket reduce to  $D_e x_0$ , similar to passive particles; and this flux is due to the particles with instantaneous negative spin. For nonzero but small  $D$ ,  $e^{-k_a x_0}$  is still negligible and the flux gains a contribution that couples thermal noise and activity:  $\frac{Dv}{2\omega} \left( m_0 + \frac{v}{\sqrt{2\omega D_e}} \right) \approx \frac{Dv}{2\omega} \left( m_0 + 1 - \frac{\omega D}{v^2} \right) = D \left( \frac{v}{\omega} s_+ - \frac{D}{2v} \right)$ . If  $s_+ > 0$  the change is proportional to  $\frac{Dv}{\omega}$  which is positive and the particles with *initial positive spin* largely contribute to an additional flux at the origin. This is consistent with the picture that at short times the particles with initial positive spin move away from the origin, which makes them available in the system to contribute to the noise driven flux at large times. There is also a spin-independent reduction

of the flux amounting to  $-\frac{D^2}{2\nu}$ , which is much smaller but gives the dominating correction for  $s_+ = 0$ . It pertains to the diffusive spread  $\sim O(\frac{D}{\nu})$  over the persistent motion.

The magnetisation also gives important insights to the dynamics at large times. As expected,  $Q_{\text{tr}}(x, t)$  vanishes in the passive case ( $\nu = 0$ ). It is interesting to note that the magnetisation profile in Eq. (61) is independent of the initial magnetisation  $m_0$ , although the probability density strongly depends on it. At large times the total magnetisation in the system is given by,

$$Q^\infty(t) = \int_0^\infty Q_{\text{tr}}^\infty(x, t) dx = \frac{\nu D}{\omega \sqrt{2\omega D_e}} \frac{2\pi}{(4\pi D_e t)^{3/2}} x_0 e^{-\frac{x_0^2}{4D_e t}}, \quad (65)$$

which is positive and undergoes a slow algebraic decay with time. It is tempting to relate this result to the simple fact that the particles with negative spin are more prone to go out of the absorbing boundary at  $x = 0$ . However, quite remarkably,  $Q^\infty(t)$  vanishes in the absence of thermal diffusion, implying that there are equal numbers of positive and negative spin particles in the system. As shown in the Figure 9, the magnetisation is positive at larger  $x$

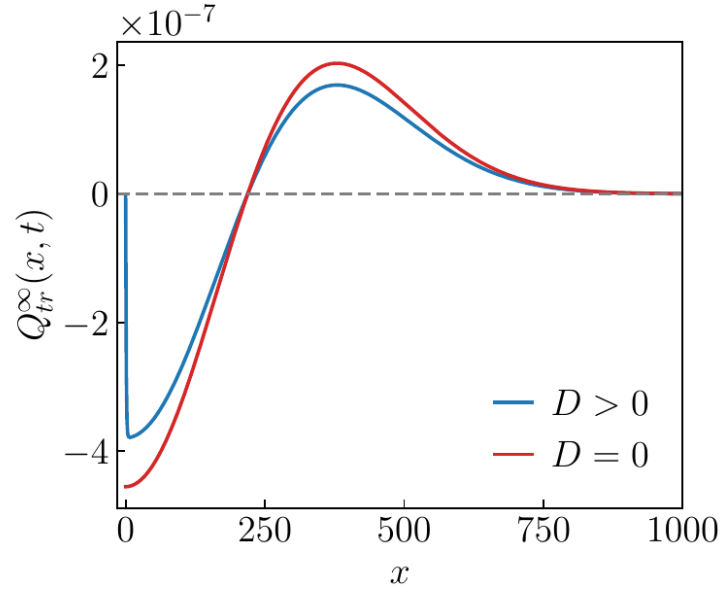


Figure 9: Comparison of magnetisation density using Eq. (61) for  $D > 0$  and  $D = 0$  cases. Here  $\nu = 1$  and  $x_0 = 1$ . We have taken  $\{D = 1, \omega = 0.1\}$ , and  $\{D = 0, \omega = 1/12\}$  such that both cases are activity dominated and the effective diffusivities are also the same ( $D_e = 6$ ).

values; as we reduce  $x$  beyond some point it becomes negative and for  $D = 0$  monotonically decreases to its minimum at  $x = 0$ , signifying a discontinuity at the absorbing end. Further from Eqs. (60)-(61),  $Q_{\text{tr}}(0, t) = -P_{\text{tr}}(0, t)$  for  $D = 0$ , which implies that only the negative spin particles cross the origin. Consequently  $P^+(0, t) = 0$  [49, 50], and discontinuity arises in  $P$  (more specifically  $P^-$ ). The above corroborates the general prevalence of negative spin particles near the origin before these are eventually absorbed. Yet, the net balance of positive and negative spin particles is maintained in the system.

The picture is considerably revised in the presence of even a small thermal diffusion. For very small  $x$  the time to hit the absorbing boundary is small. But at such small times diffusion dominates over drift, and therefore it compels a larger fraction of the particles near  $x = 0$  to get absorbed. However the region near  $x = 0$  is majorly populated by the negative spin particles, implying their enhanced elimination from the system because of the diffusion induced

absorption consistent with the flux reduction for  $s_+ = 0$  ( $m_0 = -1$ ) discussed after Eq. (64). This leads to an imbalance of the particles with different orientations. This also results in removing the discontinuity at the boundary through the formation of the kinetic boundary layer, which in this case is a depletion layer, of width  $k_a^{-1} \sim D/\nu$  for small  $D$ . As a whole, diffusion has a complex and often nonnegligible effect on the behaviour of active particle dynamics. Notably, the bulk density which is referred to as the ‘scaling’ part of the profile in Eq. (60), when extended to  $x = 0$ , deviates from the absorbing condition by an amount proportional to  $l_M = \frac{\nu^2}{\omega \sqrt{2\omega D_e}}$ . This deviation is precisely compensated by the boundary layer.

