

Elements of cosmology beyond FLRW

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Abstract

Modern cosmology is based on the cosmological principle, which states that the Universe is statistically homogeneous and isotropic. When applied in its strict – rather than statistical – sense, the cosmological principle leads to the Friedmann–Lemaître–Robertson–Walker (FLRW) model, which serves as background spacetime. This background is used to predict: (1) the dynamics of cosmic expansion; and (2) the kinematics of light propagation through the Universe, which dictates the interpretation of cosmological observations. In this lecture, we shall discuss the performance of the FLRW model for those purposes, and present some results on the so-called backreaction and fitting problems.

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1 Motivation

When proposing, in 1917, the very first cosmological model based on general relativity, Einstein made three simplifying assumptions: staticity, homogeneity and isotropy [1]. While staticity was falsified by the discovery of cosmic expansion in 1929, the premises of homogeneity and isotropy, which we may collectively refer to as the *cosmological principle*, remain part of the foundations of modern cosmology. The later observations of the tiny anisotropies of the cosmic microwave background (CMB), and of the distribution of galaxies on large scales, together with the belief that we do not occupy a special place in the Universe, came to confirm the relevance of the cosmological principle.

Of course we know that the Universe is not strictly homogeneous and isotropic, and the cosmological principle must be understood in a *statistical* sense, when structures are smoothed out on scales of hundreds of megaparsecs. But for simplicity, the standard description of the cosmos is built upon a background spacetime where the cosmological principle is applied in its strict sense, namely the Friedmann–Lemaître–Robertson–Walker (FLRW) model. Most of the interesting physics happening in the Universe, such as the dynamics of the primordial plasma, or the formation of structures, is then described as perturbations over the FLRW background.

There are, however, two classes of predictions that directly follow from the FLRW model – without accounting for perturbations. The first one is the dynamics of cosmic expansion; in other words, standard cosmology assumes that inhomogeneities have no impact on how the Universe expands on the largest scales. Assessing the validity of this assumption is known as the *backreaction problem*, which we shall discuss in the first part (section 2) of the lecture. The second class of direct predictions of the FLRW model is the propagation of light through the Universe, and in particular the relation between redshift and distance measures, which are used in the interpretation of all cosmological observations. Whether or not the FLRW is the right framework to perform such an interpretation is known as the *fitting problem*, and will be the topic of the second part (section 3) of the lecture.

2 Dynamics: the backreaction hypothesis

2.1 General idea of backreaction

The FLRW model describes the dynamics and time evolution of an idealised universe where all inhomogeneities have been smoothed out. In other words, smoothing is performed first, and time evolution follows. Yet, in principle we would rather be interested in the converse procedure, whereby smoothing would be performed after time evolution. This raises the question of the *commutation* between smoothing and time evolution.

As emphasised by Ellis in 1984 [2], smoothing is implicit in cosmological modelling, and defining precisely what a smoothed spacetime means turns out to be quite challenging. A key argument for questioning the commutation of smoothing and dynamics stems from the non-linearity of Einstein’s equation. Let $\langle \dots \rangle$ denote some smoothing or averaging procedure. Since the Einstein tensor $E_{\mu\nu}$ is a non-linear functional of the spacetime metric $g_{\mu\nu}$, the smoothed field equation is generally different from the field equation obeyed by the smoothed metric,

$$E_{\mu\nu}[\langle g_{\rho\sigma} \rangle] \neq \langle E_{\mu\nu}[g_{\rho\sigma}] \rangle = 8\pi G \langle T_{\mu\nu} \rangle. \quad (1)$$

As a consequence, there is a priori no guarantee that the Friedmann equations accurately describe the expansion dynamics of the actual, inhomogeneous Universe.

This is what backreaction is about. The name comes from the idea that, as structures form across cosmic history, the Universe departs more and more from the idealised FLRW model

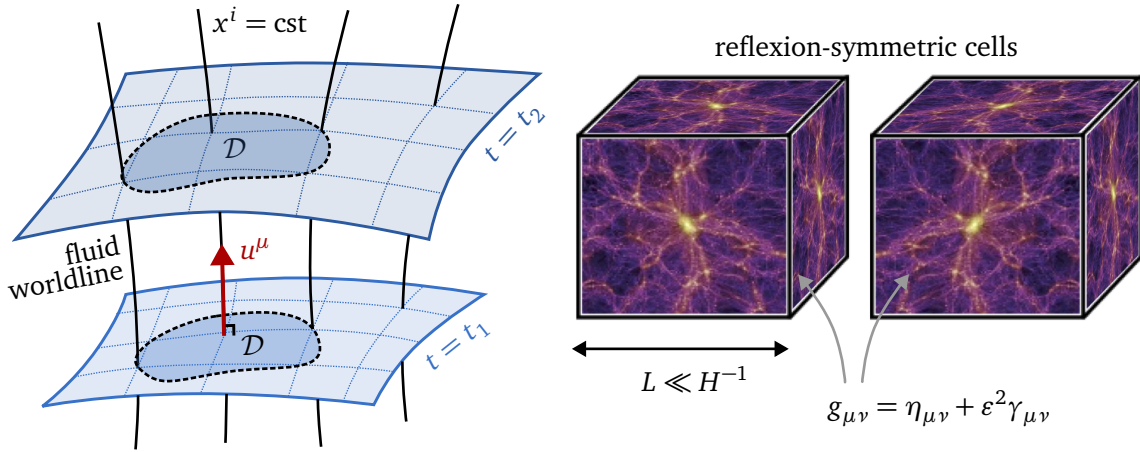


Figure 1: **Left:** geometry of an irrotational congruence of timelike geodesics. **Right:** Post-Newtonian cosmological modelling (adapted from ref. [5]).

