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The Komar charge in presence of the Holst term and the gravitational Witten effect

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Abstract

In the first-order formalism, the Einstein–Hilbert action can be modified by the addition of a Holst term multiplied by the Barbero parameter α. This modification breaks parity although it does not affect the equations of motion. We show that the standard Komar charge is also modified by the addition of a topological term multiplied by the Barbero parameter α. For the Killing vector that generates time translations, the value of the Komar integral at infinity is modified by the addition of a term proportional to the NUT charge N and the parity-breaking Barbero parameter. Thus, as in the standard Witten effect, a non-vanishing NUT charge N induces a non-vanishing mass $-\alpha N$.

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1. Introduction. The second-order Einstein–Hilbert (EH) action is the simplest non-trivial diff-invariant action one can write for the gravitational field. It leads to equations of second order in derivatives of the gravitational field (the Einstein equations) and it is natural to ask if it can be modified adding other diff-invariant terms while preserving the second-order degree of the equations of motion.

As Einstein himself soon discovered, the simplest modification that satisfies this condition is the addition of the cosmological-constant term. As a matter of fact, in $d = 4$ dimensions, the only other diff-invariant terms that one can add are either topological (the Gauss–Bonnet and the Euler 4-forms) or boundary terms that do not modify the equations of motion [1].

The first-order (Palatini [2]) formalism turns out to offer more possibilities because one can use terms that would vanish for the Levi–Civita connection but do not vanish off-shell, for a generic connection. In $d = 4$ dimensions, these are the Nieh–Yan [3, 4] and Holst (H) [5] terms. The first is quadratic in torsion and vanishes for the torsionless Levi–Civita connection. The second is proportional to a term that becomes the Bianchi identity of the Riemann curvature (*i.e.* when the connection is the Levi–Civita connection). They differ by a total derivative.

Although, out of all these possible additions, only the cosmological term modifies the Einstein equations, all of them give non-trivial contributions to the Noether–Wald charge.[1] In this letter we will only consider the H terms, not just for the sake of simplicity, but because, as we are going to show, the H term is the analogue of the parity-breaking, topological, θ-term, $\theta F \wedge F$, that one can add to the Maxwell Lagrangian $F \wedge \star F$. As we are going to show, this similarity between the H term in gravity and the θ term in electromagnetism implies that their presence also has similar effects.[2]

2. The Einstein–Hilbert and Holst terms. In the first-order formulation, the Vielbein 1-form $e^a \equiv e^a{}_\mu dx^\mu$ and the Lorentz connection $\omega^{ab} \equiv \omega_\mu{}^{ab} dx^\mu = -\omega^{ba}$ are independent variables. The Lorentz curvature 2-form is in our conventions[3]

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}. \tag{1}$$

In differential-form language, the 4-dimensional first-order EH action can be written in the form

$$S_{\rm EH}[e, \omega] = -\frac{1}{16\pi G_{\rm N}^{(4)}} \int_{\mathcal{M}} e^a \wedge e^b \wedge \tilde{R}_{ab}(\omega), \tag{2}$$

[1]See Ref. [9] for a comprehensive and pedagogical review. On the other hand, using the "gravity as a gauge theory of the Poincaré group" approach one can show that the list of terms one can add to the first-order action in $d = 4$ dimensions that we have given above is exhaustive.

[2]Interestingly, the Host term can be supersymmetrized [6, 8, 7]. In the last of these references it was also shown that, in the supergravity context, the NUT charge is associated to the supersymmetrization of the Holst term.

[3]Our conventions are those of Ref. [12]. Notice that the Lorentz connection with the Minkowski metric in tangent space η_{ab}. Thus, if its torsion $T^a \equiv -\mathcal{D}e^a = -\left(de^a - \omega^a{}_b \wedge e^b\right)$ vanishes, it is the Levi–Civita connection and the Lorentz curvature 2-form is equal to the Riemann tensor $R_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{\rho\sigma} e^a{}_\rho e^b{}_\sigma$.

where $G_{\rm N}^{(4)}$ is the 4-dimensional Newton constant and where we have defined

$$\tilde{R}_{ab} \equiv \tfrac{1}{2}\varepsilon_{abcd}R^{cd}. \tag{3}$$

Under an arbitrary variation of the Vielbein and connection

$$\delta S = \int \left\{ \mathbf{E}_a \wedge \delta e^a + \mathbf{E}_{ab} \wedge \delta\omega^{ab} + d\mathbf{\Theta} \right\}, \tag{4}$$

where

$$\mathbf{E}_a = -2e^b \wedge \tilde{R}_{ba}, \tag{5a}$$

$$\mathbf{E}_{ab} = -\varepsilon_{cdab}T^c \wedge e^d, \tag{5b}$$

$$\mathbf{\Theta} = -\tfrac{1}{2}\varepsilon_{cdab}e^c \wedge e^d \wedge \delta\omega^{ab}. \tag{5c}$$

The equations of the Vielbein $\mathbf{E}_a = 0$ imply that the Einstein tensor vanishes. These are not yet the standard Einstein equations. The equations of the connection $\mathbf{E}_{ab} = 0$ imply that the torsion vanishes.[4] Therefore, the connection is the Levi–Civita connection and, upon the use of this solution in the equations of the Vielbein, we recover the Einstein equations.

