Classification of modular data up to rank 12

Siu-Hung Ng[1], Eric C. Rowell[2,3], Xiao-Gang Wen[4*]

1 Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
2 Department of Mathematics, Texas A&M University, College Station, TX 77843, USA
3 School of Mathematics, Leeds University, Leeds, LS2 9JT UK
4 Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
* xgwen@mit.edu

Abstract

We use the computer algebra system GAP to classify modular data up to rank 12. This extends the previously obtained classification of modular data up to rank 6. Our classification includes all the modular data from modular tensor categories up to rank 12, with a few possible exceptions at rank 12 and levels $5, 7$ and 14. Those exceptions are eliminated up to a certain bound by an extensive finite search in place of required infinite search. Our list contains a few potential unitary modular data which are not known to correspond to any unitary modular tensor categories (such as those from Kac-Moody algebra, twisted quantum doubles of finite group, as well as their Abelian anyon condensations). It remains to be shown if those potential modular data can be realized by modular tensor categories or not. We provide some evidence that all may be constructed from centers of near-group categories or gauging group symmetries of known modular tensor categories, with the exception of a total of five cases at rank 11 (with $D^2 = 1964.590$) and 12 (with $D^2 = 3926.660$). The classification of modular data corresponds to a classification of modular tensor categories (up to modular isotopes which are not expected to be present at low ranks). The classification of modular tensor categories leads to a classification of gapped quantum phases of matter in 2-dimensional space for bosonic lattice systems with no symmetry, as well as a classification of generalized symmetries in 1-dimensional space.

Contents

1 Introduction

1.1 Gapped liquid phases of quantum matter and braided fusion higher categories

Quantum states of matter can be divided into four classes:

- Gapped liquid: All excitations have a gap and there are no low energy excitations. So the gapped states appear to be trivial at low energies. Band insulator and quantum Hall states are examples of gapped liquid states.
- Gapped non-liquid: All excitations also have an energy gap. But in contrast to gapped liquid, by definition, a gapped non-liquid cannot "dissolve" product states [1, 2]. Gapped fracton states are examples of gapped non-liquid states [3, 4].
- Gapless liquid: There are finitely many types of gapless low energy excitations. Dirac/Weyl semimetal, superfluid, critical point at continuous phase transition are examples of gapless liquid states.
- Gapless non-liquid: There are infinity many types of gapless low energy excitations. Fermi metal, Bose metal, *etc.* are examples of gapless non-liquid states.

People used to believe that Landau symmetry breaking theory provides a systematic description of gapped phases of quantum matter. In this case, group theory that describes symmetry breaking patterns provides a mathematical foundation and classification of spontaneous symmetry breaking states.

The experimental discovery of fractional quantum Hall states suggested that Landau symmetry breaking theory fails to describe all gapped phases. This led to the theoretical discovery of a new order in gapped liquid states: topological order [5–7], which corresponds to pattern of long range many-body entanglement [8]. But what mathematical theory systematically describes various topological orders (*i.e.* patterns of long range entanglement)?

There are two approaches:

- **Ground state based**: the robust degenerate ground states led to the discovery and physical definition of topological order [5, 10]. It is a striking property of topological order that the degeneracy of ground states depends on the topology of the closed space where the system lives. We note that the degenerate ground states give rise to a vector bundle over the moduli space of gapped quantum systems [6, 7, 11]. The vector bundle, plus many additional conditions, can form a foundation for a general theory of topological order. We refer to this ground-state-based approach as moduli bundle theory. We note that the holonomy of the vector bundle give rise to a projective representation of the mapping class group of the space on which the ground states live. Thus the degenerate ground states form a projective representation of the mapping class group. When the space is a 2-dimensional torus, the mapping class group is $SL_2(\mathbb{Z})$ and the representation (with a particular choice of basis) is called modular data. We see that the mapping class group representations and the modular data are the key ingredients of the moduli bundle theory.
- **Excitation based**: We may also use topological excitations (which is defined as excitations that cannot be created individually) to describe topological orders. Those excitations can fuse and braid. Thus one can use a non-degenerate braided (higher) fusion category to describe topological excitations and its associated topological order [12, 13]. In 2-dimensional space, the topological excitations are point-like (called anyons). They carry fractional Abelian [14–17] or non-Abelian statistics [18, 19] described by braid group representations [20, 21]. Those anyons (and the associated

Table 1: List of topological orders (TO) (up to $E(8)$ invertible topological order and up to modular isotopes) for bosonic systems with no symmetry in 2-dimensional space, which are classified by the unitary modular data (UMD) with increasing rank (the number of anyon types). The Abelian TOs have only Abelian anyons (*i.e.* pointed simple objects). The non-Abelian TOs have at least one non-Abelian anyon. The prime TOs cannot be viewed as stacking of two non-trivial TOs with fewer anyon types. Such a classification also leads to a classification of symmetry-TOs in 2-dimensional space, which classifies the generalized global symmetries in 1-dimensional space, up to holo-equivalence [9]. Those generalized symmetries include, but can go beyond, finite-group symmetries (with potential anomalies). Note that the list includes all modular data, but also contain a few potential modular data at rank 11, and 12 whose realizations are unknown (see Table 8).

# of anyon types (rank)	1	2	3	4	5	6	7	8	9	10	11	12
# of TOs (UMD)	1	4	12	18	10	50	28	64	81	76	44	221
# of prime TOs (prime UMD)	1	4	12	8	10	10	28	20	20	40	44	33
# of Abelian TOs (pointed UMD)	1	2	2	9	2	4	2	20	4	4	2	18
# of non-Abelian TOs (non-pointed UMD)	0	2	10	9	8	46	26	44	77	72	42	203
# of symTOs (UMTC in trivial Witt class)	1	0	0	3	0	0	0	6	6	3	0	3
# of finite-group symmetries (with anomaly ω)	$1_{\mathbb{Z}_1}$	0	0	$2_{\mathbb{Z}_2^\omega}$	0	0	0	$6_{S_3^\omega}$	$3_{\mathbb{Z}_3^\omega}$	0	0	0

topological orders in 2-dimensional space) are systematically describe by modular tenor category theory. It is interesting to note that modular tensor categories were first used to systematically describe rational conformal field theories [22]. Then topological quantum field theories in 2-dimensional space (which contain anyons) were shown to be closely connected to rational conformal field theory [23]. In particular, the structure of modular tenor categories in 1+1D conformal field theory have a natural interpretation in terms of anyons in 2+1D topological quantum field theory. This led to the modular tensor category description of anyons (for a review, see Ref. [24]) and 2+1D bosonic topological orders (for a review, see Ref. [25]).

To summarize, gapped quantum liquid [1,2] phases of matter (*i.e.* topological orders) in n-dimensional space are described by moduli bundle theory [6,7,11] or braided fusion $n-1$-categories with trivial center [13,26]. For example, for bosonic systems with no symmetry, there is no non-trivial gapped quantum liquid phases (*i.e.* no non-trivial topological order) in 1-dimensional space [27–29], since the mapping class group of a circle, $\mathrm{SL}_1(\mathbb{Z})$, is trivial. The gapped quantum liquid phases (*i.e.* topological orders) in 2-dimensional space (up to stacking of $E(8)$ invertible topological orders), are classified by modular data for the torus and generalized modular data for high genus surfaces [6, 7, 11, 30], or alternatively, by unitary modular tensor categories (UMTC, which are braided fusion 1-categories with trivial center). The gapped quantum liquid phases in 3+1-dimensions are also classified. For example, those without emergent fermions are classified by a finite group G and its group cohomology classes $\omega \in H^4(G; \mathbb{R}/\mathbb{Z})$ [31].

Because moduli bundle theory and non-degenerate braided fusion (higher) category theory describe the same physical object – topological order, in this paper, we are going apply this connection in 2-dimensional space, and use the moduli bundle approach to classify modular tensor categories through modular data. In other words, we will use modular data to classify modular tensor categories, up to modular isotopes. Here modular isotopes correspond to different modular tensor categories with the same modular data. The first example of modular isotopes is given in Ref. [32], the twisted quantum double

Table 2: Fusion rule for rank-11 $D^2 \approx 1964.590$ potential modular data

$\otimes$	$\mathbb{1}$	a	b	c	d	e	f	g	h	i	j
$\mathbb{1}$	$\mathbb{1}$	a	b	c	d	e	f	g	h	i	j
a	a	$\mathbb{1}\oplus a\oplus c\oplus d\oplus e\oplus f\oplus i$	$b\oplus g\oplus h\oplus i\oplus j$	$a\oplus f\oplus g\oplus i\oplus j$	$a\oplus d\oplus e\oplus f\oplus g\oplus i\oplus j$	$a\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$a\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$b\oplus e\oplus f\oplus g\oplus h\oplus i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus 2h\oplus 2i\oplus 2j$
b	b	$b\oplus g\oplus h\oplus i\oplus j$	$\mathbb{1}\oplus a\oplus b\oplus d\oplus e\oplus g\oplus i$	$c\oplus f\oplus h\oplus i\oplus j$	$b\oplus d\oplus e\oplus g\oplus h\oplus i\oplus j$	$b\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$c\oplus e\oplus f\oplus g\oplus h\oplus i\oplus 2j$	$a\oplus b\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$a\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus c\oplus d\oplus e\oplus 2f\oplus g\oplus h\oplus 2i\oplus 2j$
c	c	$a\oplus f\oplus g\oplus i\oplus j$	$c\oplus f\oplus h\oplus i\oplus j$	$\mathbb{1}\oplus b\oplus c\oplus d\oplus e\oplus h\oplus i$	$c\oplus d\oplus e\oplus f\oplus h\oplus i\oplus j$	$c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$a\oplus b\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$a\oplus e\oplus f\oplus g\oplus h\oplus i\oplus 2j$	$b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus d\oplus e\oplus f\oplus 2g\oplus h\oplus 2i\oplus 2j$
d	d	$a\oplus d\oplus e\oplus f\oplus g\oplus i\oplus j$	$b\oplus d\oplus e\oplus g\oplus h\oplus i\oplus j$	$c\oplus d\oplus e\oplus f\oplus h\oplus i\oplus j$	$\mathbb{1}\oplus a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus j$	$a\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus 2f\oplus 2g\oplus 2h\oplus 2i\oplus 3j$
e	e	$a\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$b\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus j$	$\mathbb{1}\oplus a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus 2f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus 2g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus 2h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 2i\oplus 3j$	$a\oplus b\oplus c\oplus d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$
f	f	$a\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$c\oplus e\oplus f\oplus g\oplus h\oplus i\oplus 2j$	$a\oplus b\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$a\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus 2f\oplus g\oplus h\oplus 2i\oplus 2j$	$\mathbb{1}\oplus a\oplus b\oplus c\oplus d\oplus 2e\oplus f\oplus 2g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus 2f\oplus g\oplus 2h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus 2g\oplus 2h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$	$a\oplus 2b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$
g	g	$b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$a\oplus b\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$a\oplus e\oplus f\oplus g\oplus h\oplus i\oplus 2j$	$a\oplus b\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus 2g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus 2f\oplus g\oplus 2h\oplus 2i\oplus 2j$	$\mathbb{1}\oplus a\oplus b\oplus c\oplus d\oplus 2e\oplus f\oplus g\oplus 2h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus 2f\oplus 2g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$	$a\oplus b\oplus 2c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$
h	h	$b\oplus e\oplus f\oplus g\oplus h\oplus i\oplus 2j$	$a\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus i\oplus j$	$b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus 2h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus 2g\oplus 2h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus 2f\oplus 2g\oplus h\oplus 2i\oplus 2j$	$\mathbb{1}\oplus a\oplus b\oplus c\oplus d\oplus 2e\oplus 2f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$	$2a\oplus b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$
i	i	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 2i\oplus 3j$	$a\oplus b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$	$a\oplus b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$	$a\oplus b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$	$\mathbb{1}\oplus 2a\oplus 2b\oplus 2c\oplus 2d\oplus 2e\oplus 3f\oplus 3g\oplus 3h\oplus 4i\oplus 4j$	$2a\oplus 2b\oplus 2c\oplus 2d\oplus 3e\oplus 3f\oplus 3g\oplus 3h\oplus 4i\oplus 4j$
j	j	$b\oplus c\oplus d\oplus e\oplus f\oplus g\oplus 2h\oplus 2i\oplus 2j$	$a\oplus c\oplus d\oplus e\oplus 2f\oplus g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus d\oplus e\oplus f\oplus 2g\oplus h\oplus 2i\oplus 2j$	$a\oplus b\oplus c\oplus d\oplus e\oplus 2f\oplus 2g\oplus 2h\oplus 2i\oplus 3j$	$a\oplus b\oplus c\oplus d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$	$a\oplus 2b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$	$a\oplus b\oplus 2c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$	$2a\oplus b\oplus c\oplus 2d\oplus 2e\oplus 2f\oplus 2g\oplus 2h\oplus 3i\oplus 3j$	$2a\oplus 2b\oplus 2c\oplus 2d\oplus 3e\oplus 3f\oplus 3g\oplus 3h\oplus 4i\oplus 4j$	$\mathbb{1}\oplus 2a\oplus 2b\oplus 2c\oplus 3d\oplus 3e\oplus 3f\oplus 3g\oplus 3h\oplus 4i\oplus 4j$

$D^\omega(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5)$ of rank 49. There are no known modular isotopes at rank 12 or less. Thus, at low ranks, a classification of modular data likely corresponds to a classification of modular tensor categories. A classification of modular tensor categories in turn gives rise to a classification of all gapped quantum phases of matter in 2-dimensional space (see Table 1).

1.2 Generalized symmetry and braided fusion higher categories in trivial Witt class

After a systematic understanding of gapped phases of quantum matter, the next goal is to have a systematic understanding of gapless liquid phases of quantum matter. This is a wide open and much harder problem. One idea is to use emergent symmetries at low energies to character those gapless phases, hoping to obtain a systematic understanding via the maximal emergent symmetries [33].

It became more and more clear that emergent symmetries in gapless quantum states are generalized symmetries, which can be a combination of ordinary symmetry (described by group), higher-form symmetry [34–37] (described by higher-group with only one non-trivial layer), higher-group symmetry [36], anomalous ordinary symmetry [38–41], anomalous higher symmetry [36, 37, 42–52], beyond-anomalous symmetry [53], non-invertible 0-symmetry (in 1+1D) [54–64], non-invertible higher symmetry (which includes algebraic higher symmetry) [9, 65–67], and/or non-invertible gravitational anomaly [13, 26, 68–71]. Thus emergent symmetries can go beyond the group and higher group theory description. It was proposed [9, 53, 66, 67, 70, 72–77] that the (holo-equivalent classes of) generalized symmetries in n-dimensional space can be systematically described by topological order (TO) with gappable boundary in one higher dimension, or equivalently by non-degenerate braided fusion $n-1$-categories in trivial Witt class. Such topological order with gappable boundary (*i.e.* non-degenerate braided fusion higher category in trivial Witt class) is referred to as symmetry-TO (symTO). Thus symTO, replacing group and higher group, describes generalized symmetry, which can be anomalous or beyond anomalous.

Classification of groups is a major achievement of modern mathematics. Such a classification is important since groups describe possible symmetries in our world. However, from the above discussion, we see that holo-equivalent symmetries in our quantum world are actually described by non-degenerate braided fusion higher categories in trivial Witt class. Thus our classification of modular tensor categories also leads to a classification of emergent generalized symmetry for quantum systems in 1-dimensional space (see Table 1).

1.3 Summary of results

In this paper, we use the GAP computer algebraic system [78] to classify modular data up to rank 12. We focus on *prime* modular data, *i.e.* those that are not simply products of two smaller modular data. Those prime modular data are given in Section 7, where, for each Galois orbit, we list one modular data (an unitary one if exists). We also indicate the MTCs that realize the modular data, if realizations are known.

From our explicit classifications, we find some exotic potential modular data which are not realized by Kac-Moody algebras or by twisted quantum doubles, nor by their Deligne product, their Galois conjugations, their changes of spherical structure, and their Abelian anyon condensations [79]. We show or provide strong evidence that all but five of the exotic potential modular data are indeed modular data that can be realized by center of near group fusion category or gauging the automorphism of known modular data (followed by some condensation reductions, see Section 8 for details).

One of the five potential modular data with unknown realization is labeled by $11^{35,581}_{\frac{32}{5},1964.}$ which has rank 11, central charge $c = \frac{32}{5}$ mod 8, and total quantum dimension $D^2 \approx 1964.590$. We do not know for sure if $11^{35,581}_{\frac{32}{5},1964.}$ is a modular data or not. The topological spins and quantum dimensions for such a potential modular data are given by
$s_i = 0, \frac{2}{35}, \frac{22}{35}, \frac{32}{35}, \frac{1}{5}, 0, \frac{3}{7}, \frac{5}{7}, \frac{6}{7}, \frac{3}{5}, \frac{1}{5},$

$$\begin{aligned}d_i = 1,\ &\tfrac{1}{2}\Big(5+2\sqrt{5}+\sqrt{7(5+2\sqrt{5})}\Big),\ \tfrac{1}{2}\Big(5+2\sqrt{5}+\sqrt{7(5+2\sqrt{5})}\Big),\ \tfrac{1}{2}\Big(5+2\sqrt{5}+\sqrt{7(5+2\sqrt{5})}\Big),\\ &\tfrac{1}{4}\Big(9+5\sqrt{5}+\sqrt{14(25+11\sqrt{5})}\Big),\ \tfrac{1}{4}\Big(11+7\sqrt{5}+\sqrt{14(25+11\sqrt{5})}\Big),\ \tfrac{1}{4}\Big(15+7\sqrt{5}+\sqrt{14(25+11\sqrt{5})}\Big),\\ &\tfrac{1}{4}\Big(15+7\sqrt{5}+\sqrt{14(25+11\sqrt{5})}\Big),\ \tfrac{1}{4}\Big(15+7\sqrt{5}+\sqrt{14(25+11\sqrt{5})}\Big),\ \tfrac{1}{4}\Big(21+7\sqrt{5}+\sqrt{14(65+29\sqrt{5})}\Big),\\ &\tfrac{1}{4}\Big(19+9\sqrt{5}+\sqrt{14(65+29\sqrt{5})}\Big).\end{aligned}$$

The S matrices is given in Section 7.5, and fusion by Table 2. The higher central charges [80] are $1, \zeta_5^4, -\zeta_5^4, -\zeta_5, -\zeta_5, \frac{0}{0}, -\zeta_5^4, \zeta_5, -\zeta_5, -\zeta_5, \frac{0}{0}, \zeta_5^4, -\zeta_5^4, -\zeta_5, -\zeta_5^2, \frac{0}{0}, \zeta_5^4, -\zeta_5^4, -\zeta_5, \zeta_5, \frac{0}{0}, -\zeta_5^3, -\zeta_5^4, -\zeta_5, \zeta_5, \frac{0}{0}, -\zeta_5^4, -\zeta_5^4, \zeta_5^4, -\zeta_5, \frac{0}{0}, -\zeta_5^4, -\zeta_5^4, -\zeta_5, \zeta_5$. Such a potential modular data has no non-trivial condensable algebra. There are three other potential modular data at rank 12 whose realizations are unknown. Those potential modular data have central charge $c = 0 \bmod 8$ and total quantum dimension $D^2 \approx 3926.660$ (see Section 7.6).

2 The necessary conditions on modular data

Modular data is an important invariant of MTC. In the following, we list necessary conditions on modular data. We will use those conditions, trying to classify modular data. Many conditions are well-known and can be found in, e.g. [81].

Proposition 2.1. *The modular data (S, T) of an MTC satisfies:*

1. *S, T are symmetric complex matrices, indexed by $i, j = 1, \dots, r$.*[1]

2. *T is unitary, diagonal, and $T_{11} = 1$.*

3. *$S_{11} = 1$. Let $d_i = S_{1i}$ and $D = \sqrt{\sum_{i=1}^r d_i d_i^*}$ (the positive root). Then*
$$SS^\dagger = D^2 \mathrm{id}, \tag{1}$$
and the $d_i \in \mathbb{R}$.

4. *S_{ij} are cyclotomic integers in $\mathbb{Q}_{\mathrm{ord}(T)}$*[2] *[82]. The ratios S_{ij}/S_{1j} are cyclotomic integers for all i, j [83]. Also there is a j such that $S_{ij}/S_{1j} \in [1, +\infty)$ for all i [84].*

5. *Let $\theta_i = T_{ii}$ and $p_\pm = \sum_{i=1}^r d_i^2 (\theta_i)^{\pm 1}$. Then p_+/p_- is a root of unity, and $p_+ = D e^{i 2\pi c/8}$ for some rational number c.*[3] *Moreover, the modular data (S, T) is associated with a projective $\mathrm{SL}_2(\mathbb{Z})$ representation, since:*
$$(ST)^3 = p_+ S^2, \quad \frac{S^2}{D^2} = C, \quad C^2 = \mathrm{id}, \tag{2}$$
where C is a permutation matrix satisfying
$$\mathrm{Tr}(C) > 0. \tag{3}$$

6. *D is a cyclotomic integer. $D^5/\,\mathrm{ord}(T)$ is an algebraic integer, which is also a cyclotomic integer [85]. D/d_i are cyclotomic integers (see Lemma 5.4).*

7. *Cauchy Theorem [86]: The set of prime divisors of $\mathrm{ord}(T)$ coincides with the prime divisors of $\mathrm{norm}(D^2)$.*[4] *The prime divisors of norm(D) and $\mathrm{ord}(T)$ coincide. The prime divisors of norm(D/d_i) are part of those of $\mathrm{ord}(T)$.*

8. *Verlinde formula (cf. [87]) :*
$$N_k^{ij} = \frac{1}{D^2} \sum_{l=1}^r \frac{S_{li} S_{lj} S_{lk}^*}{d_l} \in \mathbb{N}, \tag{4}$$
where $i, j, k = 1, 2, \dots, r$ and $\mathbb{N}$ is the set of non-negative integers.[5] *The N_1^{ij} satisfy*
$$N_1^{ij} = C_{ij}, \tag{5}$$
which defines a charge conjugation $i \to \bar{i}$ via
$$N_1^{\bar{i}j} = \delta_{ij}. \tag{6}$$

[1] The index also labels the simple objects in the MTC, with $i = 1$ corresponding to the unit object, and r is the **rank** of the modular data and the MTC.
[2] Here $\mathbb{Q}_n$ denotes the field $\mathbb{Q}(\zeta_n)$ for a primitive nth root of unity ζ_n
[3] The **central charge** c of the modular data and of the MTC is only defined modulo 8.
[4] Here norm(x) is the product of the distinct Galois conjugates of the algebraic number x.
[5] The N_k^{ij} are called the fusion coefficients.

9. *Let $n \in \mathbb{N}_+$. The n^{th} Frobenius-Schur indicator of the i-th simple object*

$$\nu_n(i) = D^{-2} \sum_{j,k} N_i^{jk} (d_j \theta_j^n)(d_k \theta_k^n)^* \tag{7}$$

is a cyclotomic integer whose conductor divides n and $\mathrm{ord}(T)$ [88, 89]. The 1st Frobenius-Schur indicator satisfies $\nu_1(i) = \delta_{i,1}$ while the 2nd Frobenius-Schur indicator $\nu_2(i)$ satisfies $\nu_2(i) = 0$ if $i \neq \bar{i}$, and $\nu_2(i) = \pm 1$ if $i = \bar{i}$ (see [88, 90, 91]).

10. *The twists, fusion coefficients and S-matrix entries satisfy the balancing equation[6]:*

$$S_{ij} = \sum_k N_k^{ij} \frac{\theta_i \theta_j}{\theta_k} d_k. \tag{8}$$

Based on a physics argument, Ref. [11] conjectured that modular data also satisfy

$$c\, D_g/2 \in \mathbb{Z} \ \text{ for } g \geq 3, \qquad D_g = \sum_i \big(D/d_i\big)^{2(g-1)}, \quad g = 0, 1, 2, 3, 4, \cdots . \tag{9}$$

We note that $\frac{D}{d_i}$ are real cyclotomic integers. The Galois conjugation of $\frac{D}{d_i}$ just permutes the i-indices: $\sigma(\frac{D}{d_i}) = \frac{D}{d_{\hat{\sigma}(i)}}$. Thus D_g is a cyclotomic integer that is invariant under all Galois conjugations, which implies that D_g is a positive integer. In fact, D_g is the ground state degeneracy of the corresponding topological order on genus g Riemann surface. We find that both the unitary and non-unitary modular data that we obtained satisfy the above condition, although the condition is argued for unitary topological orders. For $g = 2$, the condition $c\, D_2/2 \in \mathbb{Z}$ is not satisfied. But a weaker condition

$$c\, D_2 \in \mathbb{Z} \tag{10}$$

is satisfied by all the unitary and non-unitary modular data that we obtained, as well as by unitary modular data constructed from Kac-Moody algebras up to rank 200. Also, Ref. [92] showed that

$$c\, D_1\, D^5/2 = c\, r\, D^5/2 = \text{cyclotomic integers}. \tag{11}$$

This generalizes the above result to $g = 1$ case. The unitary and non-unitary modular data that we obtained actually satisfy a stronger condition

$$c\, D^5/2 = \text{cyclotomic integers}. \tag{12}$$

From (2), we see that the modular data (S, T) is closely related to the $\mathrm{SL}_2(\mathbb{Z})$ representations. We are going to use this relation to classify modular data. Let us first summarize some important facts about $\mathrm{SL}_2(\mathbb{Z})$ representations. Let $\mathfrak{s} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\mathfrak{t} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be the standard generators of $\mathrm{SL}_2(\mathbb{Z})$. This admits the presentation:

$$\mathrm{SL}_2(\mathbb{Z}) = \langle \mathfrak{s}, \mathfrak{t} \mid \mathfrak{s}^4 = \mathrm{id}, (\mathfrak{s}\mathfrak{t})^3 = \mathfrak{s}^2 \rangle\,. \tag{13}$$

We note that for any positive integer n, the reduction $\mathbb{Z} \to \mathbb{Z}_n$ defines a surjective group homomorphism $\pi_n : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. Thus, a representation of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is also a representation of $\mathrm{SL}_2(\mathbb{Z})$, which will be called a *congruence* representation of $\mathrm{SL}_2(\mathbb{Z})$ in this paper. It is immediate to see that a representation of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is also a $\mathrm{SL}_2(\mathbb{Z}/mn\mathbb{Z})$ representation for any positive integer m. The smallest positive integer n such that a congruence representation ρ of $\mathrm{SL}_2(\mathbb{Z})$ factors through $\pi_n : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is called the *level* of ρ. It is known that the level $n = \mathrm{ord}(\rho(\mathfrak{t}))$ (cf. [93, Lem. A.1]). Here $\mathrm{ord}(t)$ is defined as:

[6] This holds more generally in ribbon fusion categories, i.e. premodular categories.

Definition 2.2. Let t be any matrix over $\mathbb{C}$. The smallest positive integer n such that $t^n = \mathrm{id}$ is called the *order* of t, and denoted by $\mathrm{ord}(t) := n$. If such integer does not exist, we define $\mathrm{ord}(t) := \infty$.

Similarly

Definition 2.3. Let t be any matrix over $\mathbb{C}$. The smallest positive integer n such that $t^n = \alpha\mathrm{id}$ for some $\alpha \in \mathbb{C}^\times$ is called the *projective order* of t, and denoted by $\mathrm{pord}(t) := n$. If such integer does not exist, we define $\mathrm{pord}(t) := \infty$.

We can organize the finite level irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$ by the level and the dimension of the representations. Due to the Chinese remainder theorem, if the level of a irreducible representation ρ factors as $n = \prod_i p_i^{k_i}$ where p_i are distinct primes, then $\rho \cong \bigotimes_i \rho_i$ where ρ_i are level $p_i^{k_i}$ representations. Thus we can construct all irreducible $\mathrm{SL}_2(\mathbb{Z})$ representations as tensor products of irreducible $\mathrm{SL}_2(\mathbb{Z})$ representations of prime-power levels, which in turn, yields a construction of all semisimple $\mathrm{SL}_2(\mathbb{Z})$ representations ρ via direct sums of the irreducible representations.

Define $\mathbb{Q}_n = \mathbb{Q}(\zeta_n)$ to be the cyclotomic field of order n. For any positive integer n, we can construct a faithful representation $D_n : \mathrm{Gal}(\mathbb{Q}_n) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$, which identifies the Galois group $\mathrm{Gal}(\mathbb{Q}_n) \cong \mathbb{Z}_n^\times$ with the diagonal subgroup of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ [93, Remark 4.5]. More generally, for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$, $\sigma(\mathbb{Q}_n) = \mathbb{Q}_n$ and so there exists an integer a (unique modulo n) such that $\sigma(\zeta_n) = \zeta_n^a$ and

$$D_n(\sigma) := \mathfrak{t}^a\mathfrak{s}\mathfrak{t}^b\mathfrak{s}\mathfrak{t}^a\mathfrak{s}^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})\,, \tag{14}$$

where b satisfies $ab \equiv 1 \mod n$. If ρ is a level n representation of $\mathrm{SL}_2(\mathbb{Z})$, the composition

$$D_\rho(\sigma) := \rho \circ D_n(\sigma) \tag{15}$$

defines a representation of $\mathrm{Gal}(\bar{\mathbb{Q}})$. We may also write $D_n(\sigma)$ as $D_n(a)$. Such a representation of Galois group captures the Galois conjugation action on $\mathrm{SL}_2(\mathbb{Z})$ representations of modular data, and plays a very important role in our classification. In other words, in our classification, we look for some $\mathrm{SL}_2(\mathbb{Z})$ representations such that $D_\rho(a)$ is a signed permutation matrix.

We also note that the 1-dimensional representations of $\mathrm{SL}_2(\mathbb{Z})$, denoted as $\widehat{\mathrm{SL}_2(\mathbb{Z})}$, form a cyclic group of order 12 under tensor product. We will take $\chi \in \widehat{\mathrm{SL}_2(\mathbb{Z})}$ defined by $\chi(\mathfrak{t}) = \zeta_{12}$ to be the generator, where $\zeta_n := \mathrm{e}^{2\pi \mathrm{i}/n}$ and $\zeta_n^k := \mathrm{e}^{2\pi \mathrm{i} k/n}$. Under this convention, every 1-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$ is equivalent to χ^α for some integer α, unique modulo 12:

$$\chi^\alpha(\mathfrak{s}) = \bar{\zeta}_4^\alpha, \quad \chi^\alpha(\mathfrak{t}) = \zeta_{12}^\alpha. \tag{16}$$

From a modular data, we can obtain a particular type of $\mathrm{SL}_2(\mathbb{Z})$ representations, called MD representations. From a MD representation, we can also obtain its corresponding modular data. Thus we can classify modular data by classifying MD representations. In the following, we describe the detailed relation between modular data and MD representations, and the necessary conditions for a $\mathrm{SL}_2(\mathbb{Z})$ representation to be a MD representation. Many of the following collection of results on MD representations were proved in [82, 93]:

Proposition 2.4. *Given a modular data S, T of rank r, let ρ_α be any one of its* ***MD representations****, which is defined as*

$$\rho_\alpha(\mathfrak{s}) = P\bar{\zeta}_4^\alpha \mathrm{e}^{2\pi \mathrm{i}\frac{c}{8}} \frac{S}{p_+} P^\top, \quad \rho_\alpha(\mathfrak{t}) = P\zeta_{12}^\alpha \mathrm{e}^{-2\pi \mathrm{i}\frac{c}{24}} T P^\top \quad (\alpha \in \mathbb{Z}_{12}), \tag{17}$$

where $\alpha = 0, 1, \cdots, 11$ and P is a permutation matrix. Then, there exists a rational number c (called central charge), such that ρ_α for all α has the following properties:

1. *ρ_α is an unitary and symmetric matrix representation of* $\mathrm{SL}_2(\mathbb{Z})$ *with level* $\mathrm{ord}(\rho_\alpha(\mathfrak{t}))$*, and* $\mathrm{ord}(T) \mid \mathrm{ord}(\rho_\alpha(\mathfrak{t})) \mid 12\,\mathrm{ord}(T)$ *.*

2. *The conductor of the elements of $\rho_\alpha(\mathfrak{s})$ divides* $\mathrm{ord}(\rho_\alpha(\mathfrak{t}))$*.*

3. *If ρ_α is equivalent to a direct sum of two $\mathrm{SL}_2(\mathbb{Z})$ representations*
$$\rho_\alpha \cong \rho \oplus \rho', \tag{18}$$
then the eigenvalues of $\rho(\mathfrak{t})$ and $\rho'(\mathfrak{t})$ must overlap. This implies that if $\rho_\alpha \cong \rho \oplus \chi_1 \oplus \cdots \oplus \chi_\ell$ for some 1-dimensional representations $\chi_1, \ldots, \chi_\ell$, then $\chi_1, \cdots \chi_\ell$ are the same 1-dimensional representation.

4. *Suppose that $\rho_\alpha \cong \rho \oplus \ell\chi$ for an irreducible representation ρ with non-degenerate $\rho(\mathfrak{t})$, and an 1-dimensional representation χ. If $\ell \neq 2\dim(\rho) - 1$ or $\ell > 1$, then* $(\rho(\mathfrak{s})\chi(\mathfrak{s})^{-1})^2 = \mathrm{id}$.

5. *ρ_α satisfies*
$$\rho_\alpha \not\cong n\rho \tag{19}$$
for any integer $n > 1$ and any representation ρ such that $\rho(\mathfrak{t})$ is non-degenerate.

6. *If $\rho_\alpha(s)^2 = \pm\mathrm{id}$ (i.e. if the modular data or MTC is self dual),* $\mathrm{pord}(\rho_\alpha(\mathfrak{t}))$ *is a prime and satisfies* $\mathrm{pord}(\rho_\alpha(\mathfrak{t})) = 1$ *mod 4, then the representation ρ_α cannot be a direct sum of a d-dimensional irreducible $\mathrm{SL}_2(\mathbb{Z})$ representation and two or more 1-dimensional $\mathrm{SL}_2(\mathbb{Z})$ representations with $d = (p+1)/2$.*

7. *Let $3 < p < q$ be prime such that $pq \equiv 3 \mod 4$ and $\mathrm{pord}(\rho_\alpha(\mathfrak{t})) = pq$, then the rank $r \neq \frac{p+q}{2} + 1$. Moreover, if $p > 5$, rank $r > \frac{p+q}{2} + 1$.*

8. *The number of self dual objects is greater than 0. Thus*
$$\mathrm{Tr}(\rho_\alpha(\mathfrak{s})^2) \neq 0. \tag{20}$$
Since $\mathrm{Tr}(\rho_\alpha(\mathfrak{s})^2) \neq 0$, let us introduce
$$C = \frac{\mathrm{Tr}(\rho_\alpha(\mathfrak{s})^2)}{|\mathrm{Tr}(\rho_\alpha(\mathfrak{s})^2)|}\rho_\alpha(\mathfrak{s})^2. \tag{21}$$
The above C is the charge conjugation operator of MTC, i.e. C is a permutation matrix of order 2. In particular, $\mathrm{Tr}(C)$ is the number of self dual objects. Also, for each eigenvalue $\tilde{\theta}$ of $\rho_\alpha(\mathfrak{t})$,
$$\mathrm{Tr}_{\tilde{\theta}}(C) \geq 0, \tag{22}$$
where $\mathrm{Tr}_{\tilde{\theta}}$ is the trace in the degenerate subspace of $\rho_\alpha(\mathfrak{t})$ with eigenvalue $\tilde{\theta}$.

9. *If the modular data is integral and $\mathrm{ord}(\rho_\alpha(\mathfrak{t}))$ = odd, then*
$$\mathrm{Tr}(C) = \mathrm{Tr}^2(\rho_\alpha^2(\mathfrak{s})) = 1, \tag{23}$$
i.e. the unit object is the only self-dual object.

10. *For any Galois conjugation σ in $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\rho_\alpha(\mathfrak{t}))})$, there is a permutation of the indices, $i \to \hat{\sigma}(i)$, and $\epsilon_\sigma(i) \in \{1, -1\}$, such that*
$$\sigma\big(\rho_\alpha(\mathfrak{s})_{i,j}\big) = \epsilon_\sigma(i)\rho_\alpha(\mathfrak{s})_{\hat{\sigma}(i),j} = \rho_\alpha(\mathfrak{s})_{i,\hat{\sigma}(j)}\epsilon_\sigma(j) \tag{24}$$
$$\sigma^2\big(\rho_\alpha(\mathfrak{t})_{i,i}\big) = \rho_\alpha(\mathfrak{t})_{\hat{\sigma}(i),\hat{\sigma}(i)}, \tag{25}$$
for all i, j.

11. *For any integer* a *coprime to* $n = \mathrm{ord}(\rho_\alpha(\mathfrak{t}))$*, we define*

$$D_{\rho_\alpha}(a) := \rho_\alpha(\mathfrak{t}^a\mathfrak{s}\mathfrak{t}^b\mathfrak{s}\mathfrak{t}^a\mathfrak{s}^{-1}) = D_{\rho_\alpha}(a + \mathrm{ord}(\rho_\alpha(\mathfrak{t}))), \quad \text{where } ab \equiv 1 \mod \mathrm{ord}(\rho_\alpha(\mathfrak{t}))\,. \tag{26}$$

For any $\sigma \in \mathrm{Gal}(\mathbb{Q}_n)$*,* $\sigma(\zeta_n) = \zeta_n^a$ *for some unique integer* a *modulo* n*. We define*

$$D_{\rho_\alpha}(\sigma) := D_{\rho_\alpha}(a)\,. \tag{27}$$

By [93, Theorem II], $D_{\rho_\alpha} : \mathrm{Gal}(\mathbb{Q}_n) = (\mathbb{Z}_n)^\times \to \mathrm{GL}_r(\mathbb{C})$ *is a representation equivalent to the restriction of* ρ_α *on the diagonal subgroup of* $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$*.* $D_{\rho_\alpha}(\sigma)$ *in (27) must be a* ***signed permutation***

$$(D_{\rho_\alpha}(\sigma))_{i,j} = \epsilon_\sigma(i)\delta_{\hat\sigma(i),j}. \tag{28}$$

and satisfies

$$\sigma(\rho_\alpha(\mathfrak{s})) = D_{\rho_\alpha}(\sigma)\rho_\alpha(\mathfrak{s}) = \rho_\alpha(\mathfrak{s})D^\top_{\rho_\alpha}(\sigma), \quad \sigma^2(\rho_\alpha(\mathfrak{t})) = D_{\rho_\alpha}(\sigma)\rho_\alpha(\mathfrak{t})D^\top_{\rho_\alpha}(\sigma) \tag{29}$$

12. *There exists a* u *such that* $\rho_\alpha(\mathfrak{s})_{uu} \neq 0$ *and*

$$\rho_\alpha(\mathfrak{s})_{ui} \neq 0, \quad \frac{\rho_\alpha(\mathfrak{s})_{ij}}{\rho_\alpha(\mathfrak{s})_{uu}}, \frac{\rho_\alpha(\mathfrak{s})_{ij}}{\rho_\alpha(\mathfrak{s})_{uj}} \in \mathbb{O}_{\mathrm{ord}(T)}, \quad \frac{\rho_\alpha(\mathfrak{s})_{ij}}{\rho_\alpha(\mathfrak{s})_{i'j'}} \in \mathbb{Q}_{\mathrm{ord}(T)}, \quad N^{ij}_k = \sum_{l=0}^{r-1} \frac{\rho_\alpha(\mathfrak{s})_{li}\rho_\alpha(\mathfrak{s})_{lj}\rho_\alpha(\mathfrak{s}^{-1})_{lk}}{\rho_\alpha(\mathfrak{s})_{lu}} \in \mathbb{N}. \quad \forall\ i,j,k = 1,2,\dots,r. \tag{30}$$

(The index u *corresponds the unit object of MTC). Here,* $\mathbb{Q}_{\mathrm{ord}(T)}$ *is the field of cyclotomic number and* $\mathbb{O}_{\mathrm{ord}(T)}$ *is the ring of cyclotomic integer. Also,* $\rho_\alpha(\mathfrak{s})_{ui}$ *for* $i \in \{1,\cdots,r\}$ *are either all real or all imaginary.*

13. *Let* $n \in \mathbb{N}_+$*. The* n^{th} *Frobenius-Schur indicator of the* i*-th simple object*

$$\begin{aligned}\nu_n(i) &= \sum_{j,k=0}^{r-1} N^{jk}_i \rho_\alpha(\mathfrak{s})_{ju}\theta^n_j[\rho_\alpha(\mathfrak{s})_{ku}\theta^n_k]^* = \sum_{j,k=0}^{r-1} N^{jk}_i \rho_\alpha(\mathfrak{t}^n\mathfrak{s})_{ju}\rho_\alpha(\mathfrak{t}^{-n}\mathfrak{s}^{-1})_{ku}\\ &= \sum_{j,k,l=0}^{r-1} \frac{\rho_\alpha(\mathfrak{s})_{lj}\rho_\alpha(\mathfrak{s})_{lk}\rho^*_\alpha(\mathfrak{s})_{li}}{\rho_\alpha(\mathfrak{s})_{lu}}\rho_\alpha(\mathfrak{t}^n\mathfrak{s})_{ju}\rho_\alpha(\mathfrak{t}^{-n}\mathfrak{s}^{-1})_{ku}\\ &= \sum_{l=0}^{r-1} \frac{\rho_\alpha(\mathfrak{s}\mathfrak{t}^n\mathfrak{s})_{lu}\rho_\alpha(\mathfrak{s}\mathfrak{t}^{-n}\mathfrak{s}^{-1})_{lu}\rho_\alpha(\mathfrak{s}^{-1})_{li}}{\rho_\alpha(\mathfrak{s})_{lu}}\end{aligned} \tag{31}$$

is a cyclotomic integer whose conductor divides n *and* $\mathrm{ord}(T)$*. The 1st Frobenius-Schur indicator satisfies* $\nu_1(i) = \delta_{iu}$ *while the 2nd Frobenius-Schur indicator* $\nu_2(i)$ *satisfies* $\nu_2(i) = \pm\rho_\alpha(\mathfrak{s}^2)_{ii}$ *(see [89–91]).*

The above condition can also be rewritten as

$$\rho_\alpha(\mathfrak{s})_{lu}\sum_i \rho_\alpha(\mathfrak{s})_{li}\nu_n(i) = \rho_\alpha(\mathfrak{s}\mathfrak{t}^n\mathfrak{s})_{lu}\rho_\alpha(\mathfrak{s}\mathfrak{t}^{-n}\mathfrak{s}^{-1})_{lu} \tag{32}$$

Summing over l*, we obtain*

$$\sum_i \widetilde{C}_{ui}\nu_n(i) = \widetilde{C}_{uu} \quad\to\quad \widetilde{C}_{uu}\nu_n(u) = \widetilde{C}_{uu} \quad\to\quad \nu_n(u) = 1. \tag{33}$$

After a signed-diagonal conjugation $V_{ij} = v_i\delta_{ij}$ that changes $\rho_\alpha(\mathfrak{s})_{ij}$ to $\rho_{\text{pMD}}(\mathfrak{s})_{ij}$, we find

$$\rho_{\text{pMD}}(\mathfrak{s})_{lu} \sum_i \rho_{\text{pMD}}(\mathfrak{s})_{li}\nu_n(i)v_iv_u = \rho_{\text{pMD}}(\mathfrak{s}\mathfrak{t}^n\mathfrak{s})_{lu}\rho_{\text{pMD}}(\mathfrak{s}\mathfrak{t}^{-n}\mathfrak{s}^{-1})_{lu}$$
$$\rho_{\text{pMD}}(\mathfrak{s})_{lu} \sum_i \rho_{\text{pMD}}(\mathfrak{s})_{li}\nu_n^{pMD}(i) = \rho_{\text{pMD}}(\mathfrak{s}\mathfrak{t}^n\mathfrak{s})_{lu}\rho_{\text{pMD}}(\mathfrak{s}\mathfrak{t}^{-n}\mathfrak{s}^{-1})_{lu}$$
$$\nu_n^{pMD}(i)v_iv_u = \nu_n(i)v_iv_u, \quad \nu_n^{pMD}(u) = 1, \quad \nu_n^{pMD}(u) = \pm\rho_{\text{pMD}}(\mathfrak{s}^2)_{ii} \tag{34}$$

Here we like to remark that the condition involving central charge

$$p_\pm = \sum_{i=1}^r d_i^2\theta_i^{\pm 1}, \quad p_\pm = D\,\mathrm{e}^{\pm \mathrm{i}2\pi c/8}, \tag{35}$$

is not a new condition. It comes from the $\text{SL}_2(\mathbb{Z})$ condition. First, (17) can be rewritten as

$$\rho_\alpha(\mathfrak{s}) = \mathrm{e}^{2\pi\mathrm{i}\frac{\tilde c}{8}}\frac{\rho_\alpha(\mathfrak{s})/\rho_\alpha(\mathfrak{s})_{uu}}{p_+}, \quad \rho_\alpha(\mathfrak{t}) = \mathrm{e}^{-2\pi\mathrm{i}\frac{\tilde c}{24}}\rho_\alpha(\mathfrak{t})/\rho_\alpha(\mathfrak{t})_{uu}, \tag{36}$$

for a $\tilde c \in \mathbb{Q}$. We find $\rho_\alpha(\mathfrak{t})_{uu} = \mathrm{e}^{-2\pi\mathrm{i}\frac{\tilde c}{24}}$, which allows us to rewrite (36) as

$$\rho_\alpha(\mathfrak{s})_{uu}\rho_\alpha^3(\mathfrak{t})_{uu}p_+ = \rho_\alpha(\mathfrak{s})_{uu}\rho_\alpha^3(\mathfrak{t})_{uu}\sum_{i=1}^r\Big(\frac{\rho_\alpha(\mathfrak{s})_{iu}}{\rho_\alpha(\mathfrak{s})_{uu}}\Big)^2\frac{\rho_\alpha(\mathfrak{t})_{ii}}{\rho_\alpha(\mathfrak{t})_{uu}} = 1, \tag{37}$$

and becomes an condition on $\rho_\alpha(\mathfrak{s}), \rho_\alpha(\mathfrak{t})$. But this is not a new condition, since the above can be rewritten as

$$\rho_\alpha^2(\mathfrak{t})_{uu}\sum_{i=1}^r \rho_\alpha(\mathfrak{s})_{ui}\rho_\alpha(\mathfrak{s})_{iu}\rho_\alpha(\mathfrak{t})_{ii} = \rho_\alpha(\mathfrak{s})_{uu} \quad \text{or} \quad \big(\rho_\alpha(\mathfrak{t})\rho_\alpha(\mathfrak{s})\rho_\alpha(\mathfrak{t})\rho_\alpha(\mathfrak{s})\rho_\alpha(\mathfrak{t})\big)_{uu} = \rho_\alpha(\mathfrak{s})_{uu}, \tag{38}$$

which is a consequence of $\text{SL}_2(\mathbb{Z})$ representation.

Using the irreducible $\text{SL}_2(\mathbb{Z})$ representations obtained by GAP package SL2Reps, we can explicitly constructed all unitary representations of $\text{SL}_2(\mathbb{Z})$ (up to unitary equivalence). However, this only gives the $\text{SL}_2(\mathbb{Z})$ representations in some arbitrary basis, not in the basis yielding MD representations (*i.e.* satisfying Proposition 2.4), since MD representations are $\text{SL}_2(\mathbb{Z})$ representations in a particular basis.

We can improve the situation by choosing a basis to make $\rho(\mathfrak{t})$ diagonal and $\rho(\mathfrak{s})$ symmetric. We can choose more special bases to make the $\text{SL}_2(\mathbb{Z})$ representations closer to the basis of MD representations. Since we are going to use several types of bases, let us define these choices:

Definition 2.5. *An unitary* $\text{SL}_2(\mathbb{Z})$ *representations* $\tilde\rho$ *is called a* ***general*** $\text{SL}_2(\mathbb{Z})$ *matrix representations if* $\tilde\rho(\mathfrak{t})$ *is diagonal* [7]*. A general* $\text{SL}_2(\mathbb{Z})$ *matrix representation* $\tilde\rho$ *is called* ***symmetric*** *if* $\tilde\rho(\mathfrak{s})$ *is symmetric. An general* $\text{SL}_2(\mathbb{Z})$ *matrix representation* $\tilde\rho$ *is called* ***irrep-sum*** *if* $\tilde\rho(\mathfrak{s}), \tilde\rho(\mathfrak{t})$ *are matrix-direct sum of irreducible* $\text{SL}_2(\mathbb{Z})$ *representations. An* $\text{SL}_2(\mathbb{Z})$ *matrix representations* $\tilde\rho$ *is called an* $\text{SL}_2(\mathbb{Z})$ *representation* ***of modular data*** S, T*, if* $\tilde\rho$ *is unitarily equivalent to an MD representation of the modular data, i.e.,*

$$\tilde\rho(\mathfrak{s}) = \mathrm{e}^{-2\pi\mathrm{i}\frac{\alpha}{4}}\frac{1}{D}USU^\dagger, \quad \tilde\rho(\mathfrak{t}) = UTU^\dagger\mathrm{e}^{2\pi\mathrm{i}(\frac{-c}{24}+\frac{\alpha}{12})}, \tag{39}$$

[7] We will consider only $\text{SL}_2(\mathbb{Z})$ matrix representations with diagonal $\tilde\rho(\mathfrak{t})$ in this paper.

for some unitary matrix U and $\alpha \in \mathbb{Z}_{12}$, where c is the central charge.[8] *An $\mathrm{SL}_2(\mathbb{Z})$ matrix representations ρ_{pMD} is called **pseudo MD (pMD)** representation, if ρ_{pMD} is related to an MD representation of the modular data via a conjugation of a **signed diagonal** matrix V:*[9]

$$\rho_{pMD}(\mathfrak{s}) = \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{\alpha}{4}}\frac{1}{D}VSV, \qquad \rho_{pMD}(\mathfrak{t}) = VTV\,\mathrm{e}^{2\pi\,\mathrm{i}\,(\frac{-c}{24}+\frac{\alpha}{12})}. \tag{40}$$

We find that all irreducible unitary representations of $\mathrm{SL}_2(\mathbb{Z})$ are unitarily equivalent to symmetric matrix representations of $\mathrm{SL}_2(\mathbb{Z})$. We will start with those symmetric matrix representations to obtain a classification of modular data.

3 Our strategy

We will use the following strategy to classify modular data of a given rank.

1. Obtain all the irreducible representations of dimensions up to the rank r, using GAP package SL2Reps created by Siu-Hung Ng, Yilong Wang, and Samuel Wilson. Then construct all the dimension-r representations, ρ_{isum}, from those irreducible representations.

2. Using some conditions (see Section 4), to reject the representations that are not unitarily equivalent to MD representations.

3. For the remain representations ρ_{isum}, find all the unitary matrices U that transform them to pMD representations (see Section 5):

$$\rho_{\mathrm{pMD}} = U\rho_{\mathrm{isum}}U^{\dagger}. \tag{41}$$

   Reject those representations for which the unitary matrices U do not exist. This is the most difficult step, since we need to find finite solutions from infinite possibilities of unitary U.

   The key is to generate many conditions on the unitary U, so that the number of the solutions of those condition is finite. To achieve this, we first consider all the possible $D_\rho(\sigma)$'s (see Section 5.6.1) and unit-row index u. Since $D_\rho(\sigma)$'s are signed permutations, those possibilities are finite. Once $D_\rho(\sigma)$ and u are known, we can use them to obtain many conditions on U, in addition to the unitary conditions.

   Also, from $D_\rho(\sigma)$, we can determine if the corresponding MTC is integral or not. This allows us to use two different approaches to handle integral cases (see Section 6) and non-integral cases (see Section 5) separately. The integral and non-integral cases are quite different, and require different approaches to handle them.