## 5 Proposed universality in the large time distribution

A natural question is whether the new features in the distribution is specific to RTPs or is more general to active processes. We propose that not only the new features are general, but in fact the large time distribution in Eq. (60) excepting the boundary layer is universal for particle motion subject to exponentially correlated noise.

Let us consider for example the absorbing boundary problem of underdamped passive dynamics. The solution for the underdamped case is complicated and originally given in the Laplace space [60]. However it is relatively simpler to extract the approximate large time behaviour, which is shown in the Appendix E. Quite remarkably the density profile in that case is precisely of the form given in Eq. (60), except with a richer boundary layer structure (Eq. (E.3)) and a boundary discontinuity. Note that the underdamped passive problem is exactly equivalent to the absorbing boundary problem of active Ornstein-Uhlenbeck process (AOUP) in the absence of translational noise and external forces, and so are the solutions. The striking resemblance of such disparate processes, one of which is a passive process, with the only common point being the presence of an additive exponentially correlated noise evokes a general form for the large time distribution in presence of an absorbing barrier,

$$\begin{aligned} P_{\text{tr}}^{\infty}(x, t; x_0) = & \frac{1}{\sqrt{4\pi D_e t}} \left\{ e^{-\frac{(x-x_0)^2}{4D_e t}} - e^{-\frac{(x+x_0)^2}{4D_e t}} \right\} \\ & + \frac{l_M}{2\sqrt{\pi} (D_e t)^{3/2}} (x+x_0) e^{-\frac{(x+x_0)^2}{4D_e t}} \\ & + \frac{u_0 \tau_p}{4\sqrt{\pi} (D_e t)^{3/2}} \left\{ (x-x_0) e^{-\frac{(x-x_0)^2}{4D_e t}} + (x+x_0) e^{-\frac{(x+x_0)^2}{4D_e t}} \right\} \\ & + P_{\text{BL}}, \end{aligned} \quad (66)$$

where  $u_0$  is the average initial velocity,  $\tau_p$  is the noise correlation time or persistence time,  $l_M$  is the Milne length, and  $P_{\text{BL}}$ , which is nonuniversal and generically falls off exponentially with distance, represents the boundary layers. Clearly, the distribution has three components,  $P_{\text{tr}}^{\infty}(x, t) = P_{\text{BM}} + P_{\text{CS}} + P_{\text{BL}}$ . Here  $P_{\text{BM}}$  is the passive-like contribution with an effective diffusivity,  $P_{\text{BM}} = \frac{1}{\sqrt{4\pi D_e t}} \left\{ e^{-\frac{(x-x_0)^2}{4D_e t}} - e^{-\frac{(x+x_0)^2}{4D_e t}} \right\}$  and  $P_{\text{CS}}$  is proposed as a universal scaling function due to the coloured noise,  $P_{\text{CS}}(x, t; x_0) = \frac{l_M}{D_e t} \psi\left(\frac{x}{\sqrt{D_e t}}; \frac{x_0}{\sqrt{D_e t}}\right)$ ,

$$\psi(y; y_0) = \frac{u_0 \tau_p / l_M}{4\sqrt{\pi}} \left\{ (y-y_0) e^{-\frac{(y-y_0)^2}{4}} + (y+y_0) e^{-\frac{(y+y_0)^2}{4}} \right\} + \frac{y+y_0}{2\sqrt{\pi}} e^{-\frac{(y+y_0)^2}{4}}. \quad (67)$$

The scaling function, however, depends on a parameter  $\frac{u_0 \tau_p}{l_M}$  that takes different values for different initial conditions and nature of the noise. For RTP,  $u_0 = m_0 \nu$ ,  $\tau_p = (2\omega)^{-1}$ , and

$l_M = \frac{v}{\sqrt{2\omega D_e}} l_p$  where  $l_p = \frac{v}{2\omega}$  is the persistent length scale. For the nonthermal ( $D = 0$ ) RTP,  $l_M = l_p$ , the boundary layer vanishes and a boundary discontinuity is created.  $l_M$  changes nontrivially in the presence of diffusion which is related to the strength of the boundary layer (Eq. (60)). For the underdamped passive motion,  $D_e = D$ ,  $\tau_p = \frac{m}{\gamma}$  and  $l_M \approx 1.46 l_p$  with  $l_p = \sqrt{D \tau_p}$ , where  $m$  is the mass,  $\gamma$  is the damping coefficient, and  $D$  is the thermal diffusivity. For initial conditions such that  $u_0 = 0$ , the scaling function does become universal.

At this stage we discuss whether  $l_M$  indeed is the Milne length, which would imply that the particle far from the boundary sees the absorbing condition at  $x = -l_M$  instead of  $x = 0$ , as in the steady state with  $Q_0 = Q_1 = 0$  (section 3.1). In Eq. (60) or Eq. (66), the quantity  $P_{\text{tr}}^\infty - P_{\text{BL}}$  doesn't exactly vanish at  $-l_M$  and is off from zero by a tiny amount  $\sim l_M^2 t^{-3/2}$  or  $l_M u_0 \tau_p t^{-3/2}$ . For RTP such terms occur from  $O(L^{-2})$  corrections in the large- $L$  expansion of the spectrum and were omitted. Motivated from Eq. (E.2) for the underdamped passive case, we incorporate these in Eq. (66) to write the distribution at large times  $\sqrt{D_e} t \gg u_0 \tau_p, l_M$  in a particularly simple form,

$$P_{\text{tr}}^\infty(x, t; x_0) - P_{\text{BL}} \approx \frac{1}{\sqrt{4\pi D_e t}} \left[ e^{-\frac{(|x-x_0| \pm u_0 \tau_p)^2}{4D_e t}} - e^{-\frac{(x+x_0+2l_M+u_0 \tau_p)^2}{4D_e t}} \right], \quad (68)$$

where  $\pm$  is for  $x \lessgtr x_0$  respectively. This particular form correctly takes into account the absorbing condition at  $x = -l_M$ , and is proposed as the *universal distribution* for the absorbing boundary problem for dynamics with colored noise with short-ranged correlations. Using simulations we have checked the validity of Eq. (68) for nonthermal ABP as well, which is shown in the Appendix F.