The H action [5] was originally introduced to provide a Lagrangian origin for the Hamiltonian variables introduced by Barbero in Ref. [13] to solve some of the problems of Ashtekar's variables [14, 15]. It can be obtained by replacing in the EH R_{ab} by $\tilde{R}_{ab}$ ($\tilde{\tilde{R}}_{ab} = -R_{ab}$)

$$S_{\rm EH}[e,\omega] = \frac{1}{16\pi G_{\rm N}^{(4)}} \int_{\mathcal{M}} e^a \wedge e^b \wedge R_{ab}(\omega), \tag{8}$$

and it leads to

$$\mathbf{E}_a = 2e^b \wedge R_{ba}, \tag{9a}$$

$$\mathbf{E}^{ab} = 2T^{[a} \wedge e^{b]}, \tag{9b}$$

$$\mathbf{\Theta} = e^a \wedge e^b \wedge \delta\omega_{ab}. \tag{9c}$$

The equation of the connection has the same solution as in the EH action but, substituting into the Vielbein equations, these (and the whole H action) vanish identically

[4]In components, they read

$$\varepsilon_{cdab}T_{[ef}{}^{c}\eta_{g]}{}^{d} = 0. \tag{6}$$

Contracting with ε^{efgh} they reduce to

$$T_{[cd}{}^{a}\eta_{a]}{}^{h} = 0. \tag{7}$$

Setting $d = h$ we get $T_{cd}{}^{d} = 0$ and substituting back into the above equation this result we get $T = 0$.

by virtue of the Bianchi identity[5]

$$e^b \wedge R_{ba} = 0. \tag{11}$$

As noticed in Ref. [11], the replacement $R^{ab} \to \tilde{R}^{ab}$ interchanges equations of motion and Bianchi identities and can be seen as gravitational electric-magnetic duality transformation. It also changes the parity of the action: the EH action is parity-even while the Holst action is parity-odd. If the EH term is seen as the kinetic term of the gravitational field, it is natural to regard the H term as the analog of the θ-term in the Yang–Mills theory. They can only be defined in 4 dimensions.

In the first-order formalism we can combine these two actions, breaking parity, introducing the so-called *Barbero parameter* α[6]

$$S[e,\omega] = \frac{1}{16\pi G_N^{(4)}} \int \left\{ -\tfrac{1}{2}\varepsilon_{abcd} e^a \wedge e^b \wedge R^{cd} + \alpha \eta_{ab\,cd} e^a \wedge e^b \wedge R^{cd} \right\}. \tag{13}$$

As we are going to see, this action leads, again, to the Einstein equations. However, the H term will modify the definition of some of the conserved charges and our goal is to find the Komar charge for this combined action. To this end, we are going to use the standard techniques, summarized, for instance, in Ref. [16]

3. The Komar charge. Under a generic variation of the fields, the action Eq. (13) transforms as

$$\delta S = \int \left\{ \mathbf{E}_a \wedge \delta e^a + \mathbf{E}_{ab} \wedge \delta\omega^{ab} + d\mathbf{\Theta}(e,\omega,\delta\omega) \right\}, \tag{14}$$

where, defining

$$X_{abcd} \equiv -\tfrac{1}{2}\varepsilon_{abcd} + \alpha \eta_{ab\,cd}, \tag{15}$$

and, ignoring the overall normalization factor $\left(16\pi G_N^{(4)}\right)^{-1}$, the equations of motion of the Vielbein and spin connection are

$$\mathbf{E}_a = -2X_{abcd} e^b \wedge R^{cd}, \tag{16a}$$

$$\mathbf{E}_{ab} = 2X_{abcd} T^c \wedge e^d, \tag{16b}$$

while the presymplectic potential is given by

$$\mathbf{\Theta}(e,\omega,\delta\omega) = X_{abcd} e^a \wedge e^b \wedge \delta\omega^{cd}. \tag{17}$$

[5] This identity follows trivially from the Ricci identity

$$[\mathcal{D},\mathcal{D}]e^a = -R^a{}_b \wedge e^b, \tag{10}$$

and from the absence of torsion.

[6] Notice that

$$\eta_{ab\,cd} = \tfrac{1}{2}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}), \quad \Rightarrow \quad \eta_{ab\,cd} e^a \wedge e^b \wedge R^{cd} = e^a \wedge e^b \wedge R_{ab}. \tag{12}$$

The equation $\mathbf{E}_{ab} = 0$ is solved by $T^{[a} \wedge e^{b]} = 0$[7] which, as we have see, is solved, in its turn, by $T^a = 0$ so that the connection is, again, Levi–Civita connection. If we substitute this result into the Vielbein equations $\mathbf{E}_a$, the term coming from the H action vanishes identically by virtue of the Bianchi identity Eq. (11) and we end up with the usual Einstein equations.

The H term in the symplectic prepotential does not vanish, though, and it will generate a new term in the Komar charge.

The action Eq. (13) is invariant under general coordinate transformations up to a total derivative:

$$\delta_\xi S = -\int \imath_\xi \mathbf{L}. \tag{18}$$

The left-hand side of this equation can be computed particularizing the general variation of the action in Eq. (14)

$$\delta_\xi S = \int \left\{ \mathbf{E}_a \wedge \delta_\xi e^a + \mathbf{E}_{ab} \wedge \delta_\xi \omega^{ab} + d\mathbf{\Theta}(e, \omega, \delta_\xi \omega) \right\}, \tag{19}$$

and using in it the transformations of the Vielbein and connection $\delta_\xi e^a$ and $\delta_\xi \omega^{ab}$. As explained, for instance, in Ref. [17], these transformations must take into account the induced (or compensating) local Lorentz transformations and, taking into account the possible torsion of the connection as in Refs. [18–20],[8] can be written in the form Eq. (61) and (62), which we rewrite here for convenience

$$\delta_\xi e^a = -\left(\mathcal{D}\xi^a - \imath_\xi T^a + P_\xi{}^a{}_b e^b \right), \tag{20a}$$

$$\delta_\xi \omega^{ab} = -\left(\imath_\xi R^{ab} + \mathcal{D} P_\xi{}^{ab} \right), \tag{20b}$$

where $P_\xi{}^{ab}$ is a local Lorentz parameter given in Eq. (59) that we also write here for convenience:

$$P_\xi{}^{ab} = \mathcal{D}^{[a}\xi^{b]} - \xi^c T_c{}^{[ab]}. \tag{21}$$

As shown in the Appendix, these definitions guarantee that, when ξ is a Killing vector k (*i.e.* it leaves the metric invariant) that also leaves the torsion field invariant, the above transformations vanish identically.