60 and those structures, or inhomogeneities, may backreact on the expansion dynamics itself.
 61 The idea got some attention with the discovery of cosmic acceleration in the late 1990s and
 62 the *coincidence argument*: is it a coincidence that dark energy starts dominating the expansion
 63 dynamics in the late Universe, when it is most inhomogeneous? Could backreaction mimic the
 64 effect of dark energy? We shall discuss this point in the present section; for further details, I
 65 recommend the short but yet comprehensive review by Clarkson et al. [3].

66 2.2 Buchert's scalar formalism

67 In the early 2000s, T. Buchert proposed a scalar formalism [4] to describe the average dynamics
 68 of an inhomogeneous universe filled with a pressureless fluid (dust). This approach being
 69 simple and intuitive, it became quite popular in the subsequent years. Although the resulting
 70 backreaction effect is likely small, the method itself remains instructive.

71 **Description of an inhomogeneous dust flow** Consider a pressureless fluid; we denote with
 72 u^μ its four-velocity field, which is everywhere tangent to the worldline of the fluid particles.
 73 Since the fluid is only subject to gravity, the fluid elements are freely falling, and hence they
 74 follow timelike geodesics, $u^\nu \nabla_\nu u^\mu = 0$.

75 In addition, we assume that the flow is irrotational, $\nabla_\mu u_\nu = \nabla_\nu u_\mu$. This is equivalent to as-
 76 suming that the family of geodesics representing the fluid's motion is hypersurface-orthogonal:
 77 spacetime can be foliated into spatial hypersurfaces that are everywhere orthogonal to u^μ (see
 78 fig. 1, left). Those hypersurfaces physically represent space in the rest frame of an observer
 79 following the fluid's motion. As such, they can be labelled by the proper time t along the fluid
 80 elements' worldlines. When seen as a field, t satisfies $u_\mu = \partial_\mu t$.

81 **Comoving-synchronous coordinates** We may choose t as our time coordinate. As for spa-
 82 tial coordinates $(x^i)_{i \in \{1,2,3\}}$, let them be labels for the fluid elements, so that any $x^i = \text{const.}$
 83 curve is the worldline of some particle in the flow. The resulting coordinate system $(x^\mu) = (t, x^i)$
 84 is called comoving-synchronous. In terms of such coordinates, the fluid's four-velocity simply
 85 reads $u_\mu = \delta_\mu^0$, and the spacetime line element is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + h_{ij}(t, x^k) dx^i dx^j, \quad (2)$$

86 where $h_{ij}(t, x^k)$ is the spatial metric of the $t = \text{const.}$ hypersurfaces.

This may be seen as the generalisation of cosmic time and comoving coordinates for the standard FLRW geometry. The difference is that, here, the rest-frame density of the fluid is not assumed to be homogeneous, $\rho(t, x^i)$, and consequently there is in general no choice for (x^i) such that h_{ij} is a function of time only.

Volume of a comoving region We aim to determine the global expansion dynamics of this inhomogeneous fluid. We are thus interested in the time evolution of the physical volume of some very large comoving spatial domain $\mathcal{D}$,

$$V_{\mathcal{D}}(t) = \int_{\mathcal{D}} d^3x \sqrt{h}, \quad h \equiv \det[h_{ij}]. \quad (3)$$

Since the coordinate system is following the fluid's motion, the latter is entirely encoded in the spatial metric h_{ij} .¹ The evolution of $V_{\mathcal{D}}$ then depends on the average dynamics of h .

Raychaudhuri equation The local dynamics of h directly follows from the geometry of geodesic congruences. We first define the *local expansion rate* θ from the time-evolution of a very small region,

$$\theta(t, x^i) = \frac{1}{\delta V} \frac{d\delta V}{dt} = \frac{1}{\sqrt{h}} \frac{\partial \sqrt{h}}{\partial t} = \frac{1}{2} h^{ij} h_{ij,t}, \quad (4)$$

which may be seen as a local generalisation of $3 \times H$ in an inhomogeneous context.