## 6 Conclusion

Analytical results discussed in this paper for noninteracting free RTPs driven by active reservoirs provide several new insights for nonequilibrium processes subject to an exponentially correlated noise. For zero boundary magnetisations, which is closest to the passive case, the steady state current satisfy the Fick's law, and in the large  $L$  limit both the bulk density profile and current go over to the usual diffusive behaviour. Notably, in contrast to the passive case, a kinetic boundary layer and finite bulk magnetisation emerges in the system. The bulk density acquires important corrections equivalent to the substitutions:  $x \rightarrow x + l_M$  and  $L \rightarrow L + 2l_M$  with  $l_M$  the Milne length, reminiscent of effective absorbing conditions outside the boundaries. The scenario with nonzero boundary magnetisation is however starkly different in almost all aspects: [The Fick's law is modified and a nontrivial particle current and bulk magnetisation induced solely due to the boundary magnetisation appears. Generating steady currents by 'pushes' localised at the boundaries, unlike thermodynamic potential having nonzero gradients in the bulk, is not obvious in such noninteracting unbiased dynamics.](#) In this case the bulk density profile can no longer be related to an effective boundary condition, and thermal diffusion acquires a profound role. While nontrivial boundary features are generic to active processes, the complex interplay of the active motion and thermal noise can lead to unexpected current reversal which in turn would cause a change in the nature of the distribution. Surprisingly, the current also shows a nonmonotonous behaviour as the diffusivity and activity parameters are tuned. Whether such current reversal and nonmonotonous transport are generic to boundary driven active systems will be worth studying. It may also provide avenues to manoeuvre the system without affecting the boundary conditions.

In presence of boundary magnetisations a new transport behaviour emerges, which manifests only at a global scale and loosely resembles a Seebeck-like effect. [A key question is the](#)

nature of magnetisation which plays a dual role. The boundary magnetisation, although does not have an associated conserved flux, induce a particle current with magnitude  $O(L^{-1})$  similar to a potential. On the other hand the small and constant bulk magnetisation teams up with the density gradient in driving the current, thus resembling a force. Whether magnetisation and such quantities can act as novel nonequilibrium thermodynamic attributes might be an intriguing question. Devising a boundary magnetisation regulator and simulating particle-boundary interaction remain open, and some of the possibilities outlined in the Appendix A will be explored in future.

In regard to the dynamics, the boundary value problem reduces to a problem with absorbing boundaries at both ends. This enables us to find the spectrum using a reflection symmetry. For large  $L$ , the eigenvalues and eigenfunctions of the time evolution operator have been found analytically and the full initial value problem is solved at large times  $t \gg \omega^{-1}$ . The spectrum is arranged in two distinct bands, one around  $\lambda = 0$  which determines the long time behaviour, and the other around  $\lambda = -2\omega$ . The relaxation rate shows an interesting crossover, being a constant for small systems and passive-like in the thermodynamic limit. It is argued that the crossover is related to the crossing of the bands [52]; a quantitative study of the band structure for finite  $L$  and analytical determination of the crossover point  $L_c$  would be of interest. At large but intermediate times ( $\omega^{-1} \ll t \ll L^2/D_e$ ) the dynamical behaviour mimics the process on a semi-infinite line. In this case the bulk distribution carries strong and often dominant *intrinsically active* signatures not captured by an effective passive description. Many of the features are present in several other systems e.g. the underdamped passive dynamics, the AOUP, and the ABP, where the main quantitative difference occurs in the detailed structure of the boundary layer and the Milne length.

The striking similarity of the distribution in the absorbing boundary problem of such diverse systems points to a new universality in the physics induced by exponentially correlated noise. The unconstrained motions of these particles exhibit diffusive behaviour at large times; similar conclusion holds with reflecting boundaries [28]. The active and passive processes are however very different in the presence of interaction, and a set of model independent features for active single-file dynamics were found recently [61]. Contrary to both of the above, in the absorbing boundary problem the proposed universal distribution in Eq. (68) is new and different from the diffusive one, and it appears to hold for any exponentially correlated noise *including* the passive underdamped motion. We further expect that the steady state of the boundary driven problem in these cases will have a structure similar to Eq. (7), i.e. sum of a linear profile and boundary layers [57, 58]. We must however be cautious that such similarities of active and underdamped passive cases might be lost in presence of trapping and other forces, other noises, or interactions<sup>9</sup>. This is because the active motion considered here is overdamped which is different from the underdamped motion in general, and it would be worthwhile to look into the interplay of inertia and active noise.

Keeping the differences in mind, it seems plausible that the formation of a boundary layer in presence of fluxes at boundaries is generic for motions with coloured noise. For that matter the noise having continuous values might be important, specifically for the athermal case. The one dimensional athermal AOUP gives rise to boundary layers (along with a boundary discontinuity), while the athermal RTP results in a boundary discontinuity only which is smoothened to a boundary layer in presence of diffusion. For the RTP in a 1D box with reflecting boundaries a delta peak occurs at the walls that is converted to an exponential boundary layer for  $D > 0$  [13]. On the other hand, in 2-dimension where the orientation of RTP takes continuous values, a boundary layer is reported for reflecting boundaries even when thermal noise is absent [63], and we would expect a kinetic boundary layer for absorbing boundary. Study-

<sup>9</sup>For example the steady state of a trapped athermal AOUP is different from that of a trapped underdamped passive particle [62].

ing absorbing boundary problems with other variations of continuous valued coloured noise would be useful in this regard.

Apart from the boundary layer which is dynamics dependent, it is instructive to investigate the extent of validity of the proposed universality of the late time density profile in Eqs. (67)-(68), e.g. in the underdamped active processes. Further, which of the emergent features are affected in presence of a translational noise is a relevant question. We speculate that the structure of the density profile will remain the same; however the additional length scales introduced because of the thermal noise would nontrivially change the Milne length and will also alter the boundary layer and particle demography so as to remove the boundary discontinuity present in the nonthermal cases. A detailed and more general understanding of the distribution in Eqs. (67)-(68) will be important in gaining insights into the nature of such processes. The ‘fine-tuning’ of the boundary layer to the Milne length in presence of diffusion, as observed both in the steady state and the late time dynamics, points to a common origin of these nonuniversal features. This curious question is left for future studies.