We get

$$\begin{aligned} \delta_\xi S = \int \Big\{ &-\xi^a \left[\mathcal{D}\mathbf{E}_a - \mathbf{E}_c \wedge \imath_a T^c + \mathbf{E}_{bc} \wedge \imath_a R^{bc} \right] - P_{\xi\, ab} \left[\mathbf{E}^a \wedge e^b + \mathcal{D}\mathbf{E}^{ab} \right] \\ &+ d \left[\mathbf{\Theta}(e, \omega, \delta_\xi \omega) + \mathbf{E}_a \xi^a + \mathbf{E}_{ab} P_\xi{}^{ab} \right] \Big\}. \end{aligned} \tag{22}$$

[7]Contracting it with ϵ^{abef} on obtains a similar equation with different coefficients multiplying $\eta_{ab\,cd}$ and ϵ_{abcd}. The new equation can be combined with the old one to get that result.

[8]We review the derivation of these transformations for the current setting in the Appendix. We stress that the connection in the covariant derivative $\mathcal{D}$ has torsion.

Then, we obtain two Noether identities:[9]

$$\mathcal{D}\mathbf{E}_a - \mathbf{E}_c \wedge \imath_a T^c + \mathbf{E}_{bc} \wedge \imath_a R^{bc} = 0, \tag{26a}$$

$$\mathbf{E}^{[a} \wedge e^{b]} + \mathcal{D}\mathbf{E}^{ab} = 0, \tag{26b}$$

and an off-shell-closed 3-form current

$$\mathbf{J}[\xi] = \mathbf{\Theta}(e, \omega, \delta_\xi \omega) + \mathbf{E}_a \xi^a + \mathbf{E}_{ab} P_\xi{}^{ab} + \imath_\xi \mathbf{L}. \tag{27}$$

The off-shell closure of $\mathbf{J}[\xi]$ implies that it is locally exact

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi], \tag{28}$$

where

$$\mathbf{Q}[\xi] = -X_{abcd} e^a \wedge e^b P_\xi{}^{cd}, \tag{29}$$

is the *Noether–Wald charge* of this theory.

By construction

$$d\mathbf{Q}[\xi] = \mathbf{J}[\xi] = \mathbf{\Theta}(e, \omega, \delta_\xi \omega) + \mathbf{E}_a \xi^a + \mathbf{E}_{ab} P_\xi{}^{ab} + \imath_\xi \mathbf{L}, \tag{30}$$

and, on-shell and for a Killing vector k it is not difficult to see that it is on-shell closed[10]

$$d\mathbf{Q}[k] \doteq 0, \tag{32}$$

where $\doteq$ indicates an identity which may only be satisfied on-shell.

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[9]Noether identities are guaranteed to be satisfied off-shell, but their proofs provide good tests of our calculations and our understanding. Using the identities

$$\mathcal{D}e^a + T^a = 0, \tag{23a}$$

$$R^a{}_b \wedge e^b - \mathcal{D}T^a = 0, \tag{23b}$$

$$\mathcal{D}R^{ab} = 0, \tag{23c}$$

the first of these identities can be cast in the form

$$-\imath_a \left(2X_{cdef} T^e \wedge e^d \wedge R^{ef}\right) = 0, \tag{24}$$

which is manifestly true. The second Noether identity is proportional to

$$2X_{[acde}g_{b]f} e^c \wedge R^{de} \wedge e^f = 0, \tag{25}$$

which is also manifestly true.

[10]The Lagrangian is proportional trace of the Einstein equations

$$e^a \wedge \mathbf{E}_a = 2X_{abcd} e^a \wedge e^b \wedge R^{cd} = 2\mathbf{L}, \tag{31}$$

and, therefore, the Lagrangian and its inner product with k vanish on-shell.

Thus, in this case, we can identify

$$\mathbf{K}[k] \doteq -\mathbf{Q}[k] \tag{33}$$

where $\mathbf{K}[k]$ is the on-shell-closed *generalized Komar charge*.[11]

This generalized Komar charge contains two pieces: the standard GR's Komar charge plus a term proportional to the Barbero parameter. The term proportional to the Barbero parameter is a sort of magnetic dual of the standard Komar charge.

In order to show this, it is convenient to rewrite the terms appearing in it. First, using the fact that the torsion vanishes on-shell, we rewrite $P_{k\,ab}$ in terms of the components of the exterior derivative of the 1-form dual to the Killing vector $\hat{k} = k_\mu dx^\mu$ as

$$P_{k\,ab} \doteq \mathcal{D}_{[a}k_{b]} = \tfrac{1}{2}(d\hat{k})_{ab}, \tag{34}$$

Then, the standard GR's Komar charge coming from the EH action is

$$-\tfrac{1}{2}\varepsilon_{abcd}e^a \wedge e^b P_k{}^{cd} = -\star d\hat{k}, \tag{35}$$

while the term coming from the H action is

$$\alpha\eta_{ab\,cd}e^a \wedge e^b P_k{}^{cd} = \alpha d\hat{k}. \tag{36}$$

Recovering the overall normalization, the total generalized Komar charge can be written in the form given by Prabhu in Ref. [22][12]

$$\mathbf{K}[k] = \frac{1}{16\pi G_N^{(4)}}\left\{-\star d\hat{k} + \alpha d\hat{k}\right\}. \tag{37}$$

The first term in this expression is the standard on-shell (*i.e.* dynamically) closed Komar charge 2-form of GR [29]. The second, new term is topologically closed and, if integrated over 2-spheres it gives zero unless the expression $d\hat{k}$ is just the local form of a closed but non-exact 2-form. The two terms are essentially identical to those whose integrals give the electric and magnetic charges in electromagnetism $\star F$ and F with $F = dA$ locally.