It is straightforward to check that $\theta = \nabla_{\mu} u^{\mu} = \nabla_i u^i$. More generally, since $u^{\nu} \nabla_{\nu} u^{\mu} = 0$ the symmetric tensor $\nabla_{\mu} u_{\nu}$ is purely spatial; we may decompose it into a pure-trace and a trace-free part as

$$\nabla_i u_j = \frac{1}{3} \theta h_{ij} + \sigma_{ij}, \quad (5)$$

where σ_{ij} is the shear tensor of the flow, which describes its rate of deformation with time. Taking the time derivative of the expansion rate, $d\theta/dt = u^{\nu} \nabla_{\nu} (\nabla_{\mu} u^{\mu}) = -\nabla_{\mu} u_{\nu} \nabla^{\mu} u^{\nu} - R_{\mu\nu} u^{\mu} u^{\nu}$ and substituting Einstein's equation then yields the Raychaudhuri equation,

$$\frac{d\theta}{dt} = -\frac{1}{3} \theta^2 - 2\sigma^2 - 4\pi G \rho + \Lambda, \quad \sigma^2 \equiv \frac{1}{2} \sigma^{ij} \sigma_{ij}. \quad (6)$$

Equation (6) is reminiscent of the second Friedmann equation, and shows that the local expansion dynamics results from a competition between the cosmological constant, which tends to accelerate it, and the matter density and shear, which tend to slow it down.

Global expansion dynamics: backreaction Let us now examine the consequences of eq. (6) on the global expansion dynamics of $\mathcal{D}$. We first define an effective scale factor $a_{\mathcal{D}}$ as

$$V_{\mathcal{D}}(t) = a_{\mathcal{D}}^3(t) V_{\mathcal{D}}(t_0). \quad (7)$$

Then, taking the second derivative of $a_{\mathcal{D}}$, with some algebra and the use of the volume-average of the Raychaudhuri equation (6), we find the following effective Friedmann equation for $\mathcal{D}$,

$$\boxed{\frac{1}{a_{\mathcal{D}}} \frac{d^2 a_{\mathcal{D}}}{dt^2} = -\frac{4\pi G}{3} \langle \rho \rangle_{\mathcal{D}} + \frac{\Lambda}{3} + \underbrace{\frac{1}{3} \left[\frac{2}{3} (\langle \theta^2 \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2) - 2 \langle \sigma^2 \rangle_{\mathcal{D}} \right]}_{\text{backreaction term } \mathcal{B}}}, \quad (8)$$

¹This does not mean that “space expands”, as often written in the public-outreach literature and sometimes, unfortunately, in cosmology textbooks [6]. Cosmic expansion is about matter, not about space which has no substance, especially in relativity where it is coordinate-dependent. The interpretation of an expanding space directly comes from the use of comoving coordinates. See [7] for a diatribe on the expanding space.

113 with the volume-average operator

$$\langle X \rangle_{\mathcal{D}} \equiv \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} d^3x \sqrt{h} X. \quad (9)$$

114 Compared to the usual Friedmann equation, eq. (8) exhibits an additional, *backreaction term* $\mathcal{B}$,
115 which could mimic the effect of dark energy [8].

116 **Caveats** Buchert's spatial averaging formalism can only deal with scalar quantities. As such,
117 part of the microscopic dynamics is lost in the process, and the resulting set of dynamical
118 equations for $\mathcal{D}$ is not closed, which makes it hard to quantify the backreaction effect.

119 The backreaction term in eq. (8) results from the competition between the variance of the
120 local expansion rate, $\langle \theta^2 \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2$, and the mean square of the shear rate, $\langle \sigma^2 \rangle_{\mathcal{D}}$. It is quite
121 easy to see that there is more shear if the expansion rate is more inhomogeneous, so that both
122 terms tend to cancel out. As it turns out, in Newtonian cosmology $\mathcal{B}$ is a boundary term over
123 $\partial\mathcal{D}$, divided by the volume $V_{\mathcal{D}}$ [9]; hence it is negligible for large domains $\mathcal{D}$. This suggests
124 that cosmological backreaction, if anything, must come from relativistic effects.

125 2.3 Relativistic backreaction

126 Other authors chose to focus specifically on the general-relativistic origin of backreaction. Here
127 is a selection of three such approaches, which all suggest a negligibly small effect.

128 **Clifton & Sanghai's post-Newtonian patchwork** This is a construction based on the post-
129 Newtonian (PN) expansion of general relativity [5]. In this approach, the spacetime metric
130 is determined in a cubic cell $\mathcal{C}$ with size $L \ll H^{-1}$ using the PN expansion $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon^2 \gamma_{\mu\nu}$,
131 where $\varepsilon \sim v/c$ is the PN expansion parameter, v being the typical velocity of matter.

132 Carefully patching together identical copies of $\mathcal{C}$ results in a lattice universe (see fig. 1, right
133 panel), whose expansion dynamics emerges from the small-scale physics. In this construction,
134 there appears a genuinely relativistic backreaction term [10],

$$\mathcal{B} \approx \frac{2}{3} \frac{(\Omega_m H_0)^2}{a^4} (H_0 L_0)^2 \sim \varepsilon^4, \quad (10)$$

135 where L_0 is the present length of $\mathcal{C}$ and the scale factor is identified as $a(t) = L(t)/L_0$. This
136 term contributes positively to the acceleration of cosmic expansion and dilutes like radiation
137 ($\propto a^{-4}$). Because it is suppressed by $(H_0 L_0)^2 \ll 1$ compared to the other terms in the Fried-
138 mann equation, its effect is negligible in practice.