In conclusion, boundary driven active dynamics provides an elementary and unifying physical ground for diverse nonequilibrium phenomena — some of which are reported here for the first time, while the others were found previously in disparate settings. This work specifically elucidates the interplay of thermal noise with activity, highlights new features in boundary driven transport, and suggests a novel universality for the motion involving coloured noise. Whether these features survive when the noise correlation is non-exponential yet short ranged, as well as the qualitative changes introduced by interactions, will be of particular interest.

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## A Comments on the boundary magnetisation control

Realising the boundary magnetisations in specific systems is important for observing the non-trivial behaviour reported in the paper, which requires modelling the reservoir and the reservoir-particle interaction at the boundaries. Active particles are known to have nontrivial boundary features in general, and it is expected that the boundary magnetisation is typically nonzero in any realistic scenario. Here we qualitatively outline few setups for its control, which is speculative at this stage. Overdamped dynamics and the presence of thermal noise is assumed.

Magnetisation quantifies the imbalance of the particles with different orientations. This may be achieved by modelling the reservoir as RTPs confined in a rectangular box of width much larger compared to that of the channel (Figure 1). The differently oriented particles accumulate at different sides of the box and provide a nonzero magnetisation at the junction with the channel. Following [13] we expect that for box length large compared to other relevant length scales, the magnetisation and density at the edges change quite differently with diffusivity. This might give a handle to access different boundary conditions in a certain range. Introducing a partially permeable sticky membrane [64] in the reservoir near the junction or imposing a bias locally, maybe useful to further tuning these quantities. These might also be useful for making an ‘active-couple’ mentioned before.

To facilitate simulations we may view the problem as follows. A density  $P_0 = P_0^+ + P_0^-$  and magnetisation  $Q_0 = P_0^+ - P_0^-$  can be rephrased as the conditional probability  $s_0^\pm = P_0^\pm / P_0$  of finding the particle with spin  $\pm 1$  respectively, at the given point; this would also induce a net local velocity  $u = v(s_0^+ - s_0^-) = vQ_0/P_0$  to the particle. One may think of enforcing it, e.g.

by manufacturing preferential ‘kicks’ or manipulating the tumble dynamics in the reservoir or at the boundaries. Similar ideas were employed to study RTP in a box with reflecting walls where the tumble occurs only at the boundaries [65]. Devising an appropriate boundary interaction in the presence of bulk tumbling, boundary fluxes and thermal noise is however challenging. Indeed, modelling particle exchange with reservoirs for systems defined on a continuum and acted on by thermal noise is a formidable task [32], which largely prevents simulating such processes. Implementing the problem on a lattice maybe better suited for simulations at present.

On the other hand, in the zero current (or ‘active-couple’) case we can avoid the particle exchange with reservoir and work in a closed setting. Now the densities  $P^\pm$  at the boundaries (junctions) are not known except their difference, and the task is to model the particle-boundary interaction properly for maintaining the boundary magnetisations. One possibility is to find the  $P_0, P_1$  at the steady state analytically by normalising Eq. (7), then enforce a time independent probability  $s^\pm = P^\pm/P$  for particle orientation at the boundaries.

## B Exit probability of RTP at the left boundary

The exit probabilities  $E_\pm(x)$  at the left boundary for an RTP initially at  $x$  are obtained by solving two coupled backward equations with boundary conditions,  $E_\pm(0) = 1, E_\pm(L) = 0$  (see Eq. (51) of [13] with signs interchanged). The equations are identical to that satisfied by  $P_\mp(x)$  in the steady state and for zero boundary magnetisations the density and magnetisation profiles can be written as,

$$P(x) = P_1 - \Delta P [E_+(x) + E_-(x)], \quad Q(x) = \Delta P [E_+(x) - E_-(x)], \quad (\text{B.1})$$

where  $\Delta P = P_1 - P_0$ . After simulating the exit probability, this relation is used for simulating the steady state distributions for boundary conditions with  $Q_0 = Q_1 = 0$ . It however turns out that, as the tumble dynamics mixes the spin states, the correspondence between the exit probabilities and the steady state distributions are lost for general boundary conditions.

Note that the results for  $E_\pm$  reported in Eqs. (C5) of [13] carry some typo. The correct result is,

$$E_m(x) = \frac{(1 + e^{-\mu L})x - \frac{v}{2\omega}(m - \frac{v}{\mu D})[1 - e^{-\mu x}] + \frac{v}{2\omega}(m + \frac{v}{\mu D})[e^{-\mu(L-x)} - e^{-\mu L}]}{L(1 + e^{-\mu L}) + \frac{v^2}{\omega \mu D}(1 - e^{-\mu L})}. \quad (\text{B.2})$$

Using  $m = \pm 1$  one can find the expressions for  $E_\pm$ .