   Once we know the unit-row index u, we can obtain conditions on U from the second Frobenius-Schur indicator (see (31)). Also $\mathrm{norm}(\rho(\mathfrak{s})_{ui}) = q_i$ are inverse of integers. If we can isolate the polynomial conditions that depend only on q_i's (see Section 5.6.4), then we can use the generalized Egyptian-fraction method to solve q_i's (see Section 5.6.7). Those are important tricks that make our calculation possible.

4. Find all the signed permutations P_{sgn} that transform the pMD representations to MD representations:

$$S = P_{\mathrm{sgn}}\frac{\rho_{\mathrm{pMD}}(\mathfrak{s})}{(\rho_{\mathrm{pMD}})_{uu}(\mathfrak{s})}P_{\mathrm{sgn}}^{\top}, \qquad T = P_{\mathrm{sgn}}\frac{\rho_{\mathrm{pMD}}(\mathfrak{t})}{(\rho_{\mathrm{pMD}})_{uu}(\mathfrak{t})}P_{\mathrm{sgn}}^{\top}. \tag{42}$$

[8] Note that D^2 is always positive and D in (39) is the positive root of D^2, even for non-unitary cases.
[9] A signed diagonal matrix is a diagonal matrix with diagonal elements ± 1.

This step is easy, since the number of possible signed permutations is finite. We just search all the signed permutations so that the resulting S, T satisfy the conditions for modular data. The signed permutations have the form $P_{\text{sgn}} = PV_{\text{sd}}$, where P is a permutation matrix and V_{sd} is a signed diagonal matrix. We may fix P and only search for V_{sd}. From the Verlinde formula (30), we know how N_k^{ij} transform under the conjugation of V_{sd}. This allows us to find V_{sd} that make N_k^{ij} non-negative.

4 Candidate representations of modular data from $\mathrm{SL}_2(\mathbb{Z})$ representations

We note that different choices of orthogonal basis give rise to different matrix representations of $\mathrm{SL}_2(\mathbb{Z})$. The modular data S, T are obtained from some particular choices of the basis. Some properties of the MD representations of a modular data do not depend on the choices of basis in the eigenspaces of $\tilde{\rho}(\mathfrak{t})$ (induced by the block-diagonal unitary transformation U in (40) that leaves $\tilde{\rho}(\mathfrak{t})$ invariant). Those properties remain valid for any general $\mathrm{SL}_2(\mathbb{Z})$ representations $\tilde{\rho}$ of the modular data. In the following, we collect the basis-independent conditions on the $\mathrm{SL}_2(\mathbb{Z})$ matrix representations of modular data. This will help us to narrow the list of $\mathrm{SL}_2(\mathbb{Z})$ representations that are related modular data.

Proposition 4.1. *Let $\tilde{\rho}$ be a general $\mathrm{SL}_2(\mathbb{Z})$ matrix representations of a modular data or a MTC. Then $\tilde{\rho}$ must satisfy the following conditions:*

1. *If $\tilde{\rho}$ is a direct sum of two $\mathrm{SL}_2(\mathbb{Z})$ representations*
$$\tilde{\rho} \cong \rho \oplus \rho', \tag{43}$$
then the diagonals entries of $\rho(\mathfrak{t})$ and $\rho'(\mathfrak{t})$ must overlap.

2. *Suppose that $\tilde{\rho} \cong \rho \oplus \ell\chi$ for an irreducible representation ρ with $\rho(\mathfrak{t})$ non-degenerate, and a character χ. If $\ell \neq 1$ and $\ell \neq 2\dim(\rho) - 1$, then $(\rho(\mathfrak{s})\chi(\mathfrak{s})^{-1})^2 = \mathrm{id}$.*

3. *If $\tilde{\rho}(\mathfrak{s})^2 = \pm\mathrm{id}$, and $\mathrm{pord}(\tilde{\rho}(\mathfrak{t})) = 1$ mod 4 and is a prime, then the representation $\tilde{\rho}$ cannot be a direct sum of a d-dimensional irreducible $\mathrm{SL}_2(\mathbb{Z})$ representation and two or more 1-dimensional $\mathrm{SL}_2(\mathbb{Z})$ representations with $d = (\mathrm{pord}(\tilde{\rho}(\mathfrak{t})) + 1)/2$.*

4. *$\tilde{\rho}$ satisfies*
$$\tilde{\rho} \ncong n\rho \tag{44}$$
for any integer $n > 1$ and any representation ρ such that $\rho(\mathfrak{t})$ is non-degenerate.

5. *Let $3 < p < q$ be primes such that $pq \equiv 3 \mod 4$ and $\mathrm{pord}(\rho(\mathfrak{t})) = pq$, then the rank $r \neq \frac{p+q}{2} + 1$. Moreover, if $p > 5$, rank $r > \frac{p+q}{2} + 1$.*

Some other properties of an MD representation do depend on the choice of basis. To make use of those properties, we can construct some combinations of $\tilde{\rho}(\mathfrak{s})$'s that are invariant under the block-diagonal unitary transformation U.

The eigenvalues of $\tilde{\rho}(\mathfrak{t})$ partition the indices of the basis vectors. To construct the invariant combinations of $\tilde{\rho}(\mathfrak{s})$, for any eigenvalue $\tilde{\theta}$ of $\tilde{\rho}(\mathfrak{t})$, let
$$I_{\tilde{\theta}} = \{i \,|\, \tilde{\rho}(\mathfrak{t})_{ii} = \tilde{\theta}\}. \tag{45}$$

Let $I = I_{\tilde{\theta}}$, $J = J_{\tilde{\theta}'}$, $K = K_{\tilde{\theta}''}$ for some eigenvalues $\tilde{\theta}$, $\tilde{\theta}'$, $\tilde{\theta}''$ of $\tilde{\rho}(\mathfrak{t})$. We see that the following uniform polynomials of $\tilde{\rho}(\mathfrak{s})$ are invariant

$$\begin{aligned} P_I(\rho(\mathfrak{s})) &= \mathrm{Tr}\tilde{\rho}(\mathfrak{s})_{II} \equiv \sum_{i\in I} \tilde{\rho}(\mathfrak{s})_{ii}, \\ P_{IJ}(\rho(\mathfrak{s})) &= \mathrm{Tr}\tilde{\rho}(\mathfrak{s})_{IJ}\tilde{\rho}(\mathfrak{s})_{JI} \equiv \sum_{i\in I, j\in J} \tilde{\rho}(\mathfrak{s})_{i,j}\tilde{\rho}(\mathfrak{s})_{ji}, \\ P_{IJK}(\rho(\mathfrak{s})) &= \mathrm{Tr}\tilde{\rho}(\mathfrak{s})_{IJ}\tilde{\rho}(\mathfrak{s})_{JK}\tilde{\rho}(\mathfrak{s})_{KI} \equiv \sum_{i\in I, j\in J, k\in K} \tilde{\rho}(\mathfrak{s})_{i,j}\tilde{\rho}(\mathfrak{s})_{j,k}\tilde{\rho}(\mathfrak{s})_{k,i}. \end{aligned} \tag{46}$$

Certainly we can construction many other invariant uniform polynomials in the similar way. Using those invariant uniform polynomials, we have the following results

Proposition 4.2. *Let $\tilde{\rho}$ be a general $\mathrm{SL}_2(\mathbb{Z})$ representations of a modular data or a MTC. Then following statements hold:*

1. *$\tilde{\rho}(\mathfrak{s})$ satisfies*

$$\mathrm{Tr}(\tilde{\rho}(\mathfrak{s})^2) \in \mathbb{Z} \setminus \{0\}. \tag{47}$$

Let

$$C = \frac{\mathrm{Tr}(\tilde{\rho}(\mathfrak{s})^2)}{|\mathrm{Tr}(\tilde{\rho}(\mathfrak{s})^2)|}\tilde{\rho}(\mathfrak{s})^2. \tag{48}$$

For all I,

$$P_I(C) \geq 0. \tag{49}$$

2. *The conductor of $P_{\text{odd}}(\tilde{\rho}(\mathfrak{s}))$ divides $\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))$ for all the invariant uniform polynomials P_{odd} with odd powers of $\tilde{\rho}(\mathfrak{s})$ (such as P_I and P_{IJK} in (46)). The conductor of $P_{\text{even}}(\tilde{\rho}(\mathfrak{s}))$ divides $\mathrm{pord}(\tilde{\rho}(\mathfrak{t}))$ for all the invariant uniform polynomials P_{even} with even powers of $\tilde{\rho}(\mathfrak{s})$ (such as P_{IJ} in (46)).*

3. *For any Galois conjugation $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\rho(\mathfrak{t}))})$, there is a permutation on the set $\{I\}$, $I \to \hat{\sigma}(I)$, such that*

$$\begin{aligned} \sigma P_{IJ}(\tilde{\rho}(\mathfrak{s})) &= P_{I\hat{\sigma}(J)}(\tilde{\rho}(\mathfrak{s})) = P_{\hat{\sigma}(I)J}(\tilde{\rho}(\mathfrak{s})) \\ \sigma^2(\tilde{\theta}_I) &= \tilde{\theta}_{\hat{\sigma}(I)}, \end{aligned} \tag{50}$$

for all I, J.

4. *For any invariant uniform polynomials P (such as those in (46))*

$$\sigma P\big(\tilde{\rho}(\mathfrak{s})\big) = P\big(\sigma\tilde{\rho}(\mathfrak{s})\big) = P\big(\tilde{\rho}(\mathfrak{t})^a\tilde{\rho}(\mathfrak{s})\tilde{\rho}(\mathfrak{t})^b\tilde{\rho}(\mathfrak{s})\tilde{\rho}(\mathfrak{t})^a\big) \tag{51}$$

where $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))})$, and a, b are given by $\sigma(\mathrm{e}^{\mathrm{i}2\pi/\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))}) = \mathrm{e}^{a\,\mathrm{i}2\pi/\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))}$ and $ab \equiv 1 \mod \mathrm{ord}(\tilde{\rho}(\mathfrak{t}))$.

Instead of constructing invariants, there is another way to make use of the properties of an MD representation that depend on the choices of basis. We can choose a more special basis, so that the basis is closer to the basis that leads to the MD representation. For example, we can choose a basis to make $\tilde{\rho}(\mathfrak{s})$ symmetric (*i.e.* to make $\tilde{\rho}$ a symmetric representation).

Now consider a symmetric $\mathrm{SL}_2(\mathbb{Z})$ matrix representation $\tilde{\rho}$ of a modular data or a MTC. We find that the restriction of the unitary U in (40) on the non-degenerate subspace (see Ref. [94] Theorem 3.4) must be a signed diagonal matrix. In this case some properties of MD representation apply to the blocks of the symmetric representation within the non-degenerate subspace. This allows us to obtain

Proposition 4.3. *Let $\tilde{\rho}$ be a symmetric $\mathrm{SL}_2(\mathbb{Z})$ representations equivalent to an MD representation. Let*

$$I_{\mathrm{ndeg}} := \{i \mid \tilde{\rho}(\mathfrak{t})_{i,i} \text{ is a non-degenerate eigenvalue}\}, \tag{52}$$

Then there exists an orthogonal U such that $U\tilde{\rho}U^\top$ is a pMD representation, and the following statements hold:

1. *The conductor of $(U\tilde{\rho}(\mathfrak{s})U^\top)_{i,j}$ divides $\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))$ for all i,j. This implies that the conductor of $(\tilde{\rho}(\mathfrak{s}))_{i,j}$ divides $\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))$ for all $i,j \in I_{\mathrm{ndeg}}$.*

2. *For any Galois conjugation σ in $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\widetilde{\rho}(\mathfrak{t}))})$, there is a permutation $i \to \hat{\sigma}(i)$, such that*

$$\begin{aligned}\sigma\big((U\tilde{\rho}(\mathfrak{s})U^\top)_{i,j}\big) &= \epsilon_\sigma(i)(U\tilde{\rho}(\mathfrak{s})U^\top)_{\hat{\sigma}(i),j} = (U\tilde{\rho}(\mathfrak{s})U^\top)_{i,\hat{\sigma}(j)}\epsilon_\sigma(j)\\ \sigma^2\big(\tilde{\rho}(\mathfrak{t})_{i,i}\big) &= \tilde{\rho}(\mathfrak{t})_{\hat{\sigma}(i),\hat{\sigma}(i)},\end{aligned} \tag{53}$$

   for all i,j, where $\epsilon_\sigma(i) \in \{1,-1\}$. This implies that

$$\begin{aligned}\sigma\big(\tilde{\rho}(\mathfrak{s})_{i,j}\big) = \tilde{\rho}(\mathfrak{s})_{\hat{\sigma}(i),j} \quad &\text{or} \quad \sigma\big(\tilde{\rho}(\mathfrak{s})_{i,j}\big) = -\tilde{\rho}(\mathfrak{s})_{\hat{\sigma}(i),j}\\ \sigma\big(\tilde{\rho}(\mathfrak{s})_{i,j}\big) = \tilde{\rho}(\mathfrak{s})_{i,\hat{\sigma}(j)} \quad &\text{or} \quad \sigma\big(\tilde{\rho}(\mathfrak{s})_{i,j}\big) = -\tilde{\rho}(\mathfrak{s})_{i,\hat{\sigma}(j)}\end{aligned} \tag{54}$$

   for all $i,j \in I_{\mathrm{ndeg}}$. This also implies that $D_{\widetilde{\rho}}(\sigma)$ defined in (27) is a signed permutation matrix in the I_{ndeg} block, i.e. $(D_{\widetilde{\rho}}(\sigma))_{i,j}$ for $i,j \in I_{\mathrm{ndeg}}$ are matrix elements of a signed permutation matrix.

3. *For all i,j,*

$$\sigma\big((U\tilde{\rho}(\mathfrak{s})U^\top)_{i,j}\big) = \big(U\tilde{\rho}(\mathfrak{t})^a\tilde{\rho}(\mathfrak{s})\tilde{\rho}(\mathfrak{t})^b\tilde{\rho}(\mathfrak{s})\tilde{\rho}(\mathfrak{t})^aU^\top\big)_{i,j} \tag{55}$$

   where $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\widetilde{\rho}(\mathfrak{t}))})$, and a,b are given by $\sigma(\mathrm{e}^{\mathrm{i}2\pi/\mathrm{ord}(\widetilde{\rho}(\mathfrak{t}))}) = \mathrm{e}^{a\,\mathrm{i}2\pi/\mathrm{ord}(\widetilde{\rho}(\mathfrak{t}))}$ and $ab \equiv 1 \bmod \mathrm{ord}(\widetilde{\rho}(\mathfrak{t}))$. This implies that

$$\sigma\big((\tilde{\rho}(\mathfrak{s}))_{i,j}\big) = \big(\tilde{\rho}(\mathfrak{t})^a\tilde{\rho}(\mathfrak{s})\tilde{\rho}(\mathfrak{t})^b\tilde{\rho}(\mathfrak{s})\tilde{\rho}(\mathfrak{t})^a\big)_{i,j}. \tag{56}$$

   for all $i,j \in I_{\mathrm{ndeg}}$.

4. *Both T and $\widetilde{\rho}(\mathfrak{t})$ are diagonal, and without loss of generality, we may assume $\widetilde{\rho}(\mathfrak{t})$ is a scalar multiple of T. In this case U in (40) is a block diagonal matrix preserving the eigenspaces of $\widetilde{\rho}(\mathfrak{t})$. Let $I_{\mathrm{nonzero}} = \{i\}$ be a set of indices such that the i^{th} row of $U\widetilde{\rho}(\mathfrak{s})U^\top$ contains no zeros for some othorgonal U satisfying $U\widetilde{\rho}(\mathfrak{t})U^\top = \widetilde{\rho}(\mathfrak{t})$. The index for the unit object of MTC must be in I_{nonzero}. Thus I_{nonzero} must be nonempty:*

$$I_{\mathrm{nonzero}} \neq \emptyset. \tag{57}$$

5. *Let $I_{\widetilde{\theta}}$ be a set of indices for an eigenspace $E_{\widetilde{\theta}}$ of $\tilde{\rho}(\mathfrak{t})$*

$$I_{\widetilde{\theta}} := \{i \mid \tilde{\rho}(\mathfrak{t})_{i,i} = \widetilde{\theta}\}. \tag{58}$$

   Then there exists a $I_{\widetilde{\theta}}$ such that

$$I_{\widetilde{\theta}} \cap I_{\mathrm{nonzero}} \neq \emptyset \quad \text{and} \quad \mathrm{Tr}_{E_{\widetilde{\theta}}} C > 0, \tag{59}$$

   where C is given in (48).

Table 3: The numbers of the candidate irrep-sum $\mathrm{SL}_2(\mathbb{Z})$ representations for each rank.

rank	2	3	4	5	6	7	8	9	10	11	12
number of reps	2	4	9	20	57	106	258	533	1210	2374	5288

It is very helpful to determine if a $\mathrm{SL}_2(\mathbb{Z})$ representation gives rise to an integral MTC or not. This is because integral MTCs satisfy more conditions.

Theorem 4.4. *Let $\widetilde{\rho}$ be a representations of $\mathrm{SL}_2(\mathbb{Z})$. If u^{th} row of $\widetilde{\rho}$ is the unit row, and $D_{\widetilde{\rho}}(\sigma)_{uu} = \pm 1$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\widetilde{\rho}(\mathfrak{t}))}/\mathbb{Q})$, then $\widetilde{\rho}$ is either equivalent to an MD representation ρ_{MD} of integral MTC or is not equivalent to any MD representation.*

The above result comes from Eq. (28). $D_{\widetilde{\rho}}(\sigma)_{uu} = \pm 1$ implies that $D_{\rho_{\mathrm{MD}}}(\sigma)_{uu} = \pm 1$, which in turn implies that $\hat{\sigma}(u) = u$. Then Eq. (24) implies that, for the corresponding MD representation ρ_{MD} of $\widetilde{\rho}$,

$$\sigma\left(\frac{\rho_{\mathrm{MD}}(\mathfrak{s})_{ij}}{\rho_{\mathrm{MD}}(\mathfrak{s})_{iu}}\right) = \frac{\rho_{\mathrm{MD}}(\mathfrak{s})_{\hat{\sigma}(i)j}}{\rho_{\mathrm{MD}}(\mathfrak{s})_{\sigma(i)u}} \tag{60}$$

for all i, j, and hence $\sigma(d_i) = d_i$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\widetilde{\rho}(\mathfrak{t}))}/\mathbb{Q})$, where $d_i = \frac{\rho_{\mathrm{MD}}(\mathfrak{s})_{ui}}{\rho_{\mathrm{MD}}(\mathfrak{s})_{uu}}$ is the quantum dimension. Thus, d_i are integral, and the corresponding MTC is integral, if it exists. If we do not know the unit row, then we have

Theorem 4.5. *Let $\{\widetilde{\theta}\}_{\mathrm{nonzero}}$ is a set the eigenvalues of $\widetilde{\rho}(\mathfrak{t})$, $\widetilde{\theta}$, such that $\widetilde{\theta}$ is a 24^{th} root of unity and $I_{\widetilde{\theta}}$ has a non-empty overlap with I_{nonzero}. If $D_{\widetilde{\rho}}(\sigma)_{I_{\widetilde{\theta}}} = \pm\mathrm{id}$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\widetilde{\rho}(\mathfrak{t}))}/\mathbb{Q})$ and for all $\widetilde{\theta}$ in $\{\widetilde{\theta}\}_{\mathrm{nonzero}}$, then $\widetilde{\rho}$ is either equivalent to an MD representation of integral MTC or is not equivalent to any MD representation.*

We also have

Theorem 4.6. *If a $\mathrm{SL}_2(\mathbb{Z})$ representation $\widetilde{\rho}$ satisfies $\mathrm{pord}(\widetilde{\rho}(\mathfrak{t})) = \mathrm{ord}(T) \in \{2, 3, 4, 6\}$, then $\widetilde{\rho}$ is either equivalent to an MD representation of integral MTC or is not equivalent to any MD representation.*

Using GAP System for Computational Discrete Algebra, we obtain a list of symmetric irrep-sum $\mathrm{SL}_2(\mathbb{Z})$ matrix representations that satisfy the conditions in Propositions 4.1, 4.2, and 4.3. Also, our list only includes one representive for each orbit generated by Galois conjugations and tensoring 1-dim $\mathrm{SL}_2(\mathbb{Z})$ representations. The numbers of those candidate irrep-sum $\mathrm{SL}_2(\mathbb{Z})$ representations for each rank are given in Table 3.

Some of those symmetric irrep-sum $\mathrm{SL}_2(\mathbb{Z})$ matrix representations are representations of modular data, while others are not. However, the list includes all the symmetric irrep-sum $\mathrm{SL}_2(\mathbb{Z})$ matrix representations of modular data or MTC's. In the next section, we will use GAP group to determine which of those irrep-sum representations can give rise to modular data, and which should be rejected. However, there are a few cases at rank-12 are hard to handle. We have to use extensive search to reject those cases with high likelyhood. Those calculations are presented in the first few sections of the Appendix.

5 Candidate pMD representations from $\mathrm{SL}_2(\mathbb{Z})$ representations

Our $\mathrm{SL}_2(\mathbb{Z})$ representation $\widetilde{\rho}$ has a form of direct sum of irreducible representations: $\widetilde{\rho} = \rho_{\mathrm{isum}}$. We have chosen a special basis in the eigenspaces of a $\mathrm{SL}_2(\mathbb{Z})$ matrix representation

ρ_{isum} to make $\rho_{\text{isum}}(\mathfrak{s})$ symmetric. But such a special basis is still not special enough to make ρ_{isum} to be a MD representation ρ_α.

We can choose a more special basis to make $\rho_{\text{isum}}(\mathfrak{s}^2)$ a signed permutation matrix, in addition to making $\rho_{\text{isum}}(\mathfrak{s})$ symmetric. We know that, for a MD representation ρ_α, $\rho(\mathfrak{s}^2)$ is a signed permutation matrix. So the new special basis makes ρ_{isum} closer to the MD representation ρ_α.

We can choose an even more special basis in the eigenspaces of $\rho_{\text{isum}}(\mathfrak{t})$ to make ρ_{isum} into a pseudo MD representation that differs from a MD representation ρ_α only by the conjugation of a signed diagonal matrix V_{sd}: $\rho_{\text{pMD}} = V_{\text{sd}}\rho_\alpha V_{\text{sd}}$. Pseudo MD representation has a property that the matrix $D_{\rho_{\text{pMD}}}(\sigma)$ defined in (26) are signed permutations.

We would like to point out that, since both ρ_{isum} and ρ_{pMD} are symmetric $\text{SL}_2(\mathbb{Z})$ matrix representations that are related by an unitary transformation, according to Theorem 3.4 in Ref. [94], they can be related by an orthogonal transformation. An generic orthogonal transformation contains continuous real parameters. This leads to infinite many potential pseudo MD representations ρ_{pMD} and infinite many potential MD representations ρ_α. This makes it impossible to check one-by-one, if those potential ρ_α's are indeed MD representations.

However, when eigenspaces of $\rho_{\text{isum}}(\mathfrak{t})$ are all 1-dimensional (*i.e.* non-degenerate), the orthogonal matrices U that transform ρ_{isum} to ρ_{pMD} must be an identity matrix, up to signed diagonal matrices. This leads to only a finite many potential MD representations ρ_α. We can then check each of the possible ρ_α's, to see if it is a MD representation.

Even if some eigenspaces of $\rho_{\text{isum}}(\mathfrak{t})$ are degenerate, under certain conditions, the number of orthogonal transformations U that transform $D_{\rho_{\text{isum}}}(\sigma)$ into signed permutations $D_{\rho_{\text{pMD}}}(\sigma)$ can still be finite. Let I_θ is the set of indices for the degenerate eigenspace of $\rho_{\text{pMD}}(\mathfrak{t})$ with eigenvalue θ. Let us consider $D_{\rho_{\text{isum}}}(\sigma)$ in the I_θ-block, which are denoted as $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$. If the common eigenspaces for $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$ with different σ's are non-degenerate, then there is only a finite number of orthogonal transformations in the I_θ block, U_{I_θ,I_θ}, that transform $D_{\rho_{\text{isum}}}(\sigma)$ into signed permutations $D_{\rho_{\text{pMD}}}(\sigma)$.

5.1 Cases where $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$ are signed diagonal matrices with non-degenerate common eigenspaces

For example, when the non-zero $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$ are signed diagonal matrices with non-degenerate common eigenspaces, the most general orthogonal matrices U_{I_θ,I_θ} have only a finite number of choices. They must be either an identity matrix, or a combinations of $\pm 45°$ rotations among various pairs of indices, up to signed diagonal matrices,

Let us consider a concrete case. In a 3-dimensional eigenspace of $\rho_{\text{isum}}(\mathfrak{t})$, the non-zero $D_{\rho_{\text{isum}}}(\sigma)$'s may generate a 3×3 matrix groups MG, given by

$$MG = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}. \tag{61}$$

To find the most general orthogonal matrices that transform the above 3×3 matrices in MG into signed permutation matrices, we first show

Theorem 5.1. *If P is a permutation matrix with $P^2 = \text{id}$, then P is a direct sum of 2×2 and 1×1 matrices. If P_{sgn} is a signed permutation matrix with $P_{\text{sgn}}^2 = \text{id}$, then P_{sgn} is a direct sum of 2×2 and 1×1 matrices. The 2×2 matrices are the $\pm 45°$ rotations mentioned above.*

Proof of Theorem 5.1. If P is a permutation matrix with $P^2 = \text{id}$, P must be a pair-wise permutation, and thus P is a direct sum of 2×2 and 1×1 matrices. The reduction from signed permutation matrix to permutation matrix by ignoring the signs is homomorphism of the matrix product. If P_{sgn} is a signed permutation matrix with $P_{\text{sgn}}^2 = \text{id}$, its reduction given rise to a permutation matrix P with $P^2 = \text{id}$. Since P is a direct sum of 2×2 and 1×1 matrices, P_{sgn} is also a direct sum of 2×2 and 1×1 matrices. □

Using the above result, we can show that the most general orthogonal matrices that transform all $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}}$'s into signed permutations must have one of the following forms

$$\begin{aligned}
&U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ or } U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\text{or } &U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \text{ or } U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \\
\text{or } &U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \text{ or } U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \\
\text{or } &U = PV_{\text{sd}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{62}$$

where V_{sd} are signed diagonal matrices, and P are permutation matrices. We note that the non-trivial part of U is a 2×2 block for index $(1, 2)$, $(1, 3)$, and $(2, 3)$. The 2×2 block has three possibilities

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{63}$$

This is a general pattern that apply for cases when the non-zero $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta, I_\theta}$ are signed diagonal matrices with non-degenerate common eigenspaces.

Some times, the non-zero $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta, I_\theta}$ are not signed diagonal matrices. We need to examine those cases individually.

5.2 Within a 2-dimensional eigenspace of $\rho_{\text{isum}}(t)$

In this case, the matrix groups MG generated by non-zero 2-by-2 matrices, $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta, I_\theta}$, can have several different forms, for those passing representations. By examine the computer results, we find that matrix groups MG can be

$$\begin{aligned}
MG &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, && \text{for } \dim(\rho_{\text{isum}}) \geq 5; \\
MG &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, && \text{for } \dim(\rho_{\text{isum}}) \geq 6.
\end{aligned} \tag{64}$$

Those $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta, I_\theta}$'s have degenerate common eigenspaces. Thus the resulting U_{I_θ, I_θ}'s have continuous parameters and are not finite many.

We also have

$$MG = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \qquad \text{for } \dim(\rho_{\text{isum}}) \geq 4;$$

$$MG = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \qquad \text{for } \dim(\rho_{\text{isum}}) \geq 6. \tag{65}$$

In those two cases

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{or} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{66}$$

will transform all $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})_{I_\theta, I_\theta}$'s into signed permutations. In general we have

Theorem 5.2. *Let*

$$MG_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

$$MG_4 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \tag{67}$$

The most general orthogonal matrices that transform all matrices in MG_2 or MG_4 into signed permutations must have one of the following forms

$$U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \text{ or } U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ or } U = PV_{\text{sd}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{68}$$

where V_{sd} are signed diagonal matrices, and P are permutation matrices. The number of the orthogonal transformations U is finite.

Proof of Theorem 5.2. We only need to consider the first matrix group MG_2, where the matrix group is isomorphic to the $\mathbb{Z}_2$ group. There are only four matrix groups formed by 2-dimensional signed permutations matrices, that are isomorphic $\mathbb{Z}_2$. The four matrix groups are generated by the following four generators respectively:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{69}$$

An orthogonal transformation U that transforms MG to one of the above matrix groups must have a from $U = VU_0$, where V transforms MG_2 into itself, and U_0 is a fixed orthogonal transformation that transforms MG_2 to one of the above matrix groups. We can choose U_0 to have the following form

$$U_0 = \frac{P}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \text{ or } U_0 = \frac{P}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ or } U_0 = P \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{70}$$

To keep MG unchanged V must satisfy

$$V \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V. \tag{71}$$

We find that V must be diagonal. Thus V, as an orthogonal matrix, must be signed diagonal. This gives us the result (68). □

If $\dim(\rho_{\text{isum}}) \geq 8$, it is possible that the matrix group of $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s is generated by the following non-diagonal matrix

$$\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{72}$$

This is because the direct sum decomposition of ρ_{isum} contains a dimension-6 irreducible representation, whose $\rho(\mathfrak{t})$ has a 2-dimensional eigenspace. The representation can give rise to this form of $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s.

The eigenvalues of the matrices are $(\mathrm{i}, -\mathrm{i})$. The most general orthogonal matrices that transform all non-zero $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s into signed permutations must have the form

$$U = PV_{\text{sd}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{73}$$

If $\dim(\rho_{\text{isum}}) \geq 8$, it is also possible that non-zero $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s form the following matrix group:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{74}$$

This is because the direct sum decomposition of ρ_{isum} contains a dimension-8 irreducible representation whose $\rho(\mathfrak{t})$ has a 2-dimensional eigenspace, which gives rise to the this form of $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s. In the above, the eigenvalues of the later two matrices are $\pm(\mathrm{e}^{\mathrm{i}\,2\pi/3}, \mathrm{e}^{-\mathrm{i}\,2\pi/3})$. Since a permutation of two elements can only have orders 1 or 2, the corresponding 2×2 signed permutation matrix can only have eigenvalues 1, -1 or $\pm\mathrm{i}$. Any other eigenvalue is not possible. Thus, there is no orthogonal matrix that can transform the above two matrices into signed permutations. Such ρ_{isum} is not a representation of any modular data.

We also find cases where a 2-by-2 $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$ takes one of the following forms (up to conjugation of signed diagonal matrices): $\begin{pmatrix} -\frac{1+\sqrt{5}}{4}, & \frac{1}{2}c_{20}^3 \\ -\frac{1}{2}c_{20}^3, & -\frac{1+\sqrt{5}}{4} \end{pmatrix}$, $\begin{pmatrix} -\frac{1-\sqrt{5}}{4}, & -\frac{1}{2}c_{20}^1 \\ \frac{1}{2}c_{20}^1, & -\frac{1-\sqrt{5}}{4} \end{pmatrix}$, $\begin{pmatrix} \frac{1}{2}, & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2}, & \frac{1}{2} \end{pmatrix}$. We find $D^4_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta} \neq \text{id}$ for those matrices. Thus, those $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$ are not similar to any 2-by-2 signed permutation.

5.3 Within a 3-dimensional eigenspace of $\rho_{\text{isum}}(\mathfrak{t})$

We find cases where a 3-by-3 $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$ is given by $\begin{pmatrix} \frac{1}{2}, & -\frac{\sqrt{3}}{2}, & 0 \\ \frac{\sqrt{3}}{2}, & \frac{1}{2}, & 0 \\ 0, & 0, & 1 \end{pmatrix}$, whose order is 3 (*i.e.* cube to identity). Since the trace of the matrix is non-zero, it cannot be similar to any 3-by-3 order-3 signed permutation matrix (since 3-by-3 order-3 signed permutation matrices all have zero trace).

There are also cases where 3-by-3 generators of $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s are given by $\begin{pmatrix} 1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & -1 \end{pmatrix}$,

$\begin{pmatrix} 1, & 0, & 0 \\ 0, & -\frac{1}{2}, & \frac{\sqrt{3}}{2} \\ 0, & -\frac{\sqrt{3}}{2}, & -\frac{1}{2} \end{pmatrix}$. The second matrix is of order-3 and can only be similar to signed $(1,2,3)$ permutations. The most general orthogonal transformations that transform the second matrix into signed permutations have a form

$$U = PV_{\text{sd}} \begin{pmatrix} \frac{\sqrt{3}}{3}, & -\frac{\sqrt{6}}{6}, & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3}, & \frac{\sqrt{6}}{3}, & 0 \\ \frac{\sqrt{3}}{3}, & -\frac{\sqrt{6}}{6}, & \frac{\sqrt{2}}{2} \end{pmatrix} \tag{75}$$

where V_{sd} is a signed diagonal matrix and P a permutation matrix. But those orthogonal transformations all fail to transform the first matrix into signed permutations. So those cases are rejected.

There are cases where a 3-by-3 $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s is $\begin{pmatrix} 1, & 0, & 0 \\ 0, & 0, & -1 \\ 0, & 1, & 0 \end{pmatrix}$. This matrix is of order-4 and can only be similar to signed $(2,3)$ permutations. The most general orthogonal matrices that conjugate this matrix into signed permutation matrices have a form PV_{sd}.

There are also cases where a 3-by-3 $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s is (up to sign and signed permutations) $\begin{pmatrix} 1, & 0, & 0 \\ 0, & -\frac{1}{2}, & \frac{\sqrt{3}}{2} \\ 0, & -\frac{\sqrt{3}}{2}, & -\frac{1}{2} \end{pmatrix}$. The most general orthogonal transformations that transform this matrix into signed permutations are given by (75).

5.4 Within a 4-dimensional or 5-dimensional eigenspace of $\rho_{\text{isum}}(t)$

Some cases have 4-by-4 $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s generated by the following generators (up to sign and signed permutations)

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & -1, & 0, & 0 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 0, & -1 \end{pmatrix}, \quad \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & -1 \\ 0, & 0, & -1, & 0 \\ 0, & 1, & 0, & 0 \end{pmatrix} \tag{76}$$

The second matrix is a signed $(1,3)(2,4)$ permutation. It can also be transformed into signed $(1,2,3,4)$ permutation by the following matrix

$$U = PV_{\text{sd}} \begin{pmatrix} \frac{1}{2}, & \frac{\sqrt{2}}{2}, & \frac{1}{2}, & 0 \\ \frac{1}{2}, & 0, & -\frac{1}{2}, & -\frac{\sqrt{2}}{2} \\ \frac{1}{2}, & -\frac{\sqrt{2}}{2}, & \frac{1}{2}, & 0 \\ \frac{1}{2}, & 0, & -\frac{1}{2}, & \frac{\sqrt{2}}{2} \end{pmatrix} \tag{77}$$

By this U fails to transform the first matrix into a signed permutation. Thus, the most general orthogonal transformations that transform the two matrices into signed permutations have a form $U = PV_{\text{sd}}$.

Some cases have 4-by-4 $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s generated by the following generators (up to sign and signed permutations)

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & -1 \\ 0, & 0, & -1, & 0 \\ 0, & 1, & 0, & 0 \end{pmatrix} \tag{78}$$

The most general orthogonal transformations that transform this matrix into signed permutations have a form $U = PV_{\text{sd}}$ or (77).

Some cases have 5-by-5 $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s generated by the following generators (up to sign and signed permutations)

$$\begin{pmatrix} -1, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{pmatrix}, \quad \begin{pmatrix} 1, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & -1 \\ 0, & 0, & 0, & -1, & 0 \\ 0, & 0, & 1, & 0, & 0 \end{pmatrix} \tag{79}$$

The most general orthogonal transformations that transform this matrix into signed permutations have a form $U = PV_{\text{sd}}$ or

$$U = PV_{\text{sd}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}, & \frac{\sqrt{2}}{2}, & \frac{1}{2}, & 0 \\ 0 & \frac{1}{2}, & 0, & -\frac{1}{2}, & -\frac{\sqrt{2}}{2} \\ 0 & \frac{1}{2}, & -\frac{\sqrt{2}}{2}, & \frac{1}{2}, & 0 \\ 0 & \frac{1}{2}, & 0, & -\frac{1}{2}, & \frac{\sqrt{2}}{2} \end{pmatrix} \tag{80}$$

Some cases have 4-by-4 $D_{\rho_{\text{isum}}}(\sigma)_{I_\theta,I_\theta}$'s generated by the following generators (up to sign and signed permutations)

$$\begin{pmatrix} -1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix}, \quad \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & -\frac{1}{2}, & \frac{\sqrt{3}}{2} \\ 0, & 0, & -\frac{\sqrt{3}}{2}, & -\frac{1}{2} \end{pmatrix} \tag{81}$$

The second matrix is of order-6 and can only be similar to signed $(2, 3, 4)$ permutations. The most general orthogonal transformations that transform the second matrix (and the first matrix) into signed permutations have a form

$$U = PV_{\text{sd}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{3}, & -\frac{\sqrt{6}}{6}, & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{3}}{3}, & \frac{\sqrt{6}}{3}, & 0 \\ 0 & \frac{\sqrt{3}}{3}, & -\frac{\sqrt{6}}{6}, & \frac{\sqrt{2}}{2} \end{pmatrix} \tag{82}$$

5.5 General degenerate cases

In general, the orthogonal matrix U that transform ρ_{isum} to a pMD representation ρ_{pMD}

$$\rho_{\text{pMD}}(\mathfrak{t}) = U\rho_{\text{isum}}(\mathfrak{t})U^\top, \quad \rho_{\text{pMD}}(\mathfrak{s}) = U\rho_{\text{isum}}(\mathfrak{s})U^\top, \tag{83}$$

contains continuous parameters. For example, U may have the following form

$$U = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & u_1 & u_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & u_2 & -u_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & u_3 & u_4 & u_5 & \cdots \\ 0 & 0 & 0 & 0 & u_6 & u_7 & u_8 & \cdots \\ 0 & 0 & 0 & 0 & u_9 & u_{10} & u_{11} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ for } \tilde\rho(\mathfrak{t}) = \begin{pmatrix} \tilde\theta_0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \tilde\theta_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \tilde\theta_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \tilde\theta_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \tilde\theta_2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \tilde\theta_2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde\theta_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where u_i's satisfy the following orthogonality conditions:

$$u_1^2 + u_2^2 - 1 = 0, \quad u_3^2 + u_4^2 + u_5^2 - 1 = 0, \quad u_3u_6 + u_4u_7 + u_5u_8 = 0, \quad \cdots \tag{84}$$

where will be called zero conditions. When $D_{\tilde\rho}(\sigma)$ are not signed diagonal matrices with non-degenerate common eigenspaces, we will assume U to have the general form (84).

Because u_i in U are real numbers, different choices of u_i's will give us infinitely many potential pMD representations ρ_{pMD}, and hence infinitely many potential MD representations ρ_{MD}, after a finite number conjugations by signed diagonal matrices. Thus we cannot check those MD representations one by one to see which of them satisfy Proposition 2.1. Thus we need to use additional conditions on u_i. If we have enough conditions which only allow a finite numbers of solutions, then we get a finite number of U's which lead to a finite number potential pMD representations.

We will use the following conditions on pMD representations to obtain equations on u_i's. Those conditions on pMD representations are derived from condition on MD representations (see Proposition 2.4), by noticing that a pMD representation is related to a MD representation via

$$(\rho_{\text{pMD}})_{ij} = v_i v_j (\rho_{\text{MD}})_{ij}, \quad v_i \in \{+1, -1\} \tag{85}$$

Proposition 5.3. *A pMD representation ρ_{pMD} has the following properties:*

1. *The conductor of the elements of $\rho_{\text{pMD}}(\mathfrak{s})$ divides $\text{ord}(\rho_{\text{pMD}}(\mathfrak{t}))$.*
2. *The number of self dual objects is greater than 0. Thus*

$$\text{Tr}(\rho^2_{\text{pMD}}(\mathfrak{s})) \neq 0. \tag{86}$$

Since $\text{Tr}(\rho^2_{\text{pMD}}(\mathfrak{s})) \neq 0$, let us introduce

$$C = \frac{\text{Tr}(\rho^2_{\text{pMD}}(\mathfrak{s}))}{|\text{Tr}(\rho^2_{\text{pMD}}(\mathfrak{s}))|}\rho^2_{\text{pMD}}(\mathfrak{s}). \tag{87}$$

The above C is the charge conjugation operator of MTC, i.e. C is a permutation matrix of order 2. In particular, $\text{Tr}(C)$ is the number of self dual objects. Also, for each eigenvalue $\tilde\theta$ of $\rho_{\text{pMD}}(\mathfrak{t})$,

$$\text{Tr}_{\tilde\theta}(C) \geq 0, \tag{88}$$

where $\text{Tr}_{\tilde\theta}$ is the trace in the degenerate subspace of $\rho_{\text{pMD}}(\mathfrak{t})$ with eigenvalue $\tilde\theta$.

3. *If the modular data is integral and* $\mathrm{ord}(\rho_{\mathrm{pMD}}(\mathfrak{t})) = $ *odd, then*

$$\mathrm{Tr}(C) = \Big(\mathrm{Tr}\big(\rho^2_{\mathrm{pMD}}(\mathfrak{s})\big)\Big)^2 = 1, \tag{89}$$

i.e. the unit object is the only self-dual object.

4. *For any Galois conjugation* σ *in* $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\rho_{\mathrm{pMD}}(\mathfrak{t}))})$*, there is a permutation of the indices,* $i \to \hat{\sigma}(i)$*, and* $\epsilon_\sigma(i) \in \{1, -1\}$*, such that*

$$\sigma\big(\rho_{\mathrm{pMD}}(\mathfrak{s})_{i,j}\big) = \epsilon_\sigma(i)\rho_{\mathrm{pMD}}(\mathfrak{s})_{\hat{\sigma}(i),j} = \rho_{\mathrm{pMD}}(\mathfrak{s})_{i,\hat{\sigma}(j)}\epsilon_\sigma(j) \tag{90}$$

$$\sigma^2\big(\rho_{\mathrm{pMD}}(\mathfrak{t})_{i,i}\big) = \rho_{\mathrm{pMD}}(\mathfrak{t})_{\hat{\sigma}(i),\hat{\sigma}(i)}, \tag{91}$$

for all i, j*.*

5. *For any integer* a *coprime to* $n = \mathrm{ord}(\rho_{\mathrm{pMD}}(\mathfrak{t}))$*, we define*

$$D_{\rho_{\mathrm{pMD}}}(a) := \rho_{\mathrm{pMD}}(\mathfrak{t}^a\mathfrak{s}\mathfrak{t}^b\mathfrak{s}\mathfrak{t}^a\mathfrak{s}^{-1}) = D_{\rho_{\mathrm{pMD}}}(a + \mathrm{ord}(\rho_{\mathrm{pMD}}(\mathfrak{t}))), \\ \textit{where } ab \equiv 1 \mod \mathrm{ord}(\rho_{\mathrm{pMD}}(\mathfrak{t}))\,. \tag{92}$$

For any $\sigma \in \mathrm{Gal}(\mathbb{Q}_n)$*,* $\sigma(\zeta_n) = \zeta_n^a$ *for some unique integer* a *modulo* n*. We define*

$$D_{\rho_{\mathrm{pMD}}}(\sigma) := D_{\rho_{\mathrm{pMD}}}(a)\,. \tag{93}$$

By [93, Theorem II], $D_{\rho_{\mathrm{pMD}}} : \mathrm{Gal}(\mathbb{Q}_n) = (\mathbb{Z}_n)^\times \to \mathrm{GL}_r(\mathbb{C})$ *is a representation equivalent to the restriction of* ρ_{pMD} *on the diagonal subgroup of* $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$*.* $D_{\rho_{\mathrm{pMD}}}(\sigma)$ *in (92) must be a* ***signed permutation***

$$(D_{\rho_{\mathrm{pMD}}}(\sigma))_{i,j} = \epsilon_\sigma(i)\delta_{\hat{\sigma}(i),j}. \tag{94}$$

and satisfies

$$\sigma(\rho_{\mathrm{pMD}}(\mathfrak{s})) = D_{\rho_{\mathrm{pMD}}}(\sigma)\rho_{\mathrm{pMD}}(\mathfrak{s}) = \rho_{\mathrm{pMD}}(\mathfrak{s})D^\top_{\rho_{\mathrm{pMD}}}(\sigma), \\ \sigma^2(\rho_{\mathrm{pMD}}(\mathfrak{t})) = D_{\rho_{\mathrm{pMD}}}(\sigma)\rho_{\mathrm{pMD}}(\mathfrak{t})D^\top_{\rho_{\mathrm{pMD}}}(\sigma) \tag{95}$$

6. *There exists a* u *such that* $\rho_{\mathrm{pMD}}(\mathfrak{s})_{uu} \neq 0$ *and*

$$\frac{\rho_{\mathrm{pMD}}(\mathfrak{s})_{ij}}{\rho_{\mathrm{pMD}}(\mathfrak{s})_{uu}}, \ \frac{\rho_{\mathrm{pMD}}(\mathfrak{s})_{ij}}{\rho_{\mathrm{pMD}}(\mathfrak{s})_{uj}} \in \mathbb{O}_{\mathrm{ord}(T)}, \qquad \frac{\rho_{\mathrm{pMD}}(\mathfrak{s})_{ij}}{\rho_{\mathrm{pMD}}(\mathfrak{s})_{i'j'}} \in \mathbb{Q}_{\mathrm{ord}(T)}, \\ \rho_{\mathrm{pMD}}(\mathfrak{s})_{ui} \neq 0, \qquad \frac{1}{\rho_{\mathrm{pMD}}(\mathfrak{s})_{ui}} \in \mathbb{O}_{\mathrm{ord}(\rho(\mathfrak{t}))}, , \\ \widetilde{N}^{ij}_k = \sum_{l=0}^{r-1} \frac{\rho_{\mathrm{pMD}}(\mathfrak{s})_{li}\rho_{\mathrm{pMD}}(\mathfrak{s})_{lj}\rho_{\mathrm{pMD}}(\mathfrak{s}^{-1})_{lk}}{\rho_{\mathrm{pMD}}(\mathfrak{s})_{lu}} \in \mathbb{Z}, \quad \forall\ i, j, k = 1, 2, \dots, r. \tag{96}$$

(u corresponds the unit object of MTC. Also see Lemma 5.4.)

7. *Let* $n \in \mathbb{N}_+$*. The* n^{th} *pseudo Frobenius-Schur indicator of the* i*-th simple object*

$$\widetilde{\nu}_n(i) = \sum_{l=1}^{r} \frac{\rho_{\mathrm{pMD}}(\mathfrak{s}\mathfrak{t}^n\mathfrak{s})_{lu}\rho_{\mathrm{pMD}}(\mathfrak{s}\mathfrak{t}^{-n}\mathfrak{s}^{-1})_{lu}\rho_{\mathrm{pMD}}(\mathfrak{s}^{-1})_{li}}{\rho_{\mathrm{pMD}}(\mathfrak{s})_{lu}} \tag{97}$$

is a cyclotomic integer whose conductor divides n *and* $\mathrm{ord}(T)$*. The 1st pseudo Frobenius-Schur indicator satisfies* $\widetilde{\nu}_1(i) = \delta_{iu}$ *while the 2nd pseudo Frobenius-Schur indicator* $\widetilde{\nu}_2(i)$ *satisfies* $\widetilde{\nu}_2(i) \in \{\rho_{\mathrm{pMD}}(\mathfrak{s}^2)_{ii}, -\rho_{\mathrm{pMD}}(\mathfrak{s}^2)_{ii}\}$ *(see [89–91]). We also have the identity* $\widetilde{\nu}_n(u) = 1$*.*

8. *$\widetilde{D} = 1/\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ is a cyclotomic integer. $\widetilde{D}^5/\operatorname{ord}(T)$ is an algebraic integer, which is also a cyclotomic integer. The prime divisors of norm$(\widetilde{D})$ and $\operatorname{ord}(T)$ coincide. $\widetilde{D}/\widetilde{d}_i = 1/\rho_{\text{pMD}}(\mathfrak{s})_{ui}$ are cyclotomic integers (see Lemma 5.4), where $\widetilde{d}_i = \rho_{\text{pMD}}(\mathfrak{s})_{ui}/\rho_{\text{pMD}}(\mathfrak{s})_{uu}$. The prime divisors of norm$(\widetilde{D}/\widetilde{d}_i) =$ norm$(1/\rho_{\text{pMD}}(\mathfrak{s})_{ui})$ are part of the prime divisors of $\operatorname{ord}(T)$. This also implies that the prime divisors of norm$(\widetilde{d}_i) =$ norm$(\rho_{\text{pMD}}(\mathfrak{s})_{ui}/\rho_{\text{pMD}}(\mathfrak{s})_{uu})$ are part of the prime divisors of $\operatorname{ord}(T)$.*

9. *If the above conditions are satisfied for choosing an index u as the unit index, then the above conditions are also satisfied for choosing another index u' as the unit index, provided that u and u' are related by an Galois conjugation: $u' = \hat{\sigma}(u)$*

Lemma 5.4. *For a pMD representation ρ_{pMD}, let $s = \rho_{\text{pMD}}(\mathfrak{s})$, the u-th row of s the unit row, $n = \operatorname{ord}(\rho_{\text{pMD}}(\mathfrak{t}))$, and $N = \operatorname{pord}(\rho_{\text{pMD}}(\mathfrak{t}))$. Then $1/s_{uj}$ is a cyclotomic integer in $\mathbb{Q}_n$ which divides $1/s_{uu}$ for all j. In particular, the prime factors of* $\operatorname{norm}(1/s_{uj})$ *can only be part of those of N.*

Proof. Let d_i and D^2 be the quantum dimensions and the total quantum dimension of the modular tensor category that corresponds to a pMD representation ρ_{pMD}. Note that $\widetilde{D} = 1/s_{uu} \in \mathbb{Q}_n$ and $\widetilde{d}_j = \frac{s_{uj}}{s_{uu}} \in \mathbb{Q}_N$, which are cyclotomic integers for all j. D and $\widetilde{D}$ differ only by some 4-th roots of unity. d_i and $\widetilde{d}_i$ differ only by ± 1. Since D/d_j is an algebraic integer and is equal to $1/s_{uj}$ up to a 4-th root of unity, $1/s_{uj}$ is an algebraic integer in $\mathbb{Q}_n$ which divides $1/s_{uu}$ as algebraic integers. By the Cauchy theorem, $\operatorname{norm}(1/s_{uu})$ has exactly the same prime factors as those of N. Since $\operatorname{norm}(1/s_{uj}) \mid \operatorname{norm}(1/s_{uu})$, the prime factors of $\operatorname{norm}(1/s_{uj})$ must be a subcollection of the prime factors of N. □

In our computer algebra calculation, the conditions on u_i are written as *zero conditions* $f_1(u_i) = 0$ and $f_2(u_i) = 0$ and $\cdots$, where $f_k(u_i)$ are multi-variable polynomials. Since those zero conditions must be satisfied simultaneously, we refer to the set of zero conditions as and-connected zero conditions.

We may have another set zero conditions that must be satisfied simultaneously. But we only require one of the two sets of zero conditions to be satisfied. Thus the two sets of zero conditions are connected by "or", which give rise to or-connected sets of and-connected zero conditions. Some time, we create and-connected sets of or-connected zero conditions, and we need to convert them to or-connected sets of and-connected zero conditions. Our computer code is designed to manage those different logically connected zero conditions.

