# Spin-hydrodynamics of electrons in graphene and thermovortical magnetization

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## Abstract

**We examine the framework of relativistic spin-hydrodynamics in the context of electron hydrodynamics in graphene. We develop a spin-hydrodynamic model for a $(2+1)$-dimensional system of fermions under the condition of small spin polarization. Our analysis confirms that thermal vorticity, which satisfies the global equilibrium condition, is also a solution to the spin-hydrodynamic equations. Additionally, we calculate the magnetization of the system in global equilibrium and introduce a novel phenomenon—thermovortical magnetization—resulting from thermal vorticity, which can be experimentally observed in graphene.**


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# 1 Introduction

Investigation of the hydrodynamic behavior of electrons in solid-state systems has drawn immense attention from a diverse group of physicists with a wide range of expertise. One such solid state system where hydrodynamic behavior of electrons has garnered significant interest is graphene [1–15]; see Refs. [16–18] for recent reviews. Owing to the simplicity of its band structure, graphene serves as an exceptionally rich arena for exploring electron hydrodynamics. Graphene consists of a honeycomb lattice formed by carbon atoms arranged in two spatial dimensions. Moreover, the quasiparticles in the honeycomb lattice exhibit a "relativistic" dispersion relation, $E(\mathbf{p}) = \pm v_{\mathrm{F}} |\mathbf{p}|$, where $v_{\mathrm{F}}$ is the Fermi velocity of graphene [19]. This linear dispersion relation, which characterizes *massless* fermions, was experimentally validated in Refs. [20, 21]. Furthermore, the Coulomb interactions between electrons in graphene can be relatively strong, which leads to the emergence of hydrodynamic behavior [16]. Therefore, electron hydrodynamics in graphene offers a suitable platform for testing theoretical predictions of relativistic hydrodynamics with experimental results.

Relativistic hydrodynamics finds applications in fields such as astrophysics, cosmology and high-energy physics. Evolution of strongly interacting, hot and dense QCD matter, formed in high-energy heavy-ion collisions, has been modeled quite successfully within the framework of relativistic hydrodynamics [22]. The recent observation of spin polarization in non-central heavy-ion collisions has prompted the development of relativistic spin-hydrodynamics, which enforces the conservation of total angular momentum in addition to the usual energy-momentum and net particle current conservation [23–50]. However, the fireball produced in heavy-ion collisions is extremely short-lived and certain key results of relativistic spin-hydrodynamics have yet to be conclusively verified through experiments.

One of the important predictions of relativistic spin-hydrodynamics is the existence of thermal vorticity as the global equilibrium solution [23, 51, 52], which has been quite successful in explaining the global spin-polarization observed in relativistic heavy-ion collisions [53–55]. However, attempts to apply thermal vorticity solution to explain longitudinal spin-polarization in relativistic heavy-ion collisions leads to opposite sign compared to the experimental results [56–61]. This may be attributed to the fact that due to the short lifetime of the fireball created in relativistic heavy-ion collisions, the system does not have sufficient time to attain global equilibrium for spin dynamics in the transverse plane [62–67]. On the other hand, such issues related to short lifetimes do not occur in the case of electron hydrodynamics in graphene, making it an ideal platform to validate the global equilibrium solution of relativistic spin hydrodynamics.

Near charge neutrality, graphene is predicted to exhibit behavior characteristic of a quantum-critical relativistic plasma, known as the "Dirac fluid," where massless electrons and holes undergo frequent collisions [68, 69]. At charge neutrality, the quantum-critical scattering rate indicative of the Dirac fluid was experimentally observed in Ref. [70]. Further, with increased doping, emergence of two distinct current-carrying modes were also observed—one with zero total momentum and the other with nonzero total momentum—revealing a hallmark of relativistic hydrodynamics [70]. Therefore in graphene, electrons and holes form a plasma governed by relativistic hydrodynamic equations, analogous to those describing plasmas of hot quarks and gluons [71].

In this article, we explore the framework of relativistic spin-hydrodynamics within the context of electron hydrodynamics in graphene. We develop the framework of spin-hydrodynamics for a $(2+1)$-dimensional system of fermions in the limit of small spin-polarization. We reaffirm that thermal vorticity, which satisfies global equilibrium condition, is also a solution of the spin-hydrodynamic equations in $(2+1)$-dimensions. We also calculate the system's magnetization in global equilibrium and propose a novel phenomenon, *thermovortical magnetization*, which arises due to thermal vorticity and can be experimentally observed in graphene. We use the convention $g_{\mu\nu} = \mathrm{diag}(+1,-1,-1)$ for the metric tensor. We employ $A \cdot B \equiv A_\mu B^\mu$ and $A : B \equiv A_{\mu\nu} B^{\mu\nu}$ to denote scalar products. Throughout the text we use natural units where $v_\mathrm{F} = \hbar = k_B = 1$.

## 2 Density operator and equilibrium

Relativistic spin hydrodynamics asserts that a complete description of a relativistic fluid, consisting of particles with intrinsic spin, necessitates the inclusion of a rank-3 spin tensor which is the expectation value of the tensor operator $\widehat{S}^{\lambda,\mu\nu}$. This spin-tensor operator (anti-symmetric in the last two indices) contributes to the total angular momentum operator as

$$\widehat{J}^{\lambda,\mu\nu} = x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} + \widehat{S}^{\lambda,\mu\nu}, \tag{1}$$

where $\widehat{T}^{\mu\nu}$ is the energy-momentum tensor operator.

Within the framework of quantum statistical mechanics, local thermodynamic equilibrium is defined as the state that maximizes the entropy $S = -\mathrm{Tr}(\widehat{\rho} \ln \widehat{\rho})$, where $\widehat{\rho}$ is the density operator. Moreover, entropy is maximized subject to the constraint that the mean values of charge, energy, momentum, and spin densities remain fixed, leading to [24, 72, 73]

$$\widehat{\rho}_\mathrm{LE} = \frac{1}{Z_\mathrm{LE}} \exp\left[-\int_\sigma d\sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \frac{1}{2}\omega_{\alpha\beta}\widehat{S}^{\mu,\alpha\beta} - \alpha\widehat{N}^\mu\right)\right]. \tag{2}$$

Here, $\sigma$ is a space-like hypersurface with element $d\sigma_\mu$, $Z_\mathrm{LE}$ is the local equilibrium partition function, $\widehat{N}^\mu$ is the conserved net particle number operator, and $\beta_\mu$, $\omega_{\alpha\beta}$, $\alpha$ are the relevant Lagrange multipliers. Using Eq. (1), the integral in Eq. (2) can be expressed in terms of the conserved angular momentum operator as [24]

$$\widehat{\mathcal{I}}_\mathrm{LE} = \int_\sigma d\sigma_\mu \left[\widehat{T}^{\mu\nu}\left(\beta_\nu - \omega_{\nu\gamma} x^\gamma\right) - \frac{1}{2}\omega_{\alpha\beta}\widehat{J}^{\mu,\alpha\beta} - \alpha\widehat{N}^\mu\right]. \tag{3}$$

It is important to note that the conservation of particle current, energy-momentum and angular momentum implies that $\partial_\mu \widehat{N}^\mu = \partial_\mu \widehat{T}^{\mu\nu} = \partial_\lambda \widehat{J}^{\lambda,\mu\nu} = 0$.