4. Generalized Komar integrals and the Smarr formula of the Taub–NUT solution. The integral over the 2-sphere at infinity S^2_∞ of standard Komar charge associated to the Killing vector that generates time translations (in a stationary solution) gives $M/2$, where M is the ADM mass. Since the second term in the generalized Komar charge Eq. (37) is the magnetic dual of the standard Komar charge, it is natural to expect that

[11]The fact that the generalized Komar charge is on-shell closed is often referred to as *conservation*, but it is more appropriate to say that it satisfies (the analog of) a Gauss law. See the discussion in the introduction of Ref. [21].

[12]the modification of teh Komar charge due to the Holst term has also been studied in Refs. [23–25]. See also Refs. [26–28].

its integral will, instead, give the magnetic dual of the ADM mass, which has been customarily identified with the NUT charge. Thus, it is natural to test this Komar form on the Lorentzian Taub–NUT solution [30, 31], which can be cast in the form[13]

$$ds^2 = \lambda(r)(dt + A)^2 - \lambda^{-1}(r)dr^2 - \left(r^2 + n^2\right) d\Omega^2_{(2)}, \tag{38a}$$

$$\lambda(r) = \frac{(r - r_+)(r - r_-)}{r^2 + n^2}, \tag{38b}$$

$$A = 2n\,(\cos\theta + s)\; d\varphi, \tag{38c}$$

$$r_\pm = m \pm r_0, \qquad r_0^2 = m^2 + n^2. \tag{38d}$$

For $s = +1, -1$, the solution has a Misner-string singularity [32] over the $z > 0$ and $z < 0$ axes, respectively, while for $s = 0$ there are Misner-string singularities in both axes. The value of the parameter s can be changed by a change of coordinates. However, solutions with a different value of s should not be considered as physically equivalent because the coordinate transformations that relate them are large gauge transformations.

Misner showed in Ref. [32] how to construct a singularity-free solution gluing the singularity-free $z \geq 0$ region of the $s = -1$ solution to the the singularity-free $z \leq 0$ region of the $s = +1$ solution along the hypersurface $z = 0$, where the $t^{(\pm)}$ time coordinates of the $s = \mp 1$ solutions are related by the coordinate transformation

$$t^{(+)} = t^{(-)} + 4n\varphi, \tag{39}$$

which, demands, by consistency, that the time coordinate is periodic with period $8\pi n$. Thus, the price of removing the string singularities in this way is the introduction of closed timelike curves among other pathologies.

It has recently been shown in Ref. [33] that the Misner-string singularities of the original Taub–NUT solutions are actually very mild (they are transparent to free-falling particles, for instance). This makes it possible to study their physics consistently. For instance, it has been shown that taking the contributions of the strings into account, one can arrive at a consistent thermodynamic description by Euclidean [34] or Lorentzian [35] methods. Here we want to revisit the derivation of the Smarr formula of [35] using the generalized Komar charge Eq. (37).

The Smarr formulae satisfied by the thermodynamic functions and charges of the stationary black holes of a given theory can be derived by integrating the (on-shell vanishing) exterior derivative of the generalized Komar charge of the theory associated to the Killing vector k that becomes null over the horizon $k^2 \stackrel{\mathcal{H}}{=} 0$, $\mathbf{K}[k]$ over a spacelike hypersurface Σ^3 connecting a section of the event horizon (preferably the bifurcation surface $\mathcal{BH}$ in bifurcate horizons) to the sphere at spatial infinity S^2_∞ and using Stokes theorem [37–41]. Since, by construction, $\partial\Sigma^3 = \mathcal{BH} \cup S^2_\infty$, taking into account the different orientations of the two components of the boundary, we get a relation between two

[13]Here and in what follows we are setting $G_N^{(4)} = 1$.

generalized Komar integrals:[14]

$$0 = \int_{\Sigma^3} d\mathbf{K}[k] = \int_{\mathcal{BH}} \mathbf{K}[k] - \int_{S^2_\infty} \mathbf{K}[k]. \tag{40}$$

In absence of H term ($\alpha = 0$), the integral over the bifurcation surface is naturally expressed in terms of thermodynamical properties of the horizon: temperature T and entropy S in the case at hands (pure GR). The result is just ST. The integral at infinity is naturally expressed in terms of the conserved charges associated to k: ADM mass M and angular momentum J. The result is $M/2 - \Omega J$ where Ω is the angular velocity of the horizon.

Substituting in the above identity we get the standard $\alpha = 0$ Smarr formula

$$M/2 - \Omega J - ST = 0. \tag{41}$$

As noticed in Ref. [35], in the $s = \pm 1, 0$ Taub–NUT spacetimes ($\Omega = J = 0$), the boundary of Σ^3 contains additional contributions associated to the string singularities and the Smarr formula[15]

$$\int_{\mathcal{BH}} \mathbf{K}[k] - \int_{S^2_\infty} \mathbf{K}[k] + \int_{\text{string}+} \mathbf{K}[k] - \int_{\text{string}-} \mathbf{K}[k] = 0, \tag{42}$$

where string $\pm$ stands for the strings in the $\pm z > 0$ semiaxes, when they are present.