When we have two zero or-connected conditions, $g_1(u_i) = 0$ or $h_1(u_i) = 0$, we could combine them into one zero condition $f_1(u_i) = g_1(u_i)h_1(u_i) = 0$. But we will not do it. We will store the zero conditions as or-connected sets of and-connected zero conditions:

$$\begin{aligned} &[g_1(u_i) = 0 \text{ and } f_2(u_i) = 0 \text{ and } \cdots] \\ \text{or} \quad &[h_1(u_i) = 0 \text{ and } f_2(u_i) = 0 \text{ and } \cdots]. \end{aligned} \tag{98}$$

Those two sets of zero conditions can be viewed as a factorization of a single set of zero conditions $f_1(u_i) = 0$ and $f_2(u_i) = 0$ and $\cdots$. So storing the u_i's conditions as two sets of and-connected zero conditions allows us to avoid factorizing some algebraic equations (such as factor $f_1(u_i) = 0$ into $g_1(u_i) = 0$ or $h_1(u_i) = 0$), which is difficult and unreliable for multi-variable polynomials. This is a strategy that we use in computer algebra calculation to construct the conditions on u_i's: *We try to construct many or-connected sets of and-connected zero conditions.*

We call this stage of calculation as u-stage, where we try to factorize a zero condition $f_1(u_i) = g_1(u_i)h_1(u_i) = 0$ into or-connected zero conditions: $g_1(u_i) = 0$ or $h_1(u_i) = 0$. To help factorizing, we write GAP code to choose Gröbner basis for the and-connected zero conditions so that some of the zero conditions to have as few variables as possible.

When a zero condition contains a single u_i variable and if this variable is known to be a cyclotonic number of a know conductor, then we can factorize the zero condition.

If after factorization, we obtain say

$$\begin{aligned} &[u_1 = 0 \text{ and } u_2 - 1 = 0 \text{ and } u_3^2 + u_4^2 - 1 = 0 \cdots] \\ \text{or} \quad &[u_1 - \frac{\sqrt{2}}{2} = 0 \text{ and } u_2 - \frac{\sqrt{2}}{2} = 0 \text{ and } u_3^2 + u_4^2 - 1 = 0 \cdots], \end{aligned} \tag{99}$$

then u_1 and u_2 are solved, but u_3 and u_4 remain unsolved. Since u_i are real, the condition $u_1^2+u_2^2 = 0$ becomes $u_1 = 0$ and $u_2 = 0$. The condition $u_1^2+u_2^2+1 = 0$ leads to a rejection.

If we fail to solve all the variables u_i, we replace the matrix elements of $\rho_{\text{pMD}}(\mathfrak{s})$ with u_i variable's by cyclotomic numbers which contain only finite rational variables r_i. This is because the conductor of the matrix elements of $\rho_{\text{pMD}}(\mathfrak{s})$ is $\text{ord}(\rho_{\text{pMD}}(\mathfrak{t}))$ which is finite. At this stage of rational variables (called r-stage), we can utilize addition conditions that involve Galois conjugations, which we cannot use at the u-stage. Also, the zero condition like $r_1^2 - \frac{2}{3} =$ will lead to a rejection at r-stage.

If we fail to solve all the variable r_i, we can go to stage of integer variables n_i, the n-stage, (or we can go to the n-stage directly from the u-stage). This is achieved by representing $\rho_{\text{pMD}}(\mathfrak{s})$ in terms of cyclotomic integers:

$$K_{ij} \equiv \frac{\rho_{\text{pMD}}(\mathfrak{s})_{ij}}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}} \in \mathbb{O}_{\text{ord}(T)}, \quad \widetilde{K}_{ij} \equiv \frac{\rho_{\text{pMD}}(\mathfrak{s})_{ij}}{\rho_{\text{pMD}}(\mathfrak{s})_{uj}} \in \mathbb{O}_{\text{ord}(T)}, \quad J_i \equiv \frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{ui}} \in \mathbb{O}_{\text{ord}(\rho(\mathfrak{t}))}. \tag{100}$$

From K_{ij}, $\widetilde{K}_{ij}$, J_i we can recover $\rho_{\text{pMD}}(\mathfrak{s})$. Since K_{ij}, $\widetilde{K}_{ij}$, J_i have finite conductors $\text{ord}(\rho_{\text{pMD}}(\mathfrak{t}))$ or $\text{ord}(T)$, we can replace the elements with r_i or u_i variables by a finite number of integer variables n_i.

Proposition 5.5. *The two matrices K, $\widetilde{K}$ and the vector J satisfy the following conditions, which we can use to solve for integer variables n_i:*

1. *From the their definition, we find*

$$K_{ij} = K_{ji}, \qquad K_{iu} = \textit{real}, \qquad K_{ij}J_j = \widetilde{K}_{ij}J_u. \tag{101}$$

2. *For any integer σ coprime to $\text{ord}(\rho_{\text{pMD}}(\mathfrak{t}))$, we have*

$$D_{\rho_{\text{pMD}}}(\sigma)KJ_u = \rho_{\text{pMD}}(\mathfrak{t})^{\sigma} K \rho_{\text{pMD}}(\mathfrak{t})^{\gamma} K \rho_{\text{pMD}}(\mathfrak{t})^{\sigma} \tag{102}$$

where γ satisfies $\sigma\gamma = 1 \mod \text{ord}(\rho_{\text{pMD}}(\mathfrak{t}))$.

$$D_{\rho_{\text{pMD}}}(-1)J_u^2 = K^2, \tag{103}$$

where $D_{\rho_{\text{pMD}}}(\sigma) = D_{\rho_{\text{pMD}}}(\sigma + \text{ord}(\rho_{\text{pMD}}(\mathfrak{t})))$ and are signed permutation matrics, which for a representation of the multiplication group $\mathbb{Z}^{\times}_{\text{ord}(\rho_{\text{pMD}}(\mathfrak{t}))}$.

3. *The Galois conguations:*

$$\begin{aligned} \sigma(J_i) &= \sum_{j=0}^{r-1} J_j(D_{\rho_{\text{pMD}}}(\sigma)^{\top})_{ji}, & \sigma(\widetilde{K}_{ij}) &= \sum_{k=0}^{r-1} \widetilde{K}_{ik}|(D_{\rho_{\text{pMD}}}(\sigma)^{\top})_{kj}|, \\ \sigma(K_{ij})J_u &= \sum_{k=0}^{r-1} D_{\rho_{\text{pMD}}}(\sigma)_{ik}K_{kj}\sigma(J_u), & \sigma(\widetilde{K}_{ij})J_j &= \sum_{k=0}^{r-1} D_{\rho_{\text{pMD}}}(\sigma)_{ik}\widetilde{K}_{kj}\sigma(J_j). \end{aligned} \tag{104}$$

4. *If we know* $|K_{iu}| = 1$ *for an index* i*, then*

$$|K_{ij}| = 1 \quad \textit{for all indices } j. \tag{105}$$

5. *Pseudo characters of* $\mathrm{SL}_2(\mathbb{Z})$ *representation:*

$$\begin{aligned} J_u \mathrm{Tr}(\rho_{\text{isum}}(\mathfrak{s})_{I_\theta,I_\theta}) &= \mathrm{Tr}(K_{I_\theta,I_\theta}), \\ J_u^2 \mathrm{Tr}\big(\rho_{\text{isum}}(\mathfrak{s})_{I_\theta,I_{\theta'}}\rho_{\text{isum}}(\mathfrak{s})_{I_{\theta'},I_\theta}\big) &= \mathrm{Tr}\big(K_{I_\theta,I_{\theta'}}K_{I_{\theta'},I_\theta}\big). \end{aligned} \tag{106}$$

Here, I_θ *is the set of indices for the degenerate eigenspace of* $\rho_{\text{pMD}}(\mathfrak{t})$ *with eigenvalue* θ*. For example, the matrix* $K_{I_\theta,I_{\theta'}}$ *is a block of the* K*-matrix that connects* I_θ *and* $I_{\theta'}$*.*

6. *Orthogonality conditions:*

$$\sum_{i=0}^{r-1} \frac{1}{|J_i|^2} = 1, \quad \sum_{k=0}^{r-1} K_{ik}K^*_{kj} = |J_u|^2\delta_{ij}, \quad \sum_{k=0}^{r-1} \widetilde{K}_{ki}\widetilde{K}^*_{kj} = |J_i|^2\delta_{ij}, \quad \sum_{k=0}^{r-1} K_{ik}\widetilde{K}^*_{kj} = J_u J_i^* \delta_{ij} \tag{107}$$

7. *The pseudo fusion coefficients* $\widetilde{N}^{ij}_k$*:*

$$\begin{aligned} \widetilde{N}^{ij}_k &= \rho_{\text{pMD}}(\mathfrak{s})_{uu}\rho_{\text{pMD}}(\mathfrak{s})^*_{uu} \sum_{l=0}^{r-1} \frac{\rho_{\text{pMD}}(\mathfrak{s})_{li}}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}} \frac{\rho_{\text{pMD}}(\mathfrak{s})_{lj}}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}} \frac{\rho_{\text{pMD}}(\mathfrak{s})^*_{lk}}{\rho_{\text{pMD}}(\mathfrak{s})^*_{lu}} \\ &= \frac{\sum_{l=0}^{r-1} K_{il}K_{jl}\widetilde{K}^*_{kl}}{|J_u|^2} \in \mathbb{Z}, \end{aligned} \tag{108}$$

where we have used the fact that $\frac{\rho_{\text{pMD}}(\mathfrak{s})_{uu}}{\rho_{\text{pMD}}(\mathfrak{s})_{ul}}$ *are real, which leads to* $\frac{\rho_{\text{pMD}}(\mathfrak{s})_{uu}}{\rho_{\text{pMD}}(\mathfrak{s})_{ul}} = \frac{\rho_{\text{pMD}}(\mathfrak{s})^*_{uu}}{\rho_{\text{pMD}}(\mathfrak{s})^*_{ul}}$*.*

8. *The pseudo Frobenius-Schur indicators* $\widetilde{\nu}_n(i)$*:*

$$\begin{aligned} \widetilde{\nu}_n(i) &= |\rho_{\text{pMD}}(\mathfrak{s})_{uu}|^4 \sum_{l=1}^{r} \frac{\rho_{\text{pMD}}(\mathfrak{s}\mathfrak{t}^n\mathfrak{s})_{lu}}{(\rho_{\text{pMD}}(\mathfrak{s})_{uu})^2} \frac{\rho_{\text{pMD}}(\mathfrak{s}\mathfrak{t}^{-n}\mathfrak{s}^{-1})_{lu}}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}\rho_{\text{pMD}}(\mathfrak{s})^*_{uu}} \frac{\rho_{\text{pMD}}(\mathfrak{s})^*_{il}}{\rho_{\text{pMD}}(\mathfrak{s})^*_{ul}} \\ &= |J_u|^{-4} \sum_{l=1}^{r} \big(K\rho^n_{\text{pMD}}(\mathfrak{t})K\big)_{lu}\big(K\rho^{-n}_{\text{pMD}}(\mathfrak{t})K^*\big)_{lu}\widetilde{K}^*_{il}, \end{aligned} \tag{109}$$

where $\widetilde{\nu}_1(i) = \delta_{iu}$ *and* $\widetilde{\nu}_2(i) \in \{\rho_{\text{pMD}}(\mathfrak{s}^2)_{ii}, -\rho_{\text{pMD}}(\mathfrak{s}^2)_{ii}\}$

At n-stage, there are more tricks to solve zero conditions. For example, a zero condition $n_1n_2 = 10$ will lead to or-connected zero conditions:

$$\begin{aligned} &[n_1 = 1 \text{ and } n_2 = 10] \quad \text{or} \quad [n_1 = -1 \text{ and } n_2 = -10] \\ \text{or} \quad &[n_1 = 2 \text{ and } n_2 = 5] \quad \text{or} \quad [n_1 = -2 \text{ and } n_2 = -5] \\ \text{or} \quad &[n_1 = 5 \text{ and } n_2 = 2] \quad \text{or} \quad [n_1 = -5 \text{ and } n_2 = -2] \\ \text{or} \quad &[n_1 = 10 \text{ and } n_2 = 1] \quad \text{or} \quad [n_1 = -10 \text{ and } n_2 = -1]. \end{aligned} \tag{110}$$

Also, a zero condition $n_1^2 - n_1n_2 + n_2^2 = 1$ of elliptic form will lead to or-connected zero conditions:

$$[n_1 = 1 \text{ and } n_2 = 1] \quad \text{or} \quad [n_1 = -1 \text{ and } n_2 = -1]. \tag{111}$$

5.6 Some details of computer calculations

5.6.1 Determining $D_{\rho_{\text{pMD}}}(\sigma)$

$D_{\rho_{\text{pMD}}}(\sigma)$ given by Eq. (92) contain the variables u_i. The non-zero elements and the elements containing variables form blocks: Let I_θ be the indices for the degenerate eigenspace of $\rho_{\text{pMD}}(\text{t})$ with eigenvalue θ. Only elements in the block $D_{\rho_{\text{pMD}}}(\sigma)_{I_\theta,I_{\theta'}}$ is non-zero, and for each fixed I_θ, there exists only one $I_{\theta'}$ such that $D_{\rho_{\text{pMD}}}(\sigma)_{I_\theta,I_{\theta'}}$ is non-zero. Also a non-zero block is always a square matrix. This is because the non-zero block $D_{\rho_{\text{pMD}}}(\sigma)_{I_\theta,I_{\theta'}}$ connects eigenvalues θ and $\theta' = \sigma^2(\theta)$, and the number degenerate eigenvalues for θ and θ', mapped into each other by a Galois conjugation, are the same.

We know that the matrix elements of $D_{\rho_{\text{pMD}}}(\sigma)$ can only take three possible values $0, \pm 1$, since $D_{\rho_{\text{pMD}}}(\sigma)$ are signed permutations. Thus, there are only a finite number of possible $D_{\rho_{\text{pMD}}}(\sigma)$. So in the first step, we list all those possible $D_{\rho_{\text{pMD}}}(\sigma)$'s. In fact, the number of possible sets of $D_{\rho_{\text{pMD}}}(\sigma)$'s are not many, since many matrix elements of $D_{\rho_{\text{pMD}}}(\sigma)$'s are either equal or differ by sign. When we assign $0, \pm 1$ to matrix elements of $D_{\rho_{\text{pMD}}}(\sigma)$, we include such correlations. Also, the resulting $D_{\rho_{\text{pMD}}}(\sigma)$'s are commuting signed permutations, which also reduces the number of possible sets of $D_{\rho_{\text{pMD}}}(\sigma)$'s. The computation load can be further reduced if we use the commuting properties of $D_{\rho_{\text{pMD}}}(\sigma)$ as early as possible during our determination of the matrix elements of $D_{\rho_{\text{pMD}}}(\sigma)$.

The number of the possible $D_{\rho_{\text{pMD}}}(\sigma)$'s can be further reduced. We note that the transformation U is block diagonal. The block that contain u_i variable's has a form U_{I_θ,I_θ}, *i.e.* U only maps the eigenspace I_θ into itself (see (84)). The transformation U generated by our code has such a property that the block U_{I_θ,I_θ} with variable's corresponds to the most general orthogonal transformations in the eigenspace I_θ. Those orthogonal transformations, acting on various eigenspaces, generate signed permutations of the index $i = 0, 1, \cdots, r-1$. We can define two sets of $D_{\rho_{\text{pMD}}}(\sigma)$'s as equivalent if they are connected by those signed permutations by conjugation. We only need to pick one representative from each equivalence class.

We pick the representative in the following way. We first pick a σ. Then we only require $D_{\rho_{\text{pMD}}}(\sigma)$ for that one σ to satisfy some additional conditions. For a diagonal block $D_{\rho_{\text{pMD}}}(\sigma)_{I_\theta,I_\theta}$ where U_{I_θ,I_θ} contain u_i variable's, we require the square matrix $D \equiv D_{\rho_{\text{pMD}}}(\sigma)_{I_\theta,I_\theta}$ to satisfy the following conditions (*i.e.* we can use signed permutations P_{sgn} and transformation $D \to P_{\text{sgn}} D P_{\text{sgn}}^{-1}$ to make D to satisfy the following conditions):

1. D is a signed permutation matrix.
2. $D_{i,j} = 0$ if $j \geq i + 2$.
3. $D_{i,i+1} \in \{0, 1\}$, *i.e.* the possibility of $D_{i,i+1} = -1$ is excluded.
4. $D_{ii} \geq D_{i+1,i+1}$.

For example, D may take the following form

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{112}$$

In other words, we can use U_{I_θ,I_θ} to make the upper triangle part of D non-negative.

Let us define an ordering of the blocks I_θ in such a way that block-level cyclic permutation $(I_{\theta_1}, I_{\theta_2}, \cdots, I_{\theta_n},)$ generated by $D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}}$ satisfies

$$I_{\theta_1} < I_{\theta_2},\ I_{\theta_2} < I_{\theta_3},\ \cdots,\ I_{\theta_n} > I_{\theta_1}. \tag{113}$$

For an variable-containing off-diagonal block $D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}}$ with $I_\theta > I_{\theta'}$ where U_{I_θ,I_θ} and $U_{I_{\theta'},I_{\theta'}}$ contain u_i variable's, we require the square matrix $D \equiv D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}}$ to satisfy the following conditions: (*i.e.* we can use two signed permutations $P_{\rm sgn}, \widetilde{P}_{\rm sgn}$ and transformation $D \to P_{\rm sgn} D \widetilde{P}_{\rm sgn}$ to make D to satisfy the following conditions):

1. D is an identity matrix.

For an variable-containing off-diagonal block $D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}}$ with $I_\theta < I_{\theta'}$ where U_{I_θ,I_θ} and $U_{I_{\theta'},I_{\theta'}}$ contain u_i variable's, we require the square matrix $D \equiv D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}}$ to satisfy the following conditions (*i.e.* we can use signed permutations $P_{\rm sgn}$ and transformation $D \to P_{\rm sgn} D P_{\rm sgn}^{-1}$ to make D to satisfy the following conditions):

1. D is a signed permutation matrix.
2. $D_{i,j} = 0$ if $j \geq i+2$.
3. $D_{i,i+1} \in \{0, 1\}$, *i.e.* the possibility of $D_{i,i+1} = -1$ is excluded.
4. $D_{ii} \geq D_{i+1,i+1}$.

We like to remark that since we have fixed $D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}}$ with $I_\theta > I_{\theta'}$ to be identity, the transformation, $D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}} \to P_{\rm sgn} D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}} \widetilde{P}_{\rm sgn}$, that keep $D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}}$ unchanged must satisfy $\widetilde{P}_{\rm sgn} = P_{\rm sgn}^{-1}$. Therefore, the transformation on $D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}}$ with $I_\theta < I_{\theta'}$ has a form $D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}} \to P_{\rm sgn} D_{\rho_{\rm pMD}}(\sigma)_{I_\theta,I_{\theta'}} P_{\rm sgn}^{-1}$. For example, D may take the following form

$$D = \begin{pmatrix} 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \\ \tilde{D} & 0 & 0 \end{pmatrix} \tag{114}$$

where $\tilde{D}$ is given by (112).

After knowing $D_{\rho_{\rm pMD}}(\sigma)$, we can obtain a set of zero conditions on u_i's from Eq. (95), as well as the zero conditions from the orthogonality conditions of U.

5.6.2 Determining if the MTC is integral

We expand the number possible cases (*i.e.* the number of or-connected sets of and-connected zero conditions) further by choosing possible rows of $\rho_{\rm pMD}(\mathfrak{s})$ to be the unit row. Certainly, only the row that contain no zero's can be a unit row.

Then for each case, we can check if $(D_{\rho_{\rm pMD}}(\sigma))_{uu} = \pm 1$ for all σ. If yes, it means $\rho_{\rm pMD}$ must correspond to a integral MTC, if any. Also, if $\mathrm{pord}(\rho_{\rm pMD}(\mathfrak{t})) \in \{2, 3, 4, 6\}$, then $\rho_{\rm pMD}$ must correspond to an integral MTC as well, if any. Using such a method, we find that $\mathrm{SL}_2(\mathbb{Z})$ representations may give rise to integral MD's only when the prime divisors of $\mathrm{pord}(\rho_{\rm pMD}(\mathfrak{t}))$ belong to the sets listed in Table 4. This result will use to construct integral MTCs (see Section 6.3 for more details).

From now on, for each case, we not only know $\rho_{\rm pMD}(\mathfrak{s})$ and $\rho_{\rm pMD}(\mathfrak{t})$, we also know a possible $D_{\rho_{\rm pMD}}(\sigma)$ of them, as well as a possible index u for the unit row, together with a set of and-connected zero conditions of u_i. This allows us to obtain many additional zero conditions on u_i's for such a case.

Table 4: We have computed $\rho_{\text{pMD}}(\mathfrak{t})$ of all $\text{SL}_2(\mathbb{Z})$ representations that may give rise to non-pointed integral MD's for each rank. This table lists all the prime divisors of $\text{pord}(\rho_{\text{pMD}}(\mathfrak{t}))$ of those ρ_{pMD}'s.

rank	prime divisors
2	[ ]
3	[ ]
4	[2]
5	[2,3]
6	[ ]
7	[ ]
8	[2,3]
9	[2,3]
10	[2], [2,3], [2,3,5]
11	[2], [2,3], [2,3,5], [2,5]
12	[2,3], [2,3,5], [2,3,7]

5.6.3 Conditions from Frobenius-Schur indicators

Since we know the index u of the unit row, we can compute the second pseudo Frobenius-Schur indicator from $\rho_{\text{pMD}}(\mathfrak{s})$:

$$\widetilde{\nu}_2(i) = \sum_{l=1}^{r} \frac{\rho_{\text{pMD}}(\mathfrak{s}\mathfrak{t}^2\mathfrak{s})_{lu}\rho_{\text{pMD}}(\mathfrak{s}\mathfrak{t}^{-2}\mathfrak{s}^{-1})_{lu}\rho_{\text{pMD}}(\mathfrak{s}^{-1})_{li}}{\rho_{\text{pMD}}(\mathfrak{s})_{lu}}. \tag{115}$$

The 2nd pseudo Frobenius-Schur indicator $\widetilde{\nu}_2(i)$ satisfies

$$\widetilde{\nu}_2(i) = \pm\rho_{\text{pMD}}(\mathfrak{s}^2)_{ii}, \quad \widetilde{\nu}_2(u) = 1. \tag{116}$$

The 2nd pseudo Frobenius-Schur indicator give us important zero conditions on u_i's, which can effectively help us to determine u_i's and r_i's. However, for more complicated cases (such as when the number of variables is more then 12), the zero conditions from the Frobenius-Schur indicator can be too complicated.

We would like to remark that the zero conditions from the Frobenius-Schur indicator have the form $f_1(u_i)/g_1(u_i) = 0$, where $f_1(u_i)$ and $g_1(u_i)$ are multi-variable polynomials. We convert the condition $f_1(u_i)/g_1(u_i) = 0$ to a zero condition $f_1(u_i) = 0$. Such a conversion may create some fake solutions of u_i. We can rule out those fake solutions later, when we check if the resulting (S, T) matrices satisfy the conditions for modular data or not.

5.6.4 Reduce the number of variables

In order to factorize zero conditions, it is important to find a Gröbner basis for and-connected zero conditions, such that some zero conditions have fewest variables, in particular single variable. We use the following strategy to find such a Gröbner basis. Consider a set of and-connected zero conditions $[f_1(u_i) = 0$ and $f_2(u_i) = 0$ and $f_2(u_i) = 0 \cdots]$. We want to use the i^{th} zero condition f_i to transform the i^{th} zero condition f_j. Assume f_i is a sum of a few monomials: $f_i = m_1(u_i) + m_2(u_i) + m_3(u_i)$. We can use the substitution $m_1 \to -m_2 - m_3$ to transform the zero condition f_j. We can also use the substitution $m_2 \to -m_1 - m_3$ to transform the zero condition f_j, *etc.*. Among all those transformed f_j, we select those with fewer variables than that of f_j, and replace f_j by those transformed zero conditions. If the one with fewer variables does not exist, we just keep original zero condition f_j.

We perform the calculation for all the pairs f_i and f_j. We then perform the above calculation for a few iterations. We have tested this approach and found that this is an effective way to reduce the number of variables in zero conditions. This method is very important for our calculations. In particular, if we can obtain zero conditions with only single variable and if the conductor of the variable is small, then there is an effective GAP/Singular function to factorize the single-variable polynomials.

We can also use the above substitution approach, try to eliminate some variables by simply reducing the number of variables that we want to eliminate in zero conditions.

5.6.5 From u-stage to r-stage

We replace the u_i-dependent matrix elements of $\rho_{\text{pMD}}(\mathfrak{s})$ with cyclotomic numbers of a conductor $\text{ord}(\rho_{\text{pMD}}(\mathfrak{t}))$. Those cyclotomic numbers are expressed in terms of r_i variables, where r_i are the expansion coefficients over a cyclotomic basis. This way, we express $\rho_{\text{pMD}}(\mathfrak{s})$ in terms of r_i variables.

Compare $\rho_{\text{pMD}}(\mathfrak{s})$ in terms of u_i and $\rho_{\text{pMD}}(\mathfrak{s})$ in terms of r_i, we obtain many u-r relations. This allows us to convert some u_i zero conditions to r_i zero conditions, by trying to eliminate the u-variables. The u_i zero conditions that cannot be converted to r_i zero conditions will be dropped. As a result, r_i-dependent $\rho_{\text{pMD}}(\mathfrak{s})$ may not be equivalent to the starting $\text{SL}_2(\mathbb{Z})$ representation $\rho_{\text{isum}}(\mathfrak{s})$. To partially fix this problem, we re-implement the $\text{SL}_2(\mathbb{Z})$ conditions and the D_ρ conditions (94), (95), on the r_i-dependent ρ_{pMD}, to obtain additional r_i zero conditions. We also compute some simple $\text{SL}_2(\mathbb{Z})$ characters

$$\text{Tr}_{\widetilde{\theta}}\big(\rho(\mathfrak{s})\big), \quad \text{Tr}_{\widetilde{\theta}}\big(\rho(\mathfrak{s})\rho(\mathfrak{t})^n\rho(\mathfrak{s})\big), \tag{117}$$

for the ρ_{pMD} and ρ_{isum} representations. Here Tr_θ is the trace in the eigenspace of $\rho(\mathfrak{t})$ with eigenvalue $\widetilde{\theta}$. Matching those simple $\text{SL}_2(\mathbb{Z})$ characters also give us additional r_i zero conditions. This complete our conversion from u_i's to r_i's.

5.6.6 Integer conditions and inverse-pair of integer conditions

In r-stage, $\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{ui}}$, $\frac{\rho_{\text{pMD}}(\mathfrak{s})_{ij}}{\rho_{\text{pMD}}(\mathfrak{s})_{uj}}$ and $\frac{\rho_{\text{pMD}}(\mathfrak{s})_{ij}}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}}$ are cyclotomic integers in term of the rational r_i variables. We can expand the cyclotomic integers in an integral basis of cyclotomic numbers. The expansion coefficients are ratios of polynomials of r_i with rational coefficients, which must be equal to integers. This gives us many r_i integer conditions of form

$$\frac{f_1(r_i)}{g_1(r_i)} \in \mathbb{Z}, \quad \frac{f_2(r_i)}{g_2(r_i)} \in \mathbb{Z}, \quad \cdots \tag{118}$$

If two r_i integer conditions, $\frac{f_1(r_i)}{g_1(r_i)} \in \mathbb{Z}$, $\frac{f_2(r_i)}{g_2(r_i)} \in \mathbb{Z}$, satisfy

$$\frac{f_1(r_i)}{g_1(r_i)}\frac{f_2(r_i)}{g_2(r_i)} = n \in \mathbb{Z}, \tag{119}$$

we will call them an inverse-pair of integer conditions. For each inverse-pair of integer conditions, we can obtain a set of r_i zero conditions:

$$\frac{f_1(r_i)}{g_1(r_i)} = m_1 \ \text{ or } \ \frac{f_1(r_i)}{g_1(r_i)} = -m_1 \ \text{ or } \ \frac{f_1(r_i)}{g_1(r_i)} = m_2 \ \text{ or } \ \frac{f_1(r_i)}{g_1(r_i)} = -m_2 \ \text{ or } \ \cdots, \tag{120}$$

where $m_1, \ m_2, \cdots$ are factors of n.

We note that when $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ contains no variables, $\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{ui}}$ and $\frac{\rho_{\text{pMD}}(\mathfrak{s})_{ui}}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}}$ are both cyclotomic integers. They produce inverse-pairs of integer conditions. In particular $\text{norm}(\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{ui}})$ and $\text{norm}(\frac{\rho_{\text{pMD}}(\mathfrak{s})_{ui}}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}})$ is an inverse-pair of integer conditions.

5.6.7 Solve $\rho_{\text{pMD}}(\mathfrak{s})_{ui}$ using generalized Egyptian-fraction method

The generalized Egyptian-fraction method described here is an important workhorse in our calculation. Let $\{p_i\}$ be the set of prime divisors of pord$(\rho_{\text{pMD}}(\mathfrak{t}))$. We note that norm$(\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{ui}})$ are integers, whose prime divisors are contained in $\{p_i\}$. Similarly, norm$(\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}})/\prod p_i$ is also an integer, whose prime divisors are contained in $\{p_i\}$. We will try to use such conditions to find possible values of $\rho_{\text{pMD}}(\mathfrak{s})_{ui}$.

The trick is to introduce variables v_i that are reciprocals of positive integers, whose prime divisors are contained in $\{p_i\}$. We use those variables to represent norm$(\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{ui}})$ and norm$(\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}})/\prod p_i$:

$$\begin{aligned} \text{norm}(\rho_{\text{pMD}}(\mathfrak{s})_{ui}) &= v_i \text{ or } -v_i, \quad \text{for } i \neq u \\ \text{norm}(\rho_{\text{pMD}}(\mathfrak{s})_{uu}) \prod p_i &= v_u \text{ or } -v_u. \end{aligned} \tag{121}$$

The choices of $\pm$ signs produce a lot of cases, and we need handle those cases one by one.

For each case, we combine the zero conditions, norm$(\rho_{\text{pMD}}(\mathfrak{s})_{ui})\pm v_i = 0$ and norm$(\rho_{\text{pMD}}(\mathfrak{s})_{uu})\prod p_i \pm v_u = 0$, of v_i's and r_i's, with other zero conditions of r_i's. Then, we use the method outlined in Section 5.6.4 to eliminate the r_i variables, trying to obtain, as many as possible, the zero conditions that contain only v_i's.

A zero condition with only v_i's has the following form

$$c_0 + c_1^+ P_1^+(v_i) + c_2^+ P_2^+(v_i) + \cdots - c_1^- P_1^-(v_i) - c_2^- P_2^-(v_i) + \cdots \tag{122}$$

where $c_i^\pm$ are positive integers, c_0 is a non-negative inter, and $P_i^\pm(v_i)$ are products of v_i's. We apply the following generalized Egyptian-fraction method to solve the above zero conditions:

1. If $c_0 = 0$, the iteration stops, and we only get a single zero condition (122).

2. If $c_0 > 0$, then $c_1^- P_1^-(v_i) + c_2^- P_2^-(v_i) + \cdots \geq c_0$. Let $c_{j_{\text{large}}}^- P_{j_{\text{large}}}^-(v_i)$ be a term among $c_j^- P_j^-(v_i)$'s, such that $c_{j_{\text{large}}}^- P_{j_{\text{large}}}^-(v_i) \geq c_0/N^-$, where N^- is the number of terms in $c_1^- P_1^-(v_i)+c_2^- P_2^-(v_i)+\cdots$. Such a term must exit. Because $P_{j_{\text{large}}}^-(v_i)$ is the inverse of a positive integer (whose prime divisors are part of those of pord$(\rho_{\text{pMD}}(\mathfrak{t}))$), $P_{j_{\text{large}}}^-(v_i)$ has only a finite number of possible values $P_{j_{\text{large}}}^-(v_i) = 1/n^-$, where

$$\begin{aligned} &n^- \in \mathbb{N}, \quad n^- \leq \frac{N^- c_{j_{\text{large}}}^-}{c_0}, \\ &\text{the prime divisors of } n^- \text{ are contained in } \{p_i\}. \end{aligned} \tag{123}$$

   For each possibility, the zero condition (122) is reduced to a zero condition with one less terms and $P_{j_{\text{large}}}^-(v_i) = 1/n^-$.

   We also need to run through different choices of large term among $c_j^- P_j^-(v_i)$'s. At the end, we obtain many or-connected sets of and-connected zero conditions. A set of and-connected zero conditions contains $P_{j_{\text{large}}}^-(v_i) = 1/n^-$ and the reduced zero condition from (122), as discussed above.

3. If $c_0 < 0$, then $c_1^+ P_1^+(v_i) + c_2^+ P_2^+(v_i) + \cdots \geq -c_0$. We repeat the above calculation for $c_1^+ P_1^+(v_i)+c_2^+ P_2^+(v_i)+\cdots$, and obtain many or-connected sets of and-connected zero conditions. A set of and-connected zero conditions contains $P_{j_{\text{large}}}^+(v_i) = 1/n^+$ and the reduced zero condition from (122).

We perform the above calculation, starting from the simplest zero condition of v_i's that has fewest terms and a non-zero c_0. Then we handle the next simplest zero condition of v_i's. At last, we process the resulting or-connected sets of and-connected zero conditions of v_i's and r_i's, trying to replace v_i's by r_i's, and convert the zero conditions of v_i's and r_i's to zero conditions of r_i's. The remaining zero conditions containing v_i's will be ignored. This will give us or-connected sets of and-connected zero conditions of r_i's.

5.6.8 Integer conditions from norm($\rho_{\text{pMD}}(\mathfrak{s})_{uu}$)

We note that $\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}} = \pm D$ or $\pm\text{i}D$, where u is the index for the unit row. Therefore $\text{norm}(\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}})$ is an integer whose prime divisors coincide with the prime divisors of $\text{pord}(\rho_{\text{pMD}}(\mathfrak{t}))$. At r-stage, $\text{norm}(\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}})$ is a ratio of two polynomials of r_i's. Sometimes, $\text{norm}(\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}}) \in \mathbb{Z}$ has only finite numbers of solutions for rational variables r_i's. We can use this property to solve $\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}}$.

In particular, there is a class of cases which is hard to solve. For those cases, $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ only depends on a single rational variable r and

$$\rho_{\text{pMD}}(\mathfrak{s})_{uu} = r - \frac{\sqrt{5}}{10}, \qquad \text{norm}(\frac{1}{\rho_{\text{pMD}}(\mathfrak{s})_{uu}}) = \frac{1}{r^2 - \frac{1}{20}} \in \mathbb{Z} \tag{124}$$

Also the prime divisors of $\text{pord}(\rho_{\text{pMD}}(\mathfrak{t}))$ are contained in $[2, 3, 5]$. This class of cases can be solved in the following way.

Recall that the ring of algebraic integers of $\mathbb{Q}(\sqrt{5})$ is $\mathbb{Z}[\phi]$ where $\phi = \frac{1+\sqrt{5}}{2}$. It is well-known that the group G of invertible elements in $\mathbb{Z}[\phi]$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$, and is given by $\{\pm\phi^n \mid n \in \mathbb{Z}\}$. $\mathbb{Z}[\sqrt{5}]$ is a subring of $\mathbb{Z}[\phi]$, and its group G' of invertible elements, which is a subgroup of G. Since $n = 3$ is the smallest positive integer n such that $\phi^n \in \mathbb{Z}[\sqrt{5}]$,

$$G' = \{\pm\phi^{3n} \mid n \in \mathbb{Z}\}\,. \tag{125}$$

Note that $\phi^3 = 2 + \sqrt{5}$ and $\text{norm}(\phi^3) = -1$. Thus, $\phi^6 = 9 + 4\sqrt{5}$ and $\text{norm}(\phi^6) = 1$. If $u, v \in \mathbb{N}$ such that $u^2 - 5v^2 = \pm 1$, then

$$u + v\sqrt{5} = \begin{cases} (2+\sqrt{5})^{2n} = (9+4\sqrt{5})^n & \text{if } u^2 - 5v^2 = 1, \\ (2+\sqrt{5})^{2n+1} = (2+\sqrt{5})(9+4\sqrt{5})^n & \text{if } u^2 - 5v^2 = -1\,. \end{cases} \tag{126}$$

for some positive integer n.

Lemma 5.6. *Let u, v be nonzero integers such that $v > 0$, $(u^2 - 5v^2) \mid 5$ and $u = 2^a 3^b 5^c$ for some nonnegative integers a, b, c. Then (u, v) can only be one of the following pairs:*

(i) $(2, \pm 1), (9, \pm 4)$ *which are the solutions of* $u^2 - 5v^2 = \pm 1$.

(ii) $(5, \pm 2), (20, \pm 9), (360, \pm 161)$ *which are the solutions of* $u^2 - 5v^2 = \pm 5$.

Proof. Since $u^2 - 5v^2 \mid 5$, we have two cases: $u^2 - 5v^2 = \epsilon$ or 5ϵ for some $\epsilon = \pm 1$.
(i) $u^2 - 5v^2 = \epsilon$. It is immediate to see that $5 \nmid u$ and so $c = 0$. Next, we show that $a \le 2$. Assume to the contrary that $a > 2$. Then $u^2 \equiv 0 \mod 8$, and v must be odd. Thus, $u^2 - 5v^2 \equiv 3 \not\equiv \epsilon \mod 8$. Therefore, $a \le 2$.

Now, we show that $b \le 2$. Suppose not. Then $b \ge 3$ which implies $3 \nmid v$ and $u \equiv 0 \mod 27$. Since $u^2 - 5v^2 \equiv -5 \equiv 1 \mod 3$, $\epsilon = 1$. By (126), we find

$$2^a 3^b = u = \frac{1}{2}\left((9+4\sqrt{5})^n + (9-4\sqrt{5})^n\right) = \sum_{0\le 2i\le n} \binom{n}{2i} 9^{n-2i} \cdot 4^{2i} \cdot 5^i \tag{127}$$

$$\equiv \begin{cases} 4^n \cdot 5^{\frac{n}{2}} \mod 27 & \text{if } n \text{ is even,} \\ n \cdot 9 \cdot 4^{n-1} \cdot 5^{\frac{n-1}{2}} \mod 27 & \text{if } n \text{ is odd.} \end{cases} \tag{128}$$

Since the leftmost expression of (127) is divisible by 27, it follows from (128) that n must be odd and $3 \mid n$. Thus, $n = 3k$ for some positive odd integer k. It follows from (127) again

$$2^a 3^b = \frac{1}{2}\left((9+4\sqrt{5})^{3k} + (9-4\sqrt{5})^{3k}\right) = \frac{1}{2}\left((9+4\sqrt{5})^3 + (9-4\sqrt{5})^3\right) z = 2889z = 3^3 \cdot 107 \cdot z \tag{129}$$

for some integer z since k is odd. However, this is a contradiction since $107 \nmid 2^a 3^b$. Therefore, $b \le 2$.

Now, we can solve for the integral solutions v of the equations $u^2 - 5v^2 = \pm 1$ for $u = 2^a 3^b$ with $0 \le a, b \le 2$. The integral solutions to $u^2 - 5v^2 = \pm 1$ are $(u, v) = (2, \pm 1)$ and $(9, \pm 4)$.

(ii) $u^2 - 5v^2 = 5\epsilon$. Then 5 divides u (or $c \ge 1$), and we have $v^2 - 5\overline{u}^2 = \pm 1$ where $\overline{u} = u/5 = 2^a 3^b 5^{c-1}$.

We first show that $c = 1$. Assume to the contrary that $c > 1$. Then $\overline{u} \equiv 0 \mod 5$. By (126), there is a positive integer n such that

$$\overline{u} = \frac{1}{2\sqrt{5}}\left((2+\sqrt{5})^n - (2-\sqrt{5})^n\right) = \sum_{0\le 2i+1\le n} \binom{n}{2i+1} 2^{n-2i-1} \cdot 5^i \equiv \binom{n}{1} 2^{n-1} \mod 5\,. \tag{130}$$

This implies $\binom{n}{1} 2^{n-1} \equiv 0 \mod 5$, and hence $n = 5k$ for some integer k. Therefore,

$$2^a 3^b 5^{c-1} = \frac{1}{2\sqrt{5}}\left((2+\sqrt{5})^{5k} - (2-\sqrt{5})^{5k}\right) = \frac{1}{2\sqrt{5}}\left((2+\sqrt{5})^5 - (2-\sqrt{5})^5\right) z = 305z = 5 \cdot 31 \cdot z \tag{131}$$

for some integer z, which is absurd. Therefore, $c = 1$, and so $\overline{u} = 2^a 3^b$.

Next, we show that $a \le 3$ and $b \le 2$. If $a > 3$, then v is odd. Since $v^2 \not\equiv -1 \mod 16$, $\epsilon = 1$ and so

$$v^2 - 5\overline{u}^2 = 1\,. \tag{132}$$

Similarly, if $b > 2$, then $3 \nmid v$. Since $v^2 \not\equiv -1 \mod 27$, $\epsilon = 1$. Therefore, $(\overline{u}, v)$ can only satisfy (132). It follows from (126) that

$$\overline{u} = \frac{1}{2\sqrt{5}}\left((9+4\sqrt{5})^n - (9-4\sqrt{5})^n\right) = \sum_{0\le 2i+1\le n} \binom{n}{2i+1} 9^{n-2i-1} \cdot 4^{2i+1} \cdot 5^i \tag{133}$$

$$\equiv \begin{cases} \binom{n}{1} 9^{n-1} 4 \mod 16 & \\ 4^n \cdot 5^{(n-1)/2} \mod 27 & \text{if } n \text{ is odd;} \\ \binom{n}{1} 9 \cdot 4^{n-1} \cdot 5^{(n-1)/2} \mod 27 & \text{if } n \text{ is even.} \end{cases} \tag{134}$$

If $a > 3$, $\overline{u} \equiv 0 \mod 16$. Therefore, $n = 4k$ for some integer k by (134). It follows from (133) that

$$2^a 3^b = \overline{u} = \frac{1}{2\sqrt{5}}\left((9+4\sqrt{5})^{4k} - (9-4\sqrt{5})^{4k}\right) = \frac{1}{2\sqrt{5}}\left((9+4\sqrt{5})^4 - (9-4\sqrt{5})^4\right) \cdot z$$
$$= 2^4 3^2 7 \cdot 23 \cdot z \tag{135}$$

for some integer z but this is absurd. Therefore, $a \leq 3$.

If $b > 2$, $\overline{u} \equiv 0 \mod 27$. Therefore, $n = 6k$ for some integer k by (134). It follows from (133) that

$$2^a 3^b = \overline{u} = \frac{1}{2\sqrt{5}}\left((9+4\sqrt{5})^{6k} - (9-4\sqrt{5})^{6k}\right) = \frac{1}{2\sqrt{5}}\left((9+4\sqrt{5})^{6} - (9-4\sqrt{5})^{6}\right)\cdot z$$
$$= 2^3 3^3 17 \cdot 19 \cdot 107 \cdot z \tag{136}$$

for some integer z but this is absurd. Therefore, $b \leq 2$.

Now, we can solve for v of the equations $u^2 - 5v^2 = \pm 5$ for $u = 2^a \cdot 3^b \cdot 5$ with $0 \leq a \leq 3$, $0 \leq b \leq 2$. The integral solutions to $u^2 - 5v^2 = \pm 5$ are $(u, v) = (5, \pm 2)$, $(20, \pm 9)$ and $(360, \pm 161)$. □

Proposition 5.7. *Let $r \in \mathbb{Q}$ such that $(r^2 - \frac{1}{20})^{-1} = \epsilon \cdot 2^{\alpha+1} \cdot 3^{\beta} \cdot 5^{\gamma}$ for some nonnegative integers α, β, γ and $\epsilon = \pm 1$. Then,*

$$r \in \left\{0, \pm\frac{1}{4}, \pm\frac{1}{5}, \pm\frac{2}{9}, \pm\frac{9}{40}, \pm\frac{161}{720}\right\}. \tag{137}$$

Proof. It is clear that $r = 0$ is a solution. Now, we assume $r = \frac{x}{y}$ where x, y are nonzero coprime integers with $y > 0$. Then

$$\frac{20y^2}{y^2 - 20x^2} = \epsilon \cdot 2^{\alpha+1} \cdot 3^{\beta} \cdot 5^{\gamma}, \quad \text{or} \quad \frac{10y^2}{y^2 - 20x^2} = \epsilon \cdot 2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma}. \tag{138}$$

Thus, $(10 - \epsilon \cdot 2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma}) \cdot y^2 = -\epsilon \cdot 2^{\alpha+2} \cdot 3^{\beta} \cdot 5^{\gamma+1} \cdot x^2$. Since x, y are coprime, $y^2 \mid 2^{\alpha+2} \cdot 3^{b} \cdot 5^{\gamma+1}$ or

$$y = 2^a \cdot 3^b \cdot 5^c \tag{139}$$

for some nonnegative integers a, b, c such that $2a \leq \alpha + 2$, $2b \leq \beta$, $2c \leq \gamma + 1$.

Equations (138) also imply that $y^2 - 20x^2$ divides $10y^2$ and so $y^2 - 20x^2$ divides $200x^2$. Since x, y are coprime, $y^2 - 20x^2$ divides $\gcd(200x^2, 200y^2) = 200$. Note that if $5 \mid (y^2 - 20x^2)$, $5 \mid y$. Since x, y are coprime, $5 \nmid x$ and hence $y^2 - 20x^2 \not\equiv 0 \mod 25$. Therefore, $(y^2 - 20x^2) \mid 40$.

(i) Suppose y is odd. Then $y^2 - 20x^2$ is an odd divisor of 200. Thus, $(y^2 - 20x^2) \mid 5$, and so $(y^2 - 5\overline{x}^2) \mid 5$ where $\overline{x} = 2x$. It follows from Lemma 5.6 that $(y, \overline{x}) = (9, \pm 4)$ or $(5, \pm 2)$. Therefore, $r = \frac{x}{y} = \pm\frac{2}{9}$ or $\pm\frac{1}{5}$.

(ii) Now we assume y is even. Then $y^2 - 20x^2$ is also and x must be odd. Thus, $4 \mid (y^2 - 20x^2)$. If $8 \mid y^2 - 20x^2$, then $2 \mid \left((\frac{y}{2})^2 - 5x^2\right)$ which implies $\frac{y}{2}$ is odd. Therefore, $(\frac{y}{2})^2, 5x^2 \equiv 1 \mod 4$ and so $4 \mid \left((\frac{y}{2})^2 - 5x^2\right)$ or $16 \mid (y^2 - 20x^2)$, which cannot be a divisor of 40. As a consequence, we find

$$y^2 - 20x^2 = \pm 4 \text{ or } \pm 20. \tag{140}$$

Let $\overline{y} = y/2$. These diophantine equations are equivalent to $(\overline{y}^2 - 5x^2) \mid 5$. It follows from Lemma 5.6 that $(\overline{y}, x) = (2, \pm 1)$, $(20, \pm 9)$ or $(360, \pm 161)$. Therefore, $r = \frac{x}{y} = \pm\frac{1}{4}$, $\pm\frac{9}{40}$ or $\pm\frac{161}{720}$. □

Because r has only a finite number of solutions, thus $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ only takes a finite number of possible values, and becomes known. This makes the cases soluble using the methods of inverse-pair of integer conditions discussed in Section 5.6.6.

5.6.9 Go to n-stage

We can go to n-stage from r-stage or u-stage, by considering $K_{ij}, \tilde{K}_{ij}, J_i$ introduced in (100). $K_{ij}, \tilde{K}_{ij}, J_i$ are cyclotomic integers, and those cyclotomic integers are expressed in terms of n_i variables, where the n_i variables are the expansion coefficients of the cyclotomic integers over an integral basis of cyclotomic numbers. Those n_i variables satisfy many conditions, which are listed in the Proposition 5.5.

5.6.10 Solve bounded n_i zero conditions

A n_i zero condition is bounded if it has the following form

$$\text{const.} - \sum_{\{i,j,\cdots} c_{i,j,\cdots} n_i^{a_i} n_j^{a_j} \cdots = 0, \quad a_i, a_j, \cdots \in 2\mathbb{N}, \quad c_{i,j,\cdots} > 0. \tag{141}$$

In this case, n_i that can satisfy the zero condition have a simple finite range. We can test each set of n_i's in this finite range to see which set of n_i's are solutions.

5.6.11 Solve bounded n_i integer conditions

A simple example of bounded n_i integer condition is given by $\frac{n_1^2}{1+n_1+n_1^2} \in \mathbb{Z}$. The integer condition can be reduced to $\frac{-1-n_1}{1+n_1+n_1^2} \in \mathbb{Z}$. The reduced integer condition can be written as $\frac{-\frac{1}{n_1^2}-\frac{1}{n_1}}{\frac{1}{n_1^2}+\frac{1}{n_1}+1} \in \mathbb{Z}$, which has the form that except a constant term in the denominator, all other terms has negative powers. In this case, $|n_1|$ cannot be too big to satisfy the integer condition. Thus we check n_1 in a finite range to see which values satisfy the integer condition.

Let us consider a more general example of integer condition

$$h(n_1, n_2) = \frac{n_1 - n_2}{1 + n_1 + n_1^2 n_2} \in \mathbb{Z}, \tag{142}$$

which can be rewritten as $h(n_1, n_2) = \frac{\frac{1}{n_1 n_2} - \frac{1}{n_1^2}}{\frac{1}{n_1^2 n_2} + \frac{1}{n_1 n_2} + 1} \in \mathbb{Z}$. Except a "1" in the denominator, all other terms have negative exponents, which indicates that the integer condition is a bounded integer condition. We then introduce

$$h_{\max}(n_1, n_2) = \frac{\frac{1}{|n_1 n_2|} + \frac{1}{|n_1^2|}}{-\frac{1}{|n_1^2 n_2|} - \frac{1}{|n_1 n_2|} + 1} \tag{143}$$

which is obtained from $h(n_1, n_2)$ by replace n_i with $|n_i|$, changing the sign of the terms in the numerators to "+", and the sign of the terms in the denominator (except the "1" term) to "−", $h_{\max}$ satisfies $h_{\max}(n_1, n_2) \geq |h(n_1, n_2)|$ if $h_{\max}(n_1, n_2) > 0$.

We note that when $|n_i|$ are large, $0 < h_{\max}(n_1, n_2) < 1$. In this case $h(n_1, n_2)$ cannot be integer except $h(n_1, n_2) = 0$. Thus we only need to check n_1, n_2's in a finite range to see if $h(n_1, n_2)$ is integer or not. More specifically, we test $n_1 = \pm 1, n_2 = \pm 1$. Then $n_1 = \pm 2, n_2 = \pm 1$ and $n_1 = \pm 1, n_2 = \pm 2$, *etc*, until $0 < h_{\max}(n_1, n_2) < 1$. Those values of n_1, n_2 that make $h(n_1, n_2) \in \mathbb{Z}$ give us a set of zero conditions $[n_1 = \alpha_1$ and $n_2 = \alpha_2]$ or $[n_1 = \alpha_1'$ and $n_2 = \alpha_2']$ or $\cdots$, where α_i and α_i' are integral constant.

Since we did not check $n_1 = 0$ or $n_2 = 0$, as well as the possibility that $h(n_1, n_2) = 0$, we need to add those possible cases back to the sets of zero conditions. Thus the final sets of zero conditions are given by $[h(n_1, n_2) = 0]$ or $[n_1 = 0]$ or $[n_2 = 0]$ or $[n_1 = \alpha_1$ and $n_2 = \alpha_2]$ or $[n_1 = \alpha_1'$ and $n_2 = \alpha_2']$ or $\cdots$. Those are the sets of zero conditions obtained from the bounded integer condition $h(n_1, n_2) \in \mathbb{Z}$.

6 Classification of integral modular data

6.1 Mathematical results on integral modular categories

Various approaches to classifying *integral* modular categories have been developed, see, eg. [95–98]. Here we summarize some of these approaches with an eye towards automation.