## C Subleading correction to the band near $\lambda = 0$

We want to solve Eq. (47) to find the leading  $L$ -dependence of  $\lambda_n^{(1)}$ . Putting  $\lambda_n = \lambda_n^{(0)} + \lambda_n^{(1)}$  in Eq. (47), we get,

$$e^{-L\sqrt{\frac{\lambda_n^{(0)} + \lambda_n^{(1)}}{D_e}}} \approx \mp \left( 1 + \frac{v^2}{\omega D_e} \sqrt{\frac{\lambda_n^{(0)} + \lambda_n^{(1)}}{2\omega}} \right). \quad (\text{C.1})$$

To the first subleading order the left hand side of the Eq. (C.1) can be written as,

$$e^{-L\sqrt{\frac{\lambda_n^{(0)} + \lambda_n^{(1)}}{D_e}}} = e^{-L\sqrt{\frac{\lambda_n^{(0)}}{D_e}} \left( 1 + \frac{\lambda_n^{(1)}}{\lambda_n^{(0)}} \right)^{1/2}} \approx e^{-L\sqrt{\frac{\lambda_n^{(0)}}{D_e}}} e^{-L\sqrt{\frac{\lambda_n^{(0)}}{D_e}} \left( \frac{\lambda_n^{(1)}}{2\lambda_n^{(0)}} \right)} \quad (\text{C.2})$$

Now, using in the above the expression for  $\lambda_n^{(0)}$  from Eq. (45), the l.h.s. of Eq. (C.1) is obtained as,

$$e^{-L\sqrt{\frac{\lambda_n^{(0)}+\lambda_n^{(1)}}{D_e}}} \approx \mp 1 \times \exp\left(-i \frac{n\pi}{2} \frac{\lambda_n^{(1)}}{\lambda_n^{(0)}}\right) \approx \mp 1 \pm i \frac{n\pi}{2} \frac{\lambda_n^{(1)}}{\lambda_n^{(0)}}, \quad (\text{C.3})$$

Similarly, we find the right hand side of Eq. (C.1) to the first subleading order,

$$\mp \left(1 + \frac{v^2}{\omega D_e} \sqrt{\frac{\lambda_n^{(0)} + \lambda_n^{(1)}}{2\omega}}\right) \approx \mp \left(1 + \frac{v^2}{\omega D_e} \sqrt{\frac{\lambda_n^{(0)}}{2\omega}}\right) = \mp 1 \mp i \frac{n\pi}{L} \frac{v^2}{\omega \sqrt{2\omega D_e}}.$$

Then, equating the l.h.s. and r.h.s. of Eq. (C.1), we get Eq. (48) as the leading correction.

## D Calculation of the coefficients $\beta^{(i)}$ , expression for the left eigenvectors of $\mathcal{L}$ in the $\lambda = 0$ band and the distribution for large but finite $L$

In section 4.4 we write the form of the distribution for large but finite systems with size  $L$ . Here we first evaluate the coefficients  $\beta^{(i)}$ . Eqs. (52)-(54) gives the relation among the coefficients,

$$\beta^{(1)} = -\beta^{(3)} \frac{a^{(3)} \pm e^{k_b L}}{a^{(1)} \pm e^{k_a L}}, \quad (\text{D.1})$$

$$\beta^{(2)} = \pm \beta^{(1)} a^{(1)} e^{k_a L}, \quad (\text{D.2})$$

$$\beta^{(4)} = \pm \beta^{(3)} a^{(3)} e^{k_b L}, \quad (\text{D.3})$$

In the large  $L$  limit we find, for  $o_x = \pm 1$ ,  $e^{k_b L} \approx \mp(1 - \frac{n\pi}{L} \frac{v^2}{\omega \sqrt{2\omega D_e}})$  using Eqs. (C.3) and (48), and  $\beta^{(1)} = -\beta^{(3)} \frac{a^{(3)} \pm e^{k_b L}}{a^{(1)} \pm e^{k_a L}} \approx \mp \beta^{(3)} [a^{(3)} \pm e^{k_b L}] e^{-k_a L}$  from Eq. (D.1). In Eq. (27), keeping the first subleading term in the expression for  $a^{(3)}$  and using Eq. (49) to determine  $k_b = \sqrt{\lambda/D_e}$  to  $O(L^{-1})$ , we find,  $a^{(3)} \approx 1 - vk_b/\omega \approx 1 - i \frac{v}{\omega} \frac{n\pi}{L}$ . Putting these together in Eq. (D.1) we obtain,

$$\beta^{(1)} = \pm i \beta^{(3)} \frac{n\pi}{L} \frac{v}{\omega} \left(1 - \frac{v}{\sqrt{2\omega D_e}}\right) e^{-k_a L}, \text{ for } n = \text{odd/even}. \quad (\text{D.4})$$

Using the above in Eq. (D.2) we find,

$$\beta^{(2)} = i \beta^{(3)} \frac{n\pi}{L} a^{(1)} \frac{v}{\omega} \left(1 - \frac{v}{\sqrt{2\omega D_e}}\right); \quad (\text{D.5})$$

and finally, from Eq. (D.3)

$$\beta^{(4)} = -\beta^{(3)} a^{(3)} \left(1 - i \frac{n\pi}{L} \frac{v^2}{\omega \sqrt{2\omega D_e}}\right). \quad (\text{D.6})$$

Using the coefficients in Eq. (28), the eigenvector  $|\phi_n(x)\rangle$  is found up to a prefactor  $\beta_n^{(3)}$  (we resume the suffix  $n$ ). To determine  $\beta_n^{(3)}$  we use an orthonormality condition, for which we need to find the left eigenvectors  $\langle\psi_n|$  of  $\mathcal{L}$ . Noting that it is just the transpose of the right

eigenvector of  $\mathcal{L}^\dagger \equiv \mathcal{L}|_{\partial_x \rightarrow -\partial_x}$  we write:

$$\begin{aligned} \langle \psi_n(x) | &= | \phi_n(x, -v) \rangle^T \approx \tilde{\beta}_n^{(3)} \left( 2i \sin \frac{n\pi x}{L} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T + i \frac{n\pi v}{\omega L} \cos \frac{n\pi x}{L} \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \right. \\ &\quad - \frac{n\pi v}{\omega L} \left\{ \left( 1 - \frac{v}{\sqrt{2\omega D_e}} \right) \sin \frac{n\pi x}{L} - i \frac{v}{\sqrt{2\omega D_e}} \left( 1 - \frac{2x}{L} \right) \cos \frac{n\pi x}{L} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \\ &\quad \left. - i \frac{n\pi v}{\omega L} \left( 1 + \frac{v}{\sqrt{2\omega D_e}} \right) \left\{ (-1)^{n-1} e^{-k_a(L-x)} \begin{bmatrix} \tilde{a}_n^{(1)} \\ 1 \end{bmatrix}^T + e^{-k_a x} \begin{bmatrix} 1 \\ \tilde{a}_n^{(1)} \end{bmatrix}^T \right\} \right\}. \end{aligned} \quad (\text{D.7})$$