The integral over the string in the $\pm z > 0$ semiaxes are naturally expressed in terms of the *Misner potentials* of the strings $\psi_\pm$ times a conjugate charge $\mathcal{N}_\pm$ (the *Misner string strength*) and one arrives at the $\alpha = 0$ Smarr formula [35]

$$M/2 - ST - \psi_+ \mathcal{N}_+ - \psi_- \mathcal{N}_- = 0, \tag{43}$$

that can be checked by using the explicit values of the variables

$$M = m, \tag{44a}$$

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_+}, \tag{44b}$$

$$S = \pi(r_+^2 + n^2), \tag{44c}$$

$$\psi_\pm = \frac{\kappa_\pm}{4\pi} = \frac{1}{8\pi n(1 \pm s)}, \tag{44d}$$

$$\mathcal{N}_\pm = -\frac{2\pi n^3 (1 \pm s)^2}{r_+}. \tag{44e}$$

[14]The generalized Komar integral at infinity can actually be performed over any sphere of finite radius. Since it satisfies a Gauss law, it gives the same value, which must be independent of that radius.

[15]It has been recently shown in Ref. [36] that, if we use instead Misner's construction, the string singularities unavoidably reappear in the spacelike hypersurfaces. Thus, we must always consider their contribution.

The presence of the H term ($\alpha \neq 0$) does not modify the Smarr formula Eq. (43) because the standard homogeneity arguments tell us that there will not be a new term proportional to α.[16] The values of charges and thermodynamic potentials as functions of the integration constants m and n may be modified, though. For instance, the ADM mass will still be given by the integral

$$M \equiv \frac{1}{8\pi} \int_{S^2_\infty} \mathbf{K}[k]. \tag{45}$$

However, $\mathbf{K}[k]$ contains a new term and the relation between the ADM mass and the parameter m in the solution Eq. (38) will differ by a term proportional to α. The same will happen to other physical quantities of the Taub–NUT solution.

Since the term proportional to α in $\mathbf{K}[k]$ is closed by itself, we can simply compute the integrals of that term over the same components of the boundary. The sum of these contributions vanishes by itself (Stokes theorem) and the Smarr formula Eq. (43) will be satisfied once again.

Let us first consider the integral at spatial infinity. We can use this integral as the definition of the NUT charge N:

$$N \equiv -\frac{1}{8\pi} \int_{S^2_\infty} d\hat{k}. \tag{46}$$

For $k = \partial_t$, the integrand is the pullback of $d\hat{k} = \lambda' dr \wedge (dt + A) + \lambda dA$ and, then, at spatial infinity $d\hat{k} \to dA$. The integral of dA over the 2-sphere (at infinity or anywhere else) is identical to the integral of the electromagnetic field of a Dirac monopole over a 2-sphere and it can be performed in the same fashion: if we use the Wu–Yang description, we integrate $2nd\left[(\cos\theta - 1)d\varphi\right]$ over the string-free $\theta \leq \pi/2$ hemisphere *and* that of $2nd\left[(\cos\theta + 1)d\varphi\right]$ over the string-free $\theta \geq \pi/2$ hemisphere and sum the results. If we use the Dirac description, we integrate $2nd\left[(\cos\theta - 1)d\varphi\right]$ over the whole 2-sphere minus the region $\theta = \pi - \epsilon$ surrounding the string lying in the $z < 0$ semiaxis *or* we integrate $2nd\left[(\cos\theta + 1)d\varphi\right]$ over the whole 2-sphere minus the region $\theta = \epsilon$ surrounding the string lying in the $z > 0$ semiaxis *or* we integrate $2nd\left(\cos\theta \wedge d\varphi\right)$ over the whole 2-sphere minus the regions $\theta = \epsilon$ and $\theta = \pi - \epsilon$ both strings and we let $\epsilon \to 0$ in the result. Either way, we get

$$N = n, \tag{47}$$

so the ADM mass now becomes

$$M = m - \alpha n = m - \alpha N. \tag{48}$$

[16] There will be a term of the form $\Phi_\alpha \delta\alpha$ in the right-hand side of the first law, though. The mechanism is the same that, in theories containing scalar fields, leads to terms of the form $\Sigma\delta\phi_\infty$ [46].

This result gives rise to a gravitational version of the Witten effect [42]: the Taub-NUT solution with $m = 0$ has a non-vanishing ADM mass $M = -\alpha N$.[17]

Let us now compute the other integrals.

The integral over the bifurcation sphere of $d\hat{k}$ vanishes identically because $\lambda \stackrel{\mathcal{H}}{=} 0$ and $dr \stackrel{\mathcal{H}}{=} 0$:

$$\frac{1}{8\pi}\int_{\mathcal{BH}} d\hat{k} = 0. \tag{49}$$

When there is a string lying on the $z > 0$ semiaxis ($s = +1$), the contribution of the string must be computed as the integral over the cone $t =$ constant, $\theta = \epsilon$ of the pullback of $d\hat{k} = \lambda' dr \wedge (dt + A) + \lambda dA$, with $A = 2n(\cos\theta + 1)d\varphi$ from $r = r_+$ to $r = \infty$:

$$\begin{aligned}
\frac{1}{16\pi}\int_{\text{string}+} d\hat{k} &= \frac{1}{16\pi}\int_{\text{string}+}\left[\lambda' dr \wedge A + \lambda dA\right] \\
&= \frac{1}{16\pi}\int_{\text{string}+}\left[\lambda' 2n(\cos\theta + 1)dr \wedge d\varphi - 2n\lambda\sin\theta d\theta \wedge d\varphi\right] \\
&= \frac{n}{8\pi}(\cos\epsilon + 1)\int_{\text{string}+} d\lambda \wedge d\varphi \\
&= \frac{n}{8\pi}(\cos\epsilon + 1)\int_{\lambda=0}^{\lambda=1} d\lambda \int_{\varphi=0}^{\varphi=2\pi} d\varphi \\
&= \frac{n}{4}(\cos\epsilon + 1) \\
&\longrightarrow \frac{n}{2}.
\end{aligned} \tag{50}$$