139 **Green & Wald's effective field theory** In ref. [11], the spacetime metric is assumed to read
140 $g_{\mu\nu} = \bar{g}_{\mu\nu} + \gamma_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is a smooth effective metric (e.g. FLRW), and $\gamma_{\mu\nu}$ encodes the
141 small-scale physics. While $\gamma_{\mu\nu}$ is assumed to be small, its first derivatives $\nabla_{\rho} \gamma_{\mu\nu}$ are only
142 required to remain finite, and its second derivatives $\nabla_{\rho} \nabla_{\sigma} \gamma_{\mu\nu}$ may be arbitrarily large.

143 Under these assumptions, the authors of ref. [11] demonstrated that $\bar{g}_{\mu\nu}$ must satisfy an
144 effective Einstein equation, where the backreaction due to $\gamma_{\mu\nu}$ manifests a correction $t_{\mu\nu}$ to
145 the stress-energy tensor, with $t_{\mu}^{\mu} = 0$. This extra contribution is interpreted as the small-
146 scale gravitational radiation that is integrated out in the dynamics of $\bar{g}_{\mu\nu}$, thereby producing
147 a negligible backreaction effect in cosmology [12]. Such conclusions led to a heated debate
148 within the backreaction community [13, 14].

149 **GEVOLUTION** In 2016, the Geneva cosmology group addressed the backreaction problem
 150 from a numerical perspective. **GEVOLUTION** [15] is a relativistic N -body code in which the
 151 space-time metric reads $g_{\mu\nu} = \bar{g}_{\mu\nu} + \gamma_{\mu\nu}$, where the background $\bar{g}_{\mu\nu}(t)$ is formally similar to
 152 the FLRW metric, but contrary to other cosmological simulations, it does not pre-suppose its
 153 evolution based on the Friedmann equations; instead, the definition of the scale factor is up-
 154 dated at each time step. Computations are then performed at first order in $\gamma_{\mu\nu}$, second order
 155 in $\nabla_\rho \gamma_{\mu\nu}$, and arbitrary order in $\nabla_\rho \nabla_\sigma \gamma_{\mu\nu}$. The expansion law is then found to match the
 156 usual Friedmannian dynamics up to a part in 10^4 [16].

157 3 Observations: the fitting problem

158 Even if the backreaction of structures on cosmic expansion is negligible, this does not neces-
 159 sarily imply that the FLRW model is the right framework to accurately interpret cosmological
 160 observations, because we do not directly observe physical distances and velocities, but rather
 161 redshifts, angles and apparent brightnesses.

162 In this section, we shall focus on the relation between redshift and angular (or luminosity)
 163 distance, which is ubiquitous in the interpretation of cosmological data, from the Hubble di-
 164 agram of type-Ia supernovae (SNe) to the CMB anisotropies, including the imprint of baryon
 165 acoustic oscillations (BAO) in the distribution of galaxies.

166 3.1 The distance–redshift relation

167 **Definitions** The redshift is the change of a signal’s cyclic frequency in the observer’s frame
 168 relative to the source’s frame, $1 + z \equiv \omega_s/\omega_o$. The angular-diameter distance, or area dis-
 169 tance D_A is defined so as to translate the observed angular size Ω_o of an image (which is a
 170 solid angle) to the physical area A_s of its source,

$$A_s = D_A^2 \times \Omega_o . \quad (11)$$

171 The luminosity distance D_L , on the other hand, connects the observed flux F_o (power per unit
 172 area) of a light source to its intrinsic luminosity (power) L_s as

$$L_s = F_o \times 4\pi D_L^2 . \quad (12)$$

173 Both notions of distance are connected by the distance-duality relation, $D_L = (1 + z)^2 D_A$,
 174 which is valid if light follows null geodesics, and if there is no significant photon creation or
 175 absorption during light propagation from the source to the observer [17]. We may thus focus
 176 on the angular-diameter distance from now on.

177 **Relation in FLRW** When light is emitted and observed by comoving entities in a strictly
 178 homogeneous and isotropic universe, the distance–redshift relation is found to read

$$\bar{D}_A(z) = \frac{1}{1+z} \frac{\sinh[\sqrt{-K}\chi(z)]}{\sqrt{-K}}, \quad \chi(z) = \int_0^z \frac{dz'}{H(z')}, \quad (13)$$

179 where K denotes the curvature of the homogeneity hypersurfaces (“space”), and χ is known
 180 as the comoving distance. Barred quantities will refer to FLRW in the whole section. Equa-
 181 tion (13) is used to constrain cosmological parameters from the Hubble diagram of type-Ia
 182 SNe, but also from BAO and CMB observations.