Here integral means that the FP-dimension $\mathrm{FPdim}(X_i)$ of each simple object is a (necessarily positive) integer. Let $\mathcal{C}$ be an integral MTC. Firstly, by [84, Props. 8.23 and 8.24] every integral category is pseudo-unitary so that one may assume that the dimensions d_i are positive integers (equal to $\mathrm{FPdim}(X_i)$), by adjusting the spherical structure if necessary. Assume that $1 = d_1 \leq d_2 \leq \cdots \leq d_r$ and set $x_{r-i+1} := \frac{\dim(\mathcal{C})}{d_i^2} \in \mathbb{N}$. We have the following:

1. The x_i satisfy the (Egyptian fraction) Diophantine equation
$$1 = \sum_{i=1}^{r} \frac{1}{x_i}. \tag{144}$$
2. By a classical result of Landau [99] eqn (144) has finitely many solutions $(x_1, \ldots, x_r) \in \mathbb{N}^r$ for fixed r. Indeed $k \leq x_k \leq u_k(r-k+1)$ where u_k is Sylvester's sequence defined by $u_1 := 1$ and $u_k := u_{k-1}(u_{k-1}+1)$.
3. For any x_i, x_j we have that $\frac{x_i}{x_j} = \frac{d_j^2}{d_i^2}$ is a square rational number. In particular if for some prime p, $p^k \mid\mid x_i$ and $p^m \mid\mid x_j$ then the parity of k and m are the same (here $p^s \mid\mid b$ means that $p^s \mid b$ and $p^{s+1} \not| b$).
4. Let T be the T-matrix of $\mathcal{C}$, and let $\wp := \{p : p \mid \mathrm{ord}(T)\}$ be the set of primes dividing $\mathrm{ord}(T)$. By the Cauchy theorem [86] $\dim(\mathcal{C}) = \prod_{p\in\wp} p^{\alpha_p}$ where $1 \leq \alpha_p \in \mathbb{N}$. In particular, if $p \mid x_i$ then $p \in \wp$.
5. Suppose that $|\wp| \leq 2$. Then there are at least 2 invertible objects, i.e. $1 = d_1 = d_2$. If p is the minimal prime in $\wp$ then we have $1 = d_1 = \cdots = d_p$. This follows from the fact [100] that any fusion category of dimension $p^a q^b$ for primes p, q is *solvable*, and any solvable fusion category has a non-trivial invertible object.

While the bounds provided by Sylvester's sequence are doubly exponential, in our setting the primes dividing x_i are restricted to the finite set $\wp$. For example, if $\mathrm{ord}(T) = 2^a 3^b$ then $x_i = 2^{a_i} 3^{b_i}$ for a_i, b_i bounded by $\log_2(u_k(r-k+1))$, a much more practical bound. For modest sized r, the known bounds on the primes in $\wp$ make finding solutions to equation (144), restricted to $\wp$, reasonably efficient.

In practice we find that it is frequently the case that $|\wp| \leq 2$, so that we are assured of having invertible objects. For example by [98,101] every *odd dimensional* integral modular category of rank at most 23 must have non-trivial invertible objects–they are all pointed! The smallest rank for which we are aware of an integral modular category with no non-trivial invertible objects is 22: this comes from the Drinfeld center of $\mathcal{R}\mathrm{ep}_{A_5}$. A category is called *perfect* if it has no non-trivial invertible objects.

Generally, a fusion category $\mathcal{C}$ is **G-graded** if there is a decomposition as abelian categories $\mathcal{C} \cong \bigoplus_g \mathcal{C}_g$ such that if $\mathcal{C}_g \otimes \mathcal{C}_h \subset \mathcal{C}_{gh}$, for some group G. The grading is faithful if each $\mathcal{C}_g$ is non-trivial. A useful consequence of a faithful G-grading is that $\dim(\mathcal{C}_g)$ is constant for all g: it is equal to $\dim(\mathcal{C})/|G|$. This induces a partition of the simple objects, and hence the list of dimensions. It is clear that the monoidal unit $\mathbf{1}$ must lie in the trivial component $\mathcal{C}_e$, and therefore if $X \in \mathcal{C}_g$ then $X^* \in \mathcal{C}_{g^{-1}}$. Thus, if every object is self-dual, any faithful grading group must be an elementary abelian 2-group (but not conversely, there could be non-self-dual objects in the trivially-graded component).

For a modular category $\mathcal{C}$ the largest faithful grading group (called the universal grading) is isomorphic to the group of isomorphism classes of invertible simple objects (see [102] for details) so is abelian in this case. Moreover the trivial component with respect to the universal grading is the adjoint subcategory. A natural way to understand this grading is by defining, for $\varphi \in \hat{A}$, $\mathcal{C}_\varphi$ to be the abelian subcategory generated by objects X such that $c_{X,z}c_{z,X} = \varphi(z)Id_{z\otimes X}$ for all $z \in A$.

In general for $\mathcal{D} \subset \mathcal{C}$ ribbon categories, **the centralizer of** $\mathcal{D}$ **in** $\mathcal{C}$, denoted $C_{\mathcal{C}}(\mathcal{D})$, is the subcategory generated by those $Y \in \mathcal{C}$ so that $c_{Y,X}c_{X,Y} = Id_{X\otimes Y}$ for all $X \in \mathcal{D}$. For simple objects X, Y we have that $c_{Y,X}c_{X,Y} = Id_{X\otimes Y}$ if and only if $S_{X,Y} = d_X d_Y$. The centralizer of $\mathcal{D}$ in itself $C_{\mathcal{D}}(\mathcal{D})$ is sometimes denoted $Sym(\mathcal{D})$ (we avoid the somewhat vague notation $\mathcal{D}'$ that is sometimes found in the literature). For a MTC $\mathcal{C}$ it is known that the trivial component $\mathcal{C}_0$ with respect to the universal grading is the *adjoint* subcategory, which is precisely the centralizer $C_{\mathcal{C}_{pt}}(\mathcal{C})$ of the pointed subcategory. In particular, the symmetric center of $\mathcal{C}_0$ is $Sym(\mathcal{C}_0) := \mathcal{C}_{pt} \cap \mathcal{C}_0$ and is both pointed and *symmetric*. Thus each simple object $Z \in \mathcal{C}_{pt} \cap \mathcal{C}_0$ is either bosonic or fermionic, i.e. $\theta_Z = \pm 1$ and $c^2_{Z,Z} = Id$. If $Z \in Sym(\mathcal{C}_0)$ is fermionic so that $\theta_Z = -1$ then for any other object $X \in \mathcal{C}_0$ we have $Z \otimes X \ncong X$. Indeed, by (8) for Z invertible in $\mathcal{C}_0$ with $\theta_Z = -1$ we have $\theta_{Z^*}\theta_X S_{Z^*,X} = d_{Z\otimes X}\theta_{Z\otimes X}$ so that $-\theta_X = \theta_{Z\otimes X}$. We can often use this fact to show that $\mathcal{C}_{pt} \cap \mathcal{C}_0$ is Tannakian, i.e., every simple object is bosonic. For example, if there is only one simple object in $\mathcal{C}_0$ of some dimension d, then there can be no fermionic invertible objects in $\mathcal{C}_0$. We can then condense the bosons in $\mathcal{C}_0$ to obtain a new MTC $\mathcal{D}$ (the modularization of $\mathcal{C}_0$: recall that $C_{\mathcal{C}_0}(\mathcal{C}_0) = \mathcal{C}_{pt} \cap \mathcal{C}_0$ and see [103]) with dimension $\dim(\mathcal{C}_0)/\dim(\mathcal{C}_{pt} \cap \mathcal{C}_0)$. The modularization is a braided tensor functor $F : \mathcal{C}_0 \to \mathcal{D}$. This yields further constraints.

Another valuable fact is the following:

Lemma 6.1. *[104, Theorem 3.2] If $\mathcal{D} \subset \mathcal{C}$ is a subcategory of a modular category $\mathcal{C}$ then $\dim(\mathcal{C}) = \dim(\mathcal{D})\dim(C_{\mathcal{C}}(\mathcal{D}))$.*

Proposition 6.2. *Let $\mathcal{C}$ be an integral MTC and $G(\mathcal{C}) = A$ the group of invertible objects, so that $\mathcal{C}$ has universal grading group A. Then:*

(a) If $\mathcal{C}_0$ has no non-trivial invertible objects then $\mathcal{C}_0$ is itself modular, so that $\mathcal{C} \cong \mathcal{C}_0 \boxtimes \mathcal{C}_{pt}$. In particular $\mathcal{C}$ is not prime if A is non-trivial.

(b) If $\mathcal{C}_0$ has a fermionic invertible object then each d appearing as a dimension of a simple object in $\mathcal{C}_0$ occurs with even multiplicity.

(c) If $\mathcal{C}_0$ has a fermionic invertible then after condensing the maximal Tannakian subcategory of $\mathcal{C}_0$ the result is super-modular. Methods found in [96] can be brought to bear.

(d) If $Sym(\mathcal{C}_0)$ is Tannakian then every dimension d appearing in $\mathcal{C}_i$ for $i \neq 0$ must occur with multiplicity ≥ 2.

(e) If $Sym(\mathcal{C}_0) \cong \hat{G}$ is Tannakian (G is abelian) then the condensation $\mathcal{C}_0$ by G is modular, i.e. $(\mathcal{C}_0)_G = \mathcal{D}$ is modular (the modularization *of $\mathcal{C}_0$ in [103]). Denote by $F : \mathcal{C}_0 \to \mathcal{D}$ this modularization (or G-de-equivariantization) functor. In this case G acts on the simple objects of $\mathcal{C}_0$ by $X \mapsto g \otimes X$. If $Stab_G(X)$ is cyclic (for example if $Stab_G(X)$ is trivial) then $F(X)$ is a direct sum of $|Stab_G(X)|$ pairwise non-isomorphic simple objects [103, Lemme 4.3].*

Proof. Proof of (a): Since $Sym(\mathcal{C}_0)$ is pointed, the hypothesis imply $\mathcal{C}_0$ is modular. By [104, Corollary 3.5] $\mathcal{C} \cong \mathcal{C}_0 \boxtimes \mathcal{C}_{pt}$.

Proof of (b): If a fermionic invertible object $z \in \mathcal{C}_0$ exists and $X \in \mathcal{C}_0$ has dimension d then $z \otimes X \ncong X$: otherwise eqn (8) implies $-\theta_X d_X = \theta_{z^*}\theta_X S_{z^*,X} = d_{z^*\otimes X}\theta_{z^*\otimes X} = d_X\theta_X$, a contradiction.

Proof of (c): This is well-known, see eg. [105].

Proof of (d): Suppose that some $\mathcal{C}_i$ with $i \neq 0$ has rank 1, i.e., has a single simple object Y. Then for any simple $Z \in \mathcal{C}_{pt}$ we have $Z \otimes Y = Y$. Now the balancing equation (8) we have

$$\theta_Y S_{Z,Y} = \theta_Z \theta_{Y^*} S_{Z,Y} = \theta_Y \dim(Y),$$

so that $S_{Z,Y} = \dim(Y)\dim(Z)$ for all $Z \in \mathcal{C}_{pt}$. This implies that $Y \in \mathcal{C}_0$ contrary to assumption.

Proof of (e): This is directly taken from [103]. □

These can be implemented in the computer calculations. To give some idea of how this works we present a few examples by hand. One useful fact is the following:

Lemma 6.3. *If $\mathcal{D}$ is a symmetric pointed category of dimension 2^k with $k \geq 2$ then $\mathcal{D}$ contains an order 2 boson $b \ncong \mathbf{1}$, i.e. an object such that $b^{\otimes 2} \cong \mathbf{1}$ and $\theta_b = 1$.*

Proof. The twist of each simple (invertible) object $\theta_Z = \pm 1$ by (8). If $Z^2 \cong \mathbf{1}$ and $\theta_Z = -1$ for each $Z \ncong \mathbf{1}$ we obtain the (modular) 3 fermion theory [91] as a subcategory, contradicting $\mathcal{D}$ symmetric. So either there is an invertible object W of order 2^s with $s \geq 2$ or a non-trivial invertible object U with $\theta_U = 1$. First consider such a W. the balancing equation (8) gives:

$$1 = (\theta_W)(\theta_{W^*})S_{W,W^*} = \theta_{W^2}$$

so that $U = W^2$ is a nontrivial invertible object with $\theta_U = 1$. Thus we reduce to the second case. Let U be chosen non-trivial with minimal order 2^t. We must show $t = 1$. Again, (8) implies:

$$1 = \theta_U \theta_{U^*} S_{U,U^*} = \theta_{U^2}.$$

Thus by minimality of t we see that $U^2 \cong \mathbf{1}$, so $t = 1$. □

The following lemma is straight-forward, but useful.

Lemma 6.4. *Suppose that the trivial component $\mathcal{C}_0$ of an integral MTC $\mathcal{C}$ has an invertible object b with twist $\theta_b = 1$, and $X \in \mathcal{C}_i$ for $i \neq 0$ is the unique object of dimension d. Then b and X centralize each other, i.e. $S_{b,X} = d$. In particular, if $Sym(\mathcal{C}_0)$ is Tannakian then every component $\mathcal{C}_i$ must have multiple objects of any given dimension d.*

Proof. Since X is the only object in $\mathcal{C}_i$ of dimension d, we have $z \cdot X \cong X$ for any invertible $z \in \mathcal{C}_0$. The balancing equation (8) yields

$$\theta_b \theta_X S_{b,X} = d_{b^*X}\theta_{b^*X} = d\theta_X,$$

so that $S_{b,X} = d$. The second statement follows from the fact that $\mathcal{C}_{pt} = C_{\mathcal{C}}(\mathcal{C}_0)$. □

6.2 Applications to some examples

Later in section 6.3, we use some of the above results and $\mathrm{SL}_2(\mathbb{Z})$ representations to obtain sets of potential quantum dimensions that include all the integral modular data (see Table 7) via an automated computer calculation. In the following, we will apply the above general results trying to rule out some of those sets of potential quantum dimensions. In particular we can deal with all the unrealizable cases in Table 7 directly.

1. Rank 7

   Consider an MTC $\mathcal{C}$ with $\dim(\mathcal{C}) = 16$, with dimensions partitioned via the grading as $[1,1,1,1],[2],[2],[2]$ (cf. Table 7). This category does not exist by the results of [96], which classifies all rank 7 integral categories. To illustrate the power of our methods we will eliminate this case directly, so that our computational techniques plus the following short argument reproduces the main result of [96]. Suppose $\mathcal{C}$ is such a MTC. Since $\mathcal{C}_0$ is a symmetric pointed category it must contain an order 2 boson b by Lemma 6.3. Now by Lemma 6.4 b centralizes the 2 dimensional simple objects, and is thus central in $\mathcal{C}$. This contradicts modularity.

2. Rank 8

   Suppose we have $\mathcal{C}$ with $\dim(\mathcal{C}) = 36$, with dimensions partitioned by gradings being $[1,2,2,3],[1,2,2,3]$. By Prop. 6.2(c) the trivial component must be modular, which does not exist due to [91].

3. Rank 9

   - Suppose $\mathcal{C}$ has $\dim(\mathcal{C}) = 144$, and with dimensions partitioned via the grading as $[1,1,1,1,4,4],[6],[6],[6]$ (cf. Table 7). This can be eliminated similarly as above: let b the boson obtained from Lemma 6.3. Now by Lemma 6.4 b is central in $\mathcal{C}$, contradicting modularity.
   - Suppose $\mathcal{C}$ is a 144-dimensional MTC with simple objects of dimensions

     $$[1,1,3,3,4,6,6,6,6].$$

     Since $\mathcal{C}$ is faithfully $\mathbb{Z}_2$ graded the dimensions of the simple objects in $\mathcal{C}_0$ must be $[1,1,3,3,4,6]$. Now the non-trivial invertible $Z \in \mathcal{C}_0$ must be a boson by Prop. 6.2(b). Now let X_1 be a simple object of dimension 4 and X_2 a simple object of dimension 6. We have that $Z \otimes X_i \cong X_i$ for $i = 1,2$. By Prop. 6.2(e) we see that $F(X_i)$ sums of 2 non-isomorphic simple objects, and thus we obtain 4 simple objects of dimension $2,2,3,3$ in the modularization $\mathcal{D}$. Similarly, the objects Y_1, Y_2 of dimension 3 in $\mathcal{C}_0$ must obey $Z \otimes Y_1 \cong Y_2$ and so $F(Y_1) \cong F(Y_2)$ is a simple object of dimension 3 in $\mathcal{D}$. Finally, $F(Z) \cong F(\mathbf{1})$, so that $\mathcal{D}$ has simple objects of dimension $1,2,2,3,3,3$. But no such modular category exists: it has dimension 36 and so is solvable but only 1 invertible object–a contradiction.
   - Suppose there were a MTC with $\dim(\mathcal{C}) = 288$, and dimensions partitioned via the grading as: $[1,1,3,3,4,6,6,6],[12]$. Since there is a unique object of dimension 4 in the trivial component, the non-trivial invertible object b must be a boson. As the non-trivial component has only one simple, this contradicts Prop. 6.2(d).

4. rank 10

   - Suppose $\mathcal{C}$ is a MTC with $\dim(\mathcal{C}) = 108$ and dimensions partitioned as

     $$[1,1,2,2,2,2,6],[3,3,6].$$

     By Prop. 6.2(b) and (c) this category does not exist.
   - Suppose that $\mathcal{C}$ is a MTC with dimensions partitions as: $[1,1,1,1,2,2,2],[4],[4],[4]$ as in Table 7. By Lemma 6.3 there must be a boson b, which by Lemma 6.4 must centralize all of the simple objects of dimension 4 as well as $\mathcal{C}_0$ contradicting modularity.

5. rank 11

- Suppose $\mathcal{C}$ is a MTC of dimension 144 with dimensions partitioned via the grading as: $[1,1,1,1,4,4],[2,4,4],[6],[6]$ (cf. Table 7). By Lemma 6.3 we have a boson b in the trivial component. Since $b \otimes Y \cong Y$ for each of the objects of dimension 6 they both centralize b. Thus we find that the centralizer $C_{\mathcal{C}}(\langle b\rangle)$ has dimension at least $3 \cdot 6^2$ since b is centralized by $\mathcal{C}_0$ as well. But this contradicts Lemma 6.1 since $\dim(\langle b\rangle) \cdot C_{\mathcal{C}}(\langle b\rangle) \geq 6^3 > 144$.
- Now suppose that $\mathcal{C}$ has dimension 144 with dimensions partitioned as:

$$[1,1,1,3,6],[4,4,4],[4,4,4]$$

(cf. Table 7). Then clearly the trivial component has a Tannakian subcategory equivalent to $\mathcal{R}\mathrm{ep}_{\mathbb{Z}_3}$. By Prop. 6.2(e) the modularization of $\mathcal{C}_0$ is a rank 7 category of dimension 16 with dimensions $[1,1,1,1,2,2,2]$ which we have eliminated already.

6. rank 12

By now the methods are familiar so we quickly eliminate the following (cf. Table 7):

- $[1,1,1,1,2,2,2,2,4],[6],[6],[6]$ cannot occur as the boson afforded by Lemma 6.3 would be central, by Lemma 6.4, contradicting modularity.
- $[[1,1,1,1,4,4],[3,3,3,3],[6],[6]$ cannot occur as the boson afforded by Lemma 6.3 has a centralizer that has dimension more than half of the dimension of the category, contradicting Lemma 6.1.
- $[1,1,1,2,2,2,2,2,2,3],[6],[6]$ cannot occur as Lemma 6.4 implies that the non-trivial invertibles are central, contradicting modularity.

6.3 Computer calculation of Integral modular data

The methods outlined in Section 5 can be used to find both integral and non-integral modular data from $\mathrm{SL}_2(\mathbb{Z})$ representations. The results are summarized in Section 7. However, those methods are less effective to integral modular data. For rank 8 and above, we need to use a different and more effective approach to find integral modular data. The new approach contains a few steps.

In the first step, from a list all possible $\mathrm{SL}_2(\mathbb{Z})$ representations that have potential to produce modular data that we have obtained before, we can obtain a list of prime divisors $\wp$ of $\mathrm{pord}(\rho(\mathfrak{t}))$ of those $\mathrm{SL}_2(\mathbb{Z})$ representations (see Table 5). This information will be used in the next step.

In the second step, we note that $x_i \equiv D^2/d_i^2,\ i = 1,\dots,r$ are integers for integral modular data, which satisfy the Egyptian fraction condition (144).

For each rank r, the number of the lists of Egyptian fractions is finite. We use the following method to obtain all of them. First we assume x_i are ordered $x_1 \leq x_2, \leq \cdots \leq x_r$. Suppose we have known a partial list $x_1,\dots,x_k$. Then the next integer x_{k+1} in the list satisfies $\frac{1}{x_{k+1}} \geq \frac{1}{r-k}(1-\sum_{i=1}^k \frac{1}{x_i})$. Thus

$$x_k \leq x_{k+1} \leq \frac{r-k}{1-\sum_{i=1}^k \frac{1}{x_i}}. \tag{145}$$

We see that the integer x_{k+1} only has a finite range of choices. Since here, x_i's are given by the quantum dimensions of modular tensor category: $x_i = D^2/d_i^2$, the prime divisors of x_{k+1} must be a subset of prime divisors of $\mathrm{pord}(\rho(\mathfrak{t}))$. The range of choices can be greatly reduced by using this property. Since the known prime divisors of $\mathrm{pord}(\rho(\mathfrak{t}))$ form small sets (see Table 5), this allows us to obtain all possible lists of Egyptian fractions, $1/x_i$'s,

Table 5: Sets of prime divisors of pord($\rho(\mathfrak{t})$) of the $SL_2(\mathbb{Z})$ representations that have potential to produce modular data. For each rank, we drop the sets of prime divisors which are contained in some other sets of prime divisors at that rank.

rank	prime divisors
2	[2], [5]
3	[2], [3], [7]
4	[2,5], [3]
5	[2,3], [2,5], [7], [11]
6	[2,3], [2,5], [2,7], [3,5], [3,7], [5,7], [13]
7	[2,3], [2,5], [2,7], [3,5], [3,7], [11]
8	[2,3,5], [2,7], [3,7], [3,11], [13], [17]
9	[2,3,5], [2,3,7], [2,11], [2,13], [3,11], [19]
10	[2,3,5], [2,3,7], [2,11], [2,13], [3,11], [3,13], [5,7], [5,11], [5,13], [7,11], [17]
11	[2,3,5], [2,3,7], [2,3,11], [2,13], [2,17], [3,5,7], [3,13], [5,13], [7,11], [19], [23]
12	[2,3,5], [2,3,7], [2,3,11], [2,3,13], [2,5,7], [2,5,13], [2,17], [3,5,7], [3,7,11], [3,17], [3,19], [5,17]

for each set of prime divisors (of pord($\rho(\mathfrak{t})$)) in Table 5. (Otherwise, the load of computer calculation is too much for rank 11, 12.)

From x_i's we can get the quantum dimensions d_i via

$$d_i = \sqrt{\frac{x_r}{x_i}}, \quad D^2 = \sum_i d_i^2 = x_r. \tag{146}$$

This way we obtain sets of potential quantum dimensions of all integral modular data.

When d_i's are viewed as quantum dimensions of an integral modular data, they must satisfy many additional conditions discussed in Section 6.1. We further reduces the sets of potential quantum dimensions of integral modular data, by using some of those conditions that can be automated by computer:

Lemma 6.5.
- *$d_i = \sqrt{\frac{x_r}{x_i}}$ must an integer.*
- *The number N_{inv} of invertible objects (the number of $d_i = 1$'s) divides $x_r = D^2$. $\mathcal{C}$ can be decomposed as an Abelian category into N_{inv} components, such that each component $\mathcal{C}_i$ has the same total quantum dimension D^2/N_{inv}. Either $\mathcal{C}_0$ contains invertible objects, which must be bosons or fermions, or $\mathcal{C}_0$ is modular and hence $\mathcal{C}$ is not prime.*
- *If $\mathcal{C}_i$ contains invertible objects, then the set of quantum dimensions in $\mathcal{C}_i$ is the same as that in $\mathcal{C}_0$.*
- *Let $N_{\text{inv},0}$ be the number of invertibles in $\mathcal{C}_0$. Let $P_{\text{inv},0}$ be the product of distinct prime factors in $N_{\text{inv},0}$. Then $P_{\text{inv},0}$ divides $d_i N_{d_i}$, where d_i is a quantum dimension in $\mathcal{C}_0$ and N_{d_i} is the number of simple objects with dimension d_i. If one of N_{d_i} is odd, then invertible objects in $\mathcal{C}_0$ must be all bosons, i.e. Sym($\mathcal{C}_0$) is Tannakian.*
- *If $|\wp| < 3$ and Sym($\mathcal{C}_0$) is Tannakian, then $d_{\min}/N_{\text{inv},0} \le 1$, where $d_{\min}$ is the smallest non-unit dimension in $\mathcal{C}_0$.*

Proof. The first two statements are immediate. Let us justify the last three.

Let $Z \in \mathcal{C}_i$ be invertible. Then the map $X \mapsto Z \cdot X$ is a bijection between $\mathcal{C}_0$ and $\mathcal{C}_i$.

Table 6: Sets of prime divisors of D^2 (and of $\mathrm{pord}(\rho(\mathfrak{t}))$) of the reduced sets of the potential quantum dimensions for integral modular data.

rank	prime divisors
2	[2]
3	[3]
4	[2]
5	[5]
6	[2,3]
7	[2], [7]
8	[2], [2,3]
9	[2], [2,3]
10	[2], [2,3], [2,5], [2,3,5]
11	[2], [11], [2,3], [2,5], [2,3,5], [2,3,7]
12	[2,3], [2,3,5], [2,3,7], [2,3,11], [2,3,13]

Let p be a prime dividing $N_{\text{inv},0}$ so that $\mathbb{Z}_p$ acts on the objects of dimension d_i. If $p = 2$ and the corresponding invertible object is a fermion, f then $f \cdot X \ncong X$ for each simple X of dimension d_i so that N_i is even. If $p > 2$ the corresponding invertible object b is bosonic. If $p \nmid d_i$ then $b \cdot X \ncong X$ since if we condense b the object X would split into p objects of dimension d_i/p. Thus the orbit of X under the $\mathbb{Z}_p$ action must have size p, and thus $N_{\text{inv},0}$ is divisible by p.

The given hypotheses imply that $\mathcal{C}$ is solvable, and the de-equivariantization of $\mathcal{C}_0$ by $Sym(\mathcal{C}_0) \cong \mathcal{R}\mathrm{ep}(A)$ is modular and again solvable. Denote this functor by $F : \mathcal{C}_0 \to (\mathcal{C}_0)_A$. By solvability $(\mathcal{C}_0)_A$ has a non-trivial invertible object Z. Such an object must appear as a subobject of $F(X)$ for some non-invertible object X of dimension d, since $F(W) = \mathbf{1}$ for all invertible objects in $\mathcal{C}_0$. Using [103, Prop. 4.4] we find that $1 = \dim(Z) = \frac{\nu_{Z,X}\, d}{|Stab_A(X)|}$ where $Stab_A(X)$ is the X-stabilizer subgroup of A, and $\nu_{Z,X}$ is the multiplicity of Z in $F(X)$. Now $|A| = N_{\text{inv},0}$, so $1 = \frac{\nu_{Z,X}\, d}{|Stab_A(X)|} \geq \frac{d}{N_{\text{inv},0}} \geq \frac{d_{\min}}{N_{\text{inv},0}}$.

□

In Table 6, we list the prime divisors of D^2 of the reduced sets of the potential quantum dimensions. Note that the prime divisors of D^2 coincide with the prime divisors of $\mathrm{pord}(\rho(\mathfrak{t}))$, for a modular data (integral or non-integral).

In the third step, we go back to $\mathrm{SL}_2(\mathbb{Z})$ representations. We find all possible $D_\rho(\sigma)$'s for a $\mathrm{SL}_2(\mathbb{Z})$ representation ρ and examine all possible choices of unit index u. If $D_\rho(\sigma)_{uu} = \pm 1$ for all σ's, then the $\mathrm{SL}_2(\mathbb{Z})$ representation ρ has a potential to produce integral modular data, provided that the prime divisors of $\mathrm{pord}(\rho(\mathfrak{t}))$ is in Table 6 for the given rank. This leads to Table 4, where the possible sets of prime divisors of $\mathrm{pord}(\rho(\mathfrak{t}))$ are further reduced. In this calculation, we also obtain a list of possible $\rho(\mathfrak{t})$'s for each possible set of prime divisors of $\mathrm{pord}(\rho(\mathfrak{t}))$ in Table 4. This information is useful for the next step of calculation.

In the fourth step, we compute all possible fusion rings (described by fusion coefficients N^{ij}_k) from a set of potential quantum dimensions d_i's, via the following equation

$$d_i d_j = \sum_{k=1}^{r} N^{ij}_k d_k. \tag{147}$$

Since $N^{ij}_k \geq 0$, the number of solutions for the above equation is finite. However, for higher ranks, such as for rank 10,11,12, the calculation load is too much. We need to find ways to make the calculation doable.

One trick is to use the symmetry of N_k^{ij}:

$$N_k^{ij} = N_i^{j\bar{k}} = N_{\bar{j}}^{\bar{k}i} = N_j^{\bar{i}k} = N_{\bar{k}}^{\bar{j}\bar{i}} = N_i^{k\bar{j}} = N_k^{ji} = N_{\bar{i}}^{\bar{k}j} = N_{\bar{j}}^{i\bar{k}} = N_j^{k\bar{i}} = N_{\bar{k}}^{\bar{i}\bar{j}} = N_i^{\bar{j}k}, \quad (148)$$

to reduce the number of variable of N_k^{ij}. But to use this trick, we need to consider all possible charge conjugations $i \to \bar{i}$, and solve possible fusion rings for each choice of charge conjugation.

Eq. (147) gives a set of linear equations of the variables. We also have a set quadratic equations of variables from

$$\sum_m N_m^{ij} N_l^{mk} = \sum_n N_l^{in} N_n^{jk}. \quad (149)$$

We first solve the linear equation with minimal number of variables by search since all the variables are bounded integers. During the search, we apply the quadratic equations containing the searching variables to reduce the search. After solved some variables, we then solve the linear equation with minimal number of remaining variables. Repeating this process, we can solve all the variables.

The above trick is not enough. We need more tricks. When the potential quantum dimensions contain multiple 1's, the corresponding modular data will have multiple invertible objects. In this case, the corresponding modular tensor category is graded by an Abelian group whose order is given by the number of invertible objects. For example, $[d_i] = [1, 1, 2, 2, 2, 2, 3, 3]$ is graded as $\mathbb{Z}_2$: $\mathcal{C}_0 = [1, 1, 2, 2, 2, 2]$, $\mathcal{C}_1 = [3, 3]$. In this case the fusion between the invertible objects in the trivial component $\mathcal{C}_0$ is described by an Abelian group. The fusion between the components are also described by an Abelian group. Those properties can be used to reduce the calculation.

When the trivial component $\mathcal{C}_0$ contain more than one invertible object, those invertible objects form a symmetric fusion category. Let us first assume all the invertible objects in $\mathcal{C}_0$ are bosons. Then we can condense the bosons b_p whose order is a prime divisor of N_{inv}, where N_{inv} is the number of invertible objects in $\mathcal{C}_0$. In other words, $(b_p)^p$ is the trivial object if p is a prime divisor of N_{inv}. The group $\mathbb{Z}_p$ acts on the simple objects X of a given fixed dimension d via $X \mapsto b_p \otimes X$.

The condensation of b_p reduces $\mathcal{C}_0$ to a ribbon category $\mathcal{C}'_0$ with several possible dimension arrays $[d'_1, d'_2, \cdots]$. The dimension in $\mathcal{C}'_0$ are obtained in the following way: first, the number of invertible objects in $\mathcal{C}'_0$ is at least $N'_{\text{inv}} = N_{\text{inv}}/p$. If there are p d_i's in $\mathcal{C}_0$ with the same value d', then those p d_i's might be "combined" into a single d' in $\mathcal{C}'_0$. Also, if a quantum dimension d_i in $\mathcal{C}_0$ is divisible by p, then the single d_i in $\mathcal{C}_0$ can "split" into p degenerate d'_{i_j} in $\mathcal{C}'_0$, where $d'_{i_j} = d_i/p$ for $j = 1, \ldots, p$. All the quantum dimensions d_i in $\mathcal{C}_0$ must either combine or split, as describe above. Otherwise, the original quantum dimensions $[d_i]$ do not correspond to any modular tensor category. We can condense all b_p for all the prime divisor of N_{inv}, using the above method.

At the end, we obtain a list of possible condensation products $\mathcal{C}'_0$'s. Then we try to compute the possible fusion rings for each set of $[d'_1, d'_2, \cdots]$. If there is no valid fusion ring for all the possible condensation products $\mathcal{C}'_0$'s, then the original quantum dimensions $[d_i]$ does not corresponds to any modular tensor category where the invertible objects in the trivial components are all bosons.

If the invertible objects in the trivial components contain fermions, then N_{inv} must be even. If N_{inv} is divisible by 4, then b_2 is a boson and we can use the approach described above to obtain a list of possible condensation products $\mathcal{C}'_0$'s. But here we further require d'_i in $\mathcal{C}'_0$ all have an even degeneracy, since the invertible object in $\mathcal{C}'_0$ must contain the uncondensed fermions. If the even N_{inv} is not divisible by 4, then we condensed all the b_p's, except b_2, to obtain a list of possible condensation products. If there is no valid fusion

Table 7: Lists of all potential quantum dimensions for integral modular tensor categories. The quantum dimensions are presented as $[[d_1, d_2, \cdots], [d'_1, d'_2 \cdots], \cdots]$, where $[d_1, d_2, \cdots]$ form the trivial component $\mathcal{C}_0$, $[d'_1, d'_2, \cdots]$ form the first non-trvial component $\mathcal{C}_1$, *etc.* Each set of quantum dimensions has one or more valid fusion rings, but may not correspond to modular data.

rank	quantum dimensions
2	[1], [1]
3	[1], [1], [1]
4	[1], [1], [1], [1]
5	[1], [1], [1], [1], [1]
6	[1], [1], [1], [1], [1], [1]
7	[1], [1], [1], [1], [1], [1], [1]
	[1, 1, 1, 1], [2], [2], [2]
8	[1], [1], [1], [1], [1], [1], [1], [1]
	[1, 1, 2, 2, 2, 2], [3, 3]
9	[1], [1], [1], [1], [1], [1], [1], [1], [1]
	[1, 1, 1, 1, 4, 4], [6], [6], [6]
10	[1], [1], [1], [1], [1], [1], [1], [1], [1], [1]
	[1, 1, 1, 1, 2, 2, 2], [4], [4], [4]
	[1, 1, 1, 3], [2, 2, 2], [2, 2, 2]
11	[1], [1], [1], [1], [1], [1], [1], [1], [1], [1], [1]
	[1, 1, 1, 1, 2], [2, 2], [2, 2], [2, 2]
	[1, 1, 1, 1, 4, 4], [2, 4, 4], [6], [6]
	[1, 1, 1, 3, 6], [4, 4, 4], [4, 4, 4]
12	[1], [1], [1], [1], [1], [1], [1], [1], [1], [1], [1], [1]
	[1, 1, 1, 1, 2, 2, 2, 2, 4], [6], [6], [6]
	[1, 1, 1, 1, 4, 4], [3, 3, 3, 3], [6], [6]
	[1, 1, 1, 2, 2, 2, 2, 2, 2, 3], [6], [6]

ring for all the possible condensation products $\mathcal{C}'_0$'s, then the original quantum dimensions $[d_i]$ does not corresponds to any modular tensor category where the invertible objects in the trivial components contains fermions.

The above condensation consideration is very effective in ruling out many invalid potential sets of quantum dimensions. However, if the quantum dimensions $[d_1, d_2, \cdots]$ contain only a single invertible object (*i.e.* the unit), the corresponding modular tensor category will be called perfect. In this case, our above tricks will not apply. For rank 11 and 12, the computation of the fusion rings from the perfect quantum dimensions is too much for a current desktop computer to handle.

To solve this problem, we note that for each set of quantum dimensions, d_i's, we know D^2 and its prime divisors. For such a set of prime divisors, we know a list of all possible $\rho(\mathrm{t})$'s, and the corresponding twists $\widetilde{\theta}_i$'s, that may give rise to integral modular data. However, we do not know the matching between d_i's and $\widetilde{\theta}_i$'s. Thus, we need to find all permutations in the indices, p's, such that $(d_i, \widetilde{\theta}_{p(i)})$ are the quantum dimension and the twist of an object in a modular tensor category. In particular, the permutation p must satisfy

$$D\mathrm{e}^{\mathrm{i}\phi} = \sum_i d_i^2 \widetilde{\theta}_{p(i)}. \tag{150}$$

For all possible $\rho(\mathfrak{t})$'s, we search all possible permutations p to satisfy the above condition. We can reduce the search by noticing that the twists related by Galois conjugation, $\widetilde{\theta}_i \to \sigma^2 \widetilde{\theta}_i = \widetilde{\theta}_{\hat{\sigma}(i)}$, correspond to the same value of quantum dimensions.

If no valid permutations can be found for all possible $\rho(\mathfrak{t})$'s, then the set of quantum dimensions, d_i's, does not correspond to any valid integral modular data. This condition is very effective in ruling out many perfect quantum dimensions. The remaining sets of quantum dimensions that have not been ruled out are listed in Table 7. We manage to obtain all the fusion rings for each set of quantum dimensions in Table 7.

In the fifth step, we compute the possible topological spins s_i ($\mathrm{e}^{\mathrm{i}2\pi s_i}$ are eigenvalues of the T-matrix) from the obtained fusion ring N_k^{ij} via [92, 106–108]

$$\sum_r V_{ijkl}^r s_r = 0 \text{ mod } 1 \tag{151}$$

where

$$V_{ijkl}^r = N_r^{ij} N_{\bar{r}}^{kl} + N_r^{il} N_{\bar{r}}^{jk} + N_r^{ik} N_{\bar{r}}^{jl} - (\delta_{ir} + \delta_{jr} + \delta_{kr} + \delta_{lr}) \sum_m N_m^{ij} N_{\bar{m}}^{kl}. \tag{152}$$

There are many sets of solutions of s_i's, which can be calculated via the Smith normal form of above V-matrix. For each set of solution, s_i's, we can calculate the S-matrix via (8). We then check if the resulting S, T form a valid modular data. This way, we obtain a complete list of integral modular data, for rank 12 and below. The results are summarized in Section 7. We remark that our results coincide with those of [109] for rank 13 and below, although our methods are somewhat different.

7 Lists of modular data by Galois orbits

By a theorem of Mueger, if $\mathcal{C} \subset \mathcal{D}$ are both modular categories then we have a factorization $\mathcal{C} \cong \mathcal{D} \boxtimes \mathcal{D}'$ where $\mathcal{D}'$ is another modular subcategory. If $\mathcal{D}$ has no modular subcategories it is said to be *prime*. In this section we list all the prime modular data for ranks 7,8 9, 10, 11 and 12. To save space, we group the modular data by Galois orbits generated by Galois conjugations. We only list one representative for each Galois orbit. If a Galois orbit contains unitary modular data (defined by quantum dimensions $d_i \geq 1$), we will choose the representative to be an unitary one.

A grey entry means that the Galois orbit and the previous Galois orbit are connected by a change of spherical structure (*i.e.* the two Galois orbits each contain a modular data, such that the two modular data are connected by a change of spherical structure). If a Galois orbit contains no unitary modular data, and one of the non-unitary modular data is pseudo unitary, we will drop this Galois orbit. Such a Galois orbit is connected to a Galois orbit with unitary modular data. If a Galois orbit contains a modular data that is not prime, we will also drop this Galois orbit. The resulting lists are given below. The list is ordered by D^2.

In the list, the T-matrix of the modular data is presented as $(s_0, s_1, \cdots)$ where $T_{ii} = \mathrm{e}^{\mathrm{i}2\pi s_i}$, $0 \leq s_i < 1$. The S-matrix is presented as $(S_{11}, S_{12}, S_{13}, \cdots; \quad S_{22}, S_{23}, \cdots)$. The matrix elements of S are given in terms of the following cyclotomic numbers:

$$\begin{aligned}
\zeta_n^m &= e^{2\pi i m/n}, \quad c_n^m = \zeta_n^m + \zeta_n^{-m}, \quad s_n^m = \zeta_n^m - \zeta_n^{-m},\\
\xi_n^m &= \xi_n^{m,1}, \quad \eta_n^m = \eta_n^{m,1}, \quad \chi_n^m = \chi_n^{m,1}, \quad \lambda_n^m = \lambda_n^{m,1},\\
\xi_n^{m,l} &= (\zeta_{2n}^m - \zeta_{2n}^{-m})/(\zeta_{2n}^l - \zeta_{2n}^{-l}), \quad \eta_n^{m,l} = (\zeta_{2n}^m + \zeta_{2n}^{-m})/(\zeta_{2n}^l + \zeta_{2n}^{-l}),\\
\chi_n^{m,l} &= (\zeta_{2n}^m + \zeta_{2n}^{-m})/(\zeta_{2n}^l - \zeta_{2n}^{-l}), \quad \lambda_n^{m,l} = (\zeta_{2n}^m - \zeta_{2n}^{-m})/(\zeta_{2n}^l + \zeta_{2n}^{-l}).
\end{aligned} \tag{153}$$

Each modular data in the list is labeled by $r_{c,D^2}^{\text{ord}(T),\text{fp}}$. For example, $7_{2,7.}^{7,892}$ labels a modular tensor category with rank $r = 7$, chiral central charge $c = 2$, total quantum dimension $D^2 = 7.0$, order-T ord$(T) = 7$, and finger print fp $= 892$. Here the "finger print" is given by the first three digits of $|\sum_i (s_i^2 - \frac{1}{4})d_i|$, so that distinct modular tensor categories are more likely to have distinct labels.

In the table, we also list the realizations of each modular data. Usually, a realization is given by the modular tensor category of Kac-Moody algebra. For example, $SU(5)_5$ is the modular tensor category of $SU(5)$ level 5 Kac-Moody algebra. $PSU(3)_5$ is the modular tensor category that is the non-pointed Deligne factor of $SU(3)_5$, *i.e.* $SU(3)_5 = PSU(3)_5 \boxtimes \mathcal{C}(\mathbb{Z}_5, q)$. We also use O_n to represents the modular tensor category of the $U(1)_{2n}$ orbifold [110]. The modular tensor category from twisted quantum double is labeled by $\mathcal{D}^\omega(G)$, where G is a finite group and ω in the cocycle twist. The modular tensor category from twisted Haagerup-Izumi modular data is labeled by Haag$(n)_m$, $m = 0, \pm1, \cdots, \pm n$ [111]. Also many modular data are realized as Abelian anyon condensations [79] of the modular tensor category from Kac-Moody algebra and/or twisted quantum double. Two constructions closely related to Abelian anyon condensations called *zesting* [112, 113] and the *condensed fiber product* [114] are also useful. Most of the potential modular data in the lists are realized by modular tensor categories, and are indeed modular data. There are a few potential modular data whose realizations are not known or not sure, which will be discussed on Section 8.4.

7.1 Rank 7

1. $7_{2,7.}^{7,892}$: $d_i = (1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0)$

   $D^2 = 7.0 = 7$

   $T = (0, \frac{1}{7}, \frac{1}{7}, \frac{2}{7}, \frac{2}{7}, \frac{4}{7}, \frac{4}{7})$,

   $S = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1;\ \ -\zeta_{14}^3,\ \zeta_7^2,\ -\zeta_{14}^5,\ \zeta_7^1,\ -\zeta_{14}^1,\ \zeta_7^3;\ \ -\zeta_{14}^3,\ \zeta_7^1,\ -\zeta_{14}^5,\ \zeta_7^3,\ -\zeta_{14}^1;\ \ \zeta_7^3,\ -\zeta_{14}^1,\ \zeta_7^2,\ -\zeta_{14}^3;\ \ \zeta_7^3,\ -\zeta_{14}^3,\ \zeta_7^2;\ \ -\zeta_{14}^5,\ \zeta_7^1;\ \ -\zeta_{14}^5)$

   Realization: $U(7)_1$.

2. $7_{\frac{27}{4},27.31}^{32,396}$: $d_i = (1.0, 1.0, 1.847, 1.847, 2.414, 2.414, 2.613)$

   $D^2 = 27.313 = 16 + 8\sqrt{2}$

   $T = (0, \frac{1}{2}, \frac{1}{32}, \frac{1}{32}, \frac{1}{4}, \frac{3}{4}, \frac{21}{32})$,

   $S = (1,\ 1,\ c_{16}^1,\ c_{16}^1,\ 1+\sqrt{2},\ 1+\sqrt{2},\ c_{16}^1+c_{16}^3;\ \ 1,\ -c_{16}^1,\ -c_{16}^1,\ 1+\sqrt{2},\ 1+\sqrt{2},\ -c_{16}^1-c_{16}^3;\ \ (-c_{16}^1-c_{16}^3)\mathrm{i},\ (c_{16}^1+c_{16}^3)\mathrm{i},\ -c_{16}^1,\ c_{16}^1,\ 0;\ \ (-c_{16}^1-c_{16}^3)\mathrm{i},\ -c_{16}^1,\ c_{16}^1,\ 0;\ \ -1,\ -1,\ c_{16}^1+c_{16}^3;\ \ -1,\ -c_{16}^1-c_{16}^3;\ \ 0)$

   Realization: Abelian anyon condensation of $SU(6)_2$ or $Sp(12)_1$ or $\overline{SU(2)_6}$. One of 4 $\mathbb{Z}_2$-zestings of the spin modular category $\overline{SU(2)_6}$, see [?].

3. $7_{\frac{9}{4},27.31}^{32,918}$: $d_i = (1.0, 1.0, 1.847, 1.847, 2.414, 2.414, 2.613)$

   $D^2 = 27.313 = 16 + 8\sqrt{2}$

   $T = (0, \frac{1}{2}, \frac{3}{32}, \frac{3}{32}, \frac{1}{4}, \frac{3}{4}, \frac{15}{32})$,

   $S = (1,\ 1,\ c_{16}^1,\ c_{16}^1,\ 1+\sqrt{2},\ 1+\sqrt{2},\ c_{16}^1+c_{16}^3;\ \ 1,\ -c_{16}^1,\ -c_{16}^1,\ 1+\sqrt{2},\ 1+\sqrt{2},\ -c_{16}^1-c_{16}^3;\ \ c_{16}^1+c_{16}^3,\ -c_{16}^1-c_{16}^3,\ c_{16}^1,\ -c_{16}^1,\ 0;\ \ c_{16}^1+c_{16}^3,\ c_{16}^1,\ -c_{16}^1,\ 0;\ \ -1,\ -1,\ -c_{16}^1-c_{16}^3;\ \ -1,\ c_{16}^1+c_{16}^3;\ \ 0)$

Realization: Abelian anyon condensation of $SU(2)_6$ or $\overline{SU(6)_2}$ or $\overline{Sp(12)_1}$. One of 4 $\mathbb{Z}_2$-zestings of the spin modular category $\overline{SU(2)_6}$, see [?].

4. $7^{32,159}_{\frac{31}{4},27.31}$: $d_i = (1.0, 1.0, 1.847, 1.847, 2.414, 2.414, 2.613)$

$D^2 = 27.313 = 16 + 8\sqrt{2}$

$T = (0, \frac{1}{2}, \frac{5}{32}, \frac{5}{32}, \frac{1}{4}, \frac{3}{4}, \frac{25}{32})$,

$S = (1,\ 1,\ c^1_{16},\ c^1_{16},\ 1+\sqrt{2},\ 1+\sqrt{2},\ c^1_{16}+c^3_{16};\ \ 1,\ -c^1_{16},\ -c^1_{16},\ 1+\sqrt{2},\ 1+\sqrt{2},\ -c^1_{16}-c^3_{16};\ \ -c^1_{16}-c^3_{16},\ c^1_{16}+c^3_{16},\ -c^1_{16},\ c^1_{16},\ 0;\ \ -c^1_{16}-c^3_{16},\ -c^1_{16},\ c^1_{16},\ 0;\ \ -1,\ -1,\ c^1_{16}+c^3_{16};\ \ -1,\ -c^1_{16}-c^3_{16};\ \ 0)$

Realization: Abelian anyon condensation of $SU(6)_2$ or $Sp(12)_1$ or $\overline{SU(2)_6}$. One of 4 $\mathbb{Z}_2$-zestings of the spin modular category $\overline{SU(2)_6}$, see [?].

5. $7^{32,427}_{\frac{13}{4},27.31}$: $d_i = (1.0, 1.0, 1.847, 1.847, 2.414, 2.414, 2.613)$

$D^2 = 27.313 = 16 + 8\sqrt{2}$

$T = (0, \frac{1}{2}, \frac{7}{32}, \frac{7}{32}, \frac{1}{4}, \frac{3}{4}, \frac{19}{32})$,

$S = (1,\ 1,\ c^1_{16},\ c^1_{16},\ 1+\sqrt{2},\ 1+\sqrt{2},\ c^1_{16}+c^3_{16};\ \ 1,\ -c^1_{16},\ -c^1_{16},\ 1+\sqrt{2},\ 1+\sqrt{2},\ -c^1_{16}-c^3_{16};\ \ (-c^1_{16}-c^3_{16})\mathrm{i},\ (c^1_{16}+c^3_{16})\mathrm{i},\ c^1_{16},\ -c^1_{16},\ 0;\ \ (-c^1_{16}-c^3_{16})\mathrm{i},\ c^1_{16},\ -c^1_{16},\ 0;\ \ -1,\ -1,\ -c^1_{16}-c^3_{16};\ \ -1,\ c^1_{16}+c^3_{16};\ \ 0)$

Realization: Abelian anyon condensation of $SU(2)_6$ or $\overline{SU(6)_2}$ or $\overline{Sp(12)_1}$. One of 4 $\mathbb{Z}_2$-zestings of the spin modular category $\overline{SU(2)_6}$, see [?].

6. $7^{56,139}_{2,28.}$: $d_i = (1.0, 1.0, 2.0, 2.0, 2.0, 2.645, 2.645)$

$D^2 = 28.0 = 28$

$T = (0, 0, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}, \frac{1}{8}, \frac{5}{8})$,

$S = (1,\ 1,\ 2,\ 2,\ 2,\ \sqrt{7},\ \sqrt{7};\ \ 1,\ 2,\ 2,\ 2,\ -\sqrt{7},\ -\sqrt{7};\ \ 2c^2_7,\ 2c^1_7,\ 2c^3_7,\ 0,\ 0;\ \ 2c^3_7,\ 2c^2_7,\ 0,\ 0;\ \ 2c^1_7,\ 0,\ 0;\ \ \sqrt{7},\ -\sqrt{7};\ \ \sqrt{7})$

Realization: $\overline{SO(7)_2}$ or Abelian anyon condensation of O_7.

7. $7^{56,680}_{2,28.}$: $d_i = (1.0, 1.0, 2.0, 2.0, 2.0, 2.645, 2.645)$

$D^2 = 28.0 = 28$

$T = (0, 0, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}, \frac{3}{8}, \frac{7}{8})$,

$S = (1,\ 1,\ 2,\ 2,\ 2,\ \sqrt{7},\ \sqrt{7};\ \ 1,\ 2,\ 2,\ 2,\ -\sqrt{7},\ -\sqrt{7};\ \ 2c^2_7,\ 2c^1_7,\ 2c^3_7,\ 0,\ 0;\ \ 2c^3_7,\ 2c^2_7,\ 0,\ 0;\ \ 2c^1_7,\ 0,\ 0;\ \ -\sqrt{7},\ \sqrt{7};\ \ -\sqrt{7})$

Realization: Abelian anyon condensation or $\mathbb{Z}_2$-zesting of $SO(7)_2$ or $\overline{O}_7$.