In global equilibrium, the integral $\widehat{\mathcal{I}}_\mathrm{LE}$ in Eq. (3) must be independent of the choice of space-like hypersurface $\sigma$, ensuring that the density operator remains constant. This is achieved by requiring that the divergence of the integrand in Eq. (3) vanishes. Considering the conservation equations, the global equilibrium constraint imposes three conditions on the Lagrange multipliers

$$\partial_\mu \beta_\nu = \omega_{\nu\mu}, \quad \partial_\mu \omega_{\nu\gamma} = 0, \quad \partial_\mu \alpha = 0. \tag{4}$$

The above conditions implies that

$$\beta_\nu = b_\nu + \omega_{\nu\gamma} x^\gamma, \quad \omega_{\nu\gamma} = \text{const.}, \quad \alpha = \text{const.}, \tag{5}$$

where, $b_\nu$ is constant. Keeping in mind that $\omega_{\mu\nu}$ is anti-symmetric, the first condition in Eq. (4) implies that $\beta_\nu$ should satisfy the Killing equation $\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$. Moreover, Eq. (5) implies that

$$\omega_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right) = \text{const.}, \tag{6}$$

which is referred to as the thermal vorticity. In order to distinguish $\omega_{\mu\nu}$ from its global equilibrium value, thermal vorticity is often denoted by $\varpi_{\mu\nu} \equiv -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right)$.

## 3 Relativistic spin-hydrodynamics

In order to formulate relativistic spin-hydrodynamics, applicable to $2+1$-dimensional system of fermions, we start with single particle, phase-space distribution functions for particles and anti-particles [23, 52]. To account for spin degrees of freedom, the phase-space distribution functions at space-time position $x$ and momentum $p$ are generalized to spin density matrices

$$f^+_{rs}(x,p) = \frac{1}{2m}\,\bar{u}_r(p)X^+\,u_s(p), \tag{7}$$

$$f^-_{rs}(x,p) = -\frac{1}{2m}\,\bar{v}_s(p)X^-\,v_r(p), \tag{8}$$

for particles and anti-particles, respectively. In the above equations, $m$ is the mass of the particles and $u_r(p)$, $v_r(p)$ are bispinors with the normalization $\bar{u}_r(p)u_s(p) = 2m\,\delta_{rs}$ and $\bar{v}_r(p)\,v_s(p) = -2m\,\delta_{rs}$[1]. Here the spin indices $r$ and $s$ can take values of 1 and 2.

Adopting the notation used in Refs. [23, 52], we introduce the matrices

$$X^{\pm} = \left[\exp\left(\beta\cdot p \mp \alpha\right)\exp\left(\pm\frac{1}{2}\omega:\Sigma\right) + I\right]^{-1}, \tag{9}$$

where $\beta^\mu$, $\alpha$ and $\omega^{\mu\nu}$ are the Lagrange multipliers, as introduced in Eq. (2), and $p^\mu = (|\boldsymbol{p}|, \boldsymbol{p})$. In terms of physical quantities, we have $\alpha = \mu/T$ and $\beta^\mu = u^\mu/T$, with $T$, $\mu$ and $u^\mu$ being the temperature, chemical potential and fluid velocity, respectively, where $u^\mu$ is normalized to $u\cdot u = 1$. Further, the Lagrange multiplier $\omega^{\mu\nu}$ is termed as the polarization tensor and $\Sigma^{\mu\nu}$ is the spin operator which can be represented using the Dirac gamma matrices as $\Sigma^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$. It is important to note that the Dirac gamma matrices satisfy the relations of the Clifford algebra $\{\gamma^\mu,\gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,g^{\mu\nu}$. Further, we consider $4\times 4$ matrix representation of Dirac Gamma matrices with the Greek indices $(\mu, \nu, \cdots)$ taking values of $0, 1, 2$.

The fundamental conserved hydrodynamic quantities can be obtained by employing the distribution functions in Eqs. (7) and (8). For a $(2+1)$-dimensional system of massless fermions, the charge current and energy-momentum tensor can be expressed as [74]

$$N^\mu = \int \frac{d^2p}{2(2\pi)^2|\boldsymbol{p}|}\,p^\mu\left[\mathrm{tr}\big(X^+\big) - \mathrm{tr}\big(X^-\big)\right], \tag{10}$$

$$T^{\mu\nu} = \int \frac{d^2p}{2(2\pi)^2|\boldsymbol{p}|}\,p^\mu\,p^\nu\left[\mathrm{tr}\big(X^+\big) + \mathrm{tr}\big(X^-\big)\right]. \tag{11}$$

For the spin-tensor, we use the phenomenological form [51]

$$S^{\lambda,\mu\nu} = \int \frac{d^2p}{2(2\pi)^2|\boldsymbol{p}|}\,p^\lambda\,\mathrm{tr}\left[(X^+ - X^-)\Sigma^{\mu\nu}\right], \tag{12}$$

which is consistent with the global equilibrium solution of thermal vorticity [23].

In order to evaluate the trace and perform the integrals in Eqs. (10), (11) and (12), we consider that the spin-polarization induced in the medium is small. In the limit of small polarization, i.e., for small values of $\omega^{\mu\nu}$, Eq. (9) can be Taylor expanded up to linear order in $\omega^{\mu\nu}$ as

$$X^{\pm} = f^{\pm}_0\,I \pm \frac{1}{2}(\omega:\Sigma)\,f^{\pm}_0\,\tilde{f}_0^{\,\pm}, \tag{13}$$

[1] These normalization conditions ensure that the massless limit of Eqs. (7) and (8) are well defined.