$$\Rightarrow \int_0^L dx \langle \psi_n(x) | \phi_m(x) \rangle = -4 \tilde{\beta}_n^{(3)} \beta_n^{(3)} L \left( 1 - i(i + n\pi) \frac{2l_M}{L} \right) \delta_{mn}, \quad l_M = \frac{v^2}{2\omega \sqrt{2\omega D_e}} \quad (\text{D.8})$$

up to first sub-leading order in  $L$ . Invoking the orthonormality condition of the eigenfunctions, we obtain,

$$\tilde{\beta}_n^{(3)} \beta_n^{(3)} \approx -\frac{1}{4L} \left( 1 + i(i + n\pi) \frac{2l_M}{L} \right). \quad (\text{D.9})$$

The knowledge of the  $\tilde{\beta}\beta$  product is sufficient to determine the large time distribution, which we find next. For the initial condition,  $|P(x, 0)\rangle = \delta(x - x_0) \begin{bmatrix} s_+ \\ s_- \end{bmatrix}$ , we find,

$$\begin{aligned} c_n = \int_0^L dx \langle \psi_n(x) | P(x, 0) \rangle &\approx \tilde{\beta}_n^{(3)} \left[ 2i \sin \frac{n\pi x_0}{L} + \frac{in\pi v}{\omega L} (s_+ - s_-) \cos \frac{n\pi x_0}{L} \right. \\ &\quad - \frac{n\pi v}{\omega L} \left( 1 - \frac{v}{\sqrt{2\omega D_e}} \right) \sin \frac{n\pi x_0}{L} + \frac{in\pi v^2}{\omega L \sqrt{2\omega D_e}} \left( 1 - \frac{2x_0}{L} \right) \cos \frac{n\pi x_0}{L} \\ &\quad \left. - \frac{in\pi v}{\omega L} \left( 1 + \frac{v}{\sqrt{2\omega D_e}} \right) \left\{ (-1)^{n-1} e^{-k_a(L-x_0)} (\tilde{a}_n^{(1)} s_+ + s_-) + e^{-k_a x_0} (s_+ + \tilde{a}_n^{(1)} s_-) \right\} \right]. \end{aligned} \quad (\text{D.10})$$

Using Eqs. (D.9) and (D.10) in Eq. (56), we obtain Eq. (58) for the propagator to  $O(L^{-1})$  for times  $\omega^{-1} \ll t \ll L^2/D_e$ . To find a closed-form expression, we define the following quantities corresponding to the summations that occur in Eq. (58) and keep only the leading terms after using the Poisson summation formula:

$$\begin{aligned} A(x_1, x_2) &\equiv \sum_{n \geq 1} e^{-D_e t \frac{n^2 \pi^2}{L^2}} \sin \frac{n\pi x_1}{L} \sin \frac{n\pi x_2}{L} \approx \frac{L}{2\sqrt{4\pi D_e t}} \left[ e^{-\frac{(x_1-x_2)^2}{4D_e t}} - e^{-\frac{(x_1+x_2)^2}{4D_e t}} - e^{-\frac{(x_1+x_2-2L)^2}{4D_e t}} \right] \\ &= L a_L(x_1, x_2), \\ B(x_1, x_2) &\equiv \sum_{n \geq 1} e^{-D_e t \frac{n^2 \pi^2}{L^2}} \frac{n\pi}{L} \sin \frac{n\pi x_1}{L} \cos \frac{n\pi x_2}{L} \\ &\approx \frac{\pi L}{(4\pi D_e t)^{3/2}} \left[ (x_1 + x_2) e^{-\frac{(x_1+x_2)^2}{4D_e t}} + (x_1 - x_2) e^{-\frac{(x_1-x_2)^2}{4D_e t}} + (x_1 + x_2 - 2L) e^{-\frac{(x_1+x_2-2L)^2}{4D_e t}} \right] \\ &= L b_L(x_1, x_2) \\ C(x_1) &\equiv \sum_{n \geq 1} e^{-D_e t \frac{n^2 \pi^2}{L^2}} \frac{n\pi}{L} \sin \frac{n\pi x_1}{L} \approx \frac{\pi L}{2(4\pi D_e t)^{3/2}} \left[ x_1 e^{-\frac{x_1^2}{4D_e t}} + (x_1 - 2L) e^{-\frac{(x_1-2L)^2}{4D_e t}} \right] \\ &= L c_L(x_1). \end{aligned} \quad (\text{D.11})$$

The terms in Eq. (58) that involve  $(-1)^n$  are just  $C(L - x_1)$ , since,

$$C(L - x_1) = - \sum_{n \geq 1} (-1)^n e^{-D_e t \frac{n^2 \pi^2}{L^2}} \frac{n\pi}{L} \sin \frac{n\pi x_1}{L}.$$

The  $L$ -dependent terms in the square brackets, which are very small for  $L \gg x_1, x_2$ , are kept in view of the symmetries satisfied by the discrete summation. These also preserve the symmetries of the density and magnetisation profiles:

$$\begin{aligned} P(x, t; x_0, m_0) &= P(L - x, t; L - x_0, -m_0), \\ Q(x, t; x_0, m_0) &= -Q(L - x, t; L - x_0, -m_0). \end{aligned} \quad (\text{D.12})$$