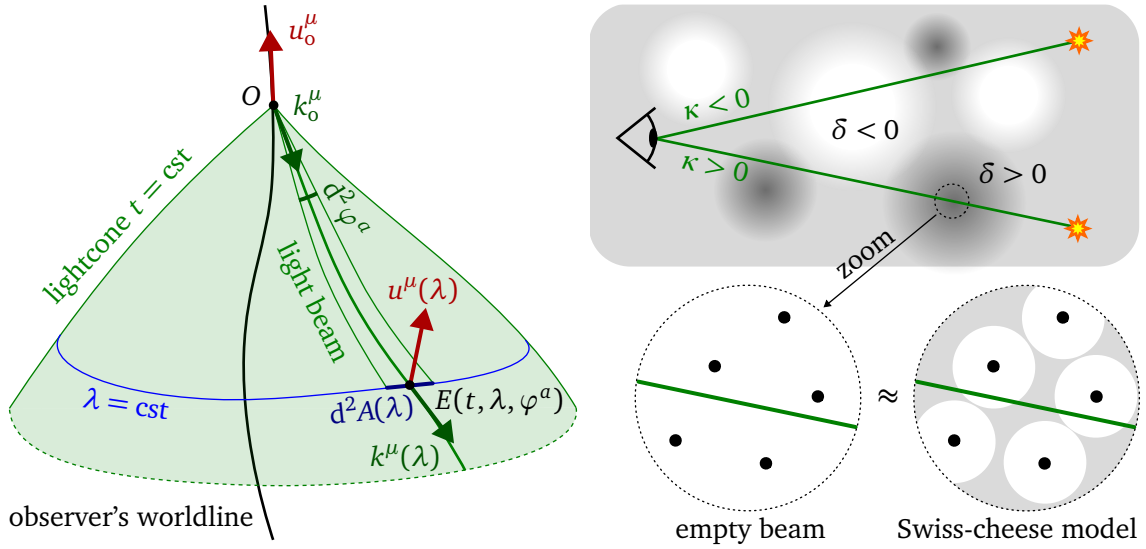


Figure 2: **Left:** Geometrical construction of the observational coordinates (t, λ, φ^a) . **Right:** Overdense (underdense) lines of sight have positive (negative) convergence κ ; in linear perturbation theory, one expects $\langle \kappa \rangle = 0$. However, accounting for the small-scale lumpiness of the matter distribution suggests instead an empty-beam model, corresponding to light propagation in, e.g., a Swiss-cheese universe.

183 **The distance–redshift relation is twofold** The standard result (13) hides the fact that $D_A(z)$
 184 is fundamentally the combination of two relations: $z(\lambda)$ and $D_A(\lambda)$, where λ is an affine
 185 parameter along the ray, i.e. the null geodesic, connecting the source to the observer.

186 Let k^μ be the past-directed wave four-vector of that ray, that is a tangent vector to the null
 187 geodesic (see fig. 2, left). The cyclic frequency measured by an observer with four-velocity u^μ
 188 is $\omega = u_\mu k^\mu$, hence $1+z(\lambda) = (u_\mu k^\mu)(\lambda)$ if we conventionally set $\omega_o = 1$. Taking the derivative
 189 with respect to λ then yields

$$\frac{dz}{d\lambda} = k^i k^j \nabla_i u_j \equiv \frac{1}{3} (1+z)^2 \theta_{||}, \quad (14)$$

190 where $\theta_{||}$ represents the local expansion rate of the matter fluid along the spatial direction of
 191 light propagation. If backreaction is small, then $\theta_{||} \approx 3H$ on average, which implies that $z(\lambda)$
 192 is mostly unaffected by inhomogeneities.

193 Regarding the second part of the relation, $D_A(\lambda)$, things are somewhat more complicated
 194 because they depend on the evolution, with light propagation, of the physical area $d^2A(\lambda)$ of
 195 a small light beam subtended by the angle $d^2\Omega_o$ at the observer.

196 3.2 Propagation of light beams in general relativity

197 **Observational coordinates** The general study of light beams is more convenient as one
 198 writes the metric in terms of a coordinate system adapted to null geodesics [18]. Those obser-
 199 vational coordinates $(t, \lambda, \varphi^1, \varphi^2)$ are defined relative to a fiducial observer (see fig. 2, left).

200 Let E be an event in spacetime, which we may see as the emission of a photon received
 201 at O by the observer. The first coordinate of E is taken to be the proper time t of O along
 202 the observer's worldline. Thus, $t = \text{const.}$ is the past lightcone of O . We may choose the
 203 second coordinate, λ , as the affine parameter of E along the null geodesic connecting it to
 204 O . The surfaces of constant (t, λ) can be seen as wavefronts converging to O . The last two
 205 coordinates, (φ^1, φ^2) , indicate the direction in which a photon emitted at E is observed at O

in the observer's frame. Curves with constant $(t, \varphi^1, \varphi^2)$ thus represent light rays converging to O . In that sense, (φ^1, φ^2) may be seen as comoving coordinates for photons.²

One can show that any spacetime metric in observational coordinates reads

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N dt^2 - dt d\lambda + k_a dt d\varphi^a + S_{ab} d\varphi^a d\varphi^b, \quad (15)$$

with $a, b \in \{1, 2\}$, and N, k_a, S_{ab} are six free functions of (t, λ, φ^a) . From now on, we shall work exclusively on the lightcone of O , hence $dt = 0$, and the metric reduces to $ds^2 = S_{ab} d\varphi^a d\varphi^b$.