where, $f_0^{\pm} \equiv [\exp(\beta \cdot p \mp \alpha) + 1]^{-1}$ and $\tilde{f}_0^{\pm} \equiv 1 - f_0^{\pm}$. The trace of operators appearing in Eqs. (10), (11) and (12) can be evaluated by noting that $\mathrm{tr}(\Sigma^{\mu\nu}) = 0$ and $\mathrm{tr}\big(\Sigma^{\mu\nu}\Sigma^{\alpha\beta}\big) = g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}$. Performing the momentum integration, we obtain

$$N^{\mu} = n u^{\mu}, \tag{14}$$

$$T^{\mu\nu} = (\varepsilon + P) u^{\mu} u^{\nu} - P g^{\mu\nu}, \tag{15}$$

$$S^{\lambda,\mu\nu} = w u^{\lambda} \omega^{\mu\nu}, \tag{16}$$

where, $n$, $\varepsilon$, $P$ and $w$ are net number density, energy density, pressure and spin density, respectively. These hydrodynamic quantities are obtained as

$$n = \frac{T^2}{\pi}\Big[\mathrm{Li}_2\left(-\mathrm{e}^{-\alpha}\right) - \mathrm{Li}_2\left(-\mathrm{e}^{\alpha}\right)\Big], \tag{17}$$

$$\varepsilon = \frac{2T^3}{\pi}\Big[-\mathrm{Li}_3\left(-\mathrm{e}^{-\alpha}\right) - \mathrm{Li}_3\left(-\mathrm{e}^{\alpha}\right)\Big], \tag{18}$$

$$P = \frac{\varepsilon}{2}, \qquad w = \frac{T^2}{4\pi} \ln\left(2 + 2\cosh\alpha\right), \tag{19}$$

where $\mathrm{Li}_s(z)$ is the polylogarithmic function of order $s$ and argument $z$.

One can also define the magnetization tensor as [75–77]

$$M^{\mu\nu} = m \int \frac{d^2p}{2(2\pi)^2 |\boldsymbol{p}|} \, \mathrm{tr}\Big[(X^+ - X^-) m^{\mu\nu}\Big] \tag{20}$$

where, the operator corresponding to the magnetic dipole moment of electrons, $m^{\alpha\beta}$, is proportional to the spin operator, i.e., $m^{\alpha\beta} = \chi\, \Sigma^{\alpha\beta}$, with $\chi = \frac{g\,e}{2m}$ resembling the gyromagnetic ratio [75, 78]. Here, $g$ is the g-factor of the electron, $e$ is the elementary charge[2] and $m$ is the mass. Note that the $1/m$ in the expression for gyromagnetic ratio cancels with the $m$ in the definition of magnetization tensor, Eq. (20), resulting in a well defined massless limit. Evaluating the trace and performing the momentum integration in Eq. (20), we obtain

$$M^{\mu\nu} = \frac{g\,e}{8\,\pi} T\, \omega^{\mu\nu}, \tag{21}$$

which is one of the main results of the present work.

## 4 Conservation laws and hydrodynamic equations

Conservation of net particle number, energy and momentum can be expressed as

$$\partial_{\mu} N^{\mu} = 0, \qquad \partial_{\mu} T^{\mu\nu} = 0. \tag{22}$$

Moreover, conservation of angular momentum implies $\partial_{\lambda} J^{\lambda,\mu\nu} = 0$, where $J^{\lambda,\mu\nu} = x^{\mu} T^{\lambda\nu} - x^{\nu} T^{\lambda\mu} + S^{\lambda,\mu\nu}$. Since energy-momentum tensor in the present work is a symmetric tensor, see Eqs. (11) and (15), the conservation of angular momentum requires that the spin tensor $S^{\lambda,\mu\nu}$ must also satisfy the conservation equation [79],

$$\partial_{\lambda} S^{\lambda,\mu\nu} = 0. \tag{23}$$

Note that Eqs. (22) and (23) constitute equations of motion for the hydrodynamic variables $T$, $\mu$, $u^{\mu}$ and $\omega^{\mu\nu}$.

[2] Not to be confused with Napier's constant/Euler's Number, e.

Using the expressions for the conserved quantities, Eqs. (14)-(16) in the conservation equations, Eqs. (22) and (23), we obtain the equations of motion

$$\dot{n} + n\,\theta = 0, \tag{24}$$
$$\dot{\varepsilon} + (\varepsilon + P)\,\theta = 0, \tag{25}$$
$$(\varepsilon + P)\,\dot{u^{\alpha}} - (g^{\alpha\beta} - u^{\alpha}u^{\beta})\,\partial_{\beta}P = 0, \tag{26}$$
$$w\,\dot{\omega}^{\mu\nu} + \dot{w}\,\omega^{\mu\nu} + w\,\theta\,\omega^{\mu\nu} = 0, \tag{27}$$

where, $\dot{A} \equiv u^{\mu}\partial_{\mu}A$ represents co-moving derivative and $\theta \equiv \partial_{\mu}u^{\mu}$ is the expansion scalar. Note that Eqs. (25) and (26) follow from projection of energy-momentum conservation equation, along and orthogonal to $u^{\mu}$, respectively. In the following, we consider the global equilibrium condition of thermal vorticity, Eq. (6), and verify that it is indeed a solution of Eqs. (24)-(27).

## 5 Thermal vorticity solution

In order to express thermal vorticity in terms of hydrodynamic variables, we consider rigid vortical motion of the fluid in the two dimensional space. In this case, components of the hydrodynamic flow velocity, $u^{\mu} = (u^0, u^1, u^2)$, is defined as

$$u^0 = \gamma, \quad u^1 = -\gamma\,\Omega\,y, \quad u^2 = \gamma\,\Omega\,x, \tag{28}$$

where $\Omega$ is a positive constant indicating rotation speed, $\gamma \equiv 1/\sqrt{1-\Omega^2 r^2}$ is the local Lorentz-$\gamma$ factor, and $r \equiv \sqrt{x^2+y^2}$ represents the distance from the center of the vortex. The profile for temperature and chemical potential is determined as

$$T = T_0\,\gamma, \quad \mu = \mu_0\,\gamma \tag{29}$$

with $T_0$ and $\mu_0$ being constants. Using the definition $\beta^{\mu} = u^{\mu}/T$ in Eq. (6), we obtain

$$\omega_{\mu\nu} = \varpi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega/T_0 \\ 0 & -\Omega/T_0 & 0 \end{pmatrix}, \tag{30}$$

which is a constant matrix.

In order to verify that Eqs. (28)-(30) satisfies the hydrodynamic equations of motion, Eqs. (24)-(27), we begin by noting that the comoving derivative takes the form

$$u^{\mu}\partial_{\mu} = -\gamma\,\Omega\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right). \tag{31}$$

Keeping in mind that $\alpha = \mu/T = \mu_0/T_0$ is a constant, and using the above expression for the comoving derivative, we obtain

$$\dot{n} = 0, \quad \dot{\varepsilon} = 0, \quad \dot{P} = 0, \quad \dot{w} = 0, \quad \theta = 0. \tag{32}$$

Further, using the equation-of-state $\varepsilon = 2\,P$, the scaled derivative of pressure can be shown to be equal to the comoving derivative of fluid velocity, i.e.,

$$\left(\frac{1}{\varepsilon + P}\right)\partial^{\alpha}P = \dot{u}^{\alpha} = -\gamma^2\Omega^2(0,\, x,\, y). \tag{33}$$

It is now apparent from Eqs. (32) and (33) that Eqs. (28)-(30) represents a solution of the hydrodynamic equations of motion, i.e., Eqs. (24)-(27).