8. $7^{15,205}_{\frac{32}{5},86.75}$: $d_i = (1.0, 1.956, 2.827, 3.574, 4.165, 4.574, 4.783)$

$D^2 = 86.750 = 30 + 15c^1_{15} + 15c^2_{15} + 15c^3_{15}$

$T = (0, \frac{1}{5}, \frac{13}{15}, 0, \frac{3}{5}, \frac{2}{3}, \frac{1}{5})$,

$S = (1,\ -c^7_{15},\ \xi^3_{15},\ \xi^{11}_{15},\ \xi^5_{15},\ \xi^9_{15},\ \xi^7_{15};\ \ -\xi^{11}_{15},\ \xi^9_{15},\ -\xi^7_{15},\ \xi^5_{15},\ -\xi^3_{15},\ 1;\ \ \xi^9_{15},\ \xi^3_{15},\ 0,\ -\xi^3_{15},\ -\xi^9_{15};\ \ 1,\ -\xi^5_{15},\ \xi^9_{15},\ c^7_{15};\ \ -\xi^5_{15},\ 0,\ \xi^5_{15};\ \ -\xi^9_{15},\ \xi^3_{15};\ \ -\xi^{11}_{15})$

Realization: $PSU(2)_{13}$.

9. $7^{8,230}_{1,93.25}$: $d_i = (1.0, 2.414, 2.414, 3.414, 3.414, 4.828, 5.828)$

$D^2 = 93.254 = 48 + 32\sqrt{2}$

$T = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{5}{8}, 0)$,

$S = (1,\ 1+\sqrt{2},\ 1+\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2},\ 2+2\sqrt{2},\ 3+2\sqrt{2};\ \ -1-2\zeta^1_8-2\zeta^2_8,\ -1-2\zeta^{-1}_8+2\zeta^2_8,\ (-2-\sqrt{2})\mathrm{i},\ (2+\sqrt{2})\mathrm{i},\ 2+2\sqrt{2},\ -1-\sqrt{2};\ \ -1-2\zeta^1_8-2\zeta^2_8,\ (2+\sqrt{2})\mathrm{i},\ (-2-\sqrt{2})\mathrm{i},\ 2+2\sqrt{2},\ -1-\sqrt{2};\ \ (2+2\sqrt{2})\zeta^3_8,\ (-2-2\sqrt{2})\zeta^1_8,\ 0,\ 2+\sqrt{2};\ \ (2+2\sqrt{2})\zeta^3_8,\ 0,\ 2+\sqrt{2};\ \ 0,\ -2-2\sqrt{2};\ \ 1)$

Realization: $PSU(3)_5$.

10. $7^{11,157}_{\frac{30}{11},135.7}$: $d_i = (1.0, 2.918, 3.513, 3.513, 4.601, 5.911, 6.742)$

$D^2 = 135.778 = 55 + 44c^1_{11} + 33c^2_{11} + 22c^3_{11} + 11c^4_{11}$

$T = (0, \frac{1}{11}, \frac{4}{11}, \frac{4}{11}, \frac{3}{11}, \frac{6}{11}, \frac{10}{11})$,

$S = (1,\ 2+c_{11}^{1}+c_{11}^{2}+c_{11}^{3}+c_{11}^{4},\ \xi_{11}^{5},\ \xi_{11}^{5},\ 2+2c_{11}^{1}+c_{11}^{2}+c_{11}^{3}+c_{11}^{4},\ 2+2c_{11}^{1}+c_{11}^{2}+c_{11}^{3},\ 2+2c_{11}^{1}+2c_{11}^{2}+c_{11}^{3};\ \ 2+2c_{11}^{1}+2c_{11}^{2}+c_{11}^{3},\ -\xi_{11}^{5},\ -\xi_{11}^{5},\ 2+2c_{11}^{1}+c_{11}^{2}+c_{11}^{3},\ 1,\ -2-2c_{11}^{1}-c_{11}^{2}-c_{11}^{3}-c_{11}^{4};\ \ s_{11}^{2}+2\zeta_{11}^{3}-\zeta_{11}^{-3}+\zeta_{11}^{4}+\zeta_{11}^{5},\ -1-c_{11}^{1}-2\zeta_{11}^{2}-2\zeta_{11}^{3}+\zeta_{11}^{-3}-\zeta_{11}^{4}-\zeta_{11}^{5},\ \xi_{11}^{5},\ -\xi_{11}^{5},\ \xi_{11}^{5};\ \ s_{11}^{2}+2\zeta_{11}^{3}-\zeta_{11}^{-3}+\zeta_{11}^{4}+\zeta_{11}^{5},\ \xi_{11}^{5},\ -\xi_{11}^{5},\ \xi_{11}^{5};\ \ -2-c_{11}^{1}-c_{11}^{2}-c_{11}^{3}-c_{11}^{4},\ -2-2c_{11}^{1}-2c_{11}^{2}-c_{11}^{3},\ 1;\ \ 2+2c_{11}^{1}+c_{11}^{2}+c_{11}^{3}+c_{11}^{4},\ 2+c_{11}^{1}+c_{11}^{2}+c_{11}^{3}+c_{11}^{4};\ \ -2-2c_{11}^{1}-c_{11}^{2}-c_{11}^{3})$

Realization: $PSO(10)_3$

7.2 Rank 8

1. $8_{1,8.}^{16,123}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0)$

   $D^2 = 8.0 = 8$

   $T = (0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{9}{16}, \frac{9}{16})$,

   $S = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1;\ \ 1,\ 1,\ 1,\ -1,\ -1,\ -1,\ -1;\ \ -1,\ -1,\ -\mathrm{i},\ \mathrm{i},\ -\mathrm{i},\ \mathrm{i};\ \ -1,\ \mathrm{i},\ -\mathrm{i},\ \mathrm{i},\ -\mathrm{i};\ \ -\zeta_8^3,\ \zeta_8^1,\ \zeta_8^3,\ -\zeta_8^1;\ \ -\zeta_8^3,\ -\zeta_8^1,\ \zeta_8^3;\ \ -\zeta_8^3,\ \zeta_8^1;\ \ -\zeta_8^3)$

   Realization: $U(8)_1$.

2. $8_{0,36.}^{6,213}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$

   $D^2 = 36.0 = 36$

   $T = (0, 0, 0, 0, \frac{1}{3}, \frac{2}{3}, 0, \frac{1}{2})$,

   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ 4,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ 4,\ -2,\ -2,\ 0,\ 0;\ \ -2,\ 4,\ 0,\ 0;\ \ -2,\ 0,\ 0;\ \ 3,\ -3;\ \ 3)$

   Realization: $\mathcal{D}(S_3)$

3. $8_{4,36.}^{6,102}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$

   $D^2 = 36.0 = 36$

   $T = (0, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 0, \frac{1}{2})$,

   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ -2,\ 4,\ -2,\ -2,\ 0,\ 0;\ \ -2,\ -2,\ -2,\ 0,\ 0;\ \ -2,\ 4,\ 0,\ 0;\ \ -2,\ 0,\ 0;\ \ -3,\ 3;\ \ -3)$

   Realization: condensation reductions of $\mathcal{Z}(\mathcal{NG}(\mathbb{Z}_3\times\mathbb{Z}_3, 0))$ (non-group-theoretical, [115]).

4. $8_{0,36.}^{12,101}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$

   $D^2 = 36.0 = 36$

   $T = (0, 0, 0, 0, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4})$,

   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ 4,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ 4,\ -2,\ -2,\ 0,\ 0;\ \ -2,\ 4,\ 0,\ 0;\ \ -2,\ 0,\ 0;\ \ -3,\ 3;\ \ -3)$

   Realization: $\mathcal{D}^3(S_3)$.

5. $8_{4,36.}^{12,972}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$

   $D^2 = 36.0 = 36$

   $T = (0, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4})$,

   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ -2,\ 4,\ -2,\ -2,\ 0,\ 0;\ \ -2,\ -2,\ -2,\ 0,\ 0;\ \ -2,\ 4,\ 0,\ 0;\ \ -2,\ 0,\ 0;\ \ 3,\ -3;\ \ 3)$

   Realization: condensation reductions of $\mathcal{Z}(\mathcal{NG}(\mathbb{Z}_3\times\mathbb{Z}_3, 0))$ (non-group-theoretical, [115]).

6. $8_{0,36.}^{18,162}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$

   $D^2 = 36.0 = 36$

   $T = (0, 0, 0, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, 0, \frac{1}{2})$,

   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ 4,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ 2c_9^2,\ 2c_9^4,\ 2c_9^1,\ 0,\ 0;\ \ 2c_9^1,\ 2c_9^2,\ 0,\ 0;\ \ 2c_9^4,\ 0,\ 0;\ \ 3,\ -3;\ \ 3)$

   Realization: $\mathcal{D}^4(S_3)$ or $SO(9)_2$.

7. $8_{0,36.}^{36,495}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$

   $D^2 = 36.0 = 36$

   $T = (0, 0, 0, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{1}{4}, \frac{3}{4})$,

$S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ 4,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ 2c^2_9,\ 2c^4_9,\ 2c^1_9,\ 0,\ 0;\ \ 2c^1_9,\ 2c^2_9,\ 0,\ 0;\ \ 2c^4_9,\ 0,\ 0;\ \ -3,\ 3;\ \ -3)$

Realization: $\mathcal{D}^1(S_3)$.

8. $8^{17,152}_{\frac{62}{17},125.8}$: $d_i = (1.0,\ 1.965,\ 2.864,\ 3.666,\ 4.342,\ 4.871,\ 5.234,\ 5.418)$

$D^2 = 125.874 = 36 + 28c^1_{17} + 21c^2_{17} + 15c^3_{17} + 10c^4_{17} + 6c^5_{17} + 3c^6_{17} + c^7_{17}$

$T = (0,\ \frac{5}{17},\ \frac{2}{17},\ \frac{8}{17},\ \frac{6}{17},\ \frac{13}{17},\ \frac{12}{17},\ \frac{3}{17}),$

$S = (1,\ -c^8_{17},\ \xi^3_{17},\ \xi^{13}_{17},\ \xi^5_{17},\ \xi^{11}_{17},\ \xi^7_{17},\ \xi^9_{17};\ \ -\xi^{13}_{17},\ \xi^{11}_{17},\ -\xi^9_{17},\ \xi^7_{17},\ -\xi^5_{17},\ \xi^3_{17},\ -1;\ \ \xi^9_{17},\ \xi^5_{17},\ -c^8_{17},\ -1,\ -\xi^{13}_{17},\ -\xi^7_{17};\ \ -1,\ -\xi^3_{17},\ \xi^7_{17},\ -\xi^{11}_{17},\ -c^8_{17};\ \ -\xi^9_{17},\ -\xi^{13}_{17},\ 1,\ \xi^{11}_{17};\ \ c^8_{17},\ \xi^9_{17},\ -\xi^3_{17};\ \ -c^8_{17},\ -\xi^5_{17};\ \ \xi^{13}_{17})$

Realization: $PSU(2)_{15}$.

9. $8^{13,370}_{\frac{36}{13},223.6}$: $d_i = (1.0,\ 2.941,\ 4.148,\ 4.148,\ 4.712,\ 6.209,\ 7.345,\ 8.55)$

$D^2 = 223.689 = 78 + 65c^1_{13} + 52c^2_{13} + 39c^3_{13} + 26c^4_{13} + 13c^5_{13}$

$T = (0,\ \frac{1}{13},\ \frac{8}{13},\ \frac{8}{13},\ \frac{3}{13},\ \frac{6}{13},\ \frac{10}{13},\ \frac{2}{13}),$

$S = (1,\ 2+c^1_{13}+c^2_{13}+c^3_{13}+c^4_{13}+c^5_{13},\ \xi^7_{13},\ \xi^7_{13},\ 2+2c^1_{13}+c^2_{13}+c^3_{13}+c^4_{13}+c^5_{13},\ 2+2c^1_{13}+c^2_{13}+c^3_{13}+c^4_{13},\ 2+2c^1_{13}+2c^2_{13}+c^3_{13}+c^4_{13},\ 2+2c^1_{13}+2c^2_{13}+c^3_{13};\ \ 2+2c^1_{13}+2c^2_{13}+c^3_{13}+c^4_{13},\ -\xi^7_{13},\ -\xi^7_{13},\ 2+2c^1_{13}+2c^2_{13}+c^3_{13},\ 2+2c^1_{13}+c^2_{13}+c^3_{13}+c^4_{13}+c^5_{13},\ -1,\ -2-2c^1_{13}-c^2_{13}-c^3_{13}-c^4_{13};\ \ -1-c^1_{13}-c^2_{13}+c^5_{13},\ 2+2c^1_{13}+2c^2_{13}+c^3_{13}-c^5_{13},\ \xi^7_{13},\ -\xi^7_{13},\ \xi^7_{13},\ -\xi^7_{13};\ \ -1-c^1_{13}-c^2_{13}+c^5_{13},\ \xi^7_{13},\ -\xi^7_{13},\ \xi^7_{13},\ -\xi^7_{13};\ \ 1,\ -2-2c^1_{13}-2c^2_{13}-c^3_{13}-c^4_{13},\ -2-2c^1_{13}-c^2_{13}-c^3_{13}-c^4_{13},\ 2+c^1_{13}+c^2_{13}+c^3_{13}+c^4_{13}+c^5_{13};\ \ -2-c^1_{13}-c^2_{13}-c^3_{13}-c^4_{13}-c^5_{13},\ 2+2c^1_{13}+2c^2_{13}+c^3_{13},\ 1;\ \ -2-c^1_{13}-c^2_{13}-c^3_{13}-c^4_{13}-c^5_{13},\ -2-2c^1_{13}-c^2_{13}-c^3_{13}-c^4_{13}-c^5_{13};\ \ 2+2c^1_{13}+2c^2_{13}+c^3_{13}+c^4_{13})$

Realization: $PSO(12)_3$.

10. $8^{15,440}_{4,308.4}$: $d_i = (1.0,\ 5.854,\ 5.854,\ 5.854,\ 5.854,\ 6.854,\ 7.854,\ 7.854)$

$D^2 = 308.434 = \frac{315+135\sqrt{5}}{2}$

$T = (0, 0, 0, \frac{1}{3}, \frac{2}{3}, 0, \frac{2}{5}, \frac{3}{5}),$

$S = (1,\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{7+3\sqrt{5}}{2},\ \frac{9+3\sqrt{5}}{2},\ \frac{9+3\sqrt{5}}{2};\ \ -5-3\sqrt{5},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ -5-3\sqrt{5},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ \frac{5+3\sqrt{5}}{2},\ -5-3\sqrt{5},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ \frac{5+3\sqrt{5}}{2},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ 1,\ \frac{9+3\sqrt{5}}{2},\ \frac{9+3\sqrt{5}}{2};\ \ \frac{3+3\sqrt{5}}{2},\ -6-3\sqrt{5};\ \ \frac{3+3\sqrt{5}}{2})$

Realization: condensation reductions of $\mathcal{Z}(\mathcal{NG}(\mathbb{Z}_5, 5))$ [116].

11. $8^{15,100}_{0,308.4}$: $d_i = (1.0,\ 5.854,\ 5.854,\ 5.854,\ 5.854,\ 6.854,\ 7.854,\ 7.854)$

$D^2 = 308.434 = \frac{315+135\sqrt{5}}{2}$

$T = (0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 0, \frac{1}{5}, \frac{4}{5}),$

$S = (1,\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{7+3\sqrt{5}}{2},\ \frac{9+3\sqrt{5}}{2},\ \frac{9+3\sqrt{5}}{2};\ \ \frac{5+3\sqrt{5}}{2},\ -5-3\sqrt{5},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ \frac{5+3\sqrt{5}}{2},\ -5-3\sqrt{5},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ \frac{5+3\sqrt{5}}{2},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ 1,\ \frac{9+3\sqrt{5}}{2},\ \frac{9+3\sqrt{5}}{2};\ \ -6-3\sqrt{5},\ \frac{3+3\sqrt{5}}{2};\ \ -6-3\sqrt{5})$

Realization: condensation reductions of $\mathcal{Z}(\mathcal{NG}(\mathbb{Z}_5, 5))$ [116].

12. $8^{45,289}_{4,308.4}$: $d_i = (1.0,\ 5.854,\ 5.854,\ 5.854,\ 5.854,\ 6.854,\ 7.854,\ 7.854)$

$D^2 = 308.434 = \frac{315+135\sqrt{5}}{2}$

$T = (0, 0, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, 0, \frac{2}{5}, \frac{3}{5}),$

$S = (1,\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{7+3\sqrt{5}}{2},\ \frac{9+3\sqrt{5}}{2},\ \frac{9+3\sqrt{5}}{2};\ \ -5-3\sqrt{5},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ \frac{5+3\sqrt{5}}{2},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ -3c^1_{45}+3c^4_{45}-4c^{10}_{45}+3c^{11}_{45},\ 3c^2_{45}+c^5_{45}+3c^7_{45}+c^{10}_{45},\ 3c^1_{45}-3c^2_{45}-3c^4_{45}-c^5_{45}-3c^7_{45}+3c^{10}_{45}-3c^{11}_{45},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ 3c^1_{45}-3c^2_{45}-3c^4_{45}-c^5_{45}-3c^7_{45}+3c^{10}_{45}-3c^{11}_{45},\ -3c^1_{45}+3c^4_{45}-4c^{10}_{45}+3c^{11}_{45},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ 3c^2_{45}+c^5_{45}+3c^7_{45}+c^{10}_{45},\ -\frac{5+3\sqrt{5}}{2},\ 0,\ 0;\ \ 1,\ \frac{9+3\sqrt{5}}{2},\ \frac{9+3\sqrt{5}}{2};\ \ \frac{3+3\sqrt{5}}{2},\ -6-3\sqrt{5};\ \ \frac{3+3\sqrt{5}}{2})$

Realization: condensation reductions of $\mathcal{Z}(\mathcal{NG}(\mathbb{Z}_5, 5))$ [116].

7.3 Rank 9

1. $9^{9,620}_{0,9.}$: $d_i = (1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0)$

   $D^2 = 9.0 = 9$

   $T = (0, 0, 0, \frac{1}{9}, \frac{1}{9}, \frac{4}{9}, \frac{4}{9}, \frac{7}{9}, \frac{7}{9})$,

   $S = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1;\ \ 1,\ 1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1;\ \ 1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1;\ -\zeta_{18}^5,\ \zeta_9^2,\ \zeta_9^4,\ -\zeta_{18}^1,\ \zeta_9^1,\ -\zeta_{18}^7;\ \ -\zeta_{18}^5,\ -\zeta_{18}^1,\ \zeta_9^4,\ -\zeta_{18}^7,\ \zeta_9^1;\ \ \zeta_9^1,\ -\zeta_{18}^7,\ -\zeta_{18}^5,\ \zeta_9^2;\ \ \zeta_9^1,\ \zeta_9^2,\ -\zeta_{18}^5;\ \zeta_9^4,\ -\zeta_{18}^1;\ \ \zeta_9^4)$

   Realization: $U(9)_1$

2. $9^{88,112}_{2,44.}$: $d_i = (1.0, 1.0, 2.0, 2.0, 2.0, 2.0, 2.0, 3.316, 3.316)$

   $D^2 = 44.0 = 44$

   $T = (0, 0, \frac{1}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}, \frac{9}{11}, \frac{1}{8}, \frac{5}{8})$,

   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ \sqrt{11},\ \sqrt{11};\ \ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ -\sqrt{11},\ -\sqrt{11};\ \ 2c_{11}^2,\ 2c_{11}^1,\ 2c_{11}^4,\ 2c_{11}^3,\ 2c_{11}^5,\ 0,\ 0;\ \ 2c_{11}^5,\ 2c_{11}^2,\ 2c_{11}^4,\ 2c_{11}^3,\ 0,\ 0;\ \ 2c_{11}^3,\ 2c_{11}^5,\ 2c_{11}^1,\ 0,\ 0;\ \ 2c_{11}^1,\ 2c_{11}^2,\ 0,\ 0;\ \ 2c_{11}^4,\ 0,\ 0;\ \ \sqrt{11},\ -\sqrt{11};\ \ \sqrt{11})$

   Realization: $SO(11)_2$ or Abelian anyon condensation of O_{11} or $\overline{SO(22)}_2$.

3. $9^{88,529}_{2,44.}$: $d_i = (1.0, 1.0, 2.0, 2.0, 2.0, 2.0, 2.0, 3.316, 3.316)$

   $D^2 = 44.0 = 44$

   $T = (0, 0, \frac{1}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}, \frac{9}{11}, \frac{3}{8}, \frac{7}{8})$,

   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ \sqrt{11},\ \sqrt{11};\ \ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ -\sqrt{11},\ -\sqrt{11};\ \ 2c_{11}^2,\ 2c_{11}^1,\ 2c_{11}^4,\ 2c_{11}^3,\ 2c_{11}^5,\ 0,\ 0;\ \ 2c_{11}^5,\ 2c_{11}^2,\ 2c_{11}^4,\ 2c_{11}^3,\ 0,\ 0;\ \ 2c_{11}^3,\ 2c_{11}^5,\ 2c_{11}^1,\ 0,\ 0;\ \ 2c_{11}^1,\ 2c_{11}^2,\ 0,\ 0;\ \ 2c_{11}^4,\ 0,\ 0;\ \ -\sqrt{11},\ \sqrt{11};\ \ -\sqrt{11})$

   Realization: Abelian anyon condensation of $SO(11)_2$ or O_{11} or $\overline{SO(22)}_2$.

4. $9^{40,304}_{\frac{12}{5},52.36}$: $d_i = (1.0, 1.0, 1.902, 1.902, 2.618, 2.618, 3.77, 3.77, 3.236)$

   $D^2 = 52.360 = 30 + 10\sqrt{5}$

   $T = (0, 0, \frac{3}{40}, \frac{23}{40}, \frac{1}{5}, \frac{1}{5}, \frac{3}{8}, \frac{7}{8}, \frac{3}{5})$,

   $S = (1,\ 1,\ c_{20}^1,\ c_{20}^1,\ \frac{3+\sqrt{5}}{2},\ \frac{3+\sqrt{5}}{2},\ c_{20}^1+c_{20}^3,\ c_{20}^1+c_{20}^3,\ 1+\sqrt{5};\ \ 1,\ -c_{20}^1,\ -c_{20}^1,\ \frac{3+\sqrt{5}}{2},\ \frac{3+\sqrt{5}}{2},\ -c_{20}^1-c_{20}^3,\ -c_{20}^1-c_{20}^3,\ 1+\sqrt{5};\ \ c_{20}^1+c_{20}^3,\ -c_{20}^1-c_{20}^3,\ -c_{20}^1-c_{20}^3,\ c_{20}^1+c_{20}^3,\ c_{20}^1,\ -c_{20}^1,\ 0;\ c_{20}^1+c_{20}^3,\ -c_{20}^1-c_{20}^3,\ c_{20}^1+c_{20}^3,\ -c_{20}^1,\ c_{20}^1,\ 0;\ \ 1,\ 1,\ c_{20}^1,\ c_{20}^1,\ -1-\sqrt{5};\ \ 1,\ -c_{20}^1,\ -c_{20}^1,\ -1-\sqrt{5};\ -c_{20}^1-c_{20}^3,\ c_{20}^1+c_{20}^3,\ 0;\ \ -c_{20}^1-c_{20}^3,\ 0;\ \ 1+\sqrt{5})$

   Realization: $SU(2)_8$. Abelian anyon condensation of $\overline{SU(8)}_2$ or $\overline{Sp(18)}_1$.

5. $9^{40,247}_{\frac{28}{5},52.36}$: $d_i = (1.0, 1.0, 1.902, 1.902, 2.618, 2.618, 3.77, 3.77, 3.236)$

   $D^2 = 52.360 = 30 + 10\sqrt{5}$

   $T = (0, 0, \frac{7}{40}, \frac{27}{40}, \frac{4}{5}, \frac{4}{5}, \frac{3}{8}, \frac{7}{8}, \frac{2}{5})$,

   $S = (1,\ 1,\ c_{20}^1,\ c_{20}^1,\ \frac{3+\sqrt{5}}{2},\ \frac{3+\sqrt{5}}{2},\ c_{20}^1+c_{20}^3,\ c_{20}^1+c_{20}^3,\ 1+\sqrt{5};\ \ 1,\ -c_{20}^1,\ -c_{20}^1,\ \frac{3+\sqrt{5}}{2},\ \frac{3+\sqrt{5}}{2},\ -c_{20}^1-c_{20}^3,\ -c_{20}^1-c_{20}^3,\ 1+\sqrt{5};\ \ -c_{20}^1-c_{20}^3,\ c_{20}^1+c_{20}^3,\ -c_{20}^1-c_{20}^3,\ c_{20}^1+c_{20}^3,\ c_{20}^1,\ -c_{20}^1,\ 0;\ -c_{20}^1-c_{20}^3,\ -c_{20}^1-c_{20}^3,\ c_{20}^1+c_{20}^3,\ -c_{20}^1,\ c_{20}^1,\ 0;\ \ 1,\ 1,\ c_{20}^1,\ c_{20}^1,\ -1-\sqrt{5};\ \ 1,\ -c_{20}^1,\ -c_{20}^1,\ -1-\sqrt{5};\ \ c_{20}^1+c_{20}^3,\ -c_{20}^1-c_{20}^3,\ 0;\ \ c_{20}^1+c_{20}^3,\ 0;\ \ 1+\sqrt{5})$

   Realization: Abelian anyon condensation of $\overline{SU(2)}_8$ or $SU(8)_2$ or $Sp(18)_1$.

6. $9^{19,574}_{\frac{120}{19},175.3}$: $d_i = (1.0, 1.972, 2.891, 3.731, 4.469, 5.86, 5.563, 5.889, 6.54)$

   $D^2 = 175.332 = 45 + 36c_{19}^1 + 28c_{19}^2 + 21c_{19}^3 + 15c_{19}^4 + 10c_{19}^5 + 6c_{19}^6 + 3c_{19}^7 + c_{19}^8$

   $T = (0, \frac{4}{19}, \frac{17}{19}, \frac{1}{19}, \frac{13}{19}, \frac{15}{19}, \frac{7}{19}, \frac{8}{19}, \frac{18}{19})$,

   $S = (1,\ -c_{19}^9,\ \xi_{19}^3,\ \xi_{19}^{15},\ \xi_{19}^5,\ \xi_{19}^{13},\ \xi_{19}^7,\ \xi_{19}^{11},\ \xi_{19}^9;\ \ -\xi_{19}^{15},\ \xi_{19}^{13},\ -\xi_{19}^{11},\ \xi_{19}^9,\ -\xi_{19}^7,\ \xi_{19}^5,\ -\xi_{19}^3,\ 1;\ \ \xi_{19}^9,\ \xi_{19}^7,\ \xi_{19}^{15},\ 1,\ c_{19}^9,\ -\xi_{19}^5,\ -\xi_{19}^{11};\ \ -\xi_{19}^3,\ -1,\ \xi_{19}^5,\ -\xi_{19}^9,\ \xi_{19}^{13},\ c_{19}^9;\ \ -\xi_{19}^{13},\ -\xi_{19}^{11},\ -\xi_{19}^3,\ -c_{19}^9,\ \xi_{19}^7;\ \ -c_{19}^9,\ \xi_{19}^{15},\ -\xi_{19}^9,\ \xi_{19}^3;\ \ \xi_{19}^{11},\ 1,\ -\xi_{19}^{13};\ \ \xi_{19}^7,\ -\xi_{19}^{15};\ \ \xi_{19}^5)$

   Realization: $\overline{PSU(2)}_{17}$. Abelian anyon condensation of $\overline{SU(2)}_{17}$ or $SU(17)_2$ or $Sp(34)_1$.

7. $9^{15,715}_{\frac{14}{5},343.2}$: $d_i = (1.0, 2.956, 4.783, 4.783, 4.783, 6.401, 7.739, 8.739, 9.357)$

$D^2 = 343.211 = 105 + 45c_{15}^1 + 75c_{15}^2 + 90c_{15}^3$

$T = (0, \frac{1}{15}, \frac{1}{5}, \frac{13}{15}, \frac{13}{15}, \frac{2}{5}, \frac{2}{3}, 0, \frac{2}{5})$,

$S = (1,\ 1+c_{15}^2+c_{15}^3,\ \xi_{15}^7,\ \xi_{15}^7,\ \xi_{15}^7,\ 2+c_{15}^1+c_{15}^2+2c_{15}^3,\ 2+c_{15}^1+2c_{15}^2+2c_{15}^3,\ 3+c_{15}^1+2c_{15}^2+2c_{15}^3,\ 3+c_{15}^1+2c_{15}^2+3c_{15}^3;\ \ 2+c_{15}^1+2c_{15}^2+2c_{15}^3,\ 2\xi_{15}^7,\ -\xi_{15}^7,\ -\xi_{15}^7,\ 2+c_{15}^1+2c_{15}^2+2c_{15}^3,\ 1+c_{15}^2+c_{15}^3,\ -1-c_{15}^2-c_{15}^3,\ -2-c_{15}^1-2c_{15}^2-2c_{15}^3;\ \ \xi_{15}^7,\ \xi_{15}^7,\ \xi_{15}^7,\ -\xi_{15}^7,\ -2\xi_{15}^7,\ -\xi_{15}^7,\ \xi_{15}^7;\ \ 2-4\zeta_{15}^1-2\zeta_{15}^{-1}+\zeta_{15}^2-2\zeta_{15}^{-2}-\zeta_{15}^3+5\zeta_{15}^{-3}-5\zeta_{15}^4,\ -3+3\zeta_{15}^1+\zeta_{15}^{-1}-2\zeta_{15}^2+\zeta_{15}^{-2}-6\zeta_{15}^{-3}+5\zeta_{15}^4,\ -\xi_{15}^7,\ \xi_{15}^7,\ -\xi_{15}^7,\ \xi_{15}^7;\ \ 2-4\zeta_{15}^1-2\zeta_{15}^{-1}+\zeta_{15}^2-2\zeta_{15}^{-2}-\zeta_{15}^3+5\zeta_{15}^{-3}-5\zeta_{15}^4,\ -\xi_{15}^7,\ \xi_{15}^7,\ -\xi_{15}^7,\ \xi_{15}^7;\ \ -3-c_{15}^1-2c_{15}^2-2c_{15}^3,\ 1+c_{15}^2+c_{15}^3,\ 3+c_{15}^1+2c_{15}^2+3c_{15}^3,\ -1;\ \ 2+c_{15}^1+2c_{15}^2+2c_{15}^3,\ -2-c_{15}^1-2c_{15}^2-2c_{15}^3,\ -1-c_{15}^2-c_{15}^3;\ \ 1,\ 2+c_{15}^1+c_{15}^2+2c_{15}^3;\ \ -3-c_{15}^1-2c_{15}^2-2c_{15}^3)$

Realization: $PSO(14)_3$.

8. $9_{7,475.1}^{24,793}$: $d_i = (1.0, 4.449, 4.449, 5.449, 5.449, 8.898, 8.898, 9.898, 10.898)$

$D^2 = 475.151 = 240 + 96\sqrt{6}$

$T = (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{7}{12}, 0, \frac{7}{8})$,

$S = (1,\ 2+\sqrt{6},\ 2+\sqrt{6},\ 3+\sqrt{6},\ 3+\sqrt{6},\ 4+2\sqrt{6},\ 4+2\sqrt{6},\ 5+2\sqrt{6},\ 6+2\sqrt{6};\ \ 2\xi_{24}^7,\ 2+2c_{24}^1-2c_{24}^2-4c_{24}^3,\ -2c_{24}^2-3c_{24}^3,\ 2c_{24}^2+3c_{24}^3,\ -4-2\sqrt{6},\ 4+2\sqrt{6},\ -2-\sqrt{6},\ 0;\ \ 2\xi_{24}^7,\ 2c_{24}^2+3c_{24}^3,\ -2c_{24}^2-3c_{24}^3,\ -4-2\sqrt{6},\ 4+2\sqrt{6},\ -2-\sqrt{6},\ 0;\ \ 3+2c_{24}^1-2c_{24}^2-4c_{24}^3,\ 3+2c_{24}^1+2c_{24}^2+2c_{24}^3,\ 0,\ 0,\ 3+\sqrt{6},\ -6-2\sqrt{6};\ \ 3+2c_{24}^1-2c_{24}^2-4c_{24}^3,\ 0,\ 0,\ 3+\sqrt{6},\ -6-2\sqrt{6};\ \ 4+2\sqrt{6},\ 4+2\sqrt{6},\ -4-2\sqrt{6},\ 0;\ \ -4-2\sqrt{6},\ -4-2\sqrt{6},\ 0;\ \ 1,\ 6+2\sqrt{6};\ \ 0)$

Realization: $G(2)_4$.

9. $9_{6,668.5}^{12,567}$: $d_i = (1.0, 6.464, 6.464, 6.464, 6.464, 6.464, 6.464, 13.928, 14.928)$

$D^2 = 668.553 = 336 + 192\sqrt{3}$

$T = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \frac{2}{3})$,

$S = (1,\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ 7+4\sqrt{3},\ 8+4\sqrt{3};\ \ -7+4\zeta_{12}^1-8\zeta_{12}^{-1}+8\zeta_{12}^2,\ 1-8\zeta_{12}^1+4\zeta_{12}^{-1}-8\zeta_{12}^2,\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ -3-2\sqrt{3},\ 0;\ \ -7+4\zeta_{12}^1-8\zeta_{12}^{-1}+8\zeta_{12}^2,\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ -3-2\sqrt{3},\ 0;\ \ 9+6\sqrt{3},\ -3-2\sqrt{3},\ -3-2\sqrt{3},\ -3-2\sqrt{3},\ -3-2\sqrt{3},\ 0;\ \ 9+6\sqrt{3},\ -3-2\sqrt{3},\ -3-2\sqrt{3},\ -3-2\sqrt{3},\ 0;\ \ 9+6\sqrt{3},\ -3-2\sqrt{3},\ -3-2\sqrt{3},\ 0;\ \ 9+6\sqrt{3},\ -3-2\sqrt{3},\ 0;\ \ 1,\ 8+4\sqrt{3};\ \ -8-4\sqrt{3})$

Realization: $\mathbb{Z}_3$-algebra condensation of $SU(3)_9$ (see also [117].)

7.4 Rank 10

1. $10_{3,24.}^{48,945}$: $d_i = (1.0, 1.0, 1.0, 1.0, 1.732, 1.732, 1.732, 1.732, 2.0, 2.0)$

$D^2 = 24.0 = 24$

$T = (0, 0, \frac{1}{4}, \frac{1}{4}, \frac{3}{16}, \frac{3}{16}, \frac{11}{16}, \frac{11}{16}, \frac{1}{3}, \frac{7}{12})$,

$S = (1,\ 1,\ 1,\ 1,\ \sqrt{3},\ \sqrt{3},\ \sqrt{3},\ \sqrt{3},\ 2,\ 2;\ \ 1,\ 1,\ 1,\ -\sqrt{3},\ -\sqrt{3},\ -\sqrt{3},\ -\sqrt{3},\ 2,\ 2;\ \ -1,\ -1,\ (-\sqrt{3})\mathrm{i},\ (\sqrt{3})\mathrm{i},\ (-\sqrt{3})\mathrm{i},\ (\sqrt{3})\mathrm{i},\ 2,\ -2;\ \ -1,\ (\sqrt{3})\mathrm{i},\ (-\sqrt{3})\mathrm{i},\ (\sqrt{3})\mathrm{i},\ (-\sqrt{3})\mathrm{i},\ 2,\ -2;\ \ -\sqrt{3}\zeta_8^3,\ \sqrt{3}\zeta_8^1,\ \sqrt{3}\zeta_8^3,\ -\sqrt{3}\zeta_8^1,\ 0,\ 0;\ \ -\sqrt{3}\zeta_8^3,\ -\sqrt{3}\zeta_8^1,\ \sqrt{3}\zeta_8^3,\ 0,\ 0;\ \ -\sqrt{3}\zeta_8^3,\ \sqrt{3}\zeta_8^1,\ 0,\ 0;\ \ -\sqrt{3}\zeta_8^3,\ 0,\ 0;\ \ -2,\ -2;\ \ 2)$

Realization: Abelian anyon condensation of $SU(2)_4$ or O_3 or $\overline{SU(4)_2}$ or $\overline{Sp(8)_1}$ or

2. $10_{7,24.}^{48,721}$: $d_i = (1.0, 1.0, 1.0, 1.0, 1.732, 1.732, 1.732, 1.732, 2.0, 2.0)$

$D^2 = 24.0 = 24$

$T = (0, 0, \frac{1}{4}, \frac{1}{4}, \frac{3}{16}, \frac{3}{16}, \frac{11}{16}, \frac{11}{16}, \frac{2}{3}, \frac{11}{12})$,

$S = (1,\ 1,\ 1,\ 1,\ \sqrt{3},\ \sqrt{3},\ \sqrt{3},\ \sqrt{3},\ 2,\ 2;\ \ 1,\ 1,\ 1,\ -\sqrt{3},\ -\sqrt{3},\ -\sqrt{3},\ -\sqrt{3},\ 2,\ 2;\ \ -1,\ -1,\ (-\sqrt{3})\mathrm{i},\ (\sqrt{3})\mathrm{i},\ (-\sqrt{3})\mathrm{i},\ (\sqrt{3})\mathrm{i},\ 2,\ -2;\ \ -1,\ (\sqrt{3})\mathrm{i},\ (-\sqrt{3})\mathrm{i},\ (\sqrt{3})\mathrm{i},\ (-\sqrt{3})\mathrm{i},\ 2,\ -2;\ \ \sqrt{3}\zeta_8^3,\ -\sqrt{3}\zeta_8^1,\ -\sqrt{3}\zeta_8^3,\ \sqrt{3}\zeta_8^1,\ 0,\ 0;\ \ \sqrt{3}\zeta_8^3,\ \sqrt{3}\zeta_8^1,\ -\sqrt{3}\zeta_8^3,\ 0,\ 0;\ \ \sqrt{3}\zeta_8^3,\ -\sqrt{3}\zeta_8^1,\ 0,\ 0;\ \ \sqrt{3}\zeta_8^3,\ 0,\ 0;\ \ -2,\ -2;\ \ 2)$

Realization: Abelian anyon condensation of $SU(4)_2$ or $Sp(8)_1$ or $\overline{SU(2)_4}$ or $\overline{O}_3$.

3. $10_{4,36.}^{6,152}$: $d_i = (1.0, 1.0, 1.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 3.0)$

$D^2 = 36.0 = 36$

$T = (0, 0, 0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{2})$,

$S = (1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 3;\ \ 1,\ 1,\ -2\zeta_6^1,\ 2\zeta_3^1,\ -2\zeta_6^1,\ 2\zeta_3^1,\ -2\zeta_6^1,\ 2\zeta_3^1,\ 3;\ \ 1,\ 2\zeta_3^1,\ -2\zeta_6^1,\ 2\zeta_3^1,\ -2\zeta_6^1,\ 2\zeta_3^1,\ -2\zeta_6^1,\ 3;\ \ -2,\ -2,\ 2\zeta_6^1,\ -2\zeta_3^1,\ -2\zeta_3^1,\ 2\zeta_6^1,\ 0;\ \ -2,\ -2\zeta_3^1,\ 2\zeta_6^1,\ 2\zeta_6^1,\ -2\zeta_3^1,\ 0;\ \ -2\zeta_3^1,\ 2\zeta_6^1,\ -2,\ -2,\ 0;\ \ -2\zeta_3^1,\ -2,\ -2,\ 0;\ \ 2\zeta_6^1,\ -2\zeta_3^1,\ 0;\ \ 2\zeta_6^1,\ 0;\ \ -3)$
Realization: $SU(3)_3$.

4. $10^{18,490}_{4,36.}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0)$
$D^2 = 36.0 = 36$
$T = (0, 0, 0, \frac{1}{9}, \frac{1}{9}, \frac{4}{9}, \frac{4}{9}, \frac{7}{9}, \frac{7}{9}, \frac{1}{2})$,
$S = (1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 3;\ \ 1,\ 1,\ -2\zeta_6^1,\ 2\zeta_3^1,\ -2\zeta_6^1,\ 2\zeta_3^1,\ -2\zeta_6^1,\ 2\zeta_3^1,\ 3;\ \ 1,\ 2\zeta_3^1,\ -2\zeta_6^1,\ 2\zeta_3^1,\ -2\zeta_6^1,\ 2\zeta_3^1,\ -2\zeta_6^1,\ 3;\ \ 2\zeta_{18}^5,\ -2\zeta_9^2,\ -2\zeta_9^4,\ 2\zeta_{18}^1,\ -2\zeta_9^1,\ 2\zeta_{18}^7,\ 0;\ \ 2\zeta_{18}^5,\ 2\zeta_{18}^1,\ -2\zeta_9^4,\ 2\zeta_{18}^7,\ -2\zeta_9^1,\ 0;\ \ -2\zeta_9^1,\ 2\zeta_{18}^7,\ 2\zeta_{18}^5,\ -2\zeta_9^2,\ 0;\ \ -2\zeta_9^1,\ -2\zeta_9^2,\ 2\zeta_{18}^5,\ 0;\ \ -2\zeta_9^4,\ 2\zeta_{18}^1,\ 0;\ \ -2\zeta_9^4,\ 0;\ \ -3)$
Realization: Zesting or Abelian anyon condensation of $SU(3)_3$ [113].

5. $10^{26,247}_{0,52.}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.605,\ 3.605)$
$D^2 = 52.0 = 52$
$T = (0, 0, \frac{1}{13}, \frac{3}{13}, \frac{4}{13}, \frac{9}{13}, \frac{10}{13}, \frac{12}{13}, 0, \frac{1}{2})$,
$S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \sqrt{13},\ \sqrt{13};\ \ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ -\sqrt{13},\ -\sqrt{13};\ \ 2c_{13}^2,\ 2c_{13}^5,\ 2c_{13}^4,\ 2c_{13}^6,\ 2c_{13}^1,\ 2c_{13}^3,\ 0,\ 0;\ \ 2c_{13}^6,\ 2c_{13}^3,\ 2c_{13}^2,\ 2c_{13}^4,\ 2c_{13}^1,\ 0,\ 0;\ \ 2c_{13}^5,\ 2c_{13}^1,\ 2c_{13}^2,\ 2c_{13}^6,\ 0,\ 0;\ \ 2c_{13}^5,\ 2c_{13}^3,\ 2c_{13}^4,\ 0,\ 0;\ \ 2c_{13}^6,\ 2c_{13}^5,\ 0,\ 0;\ \ 2c_{13}^2,\ 0,\ 0;\ \ \sqrt{13},\ -\sqrt{13};\ \ \sqrt{13})$
Realization: Abelian anyon condensation of $SO(26)_2$ or O_{13}.

6. $10^{26,862}_{4,52.}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.605,\ 3.605)$
$D^2 = 52.0 = 52$
$T = (0, 0, \frac{2}{13}, \frac{5}{13}, \frac{6}{13}, \frac{7}{13}, \frac{8}{13}, \frac{11}{13}, 0, \frac{1}{2})$,
$S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \sqrt{13},\ \sqrt{13};\ \ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ -\sqrt{13},\ -\sqrt{13};\ \ 2c_{13}^4,\ 2c_{13}^1,\ 2c_{13}^3,\ 2c_{13}^2,\ 2c_{13}^5,\ 2c_{13}^6,\ 0,\ 0;\ \ 2c_{13}^3,\ 2c_{13}^4,\ 2c_{13}^6,\ 2c_{13}^2,\ 2c_{13}^5,\ 0,\ 0;\ \ 2c_{13}^1,\ 2c_{13}^5,\ 2c_{13}^6,\ 2c_{13}^2,\ 0,\ 0;\ \ 2c_{13}^1,\ 2c_{13}^4,\ 2c_{13}^3,\ 0,\ 0;\ \ 2c_{13}^3,\ 2c_{13}^1,\ 0,\ 0;\ \ 2c_{13}^4,\ 0,\ 0;\ \ -\sqrt{13},\ \sqrt{13};\ \ -\sqrt{13})$
Realization: Abelian anyon condensation of $SO(13)_2$.

7. $10^{52,110}_{0,52.}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.605,\ 3.605)$
$D^2 = 52.0 = 52$
$T = (0, 0, \frac{1}{13}, \frac{3}{13}, \frac{4}{13}, \frac{9}{13}, \frac{10}{13}, \frac{12}{13}, \frac{1}{4}, \frac{3}{4})$,
$S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \sqrt{13},\ \sqrt{13};\ \ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ -\sqrt{13},\ -\sqrt{13};\ \ 2c_{13}^2,\ 2c_{13}^5,\ 2c_{13}^4,\ 2c_{13}^6,\ 2c_{13}^1,\ 2c_{13}^3,\ 0,\ 0;\ \ 2c_{13}^6,\ 2c_{13}^3,\ 2c_{13}^2,\ 2c_{13}^4,\ 2c_{13}^1,\ 0,\ 0;\ \ 2c_{13}^5,\ 2c_{13}^1,\ 2c_{13}^2,\ 2c_{13}^6,\ 0,\ 0;\ \ 2c_{13}^5,\ 2c_{13}^3,\ 2c_{13}^4,\ 0,\ 0;\ \ 2c_{13}^6,\ 2c_{13}^5,\ 0,\ 0;\ \ 2c_{13}^2,\ 0,\ 0;\ \ -\sqrt{13},\ \sqrt{13};\ \ -\sqrt{13})$
Realization: Abelian anyon condensation of $SO(26)_2$ or O_{13}.

8. $10^{52,489}_{4,52.}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.605,\ 3.605)$
$D^2 = 52.0 = 52$
$T = (0, 0, \frac{2}{13}, \frac{5}{13}, \frac{6}{13}, \frac{7}{13}, \frac{8}{13}, \frac{11}{13}, \frac{1}{4}, \frac{3}{4})$,
$S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \sqrt{13},\ \sqrt{13};\ \ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ -\sqrt{13},\ -\sqrt{13};\ \ 2c_{13}^4,\ 2c_{13}^1,\ 2c_{13}^3,\ 2c_{13}^2,\ 2c_{13}^5,\ 2c_{13}^6,\ 0,\ 0;\ \ 2c_{13}^3,\ 2c_{13}^4,\ 2c_{13}^6,\ 2c_{13}^2,\ 2c_{13}^5,\ 0,\ 0;\ \ 2c_{13}^1,\ 2c_{13}^5,\ 2c_{13}^6,\ 2c_{13}^2,\ 0,\ 0;\ \ 2c_{13}^1,\ 2c_{13}^4,\ 2c_{13}^3,\ 0,\ 0;\ \ 2c_{13}^3,\ 2c_{13}^1,\ 0,\ 0;\ \ 2c_{13}^4,\ 0,\ 0;\ \ \sqrt{13},\ -\sqrt{13};\ \ \sqrt{13})$
Realization: $SO(13)_2$.

9. $10^{12,311}_{6,89.56}$: $d_i = (1.0,\ 1.0,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 3.732,\ 3.732,\ 4.732)$
$D^2 = 89.569 = 48 + 24\sqrt{3}$
$T = (0, \frac{1}{2}, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{5}{6}, 0, \frac{1}{2}, \frac{3}{4})$,
$S = (1,\ 1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 3+\sqrt{3};\ \ 1,\ -1-\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -3-\sqrt{3};\ \ 0,\ -2-2\sqrt{3},\ 0,\ 0,\ 2+2\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ (-3-\sqrt{3})i,\ (3+\sqrt{3})i,\ -1-\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ (-3-\sqrt{3})i,\ -1-\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ 1,\ 1,\ 3+\sqrt{3};\ \ 1,\ -3-\sqrt{3};\ \ 0)$
Realization: Abelian anyon condensation [79] of $SO(5)_3$ or $Sp(4)_3$ or $\overline{Sp(6)_2}$.

10. $10^{12,155}_{0,89.56}$: $d_i = (1.0,\ 1.0,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 3.732,\ 3.732,\ 4.732)$

$D^2 = 89.569 = 48 + 24\sqrt{3}$

$T = (0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{5}{6}, \frac{7}{12}, \frac{7}{12}, 0, \frac{1}{2}, 0)$,

$S = (1,\ 1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 3+\sqrt{3};\ \ 1,\ 1+\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -3-\sqrt{3};\ \ 1+\sqrt{3},\ -2-2\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ 0,\ 2+2\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ (3+\sqrt{3})\mathrm{i},\ (-3-\sqrt{3})\mathrm{i},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ (3+\sqrt{3})\mathrm{i},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1,\ 1,\ 3+\sqrt{3};\ \ 1,\ -3-\sqrt{3};\ \ 0)$

Realization: Abelian anyon condensation of $SO(5)_3$ or $Sp(4)_3$ or $\overline{Sp(6)}_2$.

11. $10^{12,822}_{4,89.56}$: $d_i = (1.0,\ 1.0,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 3.732,\ 3.732,\ 4.732)$

$D^2 = 89.569 = 48 + 24\sqrt{3}$

$T = (0, \frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{5}{6}, \frac{1}{12}, \frac{1}{12}, 0, \frac{1}{2}, \frac{1}{2})$,

$S = (1,\ 1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 3+\sqrt{3};\ \ 1,\ 1+\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -3-\sqrt{3};\ \ 1+\sqrt{3},\ -2-2\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ 0,\ 2+2\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ (3+\sqrt{3})\mathrm{i},\ (-3-\sqrt{3})\mathrm{i},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ (3+\sqrt{3})\mathrm{i},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1,\ 1,\ 3+\sqrt{3};\ \ 1,\ -3-\sqrt{3};\ \ 0)$

Realization: Abelian anyon condensation of $SO(5)_3$ or $Sp(4)_3$ or $\overline{Sp(6)}_2$.

12. $10^{12,119}_{2,89.56}$: $d_i = (1.0,\ 1.0,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 3.732,\ 3.732,\ 4.732)$

$D^2 = 89.569 = 48 + 24\sqrt{3}$

$T = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, 0, \frac{1}{2}, \frac{1}{4})$,

$S = (1,\ 1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 3+\sqrt{3};\ \ 1,\ -1-\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -3-\sqrt{3};\ \ 0,\ -2-2\sqrt{3},\ 2+2\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ (-3-\sqrt{3})\mathrm{i},\ (3+\sqrt{3})\mathrm{i},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ (-3-\sqrt{3})\mathrm{i},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1,\ 1,\ 3+\sqrt{3};\ \ 1,\ -3-\sqrt{3};\ \ 0)$

Realization: Abelian anyon condensation of $SO(5)_3$ or $Sp(4)_3$ or $\overline{Sp(6)}_2$.

13. $10^{24,123}_{7,89.56}$: $d_i = (1.0,\ 1.0,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 3.732,\ 3.732,\ 4.732)$

$D^2 = 89.569 = 48 + 24\sqrt{3}$

$T = (0, \frac{1}{2}, \frac{1}{3}, \frac{5}{6}, \frac{1}{8}, \frac{11}{24}, \frac{11}{24}, 0, \frac{1}{2}, \frac{7}{8})$,

$S = (1,\ 1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 3+\sqrt{3};\ \ 1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -3-\sqrt{3};\ \ 1+\sqrt{3},\ 1+\sqrt{3},\ -2-2\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ 1+\sqrt{3},\ 2+2\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ 0,\ 0,\ 0,\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ -3-\sqrt{3},\ 3+\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ -3-\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1,\ 1,\ 3+\sqrt{3};\ \ 1,\ -3-\sqrt{3};\ \ 0)$

Realization: Abelian anyon condensation of $SO(5)_3$ or $Sp(4)_3$ or $\overline{Sp(6)}_2$.