Angular-diameter distance Equipped with the observational metric (15), we can conveniently express the cross-sectional area of a light beam, which is the portion of wavefront corresponding to some angular region $\mathcal{I}$ (an image) on the observer's celestial sphere,

$$A(\lambda) = \int_{\mathcal{I}} d^2\varphi^a \sqrt{S}, \quad S \equiv \det[S_{ab}]. \quad (16)$$

With the right choice of axes for φ^a , we have $d^2\varphi^a = d^2\Omega_o$ and hence

$$D_A^2 = \sqrt{S}. \quad (17)$$

Focusing theorem As with the case of timelike geodesics presented in section 2, we can define the local rate of expansion of the wavefront with λ ,

$$\Theta \equiv \frac{1}{\delta A} \frac{d\delta A}{d\lambda} = \frac{1}{\sqrt{S}} \frac{\partial \sqrt{S}}{\partial \lambda} = \frac{1}{2} S^{ab} S_{ab,\lambda}. \quad (18)$$

It is straightforward to check that $\Theta = \nabla_\mu k^\mu = \nabla_a k^a$; more generally, we may decompose the tensor $\nabla_a k_b$ into a pure-trace and a trace-free part as $\nabla_a k_b = (1/2)\Theta S_{ab} + \Sigma_{ab}$, where Σ_{ab} is the shear rate of the light beam. Physically speaking, while Θ is mostly sourced by the local density intercepted by the light beam, Σ_{ab} is sourced by the tidal forces produced by concentrations of matter near the beam.

Let us analyse the evolution of Θ down the observer's lightcone. Differentiating Θ with respect to λ , and using $k^\nu \nabla_\nu k^\mu = 0$, we get $d\Theta/d\lambda = k^\nu \nabla_\nu (\nabla_\mu k^\mu) = -\nabla_\mu k_\nu \nabla^\mu k^\nu - R_{\mu\nu} k^\mu k^\nu$. Then, substituting the decomposition of $\nabla_a k_b$ into expansion and shear rates, together with Einstein's equation, eventually yields the *null Raychaudhuri equation*,

$$\frac{d\Theta}{d\lambda} = -\frac{1}{2}\Theta^2 - 2\Sigma^2 - 8\pi G(1+z)^2\rho, \quad 2\Sigma^2 = \Sigma^{ab}\Sigma_{ab}. \quad (19)$$

We may further substitute $\Theta = (2/D_A) dD_A/d\lambda$ and get the *focusing theorem*,

$$\boxed{\frac{1}{D_A} \frac{d^2 D_A}{d\lambda^2} = -4\pi G(1+z)^2\rho - \Sigma^2 \leq 0.} \quad (20)$$

Equation (20) implies that $d^2 D_A/d\lambda^2 \leq 0$, so light beams cannot be defocused.³ As light propagates through some smooth matter distribution, focusing is ensured by the first term $\propto \rho$; when it passes near mass lumps, focusing occurs through the shear rate Σ^2 produced by tidal forces. One may also note that Λ is absent from eq. (20): while the cosmological constant pushes massive particles away, it has no effect on photons.

²Alternative choices exist, in particular for the coordinate used to navigate down the lightcone. An observationally interesting possibility consists in replacing λ with the observed redshift z . Another one uses instead a second time variable, based on a prior $(3+1)$ -foliation of spacetime; these so-called geodesic-lightcone coordinates [19] were used to address the fitting problem in the framework of second-order perturbation theory [20].

³A riddle for strong lensers: given eq. (20), how can the secondary image of, e.g., a point lens, be defocused?

3.3 Optics in a lumpy Universe

In the FLRW model, $\rho = \bar{\rho} = (1+z)^3 \bar{\rho}_0$ and $\Sigma = 0$. Let us now examine the consequences of eq. (20) on optics in an inhomogeneous Universe.

Linear perturbation theory As a warm-up exercise, we may consider a perturbed FLRW model with $\rho = \bar{\rho}(1 + \delta)$, where δ denotes the density contrast. At first order in δ , we may neglect the shear term $\Sigma^2 = \mathcal{O}(\delta^2)$. Solving eq. (20) then yields the following correction relative to the FLRW distance,

$$\frac{\delta D_A(z, \varphi^a)}{\bar{D}_A(z)} = -4\pi G \bar{\rho}_0 \int_0^{z(z)} d\chi' \frac{\chi' [\chi(z) - \chi']}{\chi(z)} \frac{\delta(\eta_0 - \chi', \chi', \varphi^a)}{a(\eta_0 - \chi')} \equiv -\kappa(z, \varphi^a), \quad (21)$$

where we assumed a spatially flat background ($K = 0$) for simplicity. In the above, η_0 denotes the conformal time today. This perturbative correction is known as the weak-lensing convergence κ . It is positive when the line of sight is mostly overdense and negative when it is mostly underdense (see fig. 2, top-right panel). In this framework, however, it is easy to see that the correction vanishes when averaged over the observer's sky, $\langle \kappa \rangle = 0$, which suggests that inhomogeneities cause no systematic bias in the distance–redshift relation.