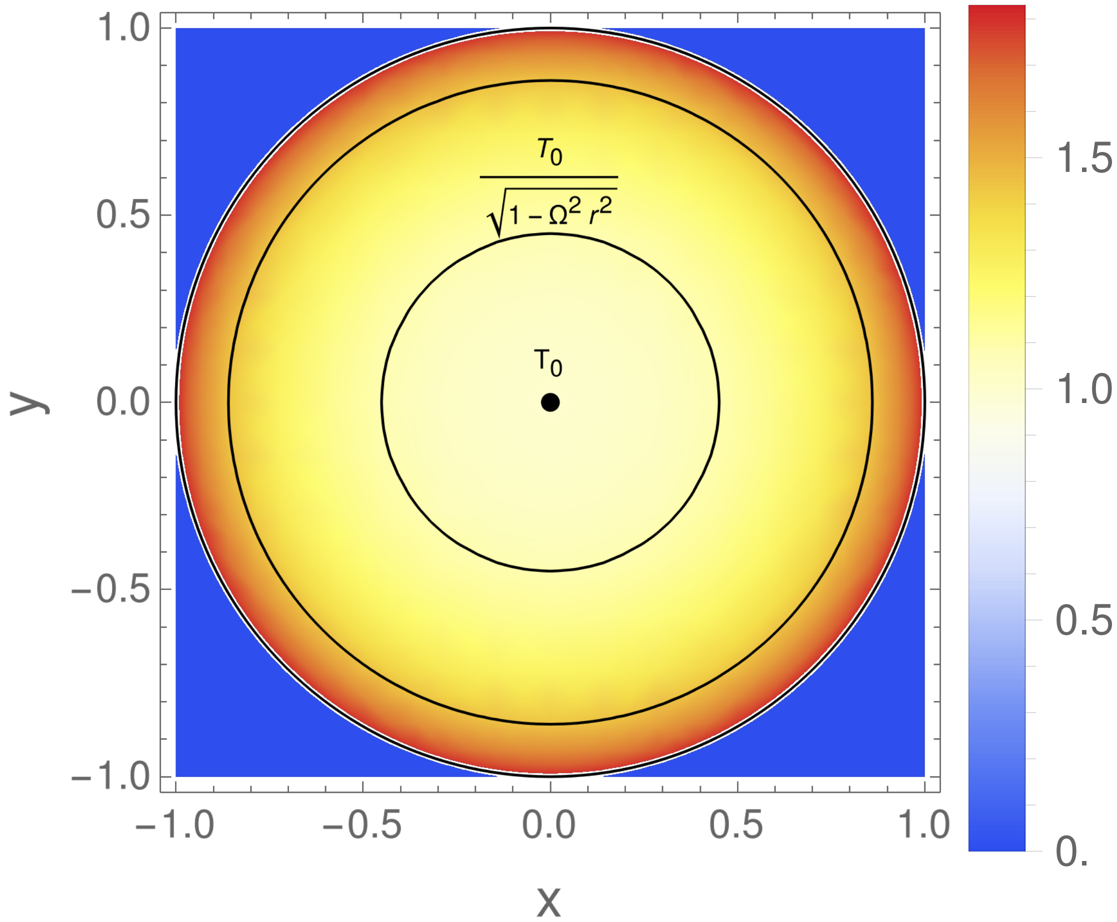


Figure 1: Schematic representation of temperature gradient on the surface of graphene, shown as a disc of radius 1 unit. The temperature at the center is $T_0$ and the temperature on the surface of graphene increases outwards as a function of radius $r$, as shown in the figure. The black concentric circles represent nanowires used to maintain the specific temperature profile radially outwards; see text for details.

# 6 Thermovortical magnetization

To understand the observable consequence of the spin hydrodynamics framework on electron hydrodynamics of graphene, we consider magnetization with thermal vorticity solution. Using Eq. (30) in Eq. (21), we obtain

$$M_{\mu\nu} = \frac{g\,e}{8\,\pi}\frac{\Omega}{\sqrt{1-\Omega^2 r^2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{34}$$

The above relation suggests that, in general, the condition of global equilibrium leads to magnetization. This phenomenon, which is reminiscent of the Barnett effect observed in solids [80], arises from the thermal vorticity solution of the hydrodynamic equations. Further, our calculation suggests that the *thermovortical magnetization* in Eq. (34) increases as a function of radial distance from the center of the vortex.

It is important to note that in order to experimentally observe thermovortical magnetization in graphene, the system has to be initialized in such a way that it relaxes to the thermal vorticity state of global equilibrium. We propose that such an initial condition can be achieved by providing a gradient to the thermodynamic quantities given in Eq. (29). For instance, an external source may be applied to maintain a radially increasing temperature gradient $T = T_0/\sqrt{1-\Omega^2 r^2}$; as shown in Fig. 1. Joule heating in nanowires can serve as an external heat source to maintain the desired temperature gradient on graphene substrate [81–83]. Once the external source maintaining the temperature gradient is switched off, the hydrodynamic evolution drives the system toward its nearest equilibrium state, which is characterized by thermal vorticity. This thermal vorticity state eventually leads to thermovortical magnetization, as given in Eq. (34). We emphasize that, while the magnetization induced by thermal vorticity may be small, it is nonetheless a measurable phenomenon, as described below.

Firstly, we note that the expression for magnetization in Eq. (34) represents the *magnetization density*. Consequently, the total magnitude of magnetization is given by $M = (ge/8\pi)\Omega\gamma(\pi R^2)$, where $R$ is the radius of the graphene sample. The central temperature of the graphene sheet, $T_0$, as illustrated in Fig. 1, can be maintained at approximately 80 K using a liquid nitrogen cold finger. On the other hand, Joule heating in nanowires can readily raise the temperature to about 400 K at the edges of the sample ($r = R$) [83, 84]. Using Eq. (29), these values corresponds to a Lorentz gamma factor of $\gamma = T/T_0 \approx 5$, indicating a substantial degree of relativistic behavior. Moreover, this leads to $\Omega R \approx v_\mathrm{F} \approx 10^6\,\mathrm{m/s}$, the typical Fermi velocity in graphene. Substituting the above values and fundamental constants into the expression for $M$, we obtain $M \approx 10^{-13} R\,[\mathrm{A{\cdot}m^2}]$, where $R$ is in meters. The minimum magnetization detectable with current technology is of the order of $\sim 10^{-17}\,\mathrm{A\cdot m^2}$ [85]. Therefore, to observe the thermovortical magnetization experimentally, the graphene sample must have a radius $R \sim 100\,\mu\mathrm{m}$, which is typical size of graphene sheets.