14. $10^{24,380}_{1,89.56}$: $d_i = (1.0,\ 1.0,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 2.732,\ 3.732,\ 3.732,\ 4.732)$

$D^2 = 89.569 = 48 + 24\sqrt{3}$

$T = (0, \frac{1}{2}, \frac{1}{3}, \frac{5}{6}, \frac{3}{8}, \frac{17}{24}, \frac{17}{24}, 0, \frac{1}{2}, \frac{1}{8})$,

$S = (1,\ 1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 3+\sqrt{3};\ \ 1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -3-\sqrt{3};\ \ 1+\sqrt{3},\ 1+\sqrt{3},\ -2-2\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ 1+\sqrt{3},\ 2+2\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ 0,\ 0,\ 0,\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 3+\sqrt{3},\ -3-\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 3+\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1,\ 1,\ 3+\sqrt{3};\ \ 1,\ -3-\sqrt{3};\ \ 0)$

Realization: $\overline{Sp(6)}_2$.

15. $10^{21,145}_{\frac{26}{7},236.3}$: $d_i = (1.0,\ 1.977,\ 2.911,\ 3.779,\ 4.563,\ 5.245,\ 5.810,\ 6.245,\ 6.541,\ 6.690)$

$D^2 = 236.341 = 42 + 42c^1_{21} + 42c^2_{21} + 21c^3_{21} + 21c^4_{21} + 21c^5_{21}$

$T = (0, \frac{2}{7}, \frac{2}{21}, \frac{3}{7}, \frac{2}{7}, \frac{2}{3}, \frac{4}{7}, 0, \frac{20}{21}, \frac{3}{7})$,

$S = (1,\ -c_{21}^{10},\ \xi_{21}^{3},\ \xi_{21}^{17},\ \xi_{21}^{5},\ \xi_{21}^{15},\ \xi_{21}^{7},\ \xi_{21}^{13},\ \xi_{21}^{9},\ \xi_{21}^{11};\ \ -\xi_{21}^{17},\ \xi_{21}^{15},\ -\xi_{21}^{13},\ \xi_{21}^{11},\ -\xi_{21}^{9},\ \xi_{21}^{7},\ -\xi_{21}^{5},\ \xi_{21}^{3},\ -1;\ \ \xi_{21}^{9},\ \xi_{21}^{9},\ \xi_{21}^{15},\ \xi_{21}^{3},\ 0,\ -\xi_{21}^{3},\ -\xi_{21}^{15},\ -\xi_{21}^{9};\ \ -\xi_{21}^{5},\ 1,\ \xi_{21}^{3},\ -\xi_{21}^{7},\ \xi_{21}^{11},\ -\xi_{21}^{15},\ -c_{21}^{10};\ \ -\xi_{21}^{17},\ -\xi_{21}^{9},\ -\xi_{21}^{7},\ c_{21}^{10},\ \xi_{21}^{3},\ \xi_{21}^{13};\ \ \xi_{21}^{15},\ 0,\ -\xi_{21}^{15},\ \xi_{21}^{9},\ -\xi_{21}^{3};\ \ \xi_{21}^{7},\ \xi_{21}^{7},\ 0,\ -\xi_{21}^{7};\ \ 1,\ -\xi_{21}^{9},\ \xi_{21}^{17};\ \ -\xi_{21}^{3},\ \xi_{21}^{15};\ \ -\xi_{21}^{5})$

Realization: $PSU(2)_{19}$.

16. $10^{17,522}_{\frac{48}{17},499.2}$: $d_i = (1.0,\ 2.965,\ 4.830,\ 5.418,\ 5.418,\ 6.531,\ 8.9,\ 9.214,\ 10.106,\ 10.653)$

$D^2 = 499.210 = 136 + 119c_{17}^{1} + 102c_{17}^{2} + 85c_{17}^{3} + 68c_{17}^{4} + 51c_{17}^{5} + 34c_{17}^{6} + 17c_{17}^{7}$

$T = (0, \frac{1}{17}, \frac{3}{17}, \frac{2}{17}, \frac{2}{17}, \frac{6}{17}, \frac{10}{17}, \frac{15}{17}, \frac{4}{17}, \frac{11}{17}),$

$S = (1,\ 2 + c_{17}^{1} + c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6} + c_{17}^{7},\ 2 + 2c_{17}^{1} + c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6} + c_{17}^{7},\ \xi_{17}^{9},\ \xi_{17}^{9},\ 2 + 2c_{17}^{1} + c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6},\ 2 + 2c_{17}^{1} + 2c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6},\ 2 + 2c_{17}^{1} + 2c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5},\ 2 + 2c_{17}^{1} + 2c_{17}^{2} + 2c_{17}^{3} + c_{17}^{4} + c_{17}^{5},\ 2 + 2c_{17}^{1} + 2c_{17}^{2} + 2c_{17}^{3} + c_{17}^{4};\ \ 2 + 2c_{17}^{1} + 2c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6},\ 2 + 2c_{17}^{1} + 2c_{17}^{2} + 2c_{17}^{3} + c_{17}^{4},\ -\xi_{17}^{9},\ -\xi_{17}^{9},\ 2 + 2c_{17}^{1} + 2c_{17}^{2} + 2c_{17}^{3} + c_{17}^{4} + c_{17}^{5},\ 2 + 2c_{17}^{1} + c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6},\ 1,\ -2 - 2c_{17}^{1} - c_{17}^{2} - c_{17}^{3} - c_{17}^{4} - c_{17}^{5} - c_{17}^{6} - c_{17}^{7},\ -2 - 2c_{17}^{1} - 2c_{17}^{2} - c_{17}^{3} - c_{17}^{4} - c_{17}^{5};\ \ 2 + 2c_{17}^{1} + 2c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6},\ \xi_{17}^{9},\ \xi_{17}^{9},\ -1,\ -2 - 2c_{17}^{1} - 2c_{17}^{2} - c_{17}^{3} - c_{17}^{4} - c_{17}^{5},\ -2 - 2c_{17}^{1} - 2c_{17}^{2} - 2c_{17}^{3} - c_{17}^{4} - c_{17}^{5},\ -2 - c_{17}^{1} - c_{17}^{2} - c_{17}^{3} - c_{17}^{4} - c_{17}^{5} - c_{17}^{6} - c_{17}^{7},\ 2 + 2c_{17}^{1} + c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6};\ \ 4 + 3c_{17}^{1} + 3c_{17}^{2} + 2c_{17}^{3} + 2c_{17}^{4} + 2c_{17}^{5} + c_{17}^{6},\ -3 - 2c_{17}^{1} - 2c_{17}^{2} - c_{17}^{3} - c_{17}^{4} - 2c_{17}^{5} - c_{17}^{6},\ -\xi_{17}^{9},\ \xi_{17}^{9},\ -\xi_{17}^{9},\ \xi_{17}^{9},\ -\xi_{17}^{9};\ \ 4 + 3c_{17}^{1} + 3c_{17}^{2} + 2c_{17}^{3} + 2c_{17}^{4} + 2c_{17}^{5} + c_{17}^{6},\ -\xi_{17}^{9},\ \xi_{17}^{9},\ -\xi_{17}^{9},\ \xi_{17}^{9},\ -\xi_{17}^{9};\ \ -2 - 2c_{17}^{1} - 2c_{17}^{2} - 2c_{17}^{3} - c_{17}^{4},\ -2 - 2c_{17}^{1} - c_{17}^{2} - c_{17}^{3} - c_{17}^{4} - c_{17}^{5} - c_{17}^{6} - c_{17}^{7},\ 2 + 2c_{17}^{1} + 2c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6},\ 2 + 2c_{17}^{1} + 2c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5},\ -2 - c_{17}^{1} - c_{17}^{2} - c_{17}^{3} - c_{17}^{4} - c_{17}^{5} - c_{17}^{6} - c_{17}^{7};\ \ 2 + 2c_{17}^{1} + 2c_{17}^{2} + 2c_{17}^{3} + c_{17}^{4} + c_{17}^{5},\ 2 + c_{17}^{1} + c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6} + c_{17}^{7},\ -2 - 2c_{17}^{1} - 2c_{17}^{2} - 2c_{17}^{3} - c_{17}^{4},\ -1;\ \ -2 - 2c_{17}^{1} - 2c_{17}^{2} - 2c_{17}^{3} - c_{17}^{4},\ 2 + 2c_{17}^{1} + c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6},\ 2 + 2c_{17}^{1} + c_{17}^{2} + c_{17}^{3} + c_{17}^{4} + c_{17}^{5} + c_{17}^{6} + c_{17}^{7};\ \ 1,\ -2 - 2c_{17}^{1} - 2c_{17}^{2} - c_{17}^{3} - c_{17}^{4} - c_{17}^{5} - c_{17}^{6};\ \ 2 + 2c_{17}^{1} + 2c_{17}^{2} + 2c_{17}^{3} + c_{17}^{4} + c_{17}^{5})$

Realization: $PSO(16)_3$. Abelian anyon condensation of $E(8)_4$.

17. $10^{14,352}_{0,537.4}$: $d_i = (1.0,\ 3.493,\ 4.493,\ 4.493,\ 5.603,\ 5.603,\ 9.97,\ 10.97,\ 10.97,\ 11.591)$

$D^2 = 537.478 = 308 + 224c_{7}^{1} + 112c_{7}^{2}$

$T = (0, 0, \frac{2}{7}, \frac{5}{7}, \frac{3}{7}, \frac{4}{7}, 0, \frac{1}{7}, \frac{6}{7}, \frac{1}{2}),$

$S = (1,\ 1 + 2c_{7}^{1},\ 2\xi_{7}^{3},\ 2\xi_{7}^{3},\ 4 + 2c_{7}^{1} + 2c_{7}^{2},\ 4 + 2c_{7}^{1} + 2c_{7}^{2},\ 5 + 4c_{7}^{1} + 2c_{7}^{2},\ 6 + 4c_{7}^{1} + 2c_{7}^{2},\ 6 + 4c_{7}^{1} + 2c_{7}^{2},\ 5 + 6c_{7}^{1} + 2c_{7}^{2};\ \ -5 - 4c_{7}^{1} - 2c_{7}^{2},\ -4 - 2c_{7}^{1} - 2c_{7}^{2},\ -4 - 2c_{7}^{1} - 2c_{7}^{2},\ 6 + 4c_{7}^{1} + 2c_{7}^{2},\ 6 + 4c_{7}^{1} + 2c_{7}^{2},\ 1,\ 2\xi_{7}^{3},\ 2\xi_{7}^{3},\ -5 - 6c_{7}^{1} - 2c_{7}^{2};\ \ -2\xi_{7}^{3},\ 6 + 6c_{7}^{1} + 2c_{7}^{2},\ 4 + 4c_{7}^{1} + 2c_{7}^{2},\ -4 - 2c_{7}^{1} - 2c_{7}^{2},\ 6 + 4c_{7}^{1} + 2c_{7}^{2},\ -6 - 4c_{7}^{1} - 2c_{7}^{2},\ -2c_{7}^{1},\ 0;\ \ -2\xi_{7}^{3},\ -4 - 2c_{7}^{1} - 2c_{7}^{2},\ 4 + 4c_{7}^{1} + 2c_{7}^{2},\ 6 + 4c_{7}^{1} + 2c_{7}^{2},\ -2c_{7}^{1},\ -6 - 4c_{7}^{1} - 2c_{7}^{2},\ 0;\ \ 6 + 4c_{7}^{1} + 2c_{7}^{2},\ 2c_{7}^{1},\ -2\xi_{7}^{3},\ -6 - 6c_{7}^{1} - 2c_{7}^{2},\ 2\xi_{7}^{3},\ 0;\ \ 6 + 4c_{7}^{1} + 2c_{7}^{2},\ -2\xi_{7}^{3},\ 2\xi_{7}^{3},\ -6 - 6c_{7}^{1} - 2c_{7}^{2},\ 0;\ \ -1 - 2c_{7}^{1},\ 4 + 2c_{7}^{1} + 2c_{7}^{2},\ 4 + 2c_{7}^{1} + 2c_{7}^{2},\ -5 - 6c_{7}^{1} - 2c_{7}^{2};\ \ -4 - 2c_{7}^{1} - 2c_{7}^{2},\ 4 + 4c_{7}^{1} + 2c_{7}^{2},\ 0;\ \ -4 - 2c_{7}^{1} - 2c_{7}^{2},\ 0;\ \ 5 + 6c_{7}^{1} + 2c_{7}^{2})$

Realization: $Sp(6)_3$.

18. $10^{77,298}_{6,684.3}$: $d_i = (1.0,\ 7.887,\ 7.887,\ 7.887,\ 7.887,\ 7.887,\ 8.887,\ 9.887,\ 9.887,\ 9.887)$

$D^2 = 684.336 = \frac{693+77\sqrt{77}}{2}$

$T = (0, \frac{1}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}, \frac{9}{11}, 0, \frac{3}{7}, \frac{5}{7}, \frac{6}{7}),$

$S = (1,\ \frac{7+\sqrt{77}}{2},\ \frac{7+\sqrt{77}}{2},\ \frac{7+\sqrt{77}}{2},\ \frac{7+\sqrt{77}}{2},\ \frac{7+\sqrt{77}}{2},\ \frac{9+\sqrt{77}}{2},\ \frac{11+\sqrt{77}}{2},\ \frac{11+\sqrt{77}}{2},\ \frac{11+\sqrt{77}}{2};\ \ -1 - 2c_{77}^{1} + c_{77}^{2} - c_{77}^{3} - c_{77}^{4} + c_{77}^{5} + 2c_{77}^{6} - c_{77}^{7} - c_{77}^{8} + c_{77}^{9} - 2c_{77}^{10} - c_{77}^{11} + c_{77}^{12} - 4c_{77}^{14} - c_{77}^{15} + 3c_{77}^{16} + 2c_{77}^{17} - c_{77}^{18} + c_{77}^{19} - c_{77}^{21} - c_{77}^{22} - c_{77}^{23} + c_{77}^{28} - c_{77}^{29},\ -2 + 2c_{77}^{2} - 2c_{77}^{3} - 2c_{77}^{4} + 2c_{77}^{5} - 2c_{77}^{6} - 6c_{77}^{7} - 2c_{77}^{8} + 2c_{77}^{9} - 2c_{77}^{11} + 2c_{77}^{12} - c_{77}^{14} - 2c_{77}^{15} - 2c_{77}^{17} - 3c_{77}^{18} + 2c_{77}^{19} - 2c_{77}^{22} + 2c_{77}^{23} - c_{77}^{26} - c_{77}^{28} - 3c_{77}^{29},\ 1 + 2c_{77}^{1} + c_{77}^{6} + c_{77}^{7} - 2c_{77}^{9} + 2c_{77}^{10} - 2c_{77}^{13} + c_{77}^{14} + c_{77}^{16} + c_{77}^{17} + 2c_{77}^{21} + 2c_{77}^{23} - 2c_{77}^{24} - 2c_{77}^{28},\ -1 + c_{77}^{1} + 2c_{77}^{9} + c_{77}^{10} + 2c_{77}^{13} - c_{77}^{14} + 2c_{77}^{18} - 4c_{77}^{21} + c_{77}^{23} + 2c_{77}^{24} + 2c_{77}^{26} - c_{77}^{28} + 2c_{77}^{29},\ 5 - 2c_{77}^{2} + 2c_{77}^{3} + 2c_{77}^{4} - 2c_{77}^{5} + 4c_{77}^{7} + 2c_{77}^{8} - c_{77}^{9} + 2c_{77}^{11} - 2c_{77}^{12} + c_{77}^{13} + 4c_{77}^{14} + 2c_{77}^{15} - 2c_{77}^{16} - 2c_{77}^{19} + 3c_{77}^{21} + 2c_{77}^{22} - 2c_{77}^{23} + c_{77}^{24} - 2c_{77}^{26} + 3c_{77}^{28},\ -\frac{7+\sqrt{77}}{2},\ 0,\ 0,\ 0;\ \ 5 - 2c_{77}^{2} + 2c_{77}^{3} + 2c_{77}^{4} - 2c_{77}^{5} + 4c_{77}^{7} + 2c_{77}^{8} - c_{77}^{9} + 2c_{77}^{11} - 2c_{77}^{12} + c_{77}^{13} + 4c_{77}^{14} + 2c_{77}^{15} - 2c_{77}^{16} - 2c_{77}^{19} + 3c_{77}^{21} + 2c_{77}^{22} - 2c_{77}^{23} + c_{77}^{24} - 2c_{77}^{26} + 3c_{77}^{28},\ -1 - 2c_{77}^{1} + c_{77}^{2} - c_{77}^{3} - c_{77}^{4} + c_{77}^{5} + 2c_{77}^{6} - c_{77}^{7} - c_{77}^{8} + c_{77}^{9} - 2c_{77}^{10} - c_{77}^{11} + c_{77}^{12} - 4c_{77}^{14} - c_{77}^{15} + 3c_{77}^{16} + 2c_{77}^{17} - c_{77}^{18} + c_{77}^{19} - c_{77}^{21} - c_{77}^{22} - c_{77}^{23} + c_{77}^{28} - c_{77}^{29},\ 1 + 2c_{77}^{1} + c_{77}^{6} + c_{77}^{7} - 2c_{77}^{9} + 2c_{77}^{10} - 2c_{77}^{13} + c_{77}^{14} + c_{77}^{16} + c_{77}^{17} + 2c_{77}^{21} + 2c_{77}^{23} - 2c_{77}^{24} - 2c_{77}^{28},\ -1 + c_{77}^{1} + 2c_{77}^{9} + c_{77}^{10} + 2c_{77}^{13} - c_{77}^{14} + 2c_{77}^{18} - 4c_{77}^{21} + c_{77}^{23} + 2c_{77}^{24} + 2c_{77}^{26} - c_{77}^{28} + 2c_{77}^{29},\ -\frac{7+\sqrt{77}}{2},\ 0,\ 0,\ 0;\ \ -1 + c_{77}^{1} + 2c_{77}^{9} + c_{77}^{10} + 2c_{77}^{13} - c_{77}^{14} + 2c_{77}^{18} - 4c_{77}^{21} + c_{77}^{23} + 2c_{77}^{24} + 2c_{77}^{26} - c_{77}^{28} + 2c_{77}^{29},\ 5 - 2c_{77}^{2} + 2c_{77}^{3} + 2c_{77}^{4} - 2c_{77}^{5} + 4c_{77}^{7} + 2c_{77}^{8} - c_{77}^{9} + 2c_{77}^{11} - 2c_{77}^{12} + c_{77}^{13} + 4c_{77}^{14} + 2c_{77}^{15} - 2c_{77}^{16} - 2c_{77}^{19} + 3c_{77}^{21} + 2c_{77}^{22} - 2c_{77}^{23} + c_{77}^{24} - 2c_{77}^{26} + 3c_{77}^{28},\ -2 + 2c_{77}^{2} - 2c_{77}^{3} - 2c_{77}^{4} + 2c_{77}^{5} - 2c_{77}^{6} - 6c_{77}^{7} - 2c_{77}^{8} + 2c_{77}^{9} - 2c_{77}^{11} + 2c_{77}^{12} - c_{77}^{14} - 2c_{77}^{15} - 2c_{77}^{17} - 3c_{77}^{18} + 2c_{77}^{19} - 2c_{77}^{22} + 2c_{77}^{23} - c_{77}^{26} - c_{77}^{28} - 3c_{77}^{29},\ -\frac{7+\sqrt{77}}{2},\ 0,\ 0,\ 0;$

$-2+2c^{2}_{77}-2c^{3}_{77}-2c^{4}_{77}+2c^{5}_{77}-2c^{6}_{77}-6c^{7}_{77}-2c^{8}_{77}+2c^{9}_{77}-2c^{11}_{77}+2c^{12}_{77}-c^{14}_{77}-2c^{15}_{77}-2c^{17}_{77}-3c^{18}_{77}+2c^{19}_{77}-2c^{22}_{77}+2c^{23}_{77}-c^{26}_{77}-c^{28}_{77}-3c^{29}_{77},\ -1-2c^{1}_{77}+c^{2}_{77}-c^{3}_{77}-c^{4}_{77}+c^{5}_{77}+2c^{6}_{77}-c^{7}_{77}-c^{8}_{77}+c^{9}_{77}-2c^{10}_{77}-c^{11}_{77}+c^{12}_{77}-4c^{14}_{77}-c^{15}_{77}+3c^{16}_{77}+2c^{17}_{77}-c^{18}_{77}+c^{19}_{77}-c^{21}_{77}-c^{22}_{77}-c^{23}_{77}+c^{28}_{77}-c^{29}_{77},\ -\frac{7+\sqrt{77}}{2},\ 0,\ 0,\ 0;\ 1+2c^{1}_{77}+c^{6}_{77}+c^{7}_{77}-2c^{9}_{77}+2c^{10}_{77}-2c^{13}_{77}+c^{14}_{77}+c^{16}_{77}+c^{17}_{77}+2c^{21}_{77}+2c^{23}_{77}-2c^{24}_{77}-2c^{28}_{77},\ -\frac{7+\sqrt{77}}{2},\ 0,\ 0,\ 0;\ 1,\ \frac{11+\sqrt{77}}{2},\ \frac{11+\sqrt{77}}{2},\ \frac{11+\sqrt{77}}{2};\ 1+3c^{4}_{77}+2c^{7}_{77}-2c^{9}_{77}+c^{10}_{77}+7c^{11}_{77}+2c^{15}_{77}-2c^{16}_{77}+c^{17}_{77}+2c^{18}_{77}-2c^{19}_{77}+c^{22}_{77}-2c^{23}_{77}+c^{24}_{77}+c^{25}_{77}+2c^{26}_{77}+2c^{29}_{77},\ -1+c^{1}_{77}+c^{2}_{77}-c^{3}_{77}-3c^{4}_{77}+c^{5}_{77}+c^{6}_{77}-c^{7}_{77}-c^{8}_{77}+c^{9}_{77}-2c^{10}_{77}-2c^{11}_{77}+c^{12}_{77}+c^{13}_{77}-c^{14}_{77}+c^{16}_{77}-2c^{17}_{77}-c^{18}_{77}+5c^{22}_{77}+c^{23}_{77}-2c^{24}_{77}-3c^{25}_{77}-c^{29}_{77},\ -4-2c^{1}_{77}-2c^{2}_{77}+2c^{3}_{77}+c^{4}_{77}-2c^{5}_{77}-2c^{6}_{77}+c^{7}_{77}+2c^{8}_{77}-c^{9}_{77}-4c^{11}_{77}-2c^{12}_{77}-2c^{13}_{77}+2c^{14}_{77}-c^{15}_{77}-c^{16}_{77}+c^{18}_{77}+c^{19}_{77}-5c^{22}_{77}-c^{23}_{77}+2c^{25}_{77}-c^{26}_{77}+c^{29}_{77};\ -4-2c^{1}_{77}-2c^{2}_{77}+2c^{3}_{77}+c^{4}_{77}-2c^{5}_{77}-2c^{6}_{77}+c^{7}_{77}+2c^{8}_{77}-c^{9}_{77}-4c^{11}_{77}-2c^{12}_{77}-2c^{13}_{77}+2c^{14}_{77}-c^{15}_{77}-c^{16}_{77}+c^{18}_{77}+c^{19}_{77}-5c^{22}_{77}-c^{23}_{77}+2c^{25}_{77}-c^{26}_{77}+c^{29}_{77},\ 1+3c^{4}_{77}+2c^{7}_{77}-2c^{9}_{77}+c^{10}_{77}+7c^{11}_{77}+2c^{15}_{77}-2c^{16}_{77}+c^{17}_{77}+2c^{18}_{77}-2c^{19}_{77}+c^{22}_{77}-2c^{23}_{77}+c^{24}_{77}+c^{25}_{77}+2c^{26}_{77}+2c^{29}_{77};\ -1+c^{1}_{77}+c^{2}_{77}-c^{3}_{77}-3c^{4}_{77}+c^{5}_{77}+c^{6}_{77}-c^{7}_{77}-c^{8}_{77}+c^{9}_{77}-2c^{10}_{77}-2c^{11}_{77}+c^{12}_{77}+c^{13}_{77}-c^{14}_{77}+c^{16}_{77}-2c^{17}_{77}-c^{18}_{77}+5c^{22}_{77}+c^{23}_{77}-2c^{24}_{77}-3c^{25}_{77}-c^{29}_{77})$

Realization: condensation reductions of $\mathcal{Z}(\mathcal{NG}(\mathbb{Z}_7, 7))$.

19. $10^{10,168}_{4,1435.}$: $d_i = (1.0,\ 9.472,\ 9.472,\ 9.472,\ 9.472,\ 9.472,\ 9.472,\ 16.944,\ 16.944,\ 17.944)$

 $D^2 = 1435.541 = 720 + 320\sqrt{5}$

 $T = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{5}, \frac{4}{5}, 0)$,

 $S = (1,\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 8+4\sqrt{5},\ 8+4\sqrt{5},\ 9+4\sqrt{5};\ 15+6\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ 15+6\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ 15+6\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ 15+6\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ 15+6\sqrt{5},\ -5-2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ 15+6\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ 14+6\sqrt{5},\ -6-2\sqrt{5},\ -8-4\sqrt{5};\ 14+6\sqrt{5},\ -8-4\sqrt{5};\ 1)$

 Realization: Condensation of $\mathbb{Z}_5$ bosons in $SU(5)_5$, see [117].

20. $10^{20,676}_{0,1435.}$: $d_i = (1.0,\ 9.472,\ 9.472,\ 9.472,\ 9.472,\ 9.472,\ 9.472,\ 16.944,\ 16.944,\ 17.944)$

 $D^2 = 1435.541 = 720 + 320\sqrt{5}$

 $T = (0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{2}{5}, \frac{3}{5}, 0)$,

 $S = (1,\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 8+4\sqrt{5},\ 8+4\sqrt{5},\ 9+4\sqrt{5};\ 15+6\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ 15+6\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ -3-6s^{1}_{20}-4c^{2}_{20}+14s^{3}_{20},\ -3+6s^{1}_{20}-4c^{2}_{20}-14s^{3}_{20},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ -3-6s^{1}_{20}-4c^{2}_{20}+14s^{3}_{20},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ -3+6s^{1}_{20}-4c^{2}_{20}-14s^{3}_{20},\ -3-6s^{1}_{20}-4c^{2}_{20}+14s^{3}_{20},\ 0,\ 0,\ 5+2\sqrt{5};\ -3+6s^{1}_{20}-4c^{2}_{20}-14s^{3}_{20},\ 0,\ 0,\ 5+2\sqrt{5};\ -6-2\sqrt{5},\ 14+6\sqrt{5},\ -8-4\sqrt{5};\ -6-2\sqrt{5},\ -8-4\sqrt{5};\ 1)$

 Realization: Condensation reductions of $\mathcal{Z}(\mathcal{NG}(\mathbb{Z}_4 \times \mathbb{Z}_4, 16))$, see [118].

21. $10^{11,372}_{\frac{60}{11},43.10}$: $d_i = (1.0,\ 0.309,\ 1.682,\ 1.830,\ 2.397,\ 2.918,\ -1.88,\ -1.309,\ -2.513,\ -3.513)$

 $D^2 = 43.108 = 33 + 11c^{1}_{11} + 11c^{2}_{11} + 11c^{3}_{11} + 11c^{4}_{11}$

 $T = (0, \frac{4}{11}, \frac{10}{11}, \frac{1}{11}, \frac{7}{11}, \frac{5}{11}, \frac{2}{11}, \frac{6}{11}, \frac{3}{11}, \frac{9}{11})$,

 $S = (1,\ -1-c^{4}_{11},\ c^{1}_{11},\ 1+c^{2}_{11},\ 1+c^{1}_{11}+c^{3}_{11},\ 2+c^{1}_{11}+c^{2}_{11}+c^{3}_{11}+c^{4}_{11},\ -1-c^{1}_{11}-c^{3}_{11}-c^{4}_{11},\ c^{4}_{11},\ -c^{1}_{11}-c^{2}_{11},\ -\xi^{5}_{11};\ -c^{4}_{11},\ \xi^{5}_{11},\ c^{1}_{11},\ 1+c^{1}_{11}+c^{3}_{11}+c^{4}_{11},\ 1+c^{2}_{11},\ 2+c^{1}_{11}+c^{2}_{11}+c^{3}_{11}+c^{4}_{11},\ 1+c^{1}_{11}+c^{3}_{11},\ 1,\ c^{1}_{11}+c^{2}_{11};\ -1-c^{1}_{11}-c^{3}_{11}-c^{4}_{11},\ -1-c^{1}_{11}-c^{3}_{11},\ -1,\ -c^{4}_{11},\ -1-c^{4}_{11},\ c^{1}_{11}+c^{2}_{11},\ 1+c^{2}_{11},\ -2-c^{1}_{11}-c^{2}_{11}-c^{3}_{11}-c^{4}_{11};\ -c^{4}_{11},\ c^{1}_{11}+c^{2}_{11},\ -1-c^{4}_{11},\ 1,\ -\xi^{5}_{11},\ 2+c^{1}_{11}+c^{2}_{11}+c^{3}_{11}+c^{4}_{11},\ 1+c^{1}_{11}+c^{3}_{11}+c^{4}_{11};\ 1+c^{2}_{11},\ -\xi^{5}_{11},\ -c^{1}_{11},\ 2+c^{1}_{11}+c^{2}_{11}+c^{3}_{11}+c^{4}_{11},\ c^{4}_{11},\ -1-c^{4}_{11};\ 1,\ -c^{1}_{11}-c^{2}_{11},\ -c^{1}_{11},\ -1-c^{1}_{11}-c^{3}_{11}-c^{4}_{11},\ 1+c^{1}_{11}+c^{3}_{11};\ \xi^{5}_{11},\ -1-c^{2}_{11},\ -1-c^{1}_{11}-c^{3}_{11},\ c^{4}_{11};\ -1-c^{1}_{11}-c^{3}_{11}-c^{4}_{11},\ 1+c^{4}_{11},\ 1;\ \xi^{5}_{11},\ -c^{1}_{11};\ 1+c^{2}_{11})$

 Realization: $PSO(5)_{\frac{5}{2}}$, *i.e.* the adjoint subcategory of the non-unitary braided fusion category $SO(5)_{\frac{5}{2}}$ corresponding to $U_q\mathfrak{so}_5$ with $q = e^{\pi i/11}$, see [119].

7.5 Rank 11

1. $11^{11,568}_{2,11.}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0)$

 $D^2 = 11.0 = 11$

$T = (0, \frac{1}{11}, \frac{1}{11}, \frac{3}{11}, \frac{3}{11}, \frac{4}{11}, \frac{4}{11}, \frac{5}{11}, \frac{5}{11}, \frac{9}{11}, \frac{9}{11})$,

$S = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1;\ \ -\zeta_{22}^{7},\ \zeta_{11}^{2},\ -\zeta_{22}^{9},\ \zeta_{11}^{1},\ -\zeta_{22}^{3},\ \zeta_{11}^{4},\ -\zeta_{22}^{5},\ \zeta_{11}^{3},\ -\zeta_{22}^{1},\ \zeta_{11}^{5};\ \ -\zeta_{22}^{7},\ \zeta_{11}^{1},\ -\zeta_{22}^{9},\ \zeta_{11}^{4},\ -\zeta_{22}^{3},\ \zeta_{11}^{3},\ -\zeta_{22}^{5},\ \zeta_{11}^{5},\ -\zeta_{22}^{1};\ \ \zeta_{11}^{5},\ -\zeta_{22}^{1},\ -\zeta_{22}^{7},\ \zeta_{11}^{2},\ \zeta_{11}^{4},\ -\zeta_{22}^{3},\ \zeta_{11}^{3},\ -\zeta_{22}^{5};\ \ \zeta_{11}^{5},\ \zeta_{11}^{2},\ -\zeta_{22}^{7},\ -\zeta_{22}^{3},\ \zeta_{11}^{4},\ -\zeta_{22}^{5},\ \zeta_{11}^{3};\ \ \zeta_{11}^{3},\ -\zeta_{22}^{5},\ \zeta_{11}^{5},\ -\zeta_{22}^{1},\ \zeta_{11}^{1},\ -\zeta_{22}^{9};\ \ \zeta_{11}^{3},\ -\zeta_{22}^{1},\ \zeta_{11}^{5},\ -\zeta_{22}^{9},\ \zeta_{11}^{1};\ \ \zeta_{11}^{1},\ -\zeta_{22}^{9},\ -\zeta_{22}^{7},\ \zeta_{11}^{2};\ \ \zeta_{11}^{1},\ \zeta_{11}^{2},\ -\zeta_{22}^{7};\ \ \zeta_{11}^{4},\ -\zeta_{22}^{3};\ \ \zeta_{11}^{4})$

Realization: $U(11)_1$.

2. $11_{1,32.}^{16,245}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0)$

$D^2 = 32.0 = 32$

$T = (0, 0, 0, 0, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{9}{16}, \frac{9}{16}, \frac{9}{16})$,

$S = (1,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2;\ \ 1,\ 1,\ 1,\ 2,\ -2,\ -2,\ 2,\ -2,\ -2,\ 2;\ \ 1,\ 1,\ 2,\ -2,\ 2,\ -2,\ -2,\ 2,\ -2;\ \ 1,\ 2,\ 2,\ -2,\ -2,\ 2,\ -2,\ -2;\ \ -4,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0;\ \ 2\sqrt{2},\ 0,\ 0,\ -2\sqrt{2},\ 0,\ 0;\ \ 2\sqrt{2},\ 0,\ 0,\ -2\sqrt{2},\ 0;\ \ 2\sqrt{2},\ 0,\ 0,\ -2\sqrt{2};\ \ 2\sqrt{2},\ 0,\ 0;\ \ 2\sqrt{2},\ 0;\ \ 2\sqrt{2})$

Realization: O_4 or $\overline{SO(16)_2}$ or Abelian anyon condensations of $\mathcal{D}^3(Q_8)$.

3. $11_{1,32.}^{16,157}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0)$

$D^2 = 32.0 = 32$

$T = (0, 0, 0, 0, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{5}{16}, \frac{9}{16}, \frac{9}{16}, \frac{13}{16})$,

$S = (1,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2;\ \ 1,\ 1,\ 1,\ 2,\ -2,\ -2,\ 2,\ -2,\ -2,\ 2;\ \ 1,\ 1,\ 2,\ -2,\ 2,\ -2,\ -2,\ 2,\ -2;\ \ 1,\ 2,\ 2,\ -2,\ -2,\ 2,\ -2,\ -2;\ \ -4,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0;\ \ 2\sqrt{2},\ 0,\ 0,\ -2\sqrt{2},\ 0,\ 0;\ \ 2\sqrt{2},\ 0,\ 0,\ -2\sqrt{2},\ 0;\ \ -2\sqrt{2},\ 0,\ 0,\ 2\sqrt{2};\ \ 2\sqrt{2},\ 0,\ 0;\ \ 2\sqrt{2},\ 0;\ \ -2\sqrt{2})$

Realization: Abelian anyon condensations of O_4 or $\mathcal{D}^1(Q_8)$ or $\overline{\mathcal{D}^3(Q_8)}$.

4. $11_{2,60.}^{120,157}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.872,\ 3.872)$

$D^2 = 60.0 = 60$

$T = (0, 0, \frac{1}{3}, \frac{1}{5}, \frac{4}{5}, \frac{2}{15}, \frac{2}{15}, \frac{8}{15}, \frac{8}{15}, \frac{1}{8}, \frac{5}{8})$,

$S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \sqrt{15},\ \sqrt{15};\ \ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ -\sqrt{15},\ -\sqrt{15};\ \ -2,\ 4,\ 4,\ -2,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ -1-\sqrt{5},\ -1+\sqrt{5},\ -1+\sqrt{5},\ -1+\sqrt{5},\ -1-\sqrt{5},\ -1-\sqrt{5},\ 0,\ 0;\ \ -1-\sqrt{5},\ -1-\sqrt{5},\ -1-\sqrt{5},\ -1+\sqrt{5},\ -1+\sqrt{5},\ 0,\ 0;\ \ 2c_{15}^{4},\ 2c_{15}^{1},\ 2c_{15}^{7},\ 2c_{15}^{2},\ 0,\ 0;\ \ 2c_{15}^{4},\ 2c_{15}^{2},\ 2c_{15}^{7},\ 0,\ 0;\ \ 2c_{15}^{1},\ 2c_{15}^{4},\ 0,\ 0;\ \ 2c_{15}^{1},\ 0,\ 0;\ \ \sqrt{15},\ -\sqrt{15};\ \ \sqrt{15})$

Connected to the orbit of $11_{2,60.}^{120,364}$ via a change of spherical structure.

Realization: Galois conjugation of $SO(15)_2$.

5. $11_{2,60.}^{120,364}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.872,\ 3.872)$

$D^2 = 60.0 = 60$

$T = (0, 0, \frac{1}{3}, \frac{1}{5}, \frac{4}{5}, \frac{2}{15}, \frac{2}{15}, \frac{8}{15}, \frac{8}{15}, \frac{3}{8}, \frac{7}{8})$,

$S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \sqrt{15},\ \sqrt{15};\ \ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ -\sqrt{15},\ -\sqrt{15};\ \ -2,\ 4,\ 4,\ -2,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ -1-\sqrt{5},\ -1+\sqrt{5},\ -1+\sqrt{5},\ -1+\sqrt{5},\ -1-\sqrt{5},\ -1-\sqrt{5},\ 0,\ 0;\ \ -1-\sqrt{5},\ -1-\sqrt{5},\ -1-\sqrt{5},\ -1+\sqrt{5},\ -1+\sqrt{5},\ 0,\ 0;\ \ 2c_{15}^{4},\ 2c_{15}^{1},\ 2c_{15}^{7},\ 2c_{15}^{2},\ 0,\ 0;\ \ 2c_{15}^{4},\ 2c_{15}^{2},\ 2c_{15}^{7},\ 0,\ 0;\ \ 2c_{15}^{1},\ 2c_{15}^{4},\ 0,\ 0;\ \ 2c_{15}^{1},\ 0,\ 0;\ \ -\sqrt{15},\ \sqrt{15};\ \ -\sqrt{15})$

Connected to the orbit of $11_{2,60.}^{120,157}$ via a change of spherical structure.

Realization: Galois conjugation of Abelian anyon condensation of $SO(30)_2$.

6. $11_{\frac{13}{2},89.56}^{48,108}$: $d_i = (1.0,\ 1.0,\ 1.931,\ 1.931,\ 2.732,\ 2.732,\ 3.346,\ 3.346,\ 3.732,\ 3.732,\ 3.863)$

$D^2 = 89.569 = 48 + 24\sqrt{3}$

$T = (0, \frac{1}{2}, \frac{1}{16}, \frac{1}{16}, \frac{1}{3}, \frac{5}{6}, \frac{13}{16}, \frac{13}{16}, 0, \frac{1}{2}, \frac{19}{48})$,

$S = (1,\ 1,\ c_{24}^{1},\ c_{24}^{1},\ 1+\sqrt{3},\ 1+\sqrt{3},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 2c_{24}^{1};\ \ 1,\ -c_{24}^{1},\ -c_{24}^{1},\ 1+\sqrt{3},\ 1+\sqrt{3},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -2c_{24}^{1};\ \ (\frac{-3-3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ (\frac{3+3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ -2c_{24}^{1},\ 2c_{24}^{1},\ (\frac{3+3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ (\frac{-3-3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ -c_{24}^{1},\ c_{24}^{1},\ 0;\ \ (\frac{-3-3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ -2c_{24}^{1},\ 2c_{24}^{1},\ (\frac{-3-3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ (\frac{3+3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ -c_{24}^{1},\ c_{24}^{1},\ 0;\ \ 1+\sqrt{3},\ 1+\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ -1-\sqrt{3},\ 2c_{24}^{1};\ \ 1+\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ -1-\sqrt{3},\ -2c_{24}^{1};\ \ (\frac{3+3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ (\frac{-3-3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 0;\ \ (\frac{3+3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 0;\ \ 1,\ 1,\ -2c_{24}^{1};\ \ 1,\ 2c_{24}^{1};\ \ 0)$

Connected to the orbit of $11_{\frac{7}{2},89.56}^{48,628}$ via a change of spherical structure.

Realization: Abelian anyon condensation of $\overline{SU}(2)_{10}$ or $SU(10)_2$ or $Sp(20)_1$.

7. $11^{48,628}_{\frac{7}{2},89.56}$: $d_i = (1.0, 1.0, 1.931, 1.931, 2.732, 2.732, 3.346, 3.346, 3.732, 3.732, 3.863)$

$D^2 = 89.569 = 48 + 24\sqrt{3}$

$T = (0, \frac{1}{2}, \frac{3}{16}, \frac{3}{16}, \frac{2}{3}, \frac{1}{6}, \frac{7}{16}, \frac{7}{16}, 0, \frac{1}{2}, \frac{41}{48})$,

$S = (1,\ 1,\ c^1_{24},\ c^1_{24},\ 1+\sqrt{3},\ 1+\sqrt{3},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 2c^1_{24};\ \ 1,\ -c^1_{24},\ -c^1_{24},\ 1+\sqrt{3},\ 1+\sqrt{3},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -2c^1_{24};\ \ (\frac{-3-3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ (\frac{3+3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ -2c^1_{24},\ 2c^1_{24},\ (\frac{3+3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ (\frac{-3-3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ -c^1_{24},\ c^1_{24},\ 0;\ \ (\frac{-3-3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ -2c^1_{24},\ 2c^1_{24},\ (\frac{-3-3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ (\frac{3+3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ -c^1_{24},\ c^1_{24},\ 0;\ \ 1+\sqrt{3},\ 1+\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ -1-\sqrt{3},\ 2c^1_{24};\ \ 1+\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ -1-\sqrt{3},\ -2c^1_{24};\ \ (\frac{3+3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ (\frac{-3-3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 0;\ \ (\frac{3+3\sqrt{3}}{\sqrt{6}})\mathrm{i},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 0;\ \ 1,\ 1,\ -2c^1_{24};\ \ 1,\ 2c^1_{24};\ \ 0)$

Connected to the orbit of $11^{48,108}_{\frac{13}{2},89.56}$ via a change of spherical structure.

Realization: Abelian anyon condensation of $SU(2)_{10}$ or $\overline{SU}(10)_2$ or $\overline{Sp}(20)_1$.

8. $11^{48,214}_{\frac{5}{2},89.56}$: $d_i = (1.0, 1.0, 1.931, 1.931, 2.732, 2.732, 3.346, 3.346, 3.732, 3.732, 3.863)$

$D^2 = 89.569 = 48 + 24\sqrt{3}$

$T = (0, \frac{1}{2}, \frac{1}{16}, \frac{1}{16}, \frac{2}{3}, \frac{1}{6}, \frac{5}{16}, \frac{5}{16}, 0, \frac{1}{2}, \frac{35}{48})$,

$S = (1,\ 1,\ c^1_{24},\ c^1_{24},\ 1+\sqrt{3},\ 1+\sqrt{3},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 2c^1_{24};\ \ 1,\ -c^1_{24},\ -c^1_{24},\ 1+\sqrt{3},\ 1+\sqrt{3},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -2c^1_{24};\ \ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ -2c^1_{24},\ 2c^1_{24},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ -c^1_{24},\ c^1_{24},\ 0;\ \ \frac{3+3\sqrt{3}}{\sqrt{6}},\ -2c^1_{24},\ 2c^1_{24},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ -c^1_{24},\ c^1_{24},\ 0;\ \ 1+\sqrt{3},\ 1+\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ -1-\sqrt{3},\ 2c^1_{24};\ \ 1+\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ -1-\sqrt{3},\ -2c^1_{24};\ \ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 0;\ \ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 0;\ \ 1,\ 1,\ -2c^1_{24};\ \ 1,\ 2c^1_{24};\ \ 0)$

Connected to the orbit of $11^{48,311}_{\frac{15}{2},89.56}$ via a change of spherical structure.

Realization: $SU(2)_{10}$ or Abelian anyon condensation of $\overline{SU}(10)_2$ or $\overline{Sp}(20)_1$.

9. $11^{48,311}_{\frac{15}{2},89.56}$: $d_i = (1.0, 1.0, 1.931, 1.931, 2.732, 2.732, 3.346, 3.346, 3.732, 3.732, 3.863)$

$D^2 = 89.569 = 48 + 24\sqrt{3}$

$T = (0, \frac{1}{2}, \frac{3}{16}, \frac{3}{16}, \frac{1}{3}, \frac{5}{6}, \frac{15}{16}, \frac{15}{16}, 0, \frac{1}{2}, \frac{25}{48})$,

$S = (1,\ 1,\ c^1_{24},\ c^1_{24},\ 1+\sqrt{3},\ 1+\sqrt{3},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 2c^1_{24};\ \ 1,\ -c^1_{24},\ -c^1_{24},\ 1+\sqrt{3},\ 1+\sqrt{3},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -2c^1_{24};\ \ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ -2c^1_{24},\ 2c^1_{24},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ -c^1_{24},\ c^1_{24},\ 0;\ \ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ -2c^1_{24},\ 2c^1_{24},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ -c^1_{24},\ c^1_{24},\ 0;\ \ 1+\sqrt{3},\ 1+\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ -1-\sqrt{3},\ 2c^1_{24};\ \ 1+\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ -1-\sqrt{3},\ -2c^1_{24};\ \ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 0;\ \ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{3+3\sqrt{3}}{\sqrt{6}},\ \frac{-3-3\sqrt{3}}{\sqrt{6}},\ 0;\ \ 1,\ 1,\ -2c^1_{24};\ \ 1,\ 2c^1_{24};\ \ 0)$

Connected to the orbit of $11^{48,214}_{\frac{5}{2},89.56}$ via a change of spherical structure.

Realization: Abelian anyon condensation of $\overline{SU}(2)_{10}$ or $SU(10)_2$ or $Sp(20)_1$.

10. $11^{23,306}_{\frac{144}{23},310.1}$: $d_i = (1.0, 1.981, 2.925, 3.815, 4.634, 5.367, 5.999, 6.520, 6.919, 7.190, 7.326)$

$D^2 = 310.117 = 66 + 55c^1_{23} + 45c^2_{23} + 36c^3_{23} + 28c^4_{23} + 21c^5_{23} + 15c^6_{23} + 10c^7_{23} + 6c^8_{23} + 3c^9_{23} + c^{10}_{23}$

$T = (0, \frac{5}{23}, \frac{21}{23}, \frac{2}{23}, \frac{17}{23}, \frac{20}{23}, \frac{11}{23}, \frac{13}{23}, \frac{3}{23}, \frac{4}{23}, \frac{16}{23})$,

$S = (1,\ -c^{11}_{23},\ \xi^3_{23},\ \xi^{19}_{23},\ \xi^5_{23},\ \xi^{17}_{23},\ \xi^7_{23},\ \xi^{15}_{23},\ \xi^9_{23},\ \xi^{13}_{23},\ \xi^{11}_{23};\ \ -\xi^{19}_{23},\ \xi^{17}_{23},\ -\xi^{15}_{23},\ \xi^{13}_{23},\ -\xi^{11}_{23},\ \xi^9_{23},\ -\xi^7_{23},\ \xi^5_{23},\ -\xi^3_{23},\ 1;\ \ \xi^9_{23},\ \xi^{11}_{23},\ \xi^{15}_{23},\ \xi^5_{23},\ -c^{11}_{23},\ -1,\ -\xi^{19}_{23},\ -\xi^7_{23},\ -\xi^{13}_{23};\ \ -\xi^7_{23},\ \xi^3_{23},\ 1,\ -\xi^5_{23},\ \xi^9_{23},\ -\xi^{13}_{23},\ \xi^{17}_{23},\ c^{11}_{23};\ \ c^{11}_{23},\ -\xi^7_{23},\ -\xi^{11}_{23},\ -\xi^{17}_{23},\ -1,\ \xi^{19}_{23},\ \xi^9_{23};\ \ \xi^{13}_{23},\ -\xi^{19}_{23},\ c^{11}_{23},\ \xi^{15}_{23},\ -\xi^9_{23},\ \xi^3_{23};\ \ \xi^3_{23},\ \xi^{13}_{23},\ \xi^{17}_{23},\ -1,\ -\xi^{15}_{23};\ \ -\xi^5_{23},\ -\xi^3_{23},\ \xi^{11}_{23},\ -\xi^{19}_{23};\ \ -\xi^{11}_{23},\ c^{11}_{23},\ \xi^7_{23};\ \ -\xi^{15}_{23},\ \xi^5_{23};\ \ -\xi^{17}_{23})$

Realization: Abelian anyon condensation of $\overline{SU}(2)_{21}$ or $SU(21)_2$ or $Sp(42)_1$.

11. $11^{19,306}_{\frac{54}{19},696.5}$: $d_i = (1.0, 2.972, 4.864, 6.54, 6.54, 6.623, 8.201, 9.556, 10.650, 11.453, 11.944)$

$D^2 = 696.547 = 171 + 152c^1_{19} + 133c^2_{19} + 114c^3_{19} + 95c^4_{19} + 76c^5_{19} + 57c^6_{19} + 38c^7_{19} + 19c^8_{19}$

$T = (0, \frac{1}{19}, \frac{3}{19}, \frac{7}{19}, \frac{7}{19}, \frac{6}{19}, \frac{10}{19}, \frac{15}{19}, \frac{2}{19}, \frac{9}{19}, \frac{17}{19})$,

$S = (1,\ 2+c^1_{19}+c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19}+c^7_{19}+c^8_{19},\ 2+2c^1_{19}+c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19}+c^7_{19}+c^8_{19},\ \xi^9_{19},$
$\xi^9_{19},\ 2+2c^1_{19}+c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19}+c^7_{19},\ 2+2c^1_{19}+2c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19}+c^7_{19},\ 2+2c^1_{19}+$
$2c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19},\ 2+2c^1_{19}+2c^2_{19}+2c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19},\ 2+2c^1_{19}+2c^2_{19}+2c^3_{19}+c^4_{19}+c^5_{19},$
$2+2c^1_{19}+2c^2_{19}+2c^3_{19}+2c^4_{19}+c^5_{19};\ \ 2+2c^1_{19}+2c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19}+c^7_{19},\ 2+2c^1_{19}+2c^2_{19}+2c^3_{19}+$
$c^4_{19}+c^5_{19},\ -\xi^9_{19},\ -\xi^9_{19},\ 2+2c^1_{19}+2c^2_{19}+2c^3_{19}+2c^4_{19}+c^5_{19},\ 2+2c^1_{19}+2c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19},$
$2+2c^1_{19}+c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19}+c^7_{19}+c^8_{19},\ -1,\ -2-2c^1_{19}-c^2_{19}-c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19}-c^7_{19},$
$-2-2c^1_{19}-2c^2_{19}-2c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19};\ \ 2+2c^1_{19}+2c^2_{19}+2c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19},\ \xi^9_{19},\ \xi^9_{19},$
$2+c^1_{19}+c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19}+c^7_{19}+c^8_{19},\ -2-2c^1_{19}-c^2_{19}-c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19}-c^7_{19},$
$-2-2c^1_{19}-2c^2_{19}-2c^3_{19}-2c^4_{19}-c^5_{19},\ -2-2c^1_{19}-2c^2_{19}-c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19},\ -1,\ 2+2c^1_{19}+2c^2_{19}+$
$c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19}+c^7_{19};\ \ s^1_{19}+s^2_{19}+s^3_{19}+s^4_{19}+2\zeta^5_{19}-\zeta^{-5}_{19}+\zeta^6_{19}+2\zeta^7_{19}-\zeta^{-7}_{19}+2\zeta^8_{19}-\zeta^{-8}_{19}+\zeta^9_{19},$
$-1-2\zeta^1_{19}-2\zeta^2_{19}-2\zeta^3_{19}-2\zeta^4_{19}-2\zeta^5_{19}+\zeta^{-5}_{19}-\zeta^6_{19}-2\zeta^7_{19}+\zeta^{-7}_{19}-2\zeta^8_{19}+\zeta^{-8}_{19}-\zeta^9_{19},\ -\xi^9_{19},\ \xi^9_{19},\ -\xi^9_{19},$
$\xi^9_{19},\ -\xi^9_{19},\ \xi^9_{19};\ \ s^1_{19}+s^2_{19}+s^3_{19}+s^4_{19}+2\zeta^5_{19}-\zeta^{-5}_{19}+\zeta^6_{19}+2\zeta^7_{19}-\zeta^{-7}_{19}+2\zeta^8_{19}-\zeta^{-8}_{19}+\zeta^9_{19},\ -\xi^9_{19},$
$\xi^9_{19},\ -\xi^9_{19},\ \xi^9_{19},\ -\xi^9_{19},\ \xi^9_{19};\ \ -2-2c^1_{19}-2c^2_{19}-c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19},\ -2-2c^1_{19}-2c^2_{19}-2c^3_{19}-$
$c^4_{19}-c^5_{19}-c^6_{19},\ 1,\ 2+2c^1_{19}+2c^2_{19}+2c^3_{19}+c^4_{19}+c^5_{19},\ 2+2c^1_{19}+2c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19}+c^7_{19},$
$-2-2c^1_{19}-c^2_{19}-c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19}-c^7_{19}-c^8_{19};\ \ 2+2c^1_{19}+c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19}+c^7_{19}+c^8_{19},$
$2+2c^1_{19}+2c^2_{19}+2c^3_{19}+c^4_{19}+c^5_{19},\ -2-c^1_{19}-c^2_{19}-c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19}-c^7_{19}-c^8_{19},$
$-2-2c^1_{19}-2c^2_{19}-2c^3_{19}-2c^4_{19}-c^5_{19},\ 1;\ \ -2-2c^1_{19}-c^2_{19}-c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19}-c^7_{19},$
$-2-2c^1_{19}-2c^2_{19}-c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19}-c^7_{19},\ 2+2c^1_{19}+2c^2_{19}+2c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19},$
$2+c^1_{19}+c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19}+c^7_{19}+c^8_{19};\ \ 2+2c^1_{19}+2c^2_{19}+2c^3_{19}+2c^4_{19}+c^5_{19},$
$-2-2c^1_{19}-c^2_{19}-c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19}-c^7_{19}-c^8_{19},\ -2-2c^1_{19}-c^2_{19}-c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19}-c^7_{19};$
$-2-c^1_{19}-c^2_{19}-c^3_{19}-c^4_{19}-c^5_{19}-c^6_{19}-c^7_{19}-c^8_{19},\ 2+2c^1_{19}+2c^2_{19}+c^3_{19}+c^4_{19}+c^5_{19}+c^6_{19};$
$-2-2c^1_{19}-2c^2_{19}-2c^3_{19}-c^4_{19}-c^5_{19})$

Realization: Abelian anyon condensation of $\overline{SO}(18)_3$.