Using Eq. (D.11) we rewrite Eq. (58) in terms of  $a_L$ ,  $b_L$  and  $c_L$ ,

$$\begin{aligned} |P_{\text{tr}}(x, t)\rangle &\approx \frac{1}{2} \left( 2a_L(x_0, x) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + m_0 \frac{v}{\omega} b_L(x, x_0) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{v}{\omega} b_L(x_0, x) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right. \\ &\quad + \frac{v^2}{\omega \sqrt{2\omega D_e}} \left\{ \left(1 - \frac{2x_0}{L}\right) b_L(x, x_0) + \left(1 - \frac{2x}{L}\right) b_L(x_0, x) \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &\quad - \frac{v}{\omega} \left\{ \left(\frac{v}{\sqrt{2\omega D_e}} + m_0\right) e^{-k_a x_0} c_L(x) + \left(\frac{v}{\sqrt{2\omega D_e}} - m_0\right) e^{-k_a(L-x_0)} c_L(L-x) \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &\quad \left. + \frac{v}{\omega} \left(1 - \frac{v}{\sqrt{2\omega D_e}}\right) \left\{ e^{-k_a x} c_L(x_0) \begin{bmatrix} 1 \\ a^{(1)} \end{bmatrix} + e^{-k_a(L-x)} c_L(L-x_0) \begin{bmatrix} a^{(1)} \\ 1 \end{bmatrix} \right\} \right), \end{aligned} \quad (\text{D.13})$$

which gives an approximate closed form expression of the propagator in the large  $L$  limit in terms of known functions. We obtain Eq. (59) in the limit  $L \rightarrow \infty$ . It is evident that unlike individual eigenstates, the ‘active’ contributions appear at the leading order in the time-dependent distribution.

## E Large time distribution of a passive underdamped Brownian motion in presence of an absorbing barrier

The absorbing boundary problem of a one dimensional underdamped Brownian motion is surprisingly challenging. The problem was posed in 1945 [66] and finally solved after four decades [60, 67]. Here we are interested in the late time behaviour of the distribution of position of the unbiased underdamped motion. The full solution is quite complex and is originally given in the Laplace space (Eq. (4.4) of [60]):

$$\begin{aligned} \tilde{P}_{x_0, v_0}(x, v; s) &= \frac{e^{-v^2/2}}{\sqrt{8\pi}} \sum_{n=0}^{\infty} \frac{e^{-q_n |x-x_0|}}{q_n} f_n^{\mp}(v) f_n^{\mp}(v_0) \\ &\quad - \frac{e^{-v^2/2}}{\sqrt{32\pi}} \sum_{m,n=0}^{\infty} \frac{\sigma_{mn}}{q_m q_n} e^{-q_m x - q_n x_0} f_m^+(v) f_n^-(v_0), \quad \mp \text{ is for } x \lessgtr x_0, \end{aligned} \quad (\text{E.1})$$

where  $x_0, v_0$  are the initial positions and velocities respectively, the absorbing boundary is at  $x = 0$ ,  $q_n = \sqrt{n+s}$ ,  $\sigma_{mn} = \frac{1}{q_m+q_n} \frac{1}{Q_m Q_n} = \sigma_{nm}$  with  $Q_n = \lim_{N \rightarrow \infty} \frac{\sqrt{n!N!} \exp(2q_n \sqrt{N+1})}{\prod_{r=0}^{N+n} (q_r + q_n)}$ , and  $f_n^+(v) = f_n(v) = \frac{e^{v^2/4}}{\sqrt{n!}} \left[ (-1)^n e^{z^2/4} \frac{d^n}{dz^n} e^{-z^2/2} \right]_{z=2q_n-v}$ ,  $f_n^-(v) = f_n(-v)$ . Here all the variables are non-dimensionalised,  $t \rightarrow t\gamma/m$ ,  $x \rightarrow x\sqrt{\gamma/(mD)}$ ,  $v \rightarrow v\sqrt{m/(D\gamma)}$ ,  $m$  is the mass of the particle,  $\gamma$  is the damping coefficient, and  $D$  is the thermal diffusivity. To find the distribution at large times we need to extract the leading behaviour of  $\tilde{P}$  in Eq. (E.1) in the  $s \rightarrow 0$  limit. In this limit, the terms in the summations with only  $m = 0$  or  $n = 0$  will contribute, and the different quantities are evaluated as,

$$\text{I. } f_0(v) \approx e^{-s+v\sqrt{s}}, \quad f_n(v) \approx \frac{\Phi_n(v)}{\sqrt{n!}} e^{-n+v\sqrt{n}} \text{ for } n > 0, \text{ where } \Phi_n(v) \text{ is a polynomial of degree } n.$$

- II. The  $Q_n$ 's are evaluated using Euler-Maclaurin formula:  $\sum_{r=1}^N f(r) = \int_1^N f(x)dx + \frac{f(1)+f(N)}{2} + \mathcal{R}_N$ , where  $\mathcal{R}_N$  is the residue. Without considering  $\mathcal{R}_N$  we find,

$$Q_0 \approx \frac{1}{2\sqrt{s}} e^{\frac{3}{2}\sqrt{s}}, \quad Q_n(s) \approx \left[ \frac{n!}{n(1+\sqrt{n})} \right]^{\frac{1}{2}} e^{\frac{n}{2} + \sqrt{n}} e^{-\sqrt{s/n}}$$

for  $n > 0$ . If we take the residue term into account, the exponent  $\frac{3}{2}$  in  $Q_0(s)$  would be slightly modified; e.g. the first correction would give an exponent  $\frac{3}{2} - \frac{1}{24} \approx 1.458$ . In  $Q_n$ , the residue would introduce a slowly varying prefactor  $b_n : Q_n \rightarrow b_n Q_n$ .