To further elaborate on the proposed experimental setup, we emphasize that Fig. 1 serves only as a conceptual schematic. The concentric nanowires illustrated in the figure can be fabricated on a substrate using established techniques [86], upon which a hexagonal boron nitride (hBN)-encapsulated graphene sample may be transferred. The thermal characteristics of the underlying substrate will play a critical role in maintaining the desired radial temperature gradient. Given the high thermal conductivity of graphene, the lattice temperature is expected to rapidly equilibrate with that of the substrate. Additionally, electron-phonon interactions will drive the electronic temperature toward the lattice temperature, thereby establishing the intended temperature profile. However, important practical considerations remain, including heat dissipation at the graphene–substrate interface [87] and the precise electrical interfacing of the nanowires. These factors must be carefully addressed in any experimental realization of the proposed setup.

To assess how closely the initial temperature profile must be in order to relax to $T = T_0\gamma$ exact configuration, it would likely require detailed numerical modeling of heat transport and hydrodynamic behavior within the graphene sample. Nevertheless, we note that there are two distinct classes of global equilibrium solutions in this context: one characterized by spatially uniform temperature, and another associated with thermal vorticity. When a radially increasing temperature profile is provided as the initial condition, the system is expected to evolve toward the nearest equilibrium state, which would correspond to a thermal vorticity configuration of the form $T' = T_0\gamma'$, possibly associated with a modified angular velocity $\Omega'$. Although this state may not quantitatively match the specific profile used in our calculations, it would still give rise to a nonzero magnetization. Thus, even if the final configuration deviates quantitatively from the idealized setup, our analysis provides a robust qualitative prediction for the emergence of magnetization driven by thermal vorticity.

# 7 Summary and outlook

In this work, the framework of relativistic spin-hydrodynamics was explored within the context of electron hydrodynamics in graphene. A spin-hydrodynamic model for a $(2+1)$-dimensional system of fermions was developed under the assumption of small spin polarization. It was confirmed that thermal vorticity, which satisfies the global equilibrium condition, also serves as a solution to the spin-hydrodynamic equations in $(2+1)$-dimensions. Furthermore, the magnetization of the system in global equilibrium was computed, leading to the introduction of a new phenomenon, *thermovortical magnetization*, which results from thermal vorticity and can be observed experimentally in graphene.

This work presents the proposal for application and observation of relativistic spin-hydrodynamics predictions in graphene. While this work is based on a non-dissipative formulation of relativistic spin hydrodynamics, it will be interesting to explore dissipative effects such as rotational viscosity and boost heat conductivity [88] in the context of electron hydrodynamics in graphene. For the application of dissipative effects, it is important to determine these new transport coefficients by connecting them to the underlying microscopic theory of electrons in graphene. Further, the effect of external magnetic field on electrons in graphene can also be included via the framework of spin magnetohydrodynamics [77]. Investigation of these aspects is left for future studies.

A generalized formulation of the current framework can be developed to describe transport phenomena in systems such as Weyl semimetals. Topological Weyl semimetals exhibit linear dispersion near specific points in momentum space, known as Dirac or Weyl nodes, serving as three-dimensional analogs of graphene. Weyl semimetals exhibit unique transport phenomena arising from their low-energy excitations, which behave as chiral, massless fermions [89–93]. These materials provide an intriguing platform to apply and extend relativistic spin-hydrodynamics due to the Lorentz-invariant structure of their effective field theory, combined with the presence of strong spin-orbit coupling, Berry curvature effects, and topological responses such as the chiral magnetic and chiral vortical effects. We leave the exploration of these promising directions to future investigations.

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## References

[1] M. J. H. Ku *et al.*, *Imaging viscous flow of the Dirac fluid in graphene*, Nature (London) **583**(7817), 537 (2020), doi:10.1038/s41586-020-2507-2.

[2] G. Varnavides, A. S. Jermyn, P. Anikeeva, C. Felser and P. Narang, *Electron hydrodynamics in anisotropic materials*, Nat. Commun. **11**(1), 4710 (2020).

[3] E. H. Hasdeo, J. Ekström, E. G. Idrisov and T. L. Schmidt, *Electron hydrodynamics of two-dimensional anomalous hall materials*, Phys. Rev. B **103**, 125106 (2021), doi:10.1103/PhysRevB.103.125106.

[4] A. Hui, V. Oganesyan and E.-A. Kim, *Beyond ohm's law: Bernoulli effect and streaming in electron hydrodynamics*, Phys. Rev. B **103**, 235152 (2021), doi:10.1103/PhysRevB.103.235152.

[5] X. Huang and A. Lucas, *Electron-phonon hydrodynamics*, Phys. Rev. B **103**, 155128 (2021), doi:10.1103/PhysRevB.103.155128.

[6] O. Tavakol and Y. B. Kim, *Artificial electric field and electron hydrodynamics*, Phys. Rev. Res. **3**, 013290 (2021), doi:10.1103/PhysRevResearch.3.013290.

[7] D. Di Sante, J. Erdmenger, M. Greiter, I. Matthaiakakis, R. Meyer, D. Rodríguez Fernández, R. Thomale, E. van Loon and T. Wehling, *Turbulent hydrodynamics in strongly correlated Kagome metals*, Nat. Commun. **11**(1), 3997 (2020), doi:10.1038/s41467-020-17663-x.

[8] B. N. Narozhny, I. V. Gornyi and M. Titov, *Hydrodynamic collective modes in graphene*, Phys. Rev. B **103**, 115402 (2021), doi:10.1103/PhysRevB.103.115402.

[9] J. A. Sulpizio, L. Ella, A. Rozen, J. Birkbeck, D. J. Perello, D. Dutta, M. Ben-Shalom, T. Taniguchi, K. Watanabe, T. Holder *et al.*, *Visualizing poiseuille flow of hydrodynamic electrons*, Nature (London) **576**(7785), 75 (2019).

[10] A. I. Berdyugin, S. G. Xu, F. M. D. Pellegrino, R. K. Kumar, A. Principi, I. Torre, M. B. Shalom, T. Taniguchi, K. Watanabe, I. V. Grigorieva, M. Polini, A. K. Geim *et al.*, *Measuring hall viscosity of graphene's electron fluid*, Science **364**(6436), 162 (2019), doi:10.1126/science.aau0685.

[11] L. Ella, A. Rozen, J. Birkbeck, M. Ben-Shalom, D. Perello, J. Zultak, T. Taniguchi, K. Watanabe, A. K. Geim, S. Ilani *et al.*, *Simultaneous voltage and current density imaging of flowing electrons in two dimensions*, Nat. Nanotechnol. **14**(5), 480 (2019).

[12] D. A. Bandurin, A. V. Shytov, L. S. Levitov, R. K. Kumar, A. I. Berdyugin, M. Ben Shalom, I. V. Grigorieva, A. K. Geim and G. Falkovich, *Fluidity onset in graphene*, Nat. Commun. **9**(1), 4533 (2018).