The same result is obtained when the string lies in the $z < 0$ semiaxis or when there are two strings, if one takes into account correctly the orientation. Thus, the values of the terms $\psi_\pm \mathcal{N}_\pm$ are shifted by $\alpha n/2$ and, since, due to their definition, $\psi_\pm$ should not be affected by the new term in the action, the change must be due to the change in the value of $\mathcal{N}_\pm$:

$$\mathcal{N}_\pm = -\frac{2\pi n^3(1 \pm s)^2}{r_+} + \alpha 4\pi n^2(1 \pm s). \tag{51}$$

As we have briefly mentioned before, now the entropy is a homogeneous function $S(M, \mathcal{N}_\pm, \alpha)$ and, although the Smarr formula does not acquire any terms proportional to α, there will be a new term in the first law

$$\delta M = T\delta S + \psi_+ \delta\mathcal{N}_+ + \psi_- \delta\mathcal{N}_- + \Phi_\alpha \delta\alpha \tag{52}$$

[17] A similar effect may be produced by the introduction of a topological Euler-density term in the action [43]. The Euler-density term would be the analog of the θ-term if the EH term was the analog of the standard Yang–Mills kinetic terms, which is not, but it leads to similar results. The combined effect of the H and Euler-density terms has been considered in [45, 44].

where the new chemical potential Φ_α is defined by

$$\Phi_\alpha = T\frac{\partial S}{\partial \alpha}. \tag{53}$$

The calculation of that partial derivative is quite involved, since we must take into account that both m and n are functions of the thermodynamical variables $M, \mathcal{N}_\pm$ and, in particular, of α. Thus,

$$\frac{\partial m}{\partial \alpha} = \frac{\partial(M - \alpha n)}{\partial \alpha} = -n - \alpha\frac{\partial n}{\partial \alpha}, \tag{54}$$

and $\partial n/\partial\alpha$ can be found by taking a partial derivative with respect to α in both sides of Eq. (51), using the above result and solving for $\partial n/\partial\alpha$. The final expression for Φ_α is a complicated and not too illuminating function of m, n and α.

5. Discussion. In this paper we have shown that the strong parallelism existing between the Holst term in first-order gravity and the θ-term in Yang–Mills theories leads to a gravitational analog of the Witten term sourced bow by the NUT charge, for which there is a very natural definition Eq. (46) that strengthens its interpretation as a "magnetic mass."

It would be interesting to apply these ideas to find a good definition of the topological charge carried by the The Gross-Perry–Sorkin KK monopole (or Euclidean Taub–NUT solution) [47, 48]. Work in this direction is currently underway.

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Appendix Under an infinitesimal GCT generated by the vector field ξ the metric transforms as

$$\delta_\xi g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu} = -\xi^\rho\partial_\rho g_{\mu\nu} - 2\partial_{(\mu}\xi^\rho g_{\nu)\rho}. \tag{55}$$

The partial derivatives can be replaced by the Levi-Civita covariant derivative, which gives the standard expression

$$\delta_\xi g_{\mu\nu} = -2\nabla^{L-C}_{(\mu}\xi_{\nu)}, \tag{56}$$

but, when we use a metric-compatible but torsionful connection we arrive to the expression

$$\delta_\xi g_{\mu\nu} = -2\nabla_{(\mu}\xi_{\nu)} + 2\xi^\rho T_{\rho(\mu\nu)}, \tag{57}$$

and the Killing vector equation takes the form

$$\nabla_{(\mu}k_{\nu)} - k^\rho T_{\rho(\mu\nu)} = 0. \tag{58}$$

In this context the momentum map associated to the vector ξ tat enters in the definition of the parameters of the compensating Lorentz transformations $\sigma_{\xi}{}^{ab}$ takes the form

$$P_{\xi\,ab} = \mathcal{D}_{[a}\xi_{b]} - \xi^c T_{c[ab]}, \tag{59}$$

and

$$\sigma_{\xi}{}^{ab} = \imath_{\xi}\omega^{ab} - P_{\xi}{}^{ab}, \tag{60}$$

where ω^{ab} is the (torsionful) lorentz (spin) connection.

Following the general recipe, the transformation of the Vielbein under GCTs is given by

$$\delta_{\xi}e^a = -\mathcal{L}_{\xi}e^a + \sigma_{\xi}{}^a{}_b e^b = -\left(\mathcal{D}\xi^a - \imath_{\xi}T^a + P_{\xi}{}^a{}_b e^b\right), \tag{61}$$

and it is not difficult to see that it vanishes identically for Killing vectors because it is proportional to the Killing vector equation (58).

Using the same rule we find

$$\delta_{\xi}\omega^{ab} = -\left(\imath_{\xi}R^{ab} + \mathcal{D}P_{\xi}{}^{ab}\right). \tag{62}$$

In the torsionless case, one can show that this expression vanishes identically for Killing vectors because it is the integrability condition of the Killing vector equation or, equivalently, of the condition $\delta_k e^a = 0$. In the torsionful case, the integrability condition of $\delta_k e^a = 0$ with $\delta_{\xi}e^a$ given by Eq. (61) is

$$\mathcal{D}\delta_k e^a = -\left(\imath_k R^a{}_b + \mathcal{D}P_k{}^a{}_b\right)\wedge e^b - \delta_k T^a, \tag{63}$$

where we have defined, following the general rule,[18]

$$\delta_{\xi}T^a = -\mathcal{L}_{\xi}T^a + \sigma_{\xi}{}^a{}_b T^b, \tag{65}$$

where $\sigma_{\xi}{}^{ab}$ is the parameter of the compensating or induced local Lorentz transformation in Eq. (60).

Thus, $\delta_k e^a = 0$ implies $\delta_k\omega^{ab} = 0$ if $\delta_k T^a = 0$. Since the torsion is an independent field, it is clearly necessary to demand its invariance as one of the conditions that ensure that δ_k is a symmetry of all the fields of the theory. In other words: we must demand Killing vectors which also leave the torsion field invariant.