Zel'dovich's empty beam An argument against the above conclusion was put forward in 1964 by Zel'dovich [21]. At the scale of the tiny light beams relevant to, e.g., SN observations, the Universe is extremely lumpy, so that such light beams should mostly propagate through vacuum ($\rho \approx 0$), even in overdense regions of the Universe (fig. 2, bottom panel). If furthermore the matter lumps making up the cosmic mean are opaque and not too compact, then shear may be neglected, $\Sigma^2 \approx 0$, which leads to the empty-beam model

$$D_A^{\text{EB}}(z) = \lambda(z) = \int_0^z \frac{dz'}{(1+z')^2 H(z')} > \bar{D}_A(z). \quad (22)$$

The relative difference between $D_A^{\text{EB}}(z)$ and $\bar{D}_A(z)$ can be significant. In a flat Λ CDM cosmology, it reaches about 10% for $z = 1$ [22].

Swiss-cheese models Zel'dovich's intuition can be made more rigorous by considering the so-called Swiss-cheese model [23] (fig. 2, bottom-right panel). The construction is the following; starting from the FLRW model, pick a sphere with comoving radius χ and concentrate the matter that it contains by a point mass $M = (4\pi/3)\bar{\rho}(a\chi)^3$, thereby making a “hole” in the otherwise homogeneous “cheese”. Spacetime inside the hole is described by the Schwarzschild geometry, which turns out to glue perfectly with FLRW at the boundary of the hole. An arbitrary amount of such holes can then be introduced as long as they do not overlap.

Light propagation in Swiss-cheese models was first investigated by Kantowski [24] and later by Dyer & Roeder [25], who concluded that light beams effectively propagate in an underdense universe with negligible shear Σ^2 ,

$$\frac{1}{D_A^{\text{KDR}}} \frac{d^2 D_A^{\text{KDR}}}{d\lambda^2} = -4\pi G (1+z)^2 \alpha \rho, \quad (23)$$

where $\alpha \equiv V_{\text{FLRW}}/V \leq 1$ is the smoothness parameter of the model. The KDR model (23) thus interpolates between the empty-beam model ($\alpha = 0$) and the FLRW prediction ($\alpha = 1$). It has been shown numerically to provide a good approximation to Swiss-cheese models [22, 26] as well as other lumpy models [27], including the PN patchwork model mentioned in section 2.3 if the opacity radius R of the matter lumps is large enough.

The crucial role of shear As the opacity radius R decreases, that is, as light beams are allowed to pass closer to the matter lumps, the contribution Σ^2 of shear increases. Using a stochastic model, ref. [28] quantified the contribution of shear to the angular distance in a Swiss-cheese model; at $z \approx 1$ this correction to the KDR distance scales as

$$\left(\frac{\delta D_A}{D_A^{\text{KDR}}} \right)_{\Sigma^2} \approx -10^{-3} \mathcal{A}, \quad \text{with} \quad \mathcal{A} \equiv \frac{2GM}{H_0 R^2} \quad (24)$$

for opaque lumps of mass M and radius R . If the matter lumps represent galaxies, then $\mathcal{A} \sim 1$, but if they represent individual stars, then $\mathcal{A} \sim 10^{10}$ so that the shear contribution blows up. This is a hint that Σ^2 should actually be able to compensate the deficit of focusing due to the smaller effective matter density, when properly taken into account.

In fact, it was shown by Weinberg [29] that the *average* of $D_A(\lambda)$ over directions in a universe filled with point masses is identical to the $\bar{D}_A(\lambda)$ of a universe homogeneously filled with matter with the same density. Although the calculation was performed at first order in the mean matter surface density (or optical depth), this indicates that the shear term Σ^2 in eq. (20) can indeed cancel out the effect of local density fluctuations.

3.4 Observational averages

Magnification theorems The aforementioned result from Weinberg can be generalised as follows. Consider an infinitesimal light source with apparent size $d^2\Omega_s$ if it were observed through an FLRW universe (unlensed size); the magnification of that source is defined as

$$\mu \equiv \frac{d^2\Omega_o}{d^2\Omega_s} = \left[\frac{\bar{D}_A}{D_A} \right]^2, \quad (25)$$

where $d^2\Omega_o$ is the observed angular size of the image as observed through the actual, inhomogeneous, Universe. A source is magnified ($\mu > 1$) if the corresponding light beam propagates through overdense regions of the Universe ($\rho > \bar{\rho}$), or regions with a high shear rate Σ .

Suppose now that such sources are homogeneously covering the celestial sphere S^2 , the mean magnification of such a set of sources then reads

$$\langle \mu \rangle_s = \frac{1}{4\pi} \int_{S^2} d^2\Omega_s \mu = \frac{1}{4\pi} \int_{S^2} d^2\Omega_o = 1, \quad (26)$$

where $\langle \dots \rangle_s$ stands for *source averaging*, because it gives the same weight to sources with the same intrinsic size $d^2\Omega_s$. If, on the contrary, we choose to give the same weight to all directions in the sky, then it is the inverse magnification that averages to unity,

$$\langle \mu^{-1} \rangle_o = \frac{1}{4\pi} \int_{S^2} d^2\Omega_o \mu^{-1} = \frac{1}{4\pi} \int_{S^2} d^2\Omega_s = 1, \quad (27)$$

where $\langle \dots \rangle_o$ stands for *directional averaging*. The magnification theorems (26) and (27) are trivial when all light sources have a single image, but they also hold in the presence of multiple imaging – see e.g. [30] and references therein.