[13] D. A. Bandurin, I. Torre, R. K. Kumar, M. B. Shalom, A. Tomadin, A. Principi, G. H. Auton, E. Khestanova, K. S. Novoselov, I. V. Grigorieva, L. A. Ponomarenko, A. K. Geim *et al.*, *Negative local resistance caused by viscous electron backflow in graphene*, Science **351**(6277), 1055 (2016), doi:10.1126/science.aad0201.

[14] C. W. Aung, T. Z. Win, G. Khandal and S. Ghosh, *Shear viscosity expression for a graphene system in relaxation time approximation*, Phys. Rev. B **108**(23), 235172 (2023), doi:10.1103/PhysRevB.108.235172, 2306.14747.

[15] E. G. Idrisov, E. H. Hasdeo, B. N. Radhakrishnan and T. L. Schmidt, *Hydrodynamic Navier-Stokes equations in two-dimensional systems with Rashba spin-orbit coupling*, Low Temperature Physics **49**(12), 1385 (2023), doi:10.1063/10.0022364.

[16] A. Lucas and K. C. Fong, *Hydrodynamics of electrons in graphene*, J. of Phys.: Condens Matter **30**(5), 053001 (2018), doi:10.1088/1361-648X/aaa274.

[17] B. N. Narozhny, *Electronic hydrodynamics in graphene*, Ann. Phys. **411**, 167979 (2019), doi:https://doi.org/10.1016/j.aop.2019.167979.

[18] B. N. Narozhny, *Hydrodynamic approach to two-dimensional electron systems*, Riv. Nuovo Cim. **45**(10), 661 (2022), doi:10.1007/s40766-022-00036-z.

[19] P. R. Wallace, *The Band Theory of Graphite*, Phys. Rev. **71**(9), 622 (1947), doi:10.1103/PhysRev.71.622.

[20] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos and A. A. Firsov, *Two-dimensional gas of massless Dirac fermions in graphene*, Nature **438**, 197 (2005), doi:10.1038/nature04233, cond-mat/0509330.

[21] Y. Zhang, Y.-W. Tan, H. L. Stormer and P. Kim, *Experimental observation of the quantum Hall effect and and Berry's phase in graphene*, Nature **438**, 201 (2005), doi:10.1038/nature04235, cond-mat/0509355.

[22] U. Heinz and R. Snellings, *Collective flow and viscosity in relativistic heavy-ion collisions*, Ann. Rev. Nucl. Part. Sci. **63**, 123 (2013), doi:10.1146/annurev-nucl-102212-170540, 1301.2826.

[23] W. Florkowski, B. Friman, A. Jaiswal and E. Speranza, *Relativistic fluid dynamics with spin*, Phys. Rev. C **97**(4), 041901 (2018), doi:10.1103/PhysRevC.97.041901, 1705.00587.

[24] W. Florkowski, A. Kumar and R. Ryblewski, *Relativistic hydrodynamics for spin-polarized fluids*, Prog. Part. Nucl. Phys. **108**, 103709 (2019), doi:10.1016/j.ppnp.2019.07.001, 1811.04409.

[25] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar and R. Ryblewski, *Dissipative Spin Dynamics in Relativistic Matter*, Phys. Rev. D **103**(1), 014030 (2021), doi:10.1103/PhysRevD.103.014030, 2008.10976.

[26] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar and R. Ryblewski, *Relativistic dissipative spin dynamics in the relaxation time approximation*, Phys. Lett. B **814**, 136096 (2021), doi:10.1016/j.physletb.2021.136096, 2002.03937.

[27] F. Becattini and M. A. Lisa, *Polarization and Vorticity in the Quark–Gluon Plasma*, Ann. Rev. Nucl. Part. Sci. **70**, 395 (2020), doi:10.1146/annurev-nucl-021920-095245, 2003.03640.

[28] J. Hu, *Relativistic first-order spin hydrodynamics via the Chapman-Enskog expansion* (2021), 2111.03571.

[29] S. Shi, C. Gale and S. Jeon, *From chiral kinetic theory to relativistic viscous spin hydrodynamics*, Phys. Rev. C **103**(4), 044906 (2021), doi:10.1103/PhysRevC.103.044906, 2008.08618.

[30] B. Fu, K. Xu, X.-G. Huang and H. Song, *Hydrodynamic study of hyperon spin polarization in relativistic heavy ion collisions*, Phys. Rev. C **103**(2), 024903 (2021), doi:10.1103/PhysRevC.103.024903, 2011.03740.

[31] E. Speranza and N. Weickgenannt, *Spin tensor and pseudo-gauges: from nuclear collisions to gravitational physics*, Eur. Phys. J. A **57**(5), 155 (2021), doi:10.1140/epja/s10050-021-00455-2, 2007.00138.

[32] E. Speranza, F. S. Bemfica, M. M. Disconzi and J. Noronha, *Challenges in Solving Chiral Hydrodynamics* (2021), 2104.02110.

[33] D. She, A. Huang, D. Hou and J. Liao, *Relativistic Viscous Hydrodynamics with Angular Momentum* (2021), 2105.04060.

[34] H.-H. Peng, J.-J. Zhang, X.-L. Sheng and Q. Wang, *Ideal Spin Hydrodynamics from the Wigner Function Approach*, Chin. Phys. Lett. **38**(11), 116701 (2021), doi:10.1088/0256-307X/38/11/116701, 2107.00448.

[35] D.-L. Wang, S. Fang and S. Pu, *Analytic solutions of relativistic dissipative spin hydrodynamics with Bjorken expansion*, Phys. Rev. D **104**(11), 114043 (2021), doi:10.1103/PhysRevD.104.114043, 2107.11726.

[36] C. Yi, S. Pu, J.-H. Gao and D.-L. Yang, *Hydrodynamic helicity polarization in relativistic heavy ion collisions* (2021), 2112.15531.

[37] D.-L. Wang, X.-Q. Xie, S. Fang and S. Pu, *Analytic solutions of relativistic dissipative spin hydrodynamics with radial expansion in Gubser flow* (2021), 2112.15535.

[38] W. Florkowski, A. Kumar, R. Ryblewski and R. Singh, *Spin polarization evolution in a boost invariant hydrodynamical background*, Phys. Rev. C **99**(4), 044910 (2019), doi:10.1103/PhysRevC.99.044910, 1901.09655.

[39] R. Singh, G. Sophys and R. Ryblewski, *Spin polarization dynamics in the Gubser-expanding background*, Phys. Rev. D **103**(7), 074024 (2021), doi:10.1103/PhysRevD.103.074024, 2011.14907.