Putting these together, and integrating out the velocity, we find the distribution of the particle's position in the Laplace space,

$$\tilde{P}_{x_0, v_0}(x; s) \approx \frac{e^{-\frac{3}{2}s}}{2\sqrt{s}} \left[ e^{-(|x-x_0| \pm v_0)\sqrt{s}} - e^{-(x+x_0+2l_M+v_0)\sqrt{s}} \right] + \tilde{P}_{BL}(x; s), \quad (\text{E.2})$$

where  $l_M \approx 1.458$ ,  $\pm$  is for  $x \lessgtr x_0$ , and

$$\begin{aligned} \tilde{P}_{BL}(x; s) \approx & \frac{1}{2} \sum_{n=0}^{\infty} \left[ \frac{1+\sqrt{n}}{n} \right]^{\frac{1}{2}} \frac{e^{-\frac{3}{2}n-\sqrt{n}}}{n! b_n} e^{-(v_0 - \frac{1}{\sqrt{n}})\sqrt{s}} \\ & \times \left[ H(n) e^{-x\sqrt{n}} e^{-(x_0+v_0)\sqrt{s}} + \Phi_n(-v_0) e^{-(x_0-v_0)\sqrt{n}} e^{-x\sqrt{s}} \right], \end{aligned} \quad (\text{E.3})$$

with  $H(n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dv \Phi_n(v) e^{-\frac{v^2}{2} + v\sqrt{n}}$ . Plugging in the units and taking the inverse transform, we find the leading time dependence of the density profile for the problem,

$$P_{x_0, v_0}(x; t) \approx \frac{1}{\sqrt{4\pi D_e t}} \left[ e^{-\frac{(|x-x_0| \pm v_0 m/\gamma)^2}{4D_e t}} - e^{-\frac{(x+x_0+2l_M+v_0 m/\gamma)^2}{4D_e t}} \right] + P_{BL}(x, t), \quad (\text{E.4})$$

where the boundary layer term  $P_{BL}(x, t)$  is given by the inverse Laplace transform of Eq. (E.3). Quite remarkably, for large  $t$ , expanding the Gaussian terms in Eq. (E.4) around  $|x-x_0|$  and  $(x+x_0)$  respectively and keeping the terms up to  $O(xt^{-3/2})$  reduces the density to a form similar to Eq. (60) obtained for RTP, except the boundary layer.

Note that the distribution is markedly different from that obtained in the overdamped problem. It contains a rich 'kinetic boundary layer' structure of a finite skin depth  $\sim l_p = \sqrt{mD/\gamma}$  near the absorbing walls. Secondly, far from the wall at large times the distribution takes a scaling-like form  $P_{sc} = P_{x_0, v_0}^{\infty}(x, t) - P_{BL}$  which corresponds to a spatial shift  $x \rightarrow x + l_M$  and  $x_0 \rightarrow x_0 + l_M + v_0 m/\gamma$ .  $P_{sc}$  satisfies the absorbing condition at  $x = -l_M$  instead of  $x = 0$  [55], i.e.  $P_{sc}(-l_M, t) = 0$ , implying that  $l_M$  is the Milne length. In [67] the Milne length is exactly determined,  $l_M = -\zeta(\frac{1}{2})l_p \approx 1.460l_p$ , a value approximated remarkably well by the Euler-Maclaurin formula. Also note the factor  $e^{-\frac{3}{2}s}$  in Eq. (E.2) that corresponds to a temporal shift,  $t \rightarrow t - \frac{3}{2}\tau_p$  with  $\tau_p = m/\gamma$ , suggesting that  $P_{sc}$  pertains to an initial condition at  $t = \frac{3}{2}\tau_p$  instead of  $t = 0$  [55].

## F Distribution of ABPs on a semi-infinite line with absorbing condition at $x = 0$

The equation of motion of an overdamped ABP is given by,

$$\dot{x} = v \cos \theta(t) + \eta(t), \quad \dot{\theta} = \sqrt{2D_r} \zeta(t), \quad (\text{F.1})$$

where  $v$  is the self-propulsion speed,  $\theta$  is the instantaneous orientation of the particle which takes on continuous values,  $D_r$  is rotational diffusivity, and  $\eta$  is the thermal noise. Since  $\langle \cos(\theta(t)) \cos(\theta(t')) \rangle = e^{-D_r |t-t'|}$ , the ABP is driven by an exponentially correlated noise.

We simulated the Eq. (F1) with  $\eta(t) = 0$  on a semi-infinite line subject to the absorbing boundary condition at  $x = 0$ . We have taken random initial orientation such that  $u_0 = 0$ . The distribution at large times is captured remarkably well with  $P^\infty(x, t) - P_{BL}$  given in Eq. (68), with  $D_e = \frac{v^2}{2D_r}$ ,  $L_M \approx 0.8 \frac{v}{D_r}$ . The result is shown in Figure F1. This corroborates the proposed

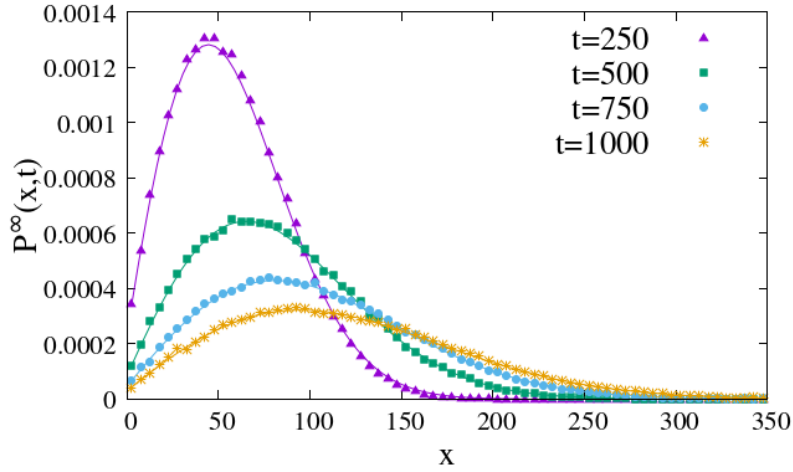


Figure F1: Distribution of athermal ABPs at different times on a semi-infinite line with an absorbing barrier at  $x = 0$ . The data is shown with particle parameters  $v = \sqrt{2}$ ,  $D_r = 0.2$  and initial condition  $x_0 = 1.0$ ,  $u_0 = \langle \cos(\theta(0)) \rangle \approx 0$ . The lines correspond to the expression in Eq. (68).

universality of the large time distribution. Note that  $P(x, t)$  has a discontinuity at  $x = 0$ . Since the noise is continuous-valued, we expect a boundary layer to be present near  $x = 0$ . This is however not taken up in this paper.

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