12. $11^{48,634}_{3,1337.}\ :\ d_i = (1.0,\ 6.464,\ 6.464,\ 7.464,\ 7.464,\ 12.928,\ 12.928,\ 12.928,\ 13.928,\ 14.928,\ 14.928)$

$D^2 = 1337.107 = 672 + 384\sqrt{3}$

$T = (0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{16}, \frac{11}{16}, 0, \frac{1}{3}, \frac{7}{12}),$

$S = (1,\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ 4+2\sqrt{3},\ 4+2\sqrt{3},\ 6+4\sqrt{3},\ 6+4\sqrt{3},\ 6+4\sqrt{3},\ 7+4\sqrt{3},\ 8+4\sqrt{3},$
$8+4\sqrt{3};\ \ -7+4\zeta^1_{12}-8\zeta^{-1}_{12}+8\zeta^2_{12},\ 1-8\zeta^1_{12}+4\zeta^{-1}_{12}-8\zeta^2_{12},\ (-6-4\sqrt{3})\mathrm{i},\ (6+4\sqrt{3})\mathrm{i},\ -6-4\sqrt{3},$
$6+4\sqrt{3},\ 6+4\sqrt{3},\ -3-2\sqrt{3},\ 0,\ 0;\ \ -7+4\zeta^1_{12}-8\zeta^{-1}_{12}+8\zeta^2_{12},\ (6+4\sqrt{3})\mathrm{i},\ (-6-4\sqrt{3})\mathrm{i},$
$-6-4\sqrt{3},\ 6+4\sqrt{3},\ 6+4\sqrt{3},\ -3-2\sqrt{3},\ 0,\ 0;\ \ (-8-4\sqrt{3})\zeta^1_6,\ (8+4\sqrt{3})\zeta^1_3,\ 0,\ 0,\ 0,\ 4+2\sqrt{3},$
$8+4\sqrt{3},\ -8-4\sqrt{3};\ \ (-8-4\sqrt{3})\zeta^1_6,\ 0,\ 0,\ 0,\ 4+2\sqrt{3},\ 8+4\sqrt{3},\ -8-4\sqrt{3};\ \ 12+8\sqrt{3},\ 0,\ 0,$
$-6-4\sqrt{3},\ 0,\ 0;\ \ \frac{24+12\sqrt{3}}{\sqrt{6}},\ \frac{-24-12\sqrt{3}}{\sqrt{6}},\ -6-4\sqrt{3},\ 0,\ 0;\ \ \frac{24+12\sqrt{3}}{\sqrt{6}},\ -6-4\sqrt{3},\ 0,\ 0;\ \ 1,\ 8+4\sqrt{3},$
$8+4\sqrt{3};\ \ -8-4\sqrt{3},\ -8-4\sqrt{3};\ \ 8+4\sqrt{3})$

Realization: may be condensation reductions of $\mathcal{Z}(\mathcal{NG}(\mathbb{Z}_{12}, 12))$.

13. $11^{35,581}_{\frac{32}{5},1964.}\ :\ d_i = (1.0,\ 8.807,\ 8.807,\ 8.807,\ 11.632,\ 13.250,\ 14.250,\ 14.250,\ 14.250,\ 19.822,\ 20.440)$

$D^2 = 1964.590 = 910-280c^1_{35}+280c^2_{35}+280c^3_{35}+175c^4_{35}+280c^5_{35}-105c^6_{35}+490c^7_{35}-280c^8_{35}+$
$175c^9_{35}+280c^{10}_{35}$

$T = (0, \frac{2}{35}, \frac{22}{35}, \frac{32}{35}, \frac{1}{5}, 0, \frac{3}{7}, \frac{5}{7}, \frac{6}{7}, \frac{3}{5}, \frac{1}{5}),$

$S = (1,\ 4-c^1_{35}+c^2_{35}+c^3_{35}+c^4_{35}+c^5_{35}+2c^7_{35}-c^8_{35}+c^9_{35}+c^{10}_{35},\ 4-c^1_{35}+c^2_{35}+c^3_{35}+c^4_{35}+c^5_{35}+2c^7_{35}-c^8_{35}+$
$c^9_{35}+c^{10}_{35},\ 4-c^1_{35}+c^2_{35}+c^3_{35}+c^4_{35}+c^5_{35}+2c^7_{35}-c^8_{35}+c^9_{35}+c^{10}_{35},\ 5-2c^1_{35}+2c^2_{35}+2c^3_{35}+c^4_{35}+2c^5_{35}-$
$c^6_{35}+3c^7_{35}-2c^8_{35}+c^9_{35}+2c^{10}_{35},\ 6-2c^1_{35}+2c^2_{35}+2c^3_{35}+c^4_{35}+2c^5_{35}-c^6_{35}+4c^7_{35}-2c^8_{35}+c^9_{35}+2c^{10}_{35},$
$7-2c^1_{35}+2c^2_{35}+2c^3_{35}+c^4_{35}+2c^5_{35}-c^6_{35}+4c^7_{35}-2c^8_{35}+c^9_{35}+2c^{10}_{35},\ 7-2c^1_{35}+2c^2_{35}+2c^3_{35}+c^4_{35}+2c^5_{35}-$
$c^6_{35}+4c^7_{35}-2c^8_{35}+c^9_{35}+2c^{10}_{35},\ 7-2c^1_{35}+2c^2_{35}+2c^3_{35}+c^4_{35}+2c^5_{35}-c^6_{35}+4c^7_{35}-2c^8_{35}+c^9_{35}+2c^{10}_{35},$
$9-3c^1_{35}+3c^2_{35}+3c^3_{35}+2c^4_{35}+3c^5_{35}-c^6_{35}+4c^7_{35}-3c^8_{35}+2c^9_{35}+3c^{10}_{35},\ 9-3c^1_{35}+3c^2_{35}+3c^3_{35}+2c^4_{35}+$
$3c^5_{35}-c^6_{35}+5c^7_{35}-3c^8_{35}+2c^9_{35}+3c^{10}_{35};\ \ 1-c^1_{35}+4c^2_{35}-2c^3_{35}+c^4_{35}+2c^5_{35}-c^6_{35}+c^7_{35}+2c^9_{35}-3c^{10}_{35}+2c^{11}_{35},$
$1+5c^1_{35}+4c^3_{35}+3c^4_{35}+c^5_{35}+4c^6_{35}+2c^8_{35}+3c^{10}_{35}+c^{11}_{35},\ 5-6c^1_{35}-2c^2_{35}-3c^4_{35}-c^5_{35}-4c^6_{35}+3c^7_{35}-$
$4c^8_{35}-c^9_{35}+2c^{10}_{35}-3c^{11}_{35},\ 7-2c^1_{35}+2c^2_{35}+2c^3_{35}+c^4_{35}+2c^5_{35}-c^6_{35}+4c^7_{35}-2c^8_{35}+c^9_{35}+2c^{10}_{35},$
$4-c^1_{35}+c^2_{35}+c^3_{35}+c^4_{35}+c^5_{35}+2c^7_{35}-c^8_{35}+c^9_{35}+c^{10}_{35},\ -1-3c^1_{35}-2c^3_{35}-2c^4_{35}-c^5_{35}-2c^6_{35}-c^8_{35}-2c^{10}_{35},$
$-3+4c^1_{35}+c^2_{35}+2c^4_{35}+c^5_{35}+2c^6_{35}-2c^7_{35}+2c^8_{35}+c^9_{35}-c^{10}_{35}+2c^{11}_{35},\ -2c^2_{35}+c^3_{35}-c^4_{35}-c^5_{35}-2c^9_{35}+$
$2c^{10}_{35}-2c^{11}_{35},\ 0,\ -7+2c^1_{35}-2c^2_{35}-2c^3_{35}-c^4_{35}-2c^5_{35}+c^6_{35}-4c^7_{35}+2c^8_{35}-c^9_{35}-2c^{10}_{35};\ \ 5-6c^1_{35}-2c^2_{35}-$
$3c^4_{35}-c^5_{35}-4c^6_{35}+3c^7_{35}-4c^8_{35}-c^9_{35}+2c^{10}_{35}-3c^{11}_{35},\ 1-c^1_{35}+4c^2_{35}-2c^3_{35}+c^4_{35}+2c^5_{35}-c^6_{35}+c^7_{35}+2c^9_{35}-$
$3c^{10}_{35}+2c^{11}_{35},\ 7-2c^1_{35}+2c^2_{35}+2c^3_{35}+c^4_{35}+2c^5_{35}-c^6_{35}+4c^7_{35}-2c^8_{35}+c^9_{35}+2c^{10}_{35},\ 4-c^1_{35}+c^2_{35}+c^3_{35}+$
$c^4_{35}+c^5_{35}+2c^7_{35}-c^8_{35}+c^9_{35}+c^{10}_{35},\ -3+4c^1_{35}+c^2_{35}+2c^4_{35}+c^5_{35}+2c^6_{35}-2c^7_{35}+2c^8_{35}+c^9_{35}-c^{10}_{35}+2c^{11}_{35},$
$-2c^2_{35}+c^3_{35}-c^4_{35}-c^5_{35}-2c^9_{35}+2c^{10}_{35}-2c^{11}_{35},\ -1-3c^1_{35}-2c^3_{35}-2c^4_{35}-c^5_{35}-2c^6_{35}-c^8_{35}-2c^{10}_{35},$
$0,\ -7+2c^1_{35}-2c^2_{35}-2c^3_{35}-c^4_{35}-2c^5_{35}+c^6_{35}-4c^7_{35}+2c^8_{35}-c^9_{35}-2c^{10}_{35};\ \ 1+5c^1_{35}+4c^3_{35}+3c^4_{35}+$

$c_{35}^{5}+4c_{35}^{6}+2c_{35}^{8}+3c_{35}^{10}+c_{35}^{11}$, $7-2c_{35}^{1}+2c_{35}^{2}+2c_{35}^{3}+c_{35}^{4}+2c_{35}^{5}-c_{35}^{6}+4c_{35}^{7}-2c_{35}^{8}+c_{35}^{9}+2c_{35}^{10}$, $4-c_{35}^{1}+c_{35}^{2}+c_{35}^{3}+c_{35}^{4}+c_{35}^{5}+2c_{35}^{7}-c_{35}^{8}+c_{35}^{9}+c_{35}^{10}$, $-2c_{35}^{2}+c_{35}^{3}-c_{35}^{4}-c_{35}^{5}-2c_{35}^{9}+2c_{35}^{10}-2c_{35}^{11}$, $-1-3c_{35}^{1}-2c_{35}^{3}-2c_{35}^{4}-c_{35}^{5}-2c_{35}^{6}-c_{35}^{8}-2c_{35}^{10}$, $-3+4c_{35}^{1}+c_{35}^{2}+2c_{35}^{4}+c_{35}^{5}+2c_{35}^{6}-2c_{35}^{7}+2c_{35}^{8}+c_{35}^{9}-c_{35}^{10}+2c_{35}^{11}$, 0, $-7+2c_{35}^{1}-2c_{35}^{2}-2c_{35}^{3}-c_{35}^{4}-2c_{35}^{5}+c_{35}^{6}-4c_{35}^{7}+2c_{35}^{8}-c_{35}^{9}-2c_{35}^{10}$; $-6+2c_{35}^{1}-2c_{35}^{2}-2c_{35}^{3}-c_{35}^{4}-2c_{35}^{5}+c_{35}^{6}-4c_{35}^{7}+2c_{35}^{8}-c_{35}^{9}-2c_{35}^{10}$, $-9+3c_{35}^{1}-3c_{35}^{2}-3c_{35}^{3}-2c_{35}^{4}-3c_{35}^{5}+c_{35}^{6}-5c_{35}^{7}+3c_{35}^{8}-2c_{35}^{9}-3c_{35}^{10}$, $-4+c_{35}^{1}-c_{35}^{2}-c_{35}^{3}-c_{35}^{4}-c_{35}^{5}-2c_{35}^{7}+c_{35}^{8}-c_{35}^{9}-c_{35}^{10}$, $-4+c_{35}^{1}-c_{35}^{2}-c_{35}^{3}-c_{35}^{4}-c_{35}^{5}-2c_{35}^{7}+c_{35}^{8}-c_{35}^{9}-c_{35}^{10}$, $-4+c_{35}^{1}-c_{35}^{2}-c_{35}^{3}-c_{35}^{4}-c_{35}^{5}-2c_{35}^{7}+c_{35}^{8}-c_{35}^{9}-c_{35}^{10}$, $9-3c_{35}^{1}+3c_{35}^{2}+3c_{35}^{3}+2c_{35}^{4}+3c_{35}^{5}-c_{35}^{6}+4c_{35}^{7}-3c_{35}^{8}+2c_{35}^{9}+3c_{35}^{10}$, 1; 1, $7-2c_{35}^{1}+2c_{35}^{2}+2c_{35}^{3}+c_{35}^{4}+2c_{35}^{5}-c_{35}^{6}+4c_{35}^{7}-2c_{35}^{8}+c_{35}^{9}+2c_{35}^{10}$, $7-2c_{35}^{1}+2c_{35}^{2}+2c_{35}^{3}+c_{35}^{4}+2c_{35}^{5}-c_{35}^{6}+4c_{35}^{7}-2c_{35}^{8}+c_{35}^{9}+2c_{35}^{10}$, $7-2c_{35}^{1}+2c_{35}^{2}+2c_{35}^{3}+c_{35}^{4}+2c_{35}^{5}-c_{35}^{6}+4c_{35}^{7}-2c_{35}^{8}+c_{35}^{9}+2c_{35}^{10}$, $-9+3c_{35}^{1}-3c_{35}^{2}-3c_{35}^{3}-2c_{35}^{4}-3c_{35}^{5}+c_{35}^{6}-4c_{35}^{7}+3c_{35}^{8}-2c_{35}^{9}-3c_{35}^{10}$, $-5+2c_{35}^{1}-2c_{35}^{2}-2c_{35}^{3}-c_{35}^{4}-2c_{35}^{5}+c_{35}^{6}-3c_{35}^{7}+2c_{35}^{8}-c_{35}^{9}-2c_{35}^{10}$; $-5+6c_{35}^{1}+2c_{35}^{2}+3c_{35}^{4}+c_{35}^{5}+4c_{35}^{6}-3c_{35}^{7}+4c_{35}^{8}+c_{35}^{9}-2c_{35}^{10}+3c_{35}^{11}$, $-1+c_{35}^{1}-4c_{35}^{2}+2c_{35}^{3}-c_{35}^{4}-2c_{35}^{5}+c_{35}^{6}-c_{35}^{7}-2c_{35}^{9}+3c_{35}^{10}-2c_{35}^{11}$, $-1-5c_{35}^{1}-4c_{35}^{3}-3c_{35}^{4}-c_{35}^{5}-4c_{35}^{6}-2c_{35}^{8}-3c_{35}^{10}-c_{35}^{11}$, 0, $4-c_{35}^{1}+c_{35}^{2}+c_{35}^{3}+c_{35}^{4}+c_{35}^{5}+2c_{35}^{7}-c_{35}^{8}+c_{35}^{9}+c_{35}^{10}$; $-1-5c_{35}^{1}-4c_{35}^{3}-3c_{35}^{4}-c_{35}^{5}-4c_{35}^{6}-2c_{35}^{8}-3c_{35}^{10}-c_{35}^{11}$, $-5+6c_{35}^{1}+2c_{35}^{2}+3c_{35}^{4}+c_{35}^{5}+4c_{35}^{6}-3c_{35}^{7}+4c_{35}^{8}+c_{35}^{9}-2c_{35}^{10}+3c_{35}^{11}$, 0, $4-c_{35}^{1}+c_{35}^{2}+c_{35}^{3}+c_{35}^{4}+c_{35}^{5}+2c_{35}^{7}-c_{35}^{8}+c_{35}^{9}+c_{35}^{10}$; $-1+c_{35}^{1}-4c_{35}^{2}+2c_{35}^{3}-c_{35}^{4}-2c_{35}^{5}+c_{35}^{6}-c_{35}^{7}-2c_{35}^{9}+3c_{35}^{10}-2c_{35}^{11}$, 0, $4-c_{35}^{1}+c_{35}^{2}+c_{35}^{3}+c_{35}^{4}+c_{35}^{5}+2c_{35}^{7}-c_{35}^{8}+c_{35}^{9}+c_{35}^{10}$; $-9+3c_{35}^{1}-3c_{35}^{2}-3c_{35}^{3}-2c_{35}^{4}-3c_{35}^{5}+c_{35}^{6}-4c_{35}^{7}+3c_{35}^{8}-2c_{35}^{9}-3c_{35}^{10}$, $9-3c_{35}^{1}+3c_{35}^{2}+3c_{35}^{3}+2c_{35}^{4}+3c_{35}^{5}-c_{35}^{6}+4c_{35}^{7}-3c_{35}^{8}+2c_{35}^{9}+3c_{35}^{10}$; $-6+2c_{35}^{1}-2c_{35}^{2}-2c_{35}^{3}-c_{35}^{4}-2c_{35}^{5}+c_{35}^{6}-4c_{35}^{7}+2c_{35}^{8}-c_{35}^{9}-2c_{35}^{10}$)

Realization: unknown

7.6 Rank 12

1. $12_{1,40.}^{80,190}$: $d_i = (1.0, 1.0, 1.0, 1.0, 2.0, 2.0, 2.0, 2.0, 2.236, 2.236, 2.236, 2.236)$

   $D^2 = 40.0 = 40$

   $T = (0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{4}{5}, \frac{1}{20}, \frac{9}{20}, \frac{1}{16}, \frac{1}{16}, \frac{9}{16}, \frac{9}{16})$,

   $S = (1,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ \sqrt{5},\ \sqrt{5},\ \sqrt{5},\ \sqrt{5};\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ -\sqrt{5},\ -\sqrt{5},\ -\sqrt{5},\ -\sqrt{5};\ -1,\ -1,\ 2,\ 2,\ -2,\ -2,\ (-\sqrt{5})\mathrm{i},\ (\sqrt{5})\mathrm{i},\ (-\sqrt{5})\mathrm{i},\ (\sqrt{5})\mathrm{i};\ -1,\ 2,\ 2,\ -2,\ -2,\ (\sqrt{5})\mathrm{i},\ (-\sqrt{5})\mathrm{i},\ (\sqrt{5})\mathrm{i},\ (-\sqrt{5})\mathrm{i};\ -1-\sqrt{5},\ -1+\sqrt{5},\ -1+\sqrt{5},\ -1-\sqrt{5},\ 0,\ 0,\ 0,\ 0;\ -1-\sqrt{5},\ -1-\sqrt{5},\ -1+\sqrt{5},\ 0,\ 0,\ 0,\ 0;\ 1+\sqrt{5},\ 1-\sqrt{5},\ 0,\ 0,\ 0,\ 0;\ 1+\sqrt{5},\ 0,\ 0,\ 0,\ 0;\ -\sqrt{5}\zeta_8^3,\ \sqrt{5}\zeta_8^1,\ \sqrt{5}\zeta_8^3,\ -\sqrt{5}\zeta_8^1;\ -\sqrt{5}\zeta_8^3,\ -\sqrt{5}\zeta_8^1,\ \sqrt{5}\zeta_8^3;\ -\sqrt{5}\zeta_8^3,\ \sqrt{5}\zeta_8^1;\ -\sqrt{5}\zeta_8^3)$

   Connected to the orbit of $12_{5,40.}^{80,348}$ via a change of spherical structure.

   Realization: $SO(10)_2$ (see [120, Section 3] for explicit data) or anyon condensation of O_5.

2. $12_{5,40.}^{80,348}$: $d_i = (1.0, 1.0, 1.0, 1.0, 2.0, 2.0, 2.0, 2.0, 2.236, 2.236, 2.236, 2.236)$

   $D^2 = 40.0 = 40$

   $T = (0, 0, \frac{1}{4}, \frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{13}{20}, \frac{17}{20}, \frac{1}{16}, \frac{1}{16}, \frac{9}{16}, \frac{9}{16})$,

   $S = (1,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ \sqrt{5},\ \sqrt{5},\ \sqrt{5},\ \sqrt{5};\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ -\sqrt{5},\ -\sqrt{5},\ -\sqrt{5},\ -\sqrt{5};\ -1,\ -1,\ 2,\ 2,\ -2,\ -2,\ (-\sqrt{5})\mathrm{i},\ (\sqrt{5})\mathrm{i},\ (-\sqrt{5})\mathrm{i},\ (\sqrt{5})\mathrm{i};\ -1,\ 2,\ 2,\ -2,\ -2,\ (\sqrt{5})\mathrm{i},\ (-\sqrt{5})\mathrm{i},\ (\sqrt{5})\mathrm{i},\ (-\sqrt{5})\mathrm{i};\ -1+\sqrt{5},\ -1-\sqrt{5},\ -1+\sqrt{5},\ -1-\sqrt{5},\ 0,\ 0,\ 0,\ 0;\ -1+\sqrt{5},\ -1-\sqrt{5},\ -1+\sqrt{5},\ 0,\ 0,\ 0,\ 0;\ 1-\sqrt{5},\ 1+\sqrt{5},\ 0,\ 0,\ 0,\ 0;\ 1-\sqrt{5},\ 0,\ 0,\ 0,\ 0;\ \sqrt{5}\zeta_8^3,\ -\sqrt{5}\zeta_8^1,\ -\sqrt{5}\zeta_8^3,\ \sqrt{5}\zeta_8^1;\ \sqrt{5}\zeta_8^3,\ \sqrt{5}\zeta_8^1,\ -\sqrt{5}\zeta_8^3;\ \sqrt{5}\zeta_8^3,\ -\sqrt{5}\zeta_8^1;\ \sqrt{5}\zeta_8^3)$

   Connected to the orbit of $12_{1,40.}^{80,190}$ via a change of spherical structure.

   Realization: Abelian anyon condensation of $SO(5)_2$.

3. $12_{0,68.}^{34,116}$: $d_i = (1.0, 1.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 4.123, 4.123)$

   $D^2 = 68.0 = 68$

   $T = (0, 0, \frac{1}{17}, \frac{2}{17}, \frac{4}{17}, \frac{8}{17}, \frac{9}{17}, \frac{13}{17}, \frac{15}{17}, \frac{16}{17}, 0, \frac{1}{2})$,

   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \sqrt{17},\ \sqrt{17};\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ -\sqrt{17},\ -\sqrt{17};\ 2c_{17}^{2},\ 2c_{17}^{5},\ 2c_{17}^{4},\ 2c_{17}^{7},\ 2c_{17}^{6},\ 2c_{17}^{1},\ 2c_{17}^{3},\ 2c_{17}^{8},\ 0,\ 0;\ 2c_{17}^{4},\ 2c_{17}^{7},\ 2c_{17}^{8},\ 2c_{17}^{2},\ 2c_{17}^{6},\ 2c_{17}^{1},\ 2c_{17}^{3},\ 0,\ 0;\ 2c_{17}^{8},\ 2c_{17}^{3},\ 2c_{17}^{5},\ 2c_{17}^{2},\ 2c_{17}^{6},\ 2c_{17}^{1},\ 0,\ 0;\ 2c_{17}^{1},\ 2c_{17}^{4},\ 2c_{17}^{5},\ 2c_{17}^{2},\ 2c_{17}^{6},\ 0,\ 0;\ 2c_{17}^{1},\ 2c_{17}^{3},\ 2c_{17}^{8},\ 2c_{17}^{7},\ 0,\ 0;\ 2c_{17}^{8},\ 2c_{17}^{7},\ 2c_{17}^{4},\ 0,\ 0;\ 2c_{17}^{4},\ 2c_{17}^{5},\ 0,\ 0;\ 2c_{17}^{2},\ 0,\ 0;\ \sqrt{17},\ -\sqrt{17};\ \sqrt{17})$

   Connected to the orbit of $12_{4,68.}^{34,824}$ via a change of spherical structure.

   Realization: $SO(17)_2$, or Abelian anyon condensation of $\overline{SO(17)_2}$, O_{17}.

4. $12^{34,824}_{4,68.}$: $d_i = (1.0, 1.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 4.123, 4.123)$

$D^2 = 68.0 = 68$

$T = (0, 0, \frac{3}{17}, \frac{5}{17}, \frac{6}{17}, \frac{7}{17}, \frac{10}{17}, \frac{11}{17}, \frac{12}{17}, \frac{14}{17}, 0, \frac{1}{2})$,

$S = (1, 1, 2, 2, 2, 2, 2, 2, 2, 2, \sqrt{17}, \sqrt{17};\ 1, 2, 2, 2, 2, 2, 2, 2, 2, -\sqrt{17}, -\sqrt{17};\ 2c^6_{17}, 2c^3_{17}, 2c^2_{17}, 2c^4_{17}, 2c^1_{17}, 2c^8_{17}, 2c^5_{17}, 2c^7_{17}, 0, 0;\ 2c^7_{17}, 2c^1_{17}, 2c^2_{17}, 2c^8_{17}, 2c^4_{17}, 2c^6_{17}, 2c^5_{17}, 0, 0;\ 2c^5_{17}, 2c^7_{17}, 2c^6_{17}, 2c^3_{17}, 2c^4_{17}, 2c^8_{17}, 0, 0;\ 2c^3_{17}, 2c^5_{17}, 2c^6_{17}, 2c^8_{17}, 2c^1_{17}, 0, 0;\ 2c^3_{17}, 2c^7_{17}, 2c^2_{17}, 2c^4_{17}, 0, 0;\ 2c^5_{17}, 2c^1_{17}, 2c^2_{17}, 0, 0;\ 2c^7_{17}, 2c^3_{17}, 0, 0;\ 2c^6_{17}, 0, 0;\ -\sqrt{17}, \sqrt{17};\ -\sqrt{17})$

Connected to the orbit of $12^{34,116}_{0,68.}$ via a change of spherical structure.

Realization: $SO(17)_2$ with a Galois conjugation and a change of spherical structure.

5. $12^{68,166}_{0,68.}$: $d_i = (1.0, 1.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 4.123, 4.123)$

$D^2 = 68.0 = 68$

$T = (0, 0, \frac{1}{17}, \frac{2}{17}, \frac{4}{17}, \frac{8}{17}, \frac{9}{17}, \frac{13}{17}, \frac{15}{17}, \frac{16}{17}, \frac{1}{4}, \frac{3}{4})$,

$S = (1, 1, 2, 2, 2, 2, 2, 2, 2, 2, \sqrt{17}, \sqrt{17};\ 1, 2, 2, 2, 2, 2, 2, 2, 2, -\sqrt{17}, -\sqrt{17};\ 2c^2_{17}, 2c^5_{17}, 2c^4_{17}, 2c^7_{17}, 2c^6_{17}, 2c^1_{17}, 2c^3_{17}, 2c^8_{17}, 0, 0;\ 2c^4_{17}, 2c^7_{17}, 2c^8_{17}, 2c^2_{17}, 2c^6_{17}, 2c^1_{17}, 2c^3_{17}, 0, 0;\ 2c^8_{17}, 2c^3_{17}, 2c^5_{17}, 2c^2_{17}, 2c^6_{17}, 2c^1_{17}, 0, 0;\ 2c^1_{17}, 2c^4_{17}, 2c^5_{17}, 2c^2_{17}, 2c^6_{17}, 0, 0;\ 2c^1_{17}, 2c^3_{17}, 2c^8_{17}, 2c^7_{17}, 0, 0;\ 2c^8_{17}, 2c^7_{17}, 2c^4_{17}, 0, 0;\ 2c^4_{17}, 2c^5_{17}, 0, 0;\ 2c^2_{17}, 0, 0;\ -\sqrt{17}, \sqrt{17};\ -\sqrt{17})$

Connected to the orbit of $12^{68,721}_{4,68.}$ via a change of spherical structure.

Realization: Abelian anyon condensation of $SO(17)_2$, $\overline{SO(17)}_2$ or O_{17}.

6. $12^{68,721}_{4,68.}$: $d_i = (1.0, 1.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 4.123, 4.123)$

$D^2 = 68.0 = 68$

$T = (0, 0, \frac{3}{17}, \frac{5}{17}, \frac{6}{17}, \frac{7}{17}, \frac{10}{17}, \frac{11}{17}, \frac{12}{17}, \frac{14}{17}, \frac{1}{4}, \frac{3}{4})$,

$S = (1, 1, 2, 2, 2, 2, 2, 2, 2, 2, \sqrt{17}, \sqrt{17};\ 1, 2, 2, 2, 2, 2, 2, 2, 2, -\sqrt{17}, -\sqrt{17};\ 2c^6_{17}, 2c^3_{17}, 2c^2_{17}, 2c^4_{17}, 2c^1_{17}, 2c^8_{17}, 2c^5_{17}, 2c^7_{17}, 0, 0;\ 2c^7_{17}, 2c^1_{17}, 2c^2_{17}, 2c^8_{17}, 2c^4_{17}, 2c^6_{17}, 2c^5_{17}, 0, 0;\ 2c^5_{17}, 2c^7_{17}, 2c^6_{17}, 2c^3_{17}, 2c^4_{17}, 2c^8_{17}, 0, 0;\ 2c^3_{17}, 2c^5_{17}, 2c^6_{17}, 2c^8_{17}, 2c^1_{17}, 0, 0;\ 2c^3_{17}, 2c^7_{17}, 2c^2_{17}, 2c^4_{17}, 0, 0;\ 2c^5_{17}, 2c^1_{17}, 2c^2_{17}, 0, 0;\ 2c^7_{17}, 2c^3_{17}, 0, 0;\ 2c^6_{17}, 0, 0;\ \sqrt{17}, -\sqrt{17};\ \sqrt{17})$

Connected to the orbit of $12^{68,166}_{0,68.}$ via a change of spherical structure.

Realization: anyon condensation of $SO(17)_2$ with a Galois conjugation and a change of spherical structure.

7. $12^{48,120}_{4,144.}$: $d_i = (1.0, 1.0, 2.0, 3.0, 3.0, 4.0, 4.0, 4.0, 4.242, 4.242, 4.242, 4.242)$

$D^2 = 144.0 = 144$

$T = (0, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{3}, \frac{2}{3}, \frac{1}{16}, \frac{7}{16}, \frac{9}{16}, \frac{15}{16})$,

$S = (1, 1, 2, 3, 3, 4, 4, 4, 3\sqrt{2}, 3\sqrt{2}, 3\sqrt{2}, 3\sqrt{2};\ 1, 2, 3, 3, 4, 4, 4, -3\sqrt{2}, -3\sqrt{2}, -3\sqrt{2}, -3\sqrt{2};\ 4, 6, 6, -4, -4, -4, 0, 0, 0, 0;\ -3, -3, 0, 0, 0, -3\sqrt{2}, 3\sqrt{2}, -3\sqrt{2}, 3\sqrt{2};\ -3, 0, 0, 0, 3\sqrt{2}, -3\sqrt{2}, 3\sqrt{2}, -3\sqrt{2};\ -8, 4, 4, 0, 0, 0, 0;\ 4, -8, 0, 0, 0, 0;\ 4, 0, 0, 0, 0;\ 0, 6, 0, -6;\ 0, -6, 0;\ 0, 6;\ 0)$

Connected to the orbit of $12^{48,650}_{4,144.}$ via a change of spherical structure.

Realization: S_3-gauging of the rank-4 3-fermion MTC $4^{2,250}_{4,4.}$ with a different minimal modular extension.

8. $12^{48,650}_{4,144.}$: $d_i = (1.0, 1.0, 2.0, 3.0, 3.0, 4.0, 4.0, 4.0, 4.242, 4.242, 4.242, 4.242)$

$D^2 = 144.0 = 144$

$T = (0, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{3}, \frac{2}{3}, \frac{3}{16}, \frac{5}{16}, \frac{11}{16}, \frac{13}{16})$,

$S = (1, 1, 2, 3, 3, 4, 4, 4, 3\sqrt{2}, 3\sqrt{2}, 3\sqrt{2}, 3\sqrt{2};\ 1, 2, 3, 3, 4, 4, 4, -3\sqrt{2}, -3\sqrt{2}, -3\sqrt{2}, -3\sqrt{2};\ 4, 6, 6, -4, -4, -4, 0, 0, 0, 0;\ -3, -3, 0, 0, 0, -3\sqrt{2}, 3\sqrt{2}, -3\sqrt{2}, 3\sqrt{2};\ -3, 0, 0, 0, 3\sqrt{2}, -3\sqrt{2}, 3\sqrt{2}, -3\sqrt{2};\ -8, 4, 4, 0, 0, 0, 0;\ 4, -8, 0, 0, 0, 0;\ 4, 0, 0, 0, 0;\ 0, 6, 0, -6;\ 0, -6, 0;\ 0, 6;\ 0)$

Connected to the orbit of $12^{48,120}_{4,144.}$ via a change of spherical structure.

Realization: S_3-gauging of the rank-4 3-fermion MTC $4^{2,250}_{4,4.}$ with a different minimal modular extension. Also constructed by condensing the diagonal copy of $\mathcal{R}ep(S_3)$ in $\mathcal{C} \boxtimes \mathcal{R}ep(D^\omega S_3)$ where $\mathcal{C}$ is $12^{144,916}_{4,144.}$.

9. $12^{144,916}_{4,144.}$: $d_i = (1.0, 1.0, 2.0, 3.0, 3.0, 4.0, 4.0, 4.0, 4.242, 4.242, 4.242, 4.242)$

$D^2 = 144.0 = 144$

$T = (0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{1}{16}, \frac{7}{16}, \frac{9}{16}, \frac{15}{16})$,

$S = (1,\ 1,\ 2,\ 3,\ 3,\ 4,\ 4,\ 4,\ 3\sqrt{2},\ 3\sqrt{2},\ 3\sqrt{2},\ 3\sqrt{2};\ \ 1,\ 2,\ 3,\ 3,\ 4,\ 4,\ 4,\ -3\sqrt{2},\ -3\sqrt{2},\ -3\sqrt{2},\ -3\sqrt{2};\ \ 4,\ 6,\ 6,\ -4,\ -4,\ -4,\ 0,\ 0,\ 0,\ 0;\ \ -3,\ -3,\ 0,\ 0,\ 0,\ -3\sqrt{2},\ 3\sqrt{2},\ -3\sqrt{2},\ 3\sqrt{2};\ \ -3,\ 0,\ 0,\ 0,\ 3\sqrt{2},\ -3\sqrt{2},\ 3\sqrt{2},\ -3\sqrt{2};\ \ -4c^2_9,\ -4c^4_9,\ -4c^1_9,\ 0,\ 0,\ 0,\ 0;\ \ -4c^1_9,\ -4c^2_9,\ 0,\ 0,\ 0,\ 0;\ \ -4c^4_9,\ 0,\ 0,\ 0,\ 0;\ \ 0,\ 6,\ 0,\ -6;\ \ 0,\ -6,\ 0;\ \ 0,\ 6;\ \ 0)$

Connected to the orbit of $12^{144,386}_{4,144.}$ via a change of spherical structure.

Realization: S_3-gauging of the rank-4 3-fermion MTC $4^{2,250}_{4,4.}$, see [121].

10. $12^{144,386}_{4,144.}$: $d_i = (1.0, 1.0, 2.0, 3.0, 3.0, 4.0, 4.0, 4.0, 4.242, 4.242, 4.242, 4.242)$

$D^2 = 144.0 = 144$

$T = (0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{3}{16}, \frac{5}{16}, \frac{11}{16}, \frac{13}{16})$,

$S = (1,\ 1,\ 2,\ 3,\ 3,\ 4,\ 4,\ 4,\ 3\sqrt{2},\ 3\sqrt{2},\ 3\sqrt{2},\ 3\sqrt{2};\ \ 1,\ 2,\ 3,\ 3,\ 4,\ 4,\ 4,\ -3\sqrt{2},\ -3\sqrt{2},\ -3\sqrt{2},\ -3\sqrt{2};\ \ 4,\ 6,\ 6,\ -4,\ -4,\ -4,\ 0,\ 0,\ 0,\ 0;\ \ -3,\ -3,\ 0,\ 0,\ 0,\ -3\sqrt{2},\ 3\sqrt{2},\ -3\sqrt{2},\ 3\sqrt{2};\ \ -3,\ 0,\ 0,\ 0,\ 3\sqrt{2},\ -3\sqrt{2},\ 3\sqrt{2},\ -3\sqrt{2};\ \ -4c^2_9,\ -4c^4_9,\ -4c^1_9,\ 0,\ 0,\ 0,\ 0;\ \ -4c^1_9,\ -4c^2_9,\ 0,\ 0,\ 0,\ 0;\ \ -4c^4_9,\ 0,\ 0,\ 0,\ 0;\ \ 0,\ 6,\ 0,\ -6;\ \ 0,\ -6,\ 0;\ \ 0,\ 6;\ \ 0)$

Connected to the orbit of $12^{144,916}_{4,144.}$ via a change of spherical structure.

Realization: S_3-gauging of the rank-4 3-fermion MTC $4^{2,250}_{4,4.}$ with a different minimal modular extension.

11. $12^{25,285}_{\frac{94}{25},397.8}$: $d_i = (1.0, 1.984, 2.937, 3.843, 4.689, 5.461, 6.147, 6.736, 7.219, 7.588, 7.837, 7.962)$

$D^2 = 397.875 = 75 + 65c^1_{25} + 55c^2_{25} + 45c^3_{25} + 35c^4_{25} + 25c^1_5 + 20c^6_{25} + 15c^7_{25} + 10c^8_{25} + 5c^9_{25}$

$T = (0, \frac{7}{25}, \frac{2}{25}, \frac{2}{5}, \frac{6}{25}, \frac{3}{5}, \frac{12}{25}, \frac{22}{25}, \frac{4}{5}, \frac{6}{25}, \frac{1}{5}, \frac{17}{25})$,

$S = (1,\ -c^{12}_{25},\ \xi^3_{25},\ \xi^{21}_{25},\ \xi^5_{25},\ \xi^{19}_{25},\ \xi^7_{25},\ \xi^{17}_{25},\ \xi^9_{25},\ \frac{1+\sqrt{5}}{2}\xi^5_{25},\ \xi^{11}_{25},\ \xi^{13}_{25};\ \ -\xi^{21}_{25},\ \xi^{19}_{25},\ -\xi^{17}_{25},\ \frac{1+\sqrt{5}}{2}\xi^5_{25},\ -\xi^{13}_{25},\ \xi^{11}_{25},\ -\xi^9_{25},\ \xi^7_{25},\ -\xi^5_{25},\ \xi^3_{25},\ -1;\ \ \xi^9_{25},\ \xi^{13}_{25},\ \frac{1+\sqrt{5}}{2}\xi^5_{25},\ \xi^7_{25},\ \xi^{21}_{25},\ 1,\ c^{12}_{25},\ -\xi^5_{25},\ -\xi^{17}_{25},\ -\xi^{11}_{25};\ \ -\xi^9_{25},\ \xi^5_{25},\ -1,\ -\xi^3_{25},\ \xi^7_{25},\ -\xi^{11}_{25},\ \frac{1+\sqrt{5}}{2}\xi^5_{25},\ -\xi^{19}_{25},\ -c^{12}_{25};\ \ 0,\ -\xi^5_{25},\ -\frac{1+\sqrt{5}}{2}\xi^5_{25},\ -\frac{1+\sqrt{5}}{2}\xi^5_{25},\ -\xi^5_{25},\ 0,\ \xi^5_{25},\ \frac{1+\sqrt{5}}{2}\xi^5_{25};\ \ \xi^{11}_{25},\ -\xi^{17}_{25},\ -c^{12}_{25},\ \xi^{21}_{25},\ -\frac{1+\sqrt{5}}{2}\xi^5_{25},\ \xi^9_{25},\ -\xi^3_{25};\ \ -1,\ \xi^{19}_{25},\ \xi^{13}_{25},\ \xi^5_{25},\ c^{12}_{25},\ -\xi^9_{25};\ \ -\xi^{11}_{25},\ \xi^3_{25},\ \xi^5_{25},\ -\xi^{13}_{25},\ \xi^{21}_{25};\ \ -\xi^{19}_{25},\ -\frac{1+\sqrt{5}}{2}\xi^5_{25},\ -1,\ \xi^{17}_{25};\ \ 0,\ \frac{1+\sqrt{5}}{2}\xi^5_{25},\ -\xi^5_{25};\ \ \xi^{21}_{25},\ -\xi^7_{25};\ \ \xi^{19}_{25})$

Realization: Abelian anyon condensation of $SU(2)_{23}$.

12. $12^{10,127}_{\frac{22}{5},495.9}$: $d_i = (1.0, 2.618, 2.618, 4.236, 4.236, 5.236, 5.236, 5.854, 8.472, 8.472, 9.472, 11.90)$

$D^2 = 495.967 = 250 + 110\sqrt{5}$

$T = (0, \frac{1}{5}, \frac{1}{5}, 0, 0, \frac{2}{5}, \frac{2}{5}, \frac{7}{10}, \frac{4}{5}, \frac{4}{5}, \frac{1}{2}, \frac{1}{5})$,

$S = (1,\ \frac{3+\sqrt{5}}{2},\ \frac{3+\sqrt{5}}{2},\ 2+\sqrt{5},\ 2+\sqrt{5},\ 3+\sqrt{5},\ 3+\sqrt{5},\ \frac{5+3\sqrt{5}}{2},\ 4+2\sqrt{5},\ 4+2\sqrt{5},\ 5+2\sqrt{5},\ \frac{11+5\sqrt{5}}{2};\ \ 1+4\zeta^1_5+4\zeta^2_5+2\zeta^3_5,\ -3-4\zeta^1_5-2\zeta^2_5,\ -3-4\zeta^1_5-\zeta^2_5+\zeta^3_5,\ 1+4\zeta^1_5+5\zeta^2_5+3\zeta^3_5,\ -(3-\sqrt{5})\zeta^2_5,\ (3+\sqrt{5})\zeta^1_{10},\ 5+2\sqrt{5},\ (4+2\sqrt{5})\zeta^3_{10},\ -(4-2\sqrt{5})\zeta^1_5,\ -\frac{5+3\sqrt{5}}{2},\ 2+\sqrt{5};\ \ 1+4\zeta^1_5+4\zeta^2_5+2\zeta^3_5,\ 1+4\zeta^1_5+5\zeta^2_5+3\zeta^3_5,\ -3-4\zeta^1_5-\zeta^2_5+\zeta^3_5,\ (3+\sqrt{5})\zeta^1_{10},\ -(3-\sqrt{5})\zeta^2_5,\ 5+2\sqrt{5},\ -(4-2\sqrt{5})\zeta^1_5,\ (4+2\sqrt{5})\zeta^3_{10},\ -\frac{5+3\sqrt{5}}{2},\ 2+\sqrt{5};\ \ 3+4\zeta^1_5+2\zeta^2_5,\ -1-4\zeta^1_5-4\zeta^2_5-2\zeta^3_5,\ -(4-2\sqrt{5})\zeta^1_5,\ (4+2\sqrt{5})\zeta^3_{10},\ \frac{5+3\sqrt{5}}{2},\ (3+\sqrt{5})\zeta^2_5,\ -(3-\sqrt{5})\zeta^1_{10},\ 5+2\sqrt{5},\ -\frac{3+\sqrt{5}}{2};\ \ 3+4\zeta^1_5+2\zeta^2_5,\ (4+2\sqrt{5})\zeta^3_{10},\ -(4-2\sqrt{5})\zeta^1_5,\ \frac{5+3\sqrt{5}}{2},\ -(3-\sqrt{5})\zeta^1_{10},\ (3+\sqrt{5})\zeta^2_5,\ 5+2\sqrt{5},\ -\frac{3+\sqrt{5}}{2};\ \ -(4-2\sqrt{5})\zeta^1_{10},\ (4+2\sqrt{5})\zeta^2_5,\ 0,\ -(3-\sqrt{5})\zeta^1_5,\ (3+\sqrt{5})\zeta^3_{10},\ 0,\ 4+2\sqrt{5};\ \ -(4-2\sqrt{5})\zeta^1_{10},\ 0,\ (3+\sqrt{5})\zeta^3_{10},\ -(3-\sqrt{5})\zeta^1_5,\ 0,\ 4+2\sqrt{5};\ \ 5+2\sqrt{5},\ 0,\ 0,\ -\frac{5+3\sqrt{5}}{2},\ -5-2\sqrt{5};\ \ -(4-2\sqrt{5})\zeta^2_5,\ (4+2\sqrt{5})\zeta^1_{10},\ 0,\ -3-\sqrt{5};\ \ -(4-2\sqrt{5})\zeta^2_5,\ 0,\ -3-\sqrt{5};\ \ -5-2\sqrt{5},\ \frac{5+3\sqrt{5}}{2};\ \ -1)$

Realization: Abelian anyon condensation of $SU(7)_3$.

13. $12^{21,324}_{\frac{20}{7},940.0}$: $d_i = (1.0, 2.977, 4.888, 6.690, 6.690, 6.690, 8.343, 9.809, 11.56, 12.56, 12.786, 13.232)$

$D^2 = 940.87 = 105 + 147c^1_{21} + 189c^2_{21} + 105c^1_7 + 126c^4_{21} + 126c^5_{21}$

$T = (0, \frac{1}{21}, \frac{1}{7}, \frac{2}{7}, \frac{13}{21}, \frac{13}{21}, \frac{10}{21}, \frac{5}{7}, 0, \frac{1}{3}, \frac{5}{7}, \frac{1}{7})$,

$S = (1,\ \xi_{42}^{3},\ \xi_{42}^{5},\ \xi_{21}^{11},\ \xi_{21}^{11},\ \xi_{21}^{11},\ \xi_{42}^{9},\ \xi_{42}^{11},\ \frac{5+\sqrt{21}}{2}\xi_{21}^{16,2},\ \xi_{42}^{15},\ \xi_{42}^{17},\ \xi_{42}^{19};\ \ \xi_{42}^{9},\ \xi_{42}^{15},\ 2\xi_{21}^{11},\ -\xi_{21}^{11},\ -\xi_{21}^{11},$ $\xi_{42}^{15},\ \xi_{42}^{9},\ \xi_{42}^{3},\ -\xi_{42}^{3},\ -\xi_{42}^{9},\ -\xi_{42}^{15};\ \ \xi_{42}^{17},\ \xi_{21}^{11},\ \xi_{21}^{11},\ \xi_{21}^{11},\ -\xi_{42}^{3},\ -\frac{5+\sqrt{21}}{2}\xi_{21}^{16,2},\ -\xi_{42}^{19},\ -\xi_{42}^{9},\ 1,\ \xi_{42}^{11};$ $-\xi_{21}^{11},\ -\xi_{21}^{11},\ -\xi_{21}^{11},\ -2\xi_{21}^{11},\ -\xi_{21}^{11},\ \xi_{21}^{11},\ 2\xi_{21}^{11},\ \xi_{21}^{11},\ -\xi_{21}^{11};\ \ \frac{1-\sqrt{21}}{2}\xi_{21}^{11},\ \frac{1+\sqrt{21}}{2}\xi_{21}^{11},\ \xi_{21}^{11},\ -\xi_{21}^{11},\ \xi_{21}^{11},$ $-\xi_{21}^{11},\ \xi_{21}^{11},\ -\xi_{21}^{11};\ \ \frac{1-\sqrt{21}}{2}\xi_{21}^{11},\ \xi_{21}^{11},\ -\xi_{21}^{11},\ \xi_{21}^{11},\ -\xi_{21}^{11},\ \xi_{21}^{11},\ -\xi_{21}^{11};\ \ -\xi_{42}^{3},\ \xi_{42}^{15},\ \xi_{42}^{9},\ -\xi_{42}^{9},\ -\xi_{42}^{15},\ \xi_{42}^{3};$ $\xi_{42}^{5},\ -\xi_{42}^{17},\ -\xi_{42}^{3},\ \xi_{42}^{19},\ 1;\ \ 1,\ \xi_{42}^{15},\ -\xi_{42}^{11},\ -\xi_{42}^{5};\ \ -\xi_{42}^{15},\ \xi_{42}^{3},\ \xi_{42}^{9};\ \ \xi_{42}^{5},\ -\frac{5+\sqrt{21}}{2}\xi_{21}^{16,2};\ \ \xi_{42}^{17})$

Realization: Abelian anyon condensation of $\overline{SO(20)}_3$.

14. $12^{39,406}_{4,1276.}$: $d_i = (1.0,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 10.908,\ 11.908,\ 11.908,\ 11.908,\ 11.908)$

$D^2 = 1276.274 = \frac{1287+351\sqrt{13}}{2}$

$T = (0,\ \frac{1}{13},\ \frac{3}{13},\ \frac{4}{13},\ \frac{9}{13},\ \frac{10}{13},\ \frac{12}{13},0,\ \frac{1}{3},\ \frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}),$

$S = (1,\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{11+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},$ $\frac{13+3\sqrt{13}}{2};\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{2},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{5},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{4},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{6},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{1},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{3},$ $-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{6},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{3},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{2},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{4},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{1},$ $-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{5},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{1},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{2},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{6},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,$ $0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{5},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{3},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{4},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{6},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{5},$ $-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{2},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ 1,\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},$ $\frac{13+3\sqrt{13}}{2};\ \ -\frac{13+3\sqrt{13}}{2},\ 13+3\sqrt{13},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2};\ \ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2};$ $-\frac{13+3\sqrt{13}}{2},\ 13+3\sqrt{13};\ \ -\frac{13+3\sqrt{13}}{2})$

Realization: May be related to condensation reductions of categories of the form $\mathcal{Z}(\mathcal{NG}(\mathbb{Z}_3 \times \mathbb{Z}_3, 9))$.