[18]Observe that this rule is consistent with

$$T_{\mu\nu}{}^a = T_{\mu\nu}{}^{\rho} e^a{}_{\rho}, \tag{64a}$$

$$\delta_{\xi}T_{\mu\nu}{}^{\rho} = -\mathcal{L}_{\xi}T_{\mu\nu}{}^{\rho}, \tag{64b}$$

$$\delta_{\xi}e^a = -\mathcal{L}_{\xi}e^a + \sigma_{\xi}{}^a{}_b e^b. \tag{64c}$$

Finally, we notice that the same rule leads to a transformation of the curvature consistent with the Palatini identity

$$\delta_\xi R^{ab} = \mathcal{D}\delta_\xi \omega^{ab} = -\mathcal{D}\left(\imath_\xi R^{ab} + \mathcal{D}P_\xi{}^{ab}\right), \tag{66}$$

which vanishes identically for Killing vectors that leave invariant the torsion field.

References

[1] D. Lovelock, "The Einstein tensor and its generalizations," J. Math. Phys. **12** (1971), 498-501 DOI:10.1063/1.1665613

[2] A. Palatini, "Deduzione invariantiva delle equazioni gravitazionali dal principio di Hamilton," Rend. Circ. Mat. Palermo **43** (1919) no.1, 203-212 DOI:10.1007/BF03014670

[3] H. T. Nieh and M. L. Yan, "An Identity in Riemann-cartan Geometry," J. Math. Phys. **23** (1982), 373 DOI:10.1063/1.525379

[4] H. T. Nieh, "A Torsional Topological Invariant," DOI:10.1142/9789812794185_0003 [arXiv:1309.0915 [gr-qc]].

[5] S. Holst, "Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action," Phys. Rev. D **53** (1996), 5966-5969 DOI:10.1103/PhysRevD.53.5966 [gr-qc/9511026 [gr-qc]].

[6] R. K. Kaul, "Holst Actions for Supergravity Theories," Phys. Rev. D **77** (2008), 045030 DOI:10.1103/PhysRevD.77.045030 [arXiv:0711.4674 [gr-qc]].

[7] B. Tomova, "Magnetic charges and phase space renormalization of gravity," Thesis: PhD Cambridge U., DAMTP (2024) DOI:10.17863/CAM.107408

[8] M. Szczachor, "Supersymmetric Holst action with matter coupling and parity violation," Int. J. Geom. Meth. Mod. Phys. **09** (2012), 1261015 DOI:10.1142/S0219887812610154 [arXiv:1202.2232 [gr-qc]].

[9] A. Corichi, I. Rubalcava-García and T. Vukašinac, "Actions, topological terms and boundaries in first-order gravity: A review," Int. J. Mod. Phys. D **25** (2016) no.04, 1630011 DOI:10.1142/S0218271816300111 [arXiv:1604.07764 [gr-qc]].

[10] J. Figueroa-O'Farrill, E. Have, S. Prohazka and J. Salzer, "The gauging procedure and carrollian gravity," JHEP **09** (2022), 243 DOI:10.1007/JHEP09(2022)243 [arXiv:2206.14178 [hep-th]].

[11] U. Kol, "Duality in Einstein's Gravity," [arXiv:2205.05752 [hep-th]].

[12] T. Ortín, "Gravity and Strings," Cambridge University Press, 2015, ISBN 978-0-521-76813-9, 978-0-521-76813-9, 978-1-316-23579-9 DOI:10.1017/CBO9781139019750

[13] J. F. Barbero G., "Real Ashtekar variables for Lorentzian signature space times," Phys. Rev. D **51** (1995), 5507-5510 DOI:10.1103/PhysRevD.51.5507 [gr-qc/9410014 [gr-qc]].

[14] A. Ashtekar, "New Variables for Classical and Quantum Gravity," Phys. Rev. Lett. **57** (1986), 2244-2247 DOI:10.1103/PhysRevLett.57.2244

[15] A. Ashtekar, "New Hamiltonian Formulation of General Relativity," Phys. Rev. D **36** (1987), 1587-1602 DOI:10.1103/PhysRevD.36.1587

[16] J.L.V. Cerdeira and T. Ortín, "On-shell Lagrangians as total derivatives and the generalized Komar charge," to appear.

[17] Z. Elgood, P. Meessen and T. Ortín, "The first law of black hole mechanics in the Einstein-Maxwell theory revisited," JHEP **09** (2020), 026 DOI:10.1007/JHEP09(2020)026 [arXiv:2006.02792 [hep-th]].

[18] Z. Elgood, T. Ortín and D. Pereñíguez, "The first law and Wald entropy formula of heterotic stringy black holes at first order in α'," JHEP **05** (2021), 110 DOI:10.1007/JHEP05(2021)110 [arXiv:2012.14892 [hep-th]].

[19] I. Bandos and T. Ortín, "Noether-Wald charge in supergravity: the fermionic contribution," JHEP **12** (2023), 095 DOI:10.1007/JHEP12(2023)095 [arXiv:2305.10617 [hep-th]].

[20] I. Bandos, P. Meessen and T. Ortín, "Komar charge of N = 2 supergravity and its superspace generalization," [arXiv:2412.18510 [hep-th]].

[21] R. Ballesteros and T. Ortín, "Generalized Komar charges and Smarr formulas for black holes and boson stars," [arXiv:2409.08268 [gr-qc]].

[22] K. Prabhu, "The First Law of Black Hole Mechanics for Fields with Internal Gauge Freedom," Class. Quant. Grav. **34** (2017) no.3, 035011 DOI:10.1088/1361-6382/aa536b [arXiv:1511.00388 [gr-qc]].