Bias on distances Directional averaging $\langle X \rangle_o$ may be understood as what an observer would measure by observing X in random directions of the sky; it is the most natural notion of average involved in numerical ray-tracing simulations. Source-averaging $\langle X \rangle_s$, on the other hand, is generally more relevant to actual observations. As such, eq. (26) can be used to derive the observational bias on the observed distance to SNe due to inhomogeneities in the Universe:

$$\langle D_A(z) \rangle_s = \bar{D}_A(z) \langle \mu^{-1/2} \rangle_s = \bar{D}_A(z) \left[1 + \frac{3}{2} \langle \kappa^2(z) \rangle + \dots \right], \quad (28)$$

where κ is the weak-lensing convergence defined in eq. (21). Note that the bias is of second order in cosmological perturbations; this is why $\langle \kappa^2 \rangle$ can be either a source average or a directional average since the difference would be of higher order. Using Limber's approximation, one shows that [31, 32]

$$\langle \kappa^2(z) \rangle = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell + 1) C_{\ell}^{\kappa}(z), \quad (29)$$

$$C_{\ell}^{\kappa}(z) \approx (4\pi G \bar{\rho}_0)^2 \int_0^{\chi(z)} d\chi' \left[\frac{\chi'(\chi - \chi')}{a(\eta_0 - \chi')\chi} \right]^2 P_m \left[\eta_0 - \chi', \frac{\ell + 1/2}{\chi'} \right], \quad (30)$$

where $P_m(\eta, k)$ denotes the matter power spectrum. In a Λ CDM cosmology, at $z = 1$ the above implies that the bias on the average distance (28) reaches about 10^{-3} , which is negligible in practice. However, for the CMB at $z_* = 1090$, the bias reaches 5%, which is significant.

The distance-to-the-CMB controversy Does this mean, as argued in ref. [33], that the standard CMB analysis, based on the FLRW expression for the angular-diameter distance, is flawed? Fortunately, it does not, for three reasons. First of all, CMB analyses do not directly depend on $\langle D_A(z_*) \rangle_s$, but on the angular power spectrum, which itself relies on directional averaging $\langle \dots \rangle_o$ rather than source-averaging [34].

Second, and most importantly, the expression (29) of $\langle \kappa^2 \rangle$ implicitly accounts for cosmological perturbations down to arbitrarily small scales ($\ell \rightarrow \infty$). This would be correct if the characteristic angular scales measured in the CMB were infinitesimal; but in reality the angular-diameter distance to the CMB is used to convert the angular size of the sound horizon $\theta_* \approx 0.6 \text{ deg}$ into a physical distance $r_s = D_A(z_*)\theta_*$. This suggests that, as far as cosmological analyses are concerned, the relevant light beams in the CMB have an aperture $\sim \theta_*$. Now, finite light beams propagating through some matter distribution effectively smooth out inhomogeneities that are smaller than the beam's size [35]; hence the sum over ℓ in eq. (29) should actually be cut above $\ell_* = \pi/\theta_* \approx 300$, which yields a bias on $D_A(z_*)$ of about 10^{-3} .

Third and finally, the impact of gravitational lensing on the interpretation CMB power spectrum at second order is already taken into account in CMB-lensing analyses [36].

4 Conclusion and outlook

The cosmological principle, which proposes that the Universe is statistically homogeneous and isotropic, is one of the fundamental premises of modern cosmology. The FLRW model is the simplest application of that principle, and yet it turns out to provide a very efficient basis to cosmological modelling. On the one hand, the *backreaction* of inhomogeneities on the expansion dynamics seems to be practically negligible in the framework of general relativity, although alternative theories of gravitation may lead to a different conclusions [37, 38]. On the other hand, inhomogeneities have a negligible impact on the *average* relation between redshift and distances, which is key to interpreting most cosmological observations, although significant corrections to the FLRW prediction can occur on individual measurements. Summarising, *if the cosmological principle is correct*, then a model built upon the FLRW background should provide an accurate description of the Universe. This conclusion highlights the necessity to test the cosmological principle itself with great precision.

There are, in my opinion, two recent observations whose results are at least intriguing in that respect. In the first one [39], the sky is divided in large pixels where the CMB anisotropies are fitted as if they were all independent; this analysis reveals three large sky patches where the cosmological parameters consistently differ by up to 30% from each other, as if we were

lying at the cross-roads of three distinct universes. The second intriguing observation is based on the *dipole test* proposed by Ellis & Baldwin in 1984 [40]. In a nutshell, if the Universe satisfies the cosmological principle, then the observed dipole in the CMB must be of kinematic origin. Because of aberration effects and kinematic redshift, our velocity with respect to the homogeneity frame also manifests itself in the apparent number density of distance sources, such as radio galaxies or quasars. Reference [41] found that this quasar dipole is twice larger than what is expected from the observed CMB dipole. To my knowledge, there is currently no explanation to those anomalies.

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