[40] R. Singh, M. Shokri and R. Ryblewski, *Spin polarization dynamics in the Bjorken-expanding resistive MHD background*, Phys. Rev. D **103**(9), 094034 (2021), doi:10.1103/PhysRevD.103.094034, 2103.02592.

[41] W. Florkowski, R. Ryblewski, R. Singh and G. Sophys, *Spin polarization dynamics in the non-boost-invariant background*, Phys. Rev. D **105**(5), 054007 (2022), doi:10.1103/PhysRevD.105.054007, 2112.01856.

[42] A. Das, W. Florkowski, A. Kumar, R. Ryblewski and R. Singh, *Semi-classical kinetic theory for massive spin-half fermions with leading-order spin effects* (2022), 2203.15562.

[43] D. Montenegro, L. Tinti and G. Torrieri, *Ideal relativistic fluid limit for a medium with polarization*, Phys. Rev. D **96**(5), 056012 (2017), doi:10.1103/PhysRevD.96.056012, [Addendum: Phys.Rev.D 96, 079901 (2017)], 1701.08263.

[44] D. Montenegro, L. Tinti and G. Torrieri, *Sound waves and vortices in a polarized relativistic fluid*, Phys. Rev. D **96**(7), 076016 (2017), doi:10.1103/PhysRevD.96.076016, 1703.03079.

[45] D. Montenegro and G. Torrieri, *Causality and dissipation in relativistic polarizable fluids*, Phys. Rev. D **100**(5), 056011 (2019), doi:10.1103/PhysRevD.100.056011, 1807.02796.

[46] D. Montenegro and G. Torrieri, *Linear response theory and effective action of relativistic hydrodynamics with spin*, Phys. Rev. D **102**(3), 036007 (2020), doi:10.1103/PhysRevD.102.036007, 2004.10195.

[47] A. D. Gallegos, U. Gürsoy and A. Yarom, *Hydrodynamics of spin currents*, SciPost Phys. **11**, 041 (2021), doi:10.21468/SciPostPhys.11.2.041, 2101.04759.

[48] M. Hongo, X.-G. Huang, M. Kaminski, M. Stephanov and H.-U. Yee, *Relativistic spin hydrodynamics with torsion and linear response theory for spin relaxation*, JHEP **11**, 150 (2021), doi:10.1007/JHEP11(2021)150, 2107.14231.

[49] C. Cartwright, M. G. Amano, M. Kaminski, J. Noronha and E. Speranza, *Convergence of hydrodynamics in rapidly spinning strongly coupled plasma* (2021), 2112.10781.

[50] A. D. Gallegos, U. Gursoy and A. Yarom, *Hydrodynamics, spin currents and torsion* (2022), 2203.05044.

[51] F. Becattini and L. Tinti, *The Ideal relativistic rotating gas as a perfect fluid with spin*, Annals Phys. **325**, 1566 (2010), doi:10.1016/j.aop.2010.03.007, 0911.0864.

[52] F. Becattini, V. Chandra, L. Del Zanna and E. Grossi, *Relativistic distribution function for particles with spin at local thermodynamical equilibrium*, Annals Phys. **338**, 32 (2013), doi:10.1016/j.aop.2013.07.004, 1303.3431.

[53] F. Becattini, I. Karpenko, M. Lisa, I. Upsal and S. Voloshin, *Global hyperon polarization at local thermodynamic equilibrium with vorticity, magnetic field and feed-down*, Phys. Rev. C **95**(5), 054902 (2017), doi:10.1103/PhysRevC.95.054902, 1610.02506.

[54] I. Karpenko and F. Becattini, *Study of $\Lambda$ polarization in relativistic nuclear collisions at $\sqrt{s_{\rm NN}} = 7.7$ –200 GeV*, Eur. Phys. J. C **77**(4), 213 (2017), doi:10.1140/epjc/s10052-017-4765-1, 1610.04717.

[55] H. Li, L.-G. Pang, Q. Wang and X.-L. Xia, *Global $\Lambda$ polarization in heavy-ion collisions from a transport model*, Phys. Rev. C **96**(5), 054908 (2017), doi:10.1103/PhysRevC.96.054908, 1704.01507.

[56] F. Becattini and I. Karpenko, *Collective Longitudinal Polarization in Relativistic Heavy-Ion Collisions at Very High Energy*, Phys. Rev. Lett. **120**(1), 012302 (2018), doi:10.1103/PhysRevLett.120.012302, 1707.07984.

[57] W. Florkowski, A. Kumar, R. Ryblewski and A. Mazeliauskas, *Longitudinal spin polarization in a thermal model*, Phys. Rev. C **100**(5), 054907 (2019), doi:10.1103/PhysRevC.100.054907, 1904.00002.

[58] B. Fu, S. Y. F. Liu, L. Pang, H. Song and Y. Yin, *Shear-Induced Spin Polarization in Heavy-Ion Collisions*, Phys. Rev. Lett. **127**(14), 142301 (2021), doi:10.1103/PhysRevLett.127.142301, 2103.10403.

[59] F. Becattini, M. Buzzegoli and A. Palermo, *Spin-thermal shear coupling in a relativistic fluid*, Phys. Lett. B **820**, 136519 (2021), doi:10.1016/j.physletb.2021.136519, 2103.10917.

[60] F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko and A. Palermo, *Local Polarization and Isothermal Local Equilibrium in Relativistic Heavy Ion Collisions*, Phys. Rev. Lett. **127**(27), 272302 (2021), doi:10.1103/PhysRevLett.127.272302, 2103.14621.

[61] W. Florkowski, A. Kumar, A. Mazeliauskas and R. Ryblewski, *Effect of thermal shear on longitudinal spin polarization in a thermal model* (2021), 2112.02799.

[62] J. I. Kapusta, E. Rrapaj and S. Rudaz, *Relaxation Time for Strange Quark Spin in Rotating Quark-Gluon Plasma*, Phys. Rev. C **101**(2), 024907 (2020), doi:10.1103/PhysRevC.101.024907, 1907.10750.

[63] A. Ayala, D. de la Cruz, L. A. Hernández and J. Salinas, *Relaxation time for the alignment between the spin of a finite-mass quark or antiquark and the thermal vorticity in relativistic heavy-ion collisions*, Phys. Rev. D **102**(5), 056019 (2020), doi:10.1103/PhysRevD.102.056019, 2003.06545.

[64] A. Kumar, B. Müller and D.-L. Yang, *Spin alignment of vector mesons by glasma fields*, Phys. Rev. D **108**(1), 016020 (2023), doi:10.1103/PhysRevD.108.016020, 2304.04181.

[65] Y. Hidaka, M. Hongo, M. Stephanov and H.-U. Yee, *Spin relaxation rate for baryons in thermal pion gas* (2023), 2312.08266.

[66] D. Wagner, M. Shokri and D. H. Rischke, *On the damping of spin waves* (2024), 2405.00533.