[23] E. De Paoli and S. Speziale, "A gauge-invariant symplectic potential for tetrad general relativity," JHEP **07** (2018), 040 DOI:10.1007/JHEP07(2018)040 [arXiv:1804.09685 [gr-qc]].

[24] R. Oliveri and S. Speziale, "Boundary effects in General Relativity with tetrad variables," Gen. Rel. Grav. **52** (2020) no.8, 83 DOI:10.1007/s10714-020-02733-8 [arXiv:1912.01016 [gr-qc]].

[25] R. Oliveri and S. Speziale, "A note on dual gravitational charges," JHEP **12** (2020), 079 DOI:10.1007/JHEP12(2020)079 [arXiv:2010.01111 [hep-th]].

[26] L. Freidel, M. Geiller and D. Pranzetti, "Edge modes of gravity. Part I. Corner potentials and charges," JHEP **11** (2020), 026 DOI:10.1007/JHEP11(2020)026 [arXiv:2006.12527 [hep-th]].

[27] R. M. Wald, "On identically closed forms locally constructed from a field," J. Math. Phys. **31** (1990) no.10, 2378 DOI:10.1063/1.528839

[28] H. Godazgar, M. Godazgar and M. J. Perry, "Hamiltonian derivation of dual gravitational charges," JHEP **09** (2020), 084 DOI:10.1007/JHEP09(2020)084 [arXiv:2007.07144 [hep-th]].

[29] A. Komar, "Covariant conservation laws in general relativity," Phys. Rev. **113** (1959), 934-936 DOI:10.1103/PhysRev.113.934

[30] A. H. Taub, "Empty space-times admitting a three parameter group of motions," Annals Math. **53** (1951), 472-490 DOI:10.2307/1969567

[31] E. Newman, L. Tamburino and T. Unti, "Empty space generalization of the Schwarzschild metric," J. Math. Phys. **4** (1963), 915 DOI:10.1063/1.1704018

[32] C. W. Misner, "The Flatter regions of Newman, Unti and Tamburino's generalized Schwarzschild space," J. Math. Phys. **4** (1963), 924-938 DOI:10.1063/1.1704019

[33] G. Clément, D. Gal'tsov and M. Guenouche, "Rehabilitating space-times with NUTs," Phys. Lett. B **750** (2015), 591-594 DOI:10.1016/j.physletb.2015.09.074 [arXiv:1508.07622 [hep-th]].

[34] R. A. Hennigar, D. Kubizňák and R. B. Mann, "Thermodynamics of Lorentzian Taub-NUT spacetimes," Phys. Rev. D **100** (2019) no.6, 064055 DOI:10.1103/PhysRevD.100.064055 [arXiv:1903.08668 [hep-th]].

[35] A. B. Bordo, F. Gray, R. A. Hennigar and D. Kubizňák, "Misner Gravitational Charges and Variable String Strengths," Class. Quant. Grav. **36** (2019) no.19, 194001 DOI:10.1088/1361-6382/ab3d4d [arXiv:1905.03785 [hep-th]].

[36] G. Barbagallo, J. L. V. Cerdeira, C. Gómez-Fayrén and T. Ortín, "A note on the calculation of the Komar integral in the Lorentzian Taub-NUT spacetime," [arXiv:2505.15349 [gr-qc]].

[37] J. M. Bardeen, B. Carter and S. W. Hawking, "The Four laws of black hole mechanics," Commun. Math. Phys. **31** (1973), 161-170 DOI:10.1007/BF01645742

[38] B. Carter, "Black holes equilibrium states," Contribution to: Les Houches Summer School of Theoretical Physics, 57-214.

[39] A. Magnon, "On Komar integrals in asymptotically anti-de Sitter space-times," J. Math. Phys. **26** (1985), 3112-3117 DOI:10.1063/1.526690

[40] S. L. Bazanski and P. Zyla, "A Gauss type law for gravity with a cosmological constant," Gen. Rel. Grav. **22** (1990), 379-387

[41] D. Kastor, S. Ray and J. Traschen, "Smarr Formula and an Extended First Law for Lovelock Gravity," Class. Quant. Grav. **27** (2010), 235014 DOI:10.1088/0264-9381/27/23/235014 [arXiv:1005.5053 [hep-th]].

[42] E. Witten, "Dyons of Charge $e\theta/2\pi$," Phys. Lett. B **86** (1979), 283-287 DOI:10.1016/0370-2693(79)90838-4

[43] O. Foda, "The gravitational analog of the Witten effect," Nucl. Phys. B **256** (1985), 353-364 DOI:10.1016/0550-3213(85)90398-0

[44] R. Durka, "The first law of black hole thermodynamics for Taub–NUT spacetime," Int. J. Mod. Phys. D **31** (2022) no.04, 2250021 DOI:10.1142/S0218271822500213 [arXiv:1908.04238 [gr-qc]].

[45] R. Durka, "Immirzi parameter and Noether charges in first order gravity," J. Phys. Conf. Ser. **343** (2012), 012032 DOI:10.1088/1742-6596/343/1/012032 [arXiv:1111.0961 [gr-qc]].

[46] G. W. Gibbons, R. Kallosh and B. Kol, "Moduli, scalar charges, and the first law of black hole thermodynamics," Phys. Rev. Lett. **77** (1996), 4992-4995 DOI:10.1103/PhysRevLett.77.4992 [hep-th/9607108 [hep-th]].

[47] D. J. Gross and M. J. Perry, "Magnetic Monopoles in Kaluza-Klein Theories," Nucl. Phys. B **226** (1983), 29-48 DOI:10.1016/0550-3213(83)90462-5

[48] R. D. Sorkin, "Kaluza-Klein Monopole," Phys. Rev. Lett. **51** (1983), 87-90 DOI:10.1103/PhysRevLett.51.87