[67] S. Banerjee, S. Bhadury, W. Florkowski, A. Jaiswal and R. Ryblewski, *Longitudinal spin polarization in a thermal model with dissipative corrections* (2024), 2405.05089.

[68] J. Crossno, J. K. Shi, K. Wang, X. Liu, A. Harzheim, A. Lucas, S. Sachdev, P. Kim, T. Taniguchi, K. Watanabe, T. A. Ohki and K. C. Fong, *Observation of the dirac fluid and the breakdown of the wiedemann-franz law in graphene*, Science **351**(6277), 1058 (2016), doi:10.1126/science.aad0343.

[69] A. Lucas, J. Crossno, K. C. Fong, P. Kim and S. Sachdev, *Transport in inhomogeneous quantum critical fluids and in the dirac fluid in graphene*, Phys. Rev. B **93**, 075426 (2016), doi:10.1103/PhysRevB.93.075426.

[70] P. Gallagher, C.-S. Yang, T. Lyu, F. Tian, R. Kou, H. Zhang, K. Watanabe, T. Taniguchi and F. Wang, *Quantum-critical conductivity of the dirac fluid in graphene*, Science **364**(6436), 158 (2019), doi:10.1126/science.aat8687.

[71] A. Lucas, *An exotic quantum fluid in graphene*, Science **364**(6436), 125 (2019), doi:10.1126/science.aaw9869.

[72] F. Becattini, L. Bucciantini, E. Grossi and L. Tinti, *Local thermodynamical equilibrium and the beta frame for a quantum relativistic fluid*, Eur. Phys. J. C **75**(5), 191 (2015), doi:10.1140/epjc/s10052-015-3384-y, 1403.6265.

[73] F. Becattini, W. Florkowski and E. Speranza, *Spin tensor and its role in non-equilibrium thermodynamics*, Phys. Lett. B **789**, 419 (2019), doi:10.1016/j.physletb.2018.12.016, 1807.10994.

[74] S. R. De Groot, *Relativistic Kinetic Theory. Principles and Applications* (1980).

[75] L. Suttorp and S. De Groot, *Covariant equations of motion, for a charged particle with a magnetic dipole moment*, Il Nuovo Cimento A (1965-1970) **65**(1), 245 (1970).

[76] C. G. van Weert, *On the relativistic kinetic theory of particles with a magnetic dipole moment in an external electromagnetic field*, Other thesis, - (1970).

[77] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar and R. Ryblewski, *Relativistic Spin Magnetohydrodynamics*, Phys. Rev. Lett. **129**(19), 192301 (2022), doi:10.1103/PhysRevLett.129.192301, 2204.01357.

[78] N. Weickgenannt, X.-L. Sheng, E. Speranza, Q. Wang and D. H. Rischke, *Kinetic theory for massive spin-1/2 particles from the Wigner-function formalism*, Phys. Rev. D **100**(5), 056018 (2019), doi:10.1103/PhysRevD.100.056018, 1902.06513.

[79] F. W. Hehl, *On the Energy Tensor of Spinning Massive Matter in Classical Field Theory and General Relativity*, Rept. Math. Phys. **9**, 55 (1976), doi:10.1016/0034-4877(76)90016-1.

[80] S. J. Barnett, *Gyromagnetic and Electron-Inertia Effects*, Rev. Mod. Phys. **7**(2), 129 (1935), doi:10.1103/RevModPhys.7.129.

[81] A. Yamaguchi, S. Nasu, H. Tanigawa, T. Ono, K. Miyake, K. Mibu and T. Shinjo, *Effect of joule heating in current-driven domain wall motion*, Applied Physics Letters **86**(1), 012511 (2004), doi:10.1063/1.1847714.

[82] G. Meier, M. Bolte, R. Eiselt, B. Krüger, D.-H. Kim and P. Fischer, *Direct imaging of stochastic domain-wall motion driven by nanosecond current pulses*, Phys. Rev. Lett. **98**, 187202 (2007), doi:10.1103/PhysRevLett.98.187202.

[83] H. Fangohr, D. S. Chernyshenko, M. Franchin, T. Fischbacher and G. Meier, *Joule heating in nanowires*, Phys. Rev. B **84**, 054437 (2011), doi:10.1103/PhysRevB.84.054437.

[84] K. Ravi Chandran, *Transient joule heating of graphene, nanowires and filaments: Analytical model for current-induced temperature evolution including substrate and end effects*, International Journal of Heat and Mass Transfer **88**, 14 (2015), doi:https://doi.org/10.1016/j.ijheatmasstransfer.2015.04.014.

[85] K. Wang, M. Shi, X. Li, N. Wang, Z. Qu and H. Du, *Compact dynamic cantilever magnetometry*, Review of Scientific Instruments **96**(4), 045104 (2025), doi:10.1063/5.0259803.

[86] C. H. Lee, D. R. Kim and X. Zheng, *Fabricating nanowire devices on diverse substrates by simple transfer-printing methods*, Proceedings of the National Academy of Sciences **107**(22), 9950 (2010), doi:10.1073/pnas.0914031107.

[87] Z. Xu and M. J. Buehler, *Heat dissipation at a graphene–substrate interface*, Journal of Physics: Condensed Matter **24**(47), 475305 (2012), doi:10.1088/0953-8984/24/47/475305.

[88] K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo and H. Taya, *Fate of spin polarization in a relativistic fluid: An entropy-current analysis*, Phys. Lett. B **795**, 100 (2019), doi:10.1016/j.physletb.2019.05.040, 1901.06615.

[89] P. Hosur and X. Qi, *Recent developments in transport phenomena in Weyl semimetals*, Comptes Rendus Physique **14**, 857 (2013), doi:10.1016/j.crhy.2013.10.010, 1309.4464.

[90] O. Vafek and A. Vishwanath, *Dirac Fermions in Solids: From High-Tc cuprates and Graphene to Topological Insulators and Weyl Semimetals*, Ann. Rev. Condensed Matter Phys. **5**, 83 (2014), doi:10.1146/annurev-conmatphys-031113-133841, 1306.2272.

[91] B. Yan and C. Felser, *Topological Materials: Weyl Semimetals*, Ann. Rev. Condensed Matter Phys. **8**, 337 (2017), doi:10.1146/annurev-conmatphys-031016-025458, 1611.04182.

[92] A. A. Burkov, *Weyl Metals*, Ann. Rev. Condensed Matter Phys. **9**, 359 (2018), doi:10.1146/annurev-conmatphys-033117-054129, 1704.06660.

[93] N. P. Armitage, E. J. Mele and A. Vishwanath, *Weyl and Dirac Semimetals in Three Dimensional Solids*, Rev. Mod. Phys. **90**(1), 015001 (2018), doi:10.1103/RevModPhys.90.015001, 1705.01111.