15. $12^{39,560}_{0,1276.}$: $d_i = (1.0,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 10.908,\ 11.908,\ 11.908,\ 11.908,\ 11.908)$

$D^2 = 1276.274 = \frac{1287+351\sqrt{13}}{2}$

$T = (0,\ \frac{2}{13},\ \frac{5}{13},\ \frac{6}{13},\ \frac{7}{13},\ \frac{8}{13},\ \frac{11}{13},0,0,0,\ \frac{1}{3},\ \frac{2}{3}),$

$S = (1,\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{11+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},$ $\frac{13+3\sqrt{13}}{2};\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{4},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{1},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{3},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{2},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{5},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{6},$ $-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{3},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{4},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{6},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{2},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{5},$ $-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{1},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{5},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{6},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{2},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,$ $0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{1},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{4},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{3},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{3},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{1},$ $-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{4},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ 1,\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},$ $\frac{13+3\sqrt{13}}{2};\ \ 13+3\sqrt{13},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2};\ \ 13+3\sqrt{13},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2};$ $-\frac{13+3\sqrt{13}}{2},\ 13+3\sqrt{13};\ \ -\frac{13+3\sqrt{13}}{2})$

Realization: $\text{Haag}(1)_0$.

16. $12^{117,251}_{0,1276.}$: $d_i = (1.0,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 10.908,\ 11.908,\ 11.908,\ 11.908,\ 11.908)$

$D^2 = 1276.274 = \frac{1287+351\sqrt{13}}{2}$

$T = (0,\ \frac{2}{13},\ \frac{5}{13},\ \frac{6}{13},\ \frac{7}{13},\ \frac{8}{13},\ \frac{11}{13},0,0,\ \frac{1}{9},\ \frac{4}{9},\ \frac{7}{9}),$

$S = (1,\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{11+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},$ $\frac{13+3\sqrt{13}}{2};\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{4},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{1},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{3},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{2},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{5},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{6},$ $-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{3},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{4},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{6},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{2},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{5},$ $-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{1},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{5},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{6},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{2},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,$ $0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{1},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{4},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{3},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{3},\ -\frac{9+3\sqrt{13}}{2}c_{13}^{1},$ $-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c_{13}^{4},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ 1,\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},$ $\frac{13+3\sqrt{13}}{2};\ \ 13+3\sqrt{13},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2};\ \ \frac{13+3\sqrt{13}}{2}c_{9}^{2},\ \frac{13+3\sqrt{13}}{2}c_{9}^{4},\ \frac{13+3\sqrt{13}}{2}c_{9}^{1};$ $\frac{13+3\sqrt{13}}{2}c_{9}^{1},\ \frac{13+3\sqrt{13}}{2}c_{9}^{2};\ \ \frac{13+3\sqrt{13}}{2}c_{9}^{4})$

Realization: $\text{Haag}(1)_1$.

17. $12^{27,115}_{\frac{70}{9},1996.}$: $d_i = (1.0, 4.932, 6.811, 7.562, 10.270, 11.585, 11.802, 14.369, 15.369, 16.734, 17.82, 21.773)$

$D^2 = 1996.556 = 243 + 243c^1_{27} + 243c^2_{27} + 243c^1_9 + 243c^4_{27} + 216c^5_{27} + 162c^2_9 + 108c^7_{27} + 54c^8_{27}$

$T = (0, \frac{2}{9}, \frac{4}{9}, \frac{23}{27}, \frac{1}{9}, \frac{14}{27}, \frac{5}{9}, \frac{2}{3}, \frac{1}{3}, \frac{8}{9}, \frac{7}{9}, \frac{5}{27})$,

$S = (1,\ \xi^5_{54},\ \xi^7_{54},\ 1+c^1_{27}+c^2_{27}+c^1_9+c^4_{27}+c^5_{27}+c^7_{27}+c^8_{27},\ \xi^{11}_{54},\ 2+c^1_{27}+c^2_{27}+c^1_9+2c^4_{27}+2c^5_{27}+c^2_9,\ \xi^{13}_{54},\ \xi^{17}_{54},\ \xi^{19}_{54},\ \xi^{23}_{54},\ \xi^{25}_{54},\ 2+3c^1_{27}+3c^2_{27}+3c^1_9+2c^4_{27}+2c^5_{27}+2c^2_9+c^7_{27}+c^8_{27};\ \ \xi^{25}_{54},\ \xi^{19}_{54},\ -2-c^1_{27}-c^2_{27}-c^1_9-2c^4_{27}-2c^5_{27}-c^2_9,\ -1,\ 2+3c^1_{27}+3c^2_{27}+3c^1_9+2c^4_{27}+2c^5_{27}+2c^2_9+c^7_{27}+c^8_{27},\ -\xi^{11}_{54},\ -\xi^{23}_{54},\ -\xi^{13}_{54},\ \xi^7_{54},\ \xi^{17}_{54},\ -1-c^1_{27}-c^2_{27}-c^1_9-c^4_{27}-c^5_{27}-c^7_{27}-c^8_{27};\ \ \xi^5_{54},\ 2+3c^1_{27}+3c^2_{27}+3c^1_9+2c^4_{27}+2c^5_{27}+2c^2_9+c^7_{27}+c^8_{27},\ -\xi^{23}_{54},\ 1+c^1_{27}+c^2_{27}+c^1_9+c^4_{27}+c^5_{27}+c^7_{27}+c^8_{27},\ -\xi^{17}_{54},\ \xi^{11}_{54},\ \xi^{25}_{54},\ 1,\ -\xi^{13}_{54},\ -2-c^1_{27}-c^2_{27}-c^1_9-2c^4_{27}-2c^5_{27}-c^2_9;\ \ 0,\ 2+3c^1_{27}+3c^2_{27}+3c^1_9+2c^4_{27}+2c^5_{27}+2c^2_9+c^7_{27}+c^8_{27},\ 0,\ -2-c^1_{27}-c^2_{27}-c^1_9-2c^4_{27}-2c^5_{27}-c^2_9,\ 1+c^1_{27}+c^2_{27}+c^1_9+c^4_{27}+c^5_{27}+c^7_{27}+c^8_{27},\ -1-c^1_{27}-c^2_{27}-c^1_9-c^4_{27}-c^5_{27}-c^7_{27}-c^8_{27},\ 2+c^1_{27}+c^2_{27}+c^1_9+2c^4_{27}+2c^5_{27}+c^2_9,\ -2-3c^1_{27}-3c^2_{27}-3c^1_9-2c^4_{27}-2c^5_{27}-2c^2_9-c^7_{27}-c^8_{27},\ 0;\ \ \xi^{13}_{54},\ 1+c^1_{27}+c^2_{27}+c^1_9+c^4_{27}+c^5_{27}+c^7_{27}+c^8_{27},\ \xi^{19}_{54},\ -\xi^{25}_{54},\ -\xi^7_{54},\ \xi^{17}_{54},\ -\xi^5_{54},\ -2-c^1_{27}-c^2_{27}-c^1_9-2c^4_{27}-2c^5_{27}-c^2_9;\ \ 0,\ 2+3c^1_{27}+3c^2_{27}+3c^1_9+2c^4_{27}+2c^5_{27}+2c^2_9+c^7_{27}+c^8_{27},\ 2+c^1_{27}+c^2_{27}+c^1_9+2c^4_{27}+2c^5_{27}+c^2_9,\ -2-c^1_{27}-c^2_{27}-c^1_9-2c^4_{27}-2c^5_{27}-c^2_9,\ -2-3c^1_{27}-3c^2_{27}-3c^1_9-2c^4_{27}-2c^5_{27}-2c^2_9-c^7_{27}-c^8_{27},\ -1-c^1_{27}-c^2_{27}-c^1_9-c^4_{27}-c^5_{27}-c^7_{27}-c^8_{27},\ 0;\ \ -\xi^7_{54},\ \xi^5_{54},\ \xi^{23}_{54},\ -\xi^{25}_{54},\ 1,\ -1-c^1_{27}-c^2_{27}-c^1_9-c^4_{27}-c^5_{27}-c^7_{27}-c^8_{27};\ \ -\xi^{19}_{54},\ -1,\ -\xi^{13}_{54},\ -\xi^7_{54},\ 2+3c^1_{27}+3c^2_{27}+3c^1_9+2c^4_{27}+2c^5_{27}+2c^2_9+c^7_{27}+c^8_{27};\ \ \xi^{17}_{54},\ \xi^5_{54},\ \xi^{11}_{54},\ -2-3c^1_{27}-3c^2_{27}-3c^1_9-2c^4_{27}-2c^5_{27}-2c^2_9-c^7_{27}-c^8_{27};\ \ -\xi^{11}_{54},\ \xi^{19}_{54},\ 1+c^1_{27}+c^2_{27}+c^1_9+c^4_{27}+c^5_{27}+c^7_{27}+c^8_{27};\ \ -\xi^{23}_{54},\ 2+c^1_{27}+c^2_{27}+c^1_9+2c^4_{27}+2c^5_{27}+c^2_9;\ \ 0)$

Realization: $G(2)_5$.

18. $12^{21,464}_{0,3926.}$: $d_i = (1.0, 12.146, 12.146, 13.146, 14.146, 15.146, 15.146, 20.887, 20.887, 20.887, 27.293, 27.293)$

$D^2 = 3926.660 = 2142 + 1701c^1_7 + 756c^2_7$

$T = (0, \frac{3}{7}, \frac{4}{7}, 0, 0, \frac{1}{7}, \frac{6}{7}, 0, \frac{1}{3}, \frac{2}{3}, \frac{2}{7}, \frac{5}{7})$,

$S = (1,\ 3\xi^5_{14},\ 3\xi^5_{14},\ 7+6c^1_7+3c^2_7,\ 8+6c^1_7+3c^2_7,\ 9+6c^1_7+3c^2_7,\ 9+6c^1_7+3c^2_7,\ 11+9c^1_7+3c^2_7,\ 11+9c^1_7+3c^2_7,\ 11+9c^1_7+3c^2_7,\ 15+12c^1_7+6c^2_7,\ 15+12c^1_7+6c^2_7;\ \ 15+12c^1_7+6c^2_7,\ 3\xi^3_7,\ -15-12c^1_7-6c^2_7,\ -9-6c^1_7-3c^2_7,\ 18+15c^1_7+6c^2_7,\ -3\xi^5_{14},\ 0,\ 0,\ 0,\ -12-9c^1_7-3c^2_7,\ 9+6c^1_7+3c^2_7;\ \ 15+12c^1_7+6c^2_7,\ -15-12c^1_7-6c^2_7,\ -9-6c^1_7-3c^2_7,\ -3\xi^5_{14},\ 18+15c^1_7+6c^2_7,\ 0,\ 0,\ 0,\ 9+6c^1_7+3c^2_7,\ -12-9c^1_7-3c^2_7;\ \ -8-6c^1_7-3c^2_7,\ -1,\ 3\xi^5_{14},\ 3\xi^5_{14},\ 11+9c^1_7+3c^2_7,\ 11+9c^1_7+3c^2_7,\ 11+9c^1_7+3c^2_7,\ -9-6c^1_7-3c^2_7,\ -9-6c^1_7-3c^2_7;\ \ 7+6c^1_7+3c^2_7,\ 15+12c^1_7+6c^2_7,\ 15+12c^1_7+6c^2_7,\ -11-9c^1_7-3c^2_7,\ -11-9c^1_7-3c^2_7,\ -11-9c^1_7-3c^2_7,\ 3\xi^5_{14},\ 3\xi^5_{14};\ \ -9-6c^1_7-3c^2_7,\ 12+9c^1_7+3c^2_7,\ 0,\ 0,\ 0,\ -15-12c^1_7-6c^2_7,\ -3\xi^3_7;\ \ -9-6c^1_7-3c^2_7,\ 0,\ 0,\ 0,\ -3\xi^3_7,\ -15-12c^1_7-6c^2_7;\ \ 22+18c^1_7+6c^2_7,\ -11-9c^1_7-3c^2_7,\ -11-9c^1_7-3c^2_7,\ 0,\ 0;\ \ -11-9c^1_7-3c^2_7,\ 22+18c^1_7+6c^2_7,\ 0,\ 0;\ \ -11-9c^1_7-3c^2_7,\ 0,\ 0;\ \ -3\xi^5_{14},\ 18+15c^1_7+6c^2_7;\ \ -3\xi^5_{14})$

Realization: unknown

19. $12^{63,774}_{0,3926.}$: $d_i = (1.0, 12.146, 12.146, 13.146, 14.146, 15.146, 15.146, 20.887, 20.887, 20.887, 27.293, 27.293)$

$D^2 = 3926.660 = 2142 + 1701c^1_7 + 756c^2_7$

$T = (0, \frac{3}{7}, \frac{4}{7}, 0, 0, \frac{1}{7}, \frac{6}{7}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{2}{7}, \frac{5}{7})$,

$S = (1,\ 3\xi^5_{14},\ 3\xi^5_{14},\ 7+6c^1_7+3c^2_7,\ 8+6c^1_7+3c^2_7,\ 9+6c^1_7+3c^2_7,\ 9+6c^1_7+3c^2_7,\ 11+9c^1_7+3c^2_7,\ 11+9c^1_7+3c^2_7,\ 11+9c^1_7+3c^2_7,\ 15+12c^1_7+6c^2_7,\ 15+12c^1_7+6c^2_7;\ \ 15+12c^1_7+6c^2_7,\ 3\xi^3_7,\ -15-12c^1_7-6c^2_7,\ -9-6c^1_7-3c^2_7,\ 18+15c^1_7+6c^2_7,\ -3\xi^5_{14},\ 0,\ 0,\ 0,\ -12-9c^1_7-3c^2_7,\ 9+6c^1_7+3c^2_7;\ \ 15+12c^1_7+6c^2_7,\ -15-12c^1_7-6c^2_7,\ -9-6c^1_7-3c^2_7,\ -3\xi^5_{14},\ 18+15c^1_7+6c^2_7,\ 0,\ 0,\ 0,\ 9+6c^1_7+3c^2_7,\ -12-9c^1_7-3c^2_7;\ \ -8-6c^1_7-3c^2_7,\ -1,\ 3\xi^5_{14},\ 3\xi^5_{14},\ 11+9c^1_7+3c^2_7,\ 11+9c^1_7+3c^2_7,\ 11+9c^1_7+3c^2_7,\ -9-6c^1_7-3c^2_7,\ -9-6c^1_7-3c^2_7;\ \ 7+6c^1_7+3c^2_7,\ 15+12c^1_7+6c^2_7,\ 15+12c^1_7+6c^2_7,\ -11-9c^1_7-3c^2_7,\ -11-9c^1_7-3c^2_7,\ -11-9c^1_7-3c^2_7,\ 3\xi^5_{14},\ 3\xi^5_{14};\ \ -9-6c^1_7-3c^2_7,\ 12+9c^1_7+3c^2_7,\ 0,\ 0,\ 0,\ -15-12c^1_7-6c^2_7,\ -3\xi^3_7;\ \ -9-6c^1_7-3c^2_7,\ 0,\ 0,\ 0,\ -3\xi^3_7,\ -15-12c^1_7-6c^2_7;\ \ 9c^1_{63}-9c^2_{63}+3c^4_{63}-9c^1_9+9c^8_{63}+6c^{10}_{63}-3c^{11}_{63}+2c^2_9-9c^{16}_{63}+9c^{17}_{63},\ -9c^1_{63}-2c^1_9-9c^8_{63}-6c^{10}_{63}-2c^2_9-6c^{17}_{63},\ 9c^2_{63}-3c^4_{63}+11c^1_9+3c^{11}_{63}+9c^{16}_{63}-3c^{17}_{63},\ 0,\ 0;\ \ 9c^2_{63}-3c^4_{63}+11c^1_9+3c^{11}_{63}+9c^{16}_{63}-3c^{17}_{63},\ 9c^1_{63}-9c^2_{63}+3c^4_{63}-9c^1_9+9c^8_{63}+6c^{10}_{63}-3c^{11}_{63}+2c^2_9-9c^{16}_{63}+9c^{17}_{63},\ 0,\ 0;\ \ -9c^1_{63}-2c^1_9-9c^8_{63}-6c^{10}_{63}-2c^2_9-6c^{17}_{63},\ 0,\ 0;\ \ -3\xi^5_{14},\ 18+15c^1_7+6c^2_7;\ \ -3\xi^5_{14})$

Realization: unknown

8 Realizations of exotic modular data

We find some potential modular data that cannot be realized by modular tensor categories from Kac-Moody algebra or twisted quantum doubles, nor from their Abelian anyon condensations [79], their Galois conjugation, and their change of spherical structure. We refer to those potential modular data as exotic potential modular data.

All the exotic potential modular data that we found are listed below:

1. two Galois orbits of rank-8 modular data represented by $8^{6,102}_{4,36.}$ and $8^{12,972}_{4,36.}$ with $D^2 = 36$,
2. three Galois orbits of rank-8 modular data with $D^2 \approx 308.434$,
3. one Galois orbit of rank-10 modular data represented by $10^{20,676}_{0,1435.}$ with $D^2 \approx 1435.541$,
4. two Galois orbits of rank-11 modular data with $D^2 \approx 1337.107$ and $D^2 \approx 1964.590$,
5. four Galois orbits of rank-12 modular data with $D^2 = 144$ (which contain six unitary modular data).
6. one Galois orbit of rank-12 modular data represented by $12^{39,406}_{4,1276.}$ with $D^2 \approx 1276.274$.
7. two Galois orbits of rank-12 modular data with $D^2 \approx 3926.660$.

In the following, we will discuss the realizations of those exotic potential modular data, to see if they are actually exotic modular data that can realized by some modular tensor categories.

8.1 Near-Group fusion categories and their centers

The main references for near-group categories and their centers are [116, 122]. We reproduce their results for the reader's convenience.

Let G be a finite group of order $|G|$ and m a non-negative integer. A **near-group category** of type $G+m$ is a rank $|G|+1$ fusion category with simple objects labeled by elements $g \in G$ and an additional simple object ρ such that the fusion rules are given by the group operation in G, $g\rho = \rho g = \rho$ for all $g \in G$, and $\rho \otimes \rho = m\rho + \sum_{g\in G} g$. While the near-group fusion rule of type $G+m$ is well-defined, not all are associated with fusion categories. We denote the (possibly empty) class of fusion categories of near-group type $G+m$ by $\mathcal{NG}(G,m)$. In the literature, one typically finds results in the *unitary* setting, so we will focus on this situation. It is known [116, Theorem 2] that, in order for $\mathcal{NG}(G,m)$ to contain an unitary fusion category with $H^2(G,\mathbb{R}/\mathbb{Z}) = 0$, the only possible values of m are $|G|-1$ or $k|G|$ for some non-negative integer k. The Tambara-Yamagami categories are the near-group fusion categories with $m = 0$, which are classified in [123].

To construct/classify unitary near-group fusion categories, one must solve a system of non-linear equations [122], which is computationally strenuous. A precise statement for G abelian can be found in [122, Theorem 5.3] and [116, Corollary 5]. In the abelian case one first chooses a non-degenerate symmetric bicharacter $\langle\ ,\ \rangle$ on G, which facilitates the solution method found in [116, 122]. Realizations have been found for near-group fusion rules of the following types, for example:

1. $A+0$ for all abelian groups A [123]
2. $A+|A|$ for A abelian and $|A| \leq 13$ [116]
3. $\mathbb{Z}_N + N$ for $N \leq 30$, except $N = 19, 29$. [124].
4. $\mathbb{Z}_3 + 6$ see [125] who attributes this to Liu and Snyder, [126].

5. $G + 2^k$ with G and extra-special 2-group of order $|G| = 2^{2k+1}$. [125, Therem 6.1].

If it is true that there are finitely many fusion categories of each rank, as is true in the braided setting [127], there should be no near-group categories of type $G+m$ for m above some bound. For example there are no unitary near-group categories of the following types:

1. $\mathbb{Z}_2 + m$ $m \geq 3$ [128],
2. $(\mathbb{Z}_2)^k + m$ with $k \geq 3$ and $m \neq 0$ [129],
3. $\mathbb{Z}_3 + m$ $m \geq 7$ [130],
4. $G+m$ where G is non-abelian, except when G is an extra-special 2-group as described above [125].

The modular data of the center $\mathcal{Z}(\mathcal{C})$ of a near group-category $\mathcal{C}$ can sometimes be obtained, but it is computationally difficult (employing tube algebra or Cuntz algebra methods, [122, 125]).

If $\mathcal{C}$ is a unitary near-group fusion category of type $A + |A|$ for an abelian group A, formulae for the modular data are found in [116], see [131] for some explicit computations. In this case we have the following facts about $\mathcal{D} := \mathcal{Z}(\mathcal{C})$, where $\mathcal{C}$ is of type $A + N$ where $|A| = N$:

1. The rank of $\mathcal{D}$ is $N(N+3)$
2. The dimensions of simple objects in $\mathcal{D}$ are $1, d, d+1, d+2$ where $d := \frac{N+\sqrt{N^2+4N}}{2}$.
3. There are N invertible objects,
4. N simple objects of dimension $d+1$,
5. $\binom{N}{2}$ simple objects of dimension $d+2$, and
6. $N(N+3)/2$ simple objects of dimension d.
7. $\dim(\mathcal{D}) = (N + d^2)^2$,
8. the pointed part of $\mathcal{D}$ is a ribbon fusion category of the form $\mathcal{C}(A,q)$ where q is a quadratic form given by $q(a) = \langle a, a\rangle$ with $\langle , \rangle$ a non-degenerate symmetric bicharacter on A. In particular the S-matrix for the pointed subcategory has entries $\langle a, b\rangle^{-2}$ and the T-matrix has entries $\delta_{a,b}\langle a, b\rangle$.

The last point completely characterizes the pointed subcategory of $\mathcal{D}$, without the need of further computation. Using the results of [116] it can be shown (cf. [131, Section 2.4]) that if $\mathcal{C}$ is a near-group unitary fusion category of type $A + |A|$ for $|A| = N$ odd, then the rank N pointed subcategory $\mathcal{C}(A,q)$ of $\mathcal{D}$ is modular, and thus $\mathcal{D} \cong \mathcal{C}(A,q) \boxtimes \mathcal{F}$ where $\mathcal{F}$ is a modular category. Moreover, $\mathcal{F}$ has rank $(N+3)$ and dimension $D^2 = \frac{N(N+4)}{2}\left(N + \sqrt{N(N+4)} + 2\right)$, with no non-trivial invertible objects. When $|A|$ is even, one often finds that $\mathcal{D}$ contains invertible bosons, which can be condensed.

In the following we address the question of realizability of several of our modular data via centers of near-group categories. In some cases this leads to definitive constructions, while in other we provide strong evidence of realizability.

1. One obtains rank 8 modular categories as the non-pointed Deligne factor of $\mathcal{Z}(\mathcal{C})$ for $\mathcal{C}$ a near-group category of type $\mathbb{Z}_3 \times \mathbb{Z}_3 + 0$. These provide **definitive** realizations for the two modular data of the form $8^{6,102}_{4,36.}$ and $8^{12,972}_{4,36.}$.

2. One also obtains rank 8 modular categories as the non-pointed Deligne factor of $\mathcal{Z}(\mathcal{C})$ for $\mathcal{C}$ a near-group category of type $\mathbb{Z}_5+5$. These provide **definitive** realizations for the three modular data of the form $8^{15,x}_{y,308.4}$ (note that there are exactly 3 near-group categories of type $\mathbb{Z}_5+5$ in [116]).

3. The case of $\mathcal{C}$ being a near-group category of type $\mathbb{Z}_3+6$ is relevant to us. It exists by [125], but we do not know many details about $\mathcal{D}=\mathcal{Z}(\mathcal{C})$ other than its dimension: $144\cdot(7+4\sqrt{3})$. One can show directly that $\mathcal{D}$ contains a modular pointed subcategory with fusion rules like $\mathbb{Z}_3$, so that $\mathcal{D}\cong\mathcal{F}\boxtimes\mathcal{C}(\mathbb{Z}_3,q)$ where $\mathcal{F}$ is a modular category of dimension $48\cdot(7+4\sqrt{3})\approx 668.553$, and hence a **candidate** for a realization of the potential modular data $9^{12,567}_{6,668.5}$. For our purposes it is sufficient to show that $\mathcal{D}$ has rank 27, which is a special case of [132, Conj. 7.6] see Example 7.1.7(1) in the same reference. We have verified that our modular data coincides with the conjectural data provided in [132]. On the other hand, it is shown in [117] the existence of a category with the same rank and dimensions as $9^{12,567}_{6,668.5}$ by condensing the $\mathbb{Z}_3$-bosons in $SU(3)_9$, providing a **definitive** realization. We find that $SU(3)_{-9}\boxtimes 9^{12,567}_{6,668.5}$ has 9 potential Lagrangian condensable algebras, supporting the above realization.

4. Similarly as the rank 8 case above, one obtains rank 10 modular categories as the non-pointed Deligne factor of $\mathcal{Z}(\mathcal{C})$ for $\mathcal{C}$ a near-group category of type $\mathbb{Z}_7+7$. These provide **definitive** realizations for the modular data $10^{77,298}_{6,684.3}$ (there is one fusion category of type $\mathbb{Z}_7+7$ in [116] up to complex conjugation).

5. Consider $\mathcal{D}=\mathcal{Z}(\mathcal{C})$ where $\mathcal{C}$ is a near-group fusion category of type $A+|A|$ for $A=\mathbb{Z}_4\times\mathbb{Z}_4$ with symmetric bicharacter $\langle(a,b),(c,d)\rangle=(\zeta_4)^{ac-bd}$ where $\zeta_4=e^{\pi i/2}$. $\mathcal{D}$ has dimension $5\cdot 2^{10}\cdot(9+4\sqrt{5})$ and rank 304. We find a Tannakian pointed subcategory $\mathcal{P}\subset\mathcal{D}$ with fusion rules like $\mathbb{Z}_4\times\mathbb{Z}_2$, generated by $(1,1)$ and $(2,0)$. Note that $\langle(1,0),(1,1)\rangle^{-2}=(\zeta_4)^{-2}=-1$, so that $(1,0)$ is not in the centralizer of $\mathcal{P}$. Thus the condensation $[\mathcal{D}_{\mathbb{Z}_4\times\mathbb{Z}_4}]_0$ has no invertible objects and has dimension $\frac{5\cdot 2^{10}\cdot(9+4\sqrt{5})}{8^2}\approx 1435.541$. This method was used to confirm that we have a realization of modular data $10^{20,676}_{0,1435.}$ in [118], providing a **definitive** construction for this modular data. It is also found to be the center of a rank 4 fusion category, see [133] On the other hand, the modular data $10^{10,168}_{4,1435.}$ looks very similar, but is realized by quantum groups: a $\mathbb{Z}_5$ boson condensation of $SU(5)_5$.

6. Another example is the following: Consider a fusion category $\mathcal{C}$ of near-group type $\mathbb{Z}_{12}+12$ (which exists, by [116]). In this case we find that $\mathcal{D}=\mathcal{Z}(\mathcal{C})$ a modular category of rank 180 and dimension $48^2\cdot(7+4\sqrt{3})$. By example 2.5 of [131] We see that $\mathcal{D}\cong\mathcal{F}\boxtimes\mathcal{C}(\mathbb{Z}_3,q)$ for some quadratic form q on $\mathbb{Z}_3$. Now $\mathcal{F}$ contains an invertible boson b with $b\otimes b\cong\mathbf{1}$ so that we obtain a modular category $[\mathcal{F}_{\mathbb{Z}_2}]_0$ by condensation. Moreover, computations as in [131] show that $[\mathcal{F}_{\mathbb{Z}_2}]_0$ factors as $\mathcal{B}\otimes\mathcal{C}(\mathbb{Z}_2,q')$ where $\mathcal{B}$ is a modular category of dimension $96\cdot(7+4\sqrt{3})\approx 1337.107510$. A more detailed analysis is necessary, but this is a strong evidence that this provides a **candidate** realization of potential modular data $11^{48,634}_{3,1337.}$.

7. The potential modular data $12^{39,406}_{4,1276.}$ is similar to that of $12^{39,560}_{0,1276.}$ which is realized by means of the center of the even part of the Haagerup subfactor. As the latter may also be obtained from the center of the near-group category $\mathcal{NG}(\mathbb{Z}_3\times\mathbb{Z}_3,9)$, this suggests a similar construction for this potential modular data, and therefore is a **candidate** realization. Indeed, in [116] there are 2 distinct near-group categories associated with $\mathbb{Z}_3\times\mathbb{Z}_3$.

8.2 Realizations via minimal modular extensions

The theory of minimal modular extensions could explain the multiplicity of a collection of similar but not identical modular tensor categories. Let $\mathcal{C}$ be a modular tensor category which contains a fusion subcategory $\mathcal{F}$ such that the Frobenius-Perron dimension of $\mathcal{C}$ is the product of FPdim($\mathcal{F}$) and the FP-dimension of the Müger center $\mathcal{E}$ of $\mathcal{F}$. In this case, $\mathcal{C}$ is called a minimal modular extension of $\mathcal{F}$ (cf. [134]). Note that $\mathcal{E}$ is a symmetric fusion category, and so $\mathcal{E}$ is equivalent as braided fusion categories to the representation category of a finite group G, which is uniquely determined by $\mathcal{E}$. According to the theorem of [135], the set of equivalent classes of the minimal modular extensions of $\mathcal{F}$ is a torsor of the 3rd cohomology group $H^3(G, U(1))$ of G. In fact, there is a faithful transitive action of $H^3(G, U(1))$ on the set of minimal modular extensions of $\mathcal{F}$ by condensing $\mathcal{E}$ of the Deligne product of $\mathcal{C}$ and the center $Z(Vec_G^\omega)$ of the pointed category Vec_G^ω for any cohomology class ω in $H^3(G, U(1))$. In particular, if a minimal modular extension of $\mathcal{F}$ shows up in the list, one should find exactly the number of classes in the group $H^3(G, U(1))$.

It is quite routine to check a given modular data is a minimal extension of one of its fusion subcategory. We consider the six rank 12 modular data of FPdim $D^2 = 144$ in our list as examples. Let $\mathcal{C}$ be an MTC whose modular data is any of these rank 12 data. One can see immediately that there are 3 bosons but one of them is nonabelian. It is can check from the fusion rules computed from the S-matrix that these three objects form a Tannakian subcategory $\mathcal{E}$ of dimension 6. It has to be the representation category of the symmetric group S_3. Using the modular data again, we see that the simple objects of dimension 3 together with the simple object of $\mathcal{E}$ form a fusion subcategory $\mathcal{F}$ of dimension 24. By considering the twists of the simple objects of $\mathcal{F}$, we can confirm that $\mathcal{E}$ is the Mg̈er center of $\mathcal{F}$ and so $\mathcal{C}$ is a minimal modular extension of $\mathcal{F}$. Since $H^3(S_3, U(1))$ is a cyclic group of order 6, there are exactly 6 minimal modular extensions of $\mathcal{F}$, and they are all there.

The two rank 12 dimension $D^2 = 68$ modular data, $12^{34,116}_{0,68.}$ and $12^{68,166}_{0,68.}$, are minimal modular extensions $\mathcal{C}$ of its integral fusion subcategory $\mathcal{F}$, whose Müger center is obviously a fusion subcategory generated by the abelian boson. A fusion category of FP-dimension 2 can only be equivalent to the representation category of the cyclic group G of order 2. Since $H^3(G, U(1))$ is also a cyclic group of order 2, there are exactly two such minimal modular extensions.

8.3 Gauging and Zesting

Some of the modular data realizations are related to other more familiar categories by means of well-studied constructions such as gauging [121] and zesting [112, 136]. For example $12^{144,916}_{4,144.}$ was first constructed by gauging, and $10^{18,490}_{4,36.}$ was first constructed by zesting. The condensed fiber product [114] is related to zesting and the minimal modular construction described above. We briefly describe gauging and zesting mathematically, referring the interested reader to *loc. cit.* for details.

Gauging is a 2 step process. One begins with a MTC $\mathcal{C}$ with an action of a finite group G by braided tensor autoequivalences. For example the rank 4 theory known as 3-fermions with fusion rules like $\mathbb{Z}_2 \times \mathbb{Z}_2$ and non-trivial objects having twist -1 admits an action of S_3, by permuting the 3 nontrivial objects. Then, assuming certain cohomological obstructions vanish, one constructs a G-crossed braided fusion category $\mathcal{D}$ as a G-extension of $\mathcal{C}$. There are typically choices to be made in this step, which are parametrized by cohomological data. Then one takes the G-equivariantization $\mathcal{D}^G$ of $\mathcal{D}$. This will be a modular category of dimension $\dim(\mathcal{C}) \cdot |G|^2$ and will contain a symmetric subcategory equivalent to $\mathcal{R}\text{ep}(G)$. It is often easy to see that a given modular category is a gauging, by looking for these signatures. Of course if one has modular data with these signatures this helps in the search for a realization, which must then be constructed explicitly by gauging.

Table 8: A list of potential modular data (one for each Galois orbit) whose realization are unknown or not definitively constructed.

$r_{c,D^2}^{\text{ord}T,\text{fp}}$	s_i	d_i
$11^{48,634}_{3,1337.}$	$0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{16}, \frac{11}{16}, 0, \frac{1}{3}, \frac{7}{12}$	$1,\ 3+2\sqrt{3},\ 3+2\sqrt{3},\ 4+2\sqrt{3},\ 4+2\sqrt{3},\ 6+4\sqrt{3},\ 6+4\sqrt{3},\ 6+4\sqrt{3},\ 7+4\sqrt{3},\ 8+4\sqrt{3},\ 8+4\sqrt{3}$ (see Section 8.1)
$11^{35,581}_{\frac{32}{5},1964.}$	$0, \frac{2}{35}, \frac{22}{35}, \frac{32}{35}, \frac{1}{5}, 0, \frac{3}{7}, \frac{5}{7}, \frac{6}{7}, \frac{3}{5}, \frac{1}{5}$	1.0, 8.807, 8.807, 8.807, 11.632, 13.250, 14.250, 14.250, 14.250, 19.822, 20.440 (for algebraic expressions, see Section 1.3)
$12^{39,406}_{4,1276.}$	$0, \frac{1}{13}, \frac{3}{13}, \frac{4}{13}, \frac{9}{13}, \frac{10}{13}, \frac{12}{13}, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$1,\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{11+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2}$
$12^{21,464}_{0,3926.}$	$0, \frac{3}{7}, \frac{4}{7}, 0, 0, \frac{1}{7}, \frac{6}{7}, 0, \frac{1}{3}, \frac{2}{3}, \frac{2}{7}, \frac{5}{7}$	$1,\ 3\xi_{14}^{5},\ 3\xi_{14}^{5},\ 7+6c_7^1+3c_7^2,\ 8+6c_7^1+3c_7^2,\ 9+6c_7^1+3c_7^2,\ 9+6c_7^1+3c_7^2,\ 11+9c_7^1+3c_7^2,\ 11+9c_7^1+3c_7^2,\ 11+9c_7^1+3c_7^2,\ 15+12c_7^1+6c_7^2,\ 15+12c_7^1+6c_7^2$
$12^{63,774}_{0,3926.}$	$0, \frac{3}{7}, \frac{4}{7}, 0, 0, \frac{1}{7}, \frac{6}{7}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{2}{7}, \frac{5}{7}$	$1,\ 3\xi_{14}^{5},\ 3\xi_{14}^{5},\ 7+6c_7^1+3c_7^2,\ 8+6c_7^1+3c_7^2,\ 9+6c_7^1+3c_7^2,\ 9+6c_7^1+3c_7^2,\ 11+9c_7^1+3c_7^2,\ 11+9c_7^1+3c_7^2,\ 11+9c_7^1+3c_7^2,\ 15+12c_7^1+6c_7^2,\ 15+12c_7^1+6c_7^2$

Zesting is similarly a process that uses cohomological data to build a new category out of a given one. In this case the set up is that one has an A-graded MTC with a pointed (i.e., abelian) subcategory $\mathcal{B}$ in the trivial component (here A is necessarily an abelian group). For example $SU(3)_3$ i.e., $10^{6,152}_{4,36.}$ is $\mathbb{Z}_3$-graded and the trivial component contains the pointed subcategory $\mathcal{R}\text{ep}(\mathbb{Z}_3)$. Then one can twist the fusion rules component-wise, using a 2-cocycle on A with values in $\mathcal{B}$. This requires adjustment of the associativity constraints component-wise by a 3-cochain. In some cases the resulting category admits a braiding, which is obtained by adjusting the braiding in $\mathcal{C}$, again component-wise. Finally, one may typically adjust the twists to obtain a ribbon fusion category–which will be modular under some mild conditions. This all requires choices at each stage, but is quite explicit.

More recently, some instances of the zesting procedure has been explained in more physically relevant terms, called the *condensed fiber product* [114]. This is also related to the minimal modular extension torsor described above. Here one takes two A-graded modular categories $\mathcal{D}$ and $\mathcal{C}$ with a common pointed symmetric subcategory $\mathcal{B}$ in the trivial components $\mathcal{D}_0$ and $\mathcal{C}_0$, and condenses the Tannakian diagonal subcategory $\mathcal{R}\text{ep}(B) \cong \Delta(\mathcal{B}) \subset \mathcal{B} \boxtimes \mathcal{B}$ in $\mathcal{D} \boxtimes \mathcal{C}$, for some finite abelian group B. The Tannakian subcategory $\Delta(\mathcal{B})$ centralizes the fiber product $\bigoplus_a \mathcal{D}_a \boxtimes \mathcal{C}_a$, so that $([\mathcal{D} \boxtimes \mathcal{C}]_B)_0$ is again a modular category. If $\mathcal{D}$ is pointed with $\dim(\mathcal{D}) = |B|^2$ then the condensed fiber product coincides with zesting. This is also related to the construction of *anyon condensation* found in [137].

8.4 Potential modular data whose realizations are unknown

Table 8 lists the potential modular data whose realizations are still unknown or unsure. Those potential modular data have different fusion rings from the the modular tensor categories generated from Kac-Moody algebra and twisted quantum doubles, plus Abelian anyon condensations. Those data may correspond to new modular tensor categories, or they are fake modular data. For the rank-11 data $11^{48,634}_{3,1337.}$, we have some evidence that they can be realized by centers of near-group fusion categories, followed by some condensation reductions.

In order to gain some understanding, in this section, we are going compute the potential condensible algebra $\mathcal{A} = \oplus_i A_i(d_i, s_i)$ (see Appendix E) for those potential modular data. Here (d_i, s_i) is the i^{th} simple object, labeled by its quantum dimension d_i and topological spin s_i.

1. **Potential modular data** $11^{48,634}_{3,1337.}$ has no Lagrangian condensible algebra since

the central charge $c \neq 0$. It has one potential condensible algebra $\mathcal{A} = (1,0) \oplus (7+4\sqrt{3},0)$. The condensation of anyon $(7+4\sqrt{3},0)$ reduces the modular data $11^{48,634}_{3,1337.}$ to the modular data $6^{12,534}_{3,6} = 2^{4,437}_{1,2} \boxtimes 3^{3,527}_{2,3}$ – a pointed $\mathbb{Z}_2 \boxtimes \mathbb{Z}_3$ MTC, since $11^{48,634}_{3,1337.} \boxtimes \overline{6^{12,534}_{3,6}}$ has Lagrangian condensible algebras.

2. **Potential modular data** $11^{35,581}_{\frac{32}{5},1964.}$ (see Section 1.3 for a more detailed descrition) has no Lagrangian condensible algebra. It has one potential condensible algebra $\mathcal{A} = (1,0) \oplus (13.250,0)$. The condensation of anyon $(13.250,0)$ should reduce $11^{35,581}_{\frac{32}{5},1964.}$ to an unitary modular data with $D^2 = \frac{35-7\sqrt{5}}{2} = 9.6737$. But there are no unitary modular data with $D^2 = \frac{35-7\sqrt{5}}{2} = 9.6737$ based on our classification. Thus $\mathcal{A} = (1,0) \oplus (13.250,0)$ is a fake condensible algebra, and $11^{35,581}_{\frac{32}{5},1964.}$ has no condensible algebra.

3. **Potential modular data** $12^{39,406}_{4,1276.}$ has no Lagrangian condensible algebra. It has one potential condensible algebra $\mathcal{A} = (1,0) \oplus (\frac{11+3\sqrt{13}}{2},0)$. The condensation of anyon $(\frac{11+3\sqrt{13}}{2},0)$ should reduce the modular data $12^{39,406}_{4,1276.}$ to the modular data $9^{3,277}_{4,9} = \overline{9^{3,277}_{4,9}} = 3^{3,527}_{2,3} \boxtimes 3^{3,527}_{2,3}$ – a pointed $\mathbb{Z}_3 \boxtimes \mathbb{Z}_3$ MTC. We remark that $12^{39,406}_{4,1276.}$ is related to Haagerup-Izumi modular data. They share the same set of quantum dimension d_i. Also, $12^{39,406}_{4,1276.}$ has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism, generated exchanging two simple objects with $(d,s) = (\frac{13+3\sqrt{13}}{2}, \frac{1}{3})$, and exchanging two simple objects with $(d,s) = (\frac{13+3\sqrt{13}}{2}, \frac{2}{3})$.

4. **Potential modular data** $12^{21,464}_{0,3926.}$ has no Lagrangian condensible algebra. It has three potential condensible algebra

$$\begin{aligned}
\mathcal{A}_1 &= (1,0) \oplus (8+6c^1_7+3c^2_7,0), \\
\mathcal{A}_2 &= (1,0) \oplus (11+9c^1_7+3c^2_7,0), \\
\mathcal{A}_3 &= (1,0) \oplus (7+6c^1_7+3c^2_7,0) \oplus (8+6c^1_7+3c^2_7,0) \oplus (11+9c^1_7+3c^2_7,0).
\end{aligned} \tag{154}$$

The condensation of $\mathcal{A}_1$ should reduce $12^{21,464}_{0,3926.}$ to an unitary modular data with $D^2 = 14 - 7c^2_7 = 17.1152$. The condensation of $\mathcal{A}_2$ should reduce $12^{21,464}_{0,3926.}$ to an unitary modular data with $D^2 = 35 - 14c^1_7 + 21c^2_7 = 8.1964$. The condensation of $\mathcal{A}_3$ should reduce $12^{21,464}_{0,3926.}$ to an unitary modular data with $D^2 = 49 - 28c^1_7 + 28c^2_7 = 1.6233$. There are some modular data with $D^2 = 14 - 7c^2_7 = 17.1152$ and $D^2 = 35 - 14c^1_7 + 21c^2_7 = 8.1964$ at rank 9. But all those modular data are not unitary. There is no unitary modular data with $D^2 = 49 - 28c^1_7 + 28c^2_7 = 1.6233$. Thus $\mathcal{A}_{1,2,3}$ are fake condensible algebras, and $12^{21,464}_{0,3926.}$ has no condensible algebra.

5. **Potential modular data** $12^{63,774}_{0,3926.}$ has no Lagrangian condensible algebra. It has one potential condensible algebra $\mathcal{A} = (1,0) \oplus (8+6c^1_7+3c^2_7,0)$. The condensation of anyon $(8+6c^1_7+3c^2_7,0)$ should reduce $12^{63,774}_{0,3926.}$ to an unitary modular data with $D^2 = 14 - 7c^2_7 = 17.1152$. There are some modular data with $D^2 = 14 - 7c^2_7 = 17.1152$ at rank 9. But all those modular data are not unitary. Thus $\mathcal{A} = (1,0) \oplus (8+6c^1_7+3c^2_7,0)$ is a fake condensible algebra, and $12^{63,774}_{0,3926.}$ has no condensible algebra.

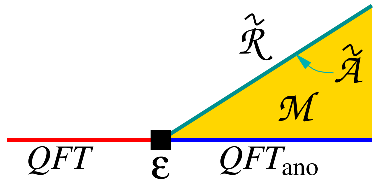

Figure 1: The isomorphism ε (*i.e.* the transparent domain wall in space-time) between two anomaly-free (gapped or gapless) quantum field theories, QFT and $QFT_{\rm ano} \boxtimes_{\mathcal{M}} \widetilde{\mathcal{R}}$, describes an low energy equivalence of the two quantum field theories, below the energy gap of the bulk topological order $\mathcal{M}$ and its gapped boundary $\widetilde{\mathcal{R}}$ [70]. Such an equivalence (called an isomorphic holographic decomposition) exposes the emergent symmetry in the quantum field theory QFT. The emergent symmetry is described by the fusion higher category $\widetilde{\mathcal{R}}$ that describes the excitations on the gapped boundary $\widetilde{\mathcal{R}}$, and will be referred to as $\widetilde{\mathcal{R}}$-category symmetry. The gapped boundary $\widetilde{\mathcal{R}}$ is induced by Lagrangian condensable algebra $\widetilde{\mathcal{A}}$. The holo-equivalent class of the emergent symmetries, by definition, is described by a braided fusion higher category $\mathcal{M}$ that describes the excitations in the bulk topological order $\mathcal{M}$. The bulk topological order $\mathcal{M}$ will be referred to as the symTO, which is a topological order with gappable boundary.

9 Classify symmetries via symTOs (*i.e.* UMTCs in the trivial Witt class)

We used to think symmetries are described by groups. In recent years, we realized that the low energy emergent symmetry in a quantum field theory QFT can be a generalized symmetry beyond group and higher group. It turns out that finite generalized symmetries can all be described by higher fusion categories.

One way to obtain such a result is through the isomorphic holographic decomposition in Fig. 1, which was introduced to define homomorphism between quantum field theories [70]. If a quantum field theory QFT has an isomorphic holographic decomposition in Fig. 1, then we say that the quantum field theory QFT has a $\widetilde{\mathcal{R}}$-category symmetry. Thus a generalized symmetry is described by a fusion higher category $\widetilde{\mathcal{R}}$, which describes the excitations on the gapped boundary $\widetilde{\mathcal{R}}$ of the bulk topological order $\mathcal{M} = \mathcal{Z}(\widetilde{\mathcal{R}})$ in Fig. 1. Here $\mathcal{Z}$ is the generalized Drinfeld center that maps the fusion higher category $\widetilde{\mathcal{R}}$ to the braided fusion higher category $\mathcal{M}$ that describes the excitations in the bulk topological order.

The connection between boundary symmetry and bulk topological order was observed in Ref. [138], where it was shown that topological entanglement entropy arises from a boundary conservation law rooted in the bulk topological order. This connection was later confirmed through numerical calculations [139]. A systematic theory of symmetry topological-order (sym/TO) correspondence was developed via holographic picture of emergent non-invertible gravitational anomaly [70, 71, 140], holographic picture of duality [141, 142], which lead to holographic picture of generalized symmetry [9, 66, 67, 72–74, 143]. Symm/TO correspondence is also closely related to *topological Wick rotation* introduced in Ref. [140, 144, 145], which summarizes a mathematical theory on how bulk can determine boundary.

A notion of holo-equivalence between symmetries was introduced in [9, 61, 66]. Holo-equivalent symmetries impose the same constraint on the dynamical properties of the associated systems within the symmetric sub Hilbert spaces. Thus holo-equivalent symmetries are indistinguishable if we ignore the sectors with non-zero total symmetry charge.

The holo-equivalent class of the above $\widetilde{\mathcal{R}}$-category symmetry is described by the bulk $\mathcal{M}$ in Fig. 1, which will be referred to as the symTO of $\widetilde{\mathcal{R}}$-category symmetry. SymTOs are classified by topological orders $\mathcal{M}$ with gappable boundary.

For bosonic systems in 1-dimensional space, the holo-equivalent classes of the generalized symmetries are classified by UMTCs in trivial Witt class. The symmetries in a holo-equivalent class described by an UMTC $\mathcal{M}$ are classified by the gapped boundaries (*i.e.* fusion categories $\widetilde{\mathcal{R}}$) of $\mathcal{M}$: $\mathcal{M} = \mathcal{Z}(\widetilde{\mathcal{R}})$. In this section, we are going to use the classification of UMTCs to classify symTOs via UMTCs in trivial Witt class, which in turn classify (generalized) symmetries in 1-dimensional space.

How to determine if a UMTC is in the trivial Witt class or not? We know some necessary conditions for a UMTC to be in the trivial Witt class. One set of necessary conditions is the higher central charges [80]. Another set of necessary conditions comes from Lagrangian condensable algebra [146], which is summarized in Appendix E. We apply these two sets of necessary conditions and find the potential Witt-trivial UMTCs. By examining them one by one, we find that those potential Witt-trivial UMTCs are actually in trivial Witt class.

Witt-trivial UMTCs do not exist for rank 2, 3, 5, 6, 7, 11. In the following we list Witt-trivial UMTCs (*i.e.* symTOs) for rank 4, 8, 9, 10, 12. We also list the composite objects $\mathcal{A} = \bigoplus_i A_i\, i$ (where i are simple objects and $A_i \in \mathbb{N}$) that give rise to the Lagrangian condensable algebras. Actually, in next a few subsections, the composite object $\mathcal{A}$ is listed by its expansion coefficients A_i. We note that a Lagrangian condensable algebra $\widetilde{\mathcal{A}}$ gives rise to a gapped boundary $\widetilde{\mathcal{R}}$ and a $\widetilde{\mathcal{R}}$-category symmetry in the holo-equivalence class of the symTO.

When $\widetilde{\mathcal{R}}$ is a local fusion category, it will describe an anomaly-free symmetry [9]. A $\widetilde{\mathcal{R}}$-category symmetry is anomaly-free if the Lagrangian condensable algebra $\widetilde{\mathcal{A}} = \bigoplus_i \widetilde{A}_i$ that induces $\widetilde{\mathcal{R}}$ has a dual Lagrangian condensable algebra $\mathcal{A} = \bigoplus_i A_i$ such that

$$\sum_i A_i \widetilde{A}_i = 1. \tag{155}$$

We will also indicate such an anomaly-free symmetry.

9.1 Rank 4

1. $4^{2,750}_{0,4.}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0)$
   $D^2 = 4.0 = 4$
   $T = (0, 0, 0, \frac{1}{2})$,
   $S = (1,\ 1,\ 1,\ 1;\ \ 1,\ -1,\ -1;\ \ 1,\ -1;\ \ 1)$
   The holo-equivalence class of two symmetries.
   Lagrangian condensible algebra A_i:
   $(1, 1, 0, 0) \to \mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}$-category symmetry $= \mathbb{Z}_2$ symmetry
   $(1, 0, 1, 0) \to \mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$-category symmetry. Isomorphic to the above symmetry.

2. $4^{4,375}_{0,4.}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0)$
   $D^2 = 4.0 = 4$
   $T = (0, 0, \frac{1}{4}, \frac{3}{4})$,
   $S = (1,\ 1,\ 1,\ 1;\ \ 1,\ -1,\ -1;\ \ -1,\ 1;\ \ -1)$
   Factors $= 2^{4,437}_{1,2.} \boxtimes 2^{4,625}_{7,2.}$
   The holo-equivalence class of one symmetry.
   Lagrangian condensible algebra A_i:
   $(1, 1, 0, 0) \to \widetilde{\mathcal{R}}_{\mathrm{semion}}$-category symmetry = anomalous $\mathbb{Z}_2$ symmetry

3. $4^{5,872}_{0,13.09}$: $d_i = (1.0,\ 1.618,\ 1.618,\ 2.618)$
   $D^2 = 13.90 = \frac{15+5\sqrt{5}}{2}$
   $T = (0, \frac{2}{5}, \frac{3}{5}, 0)$,
   $S = (1,\ \frac{1+\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2},\ \frac{3+\sqrt{5}}{2};\ \ -1,\ \frac{3+\sqrt{5}}{2},\ -\frac{1+\sqrt{5}}{2};\ \ -1,\ -\frac{1+\sqrt{5}}{2};\ \ 1)$
   Factors $= 2^{5,395}_{\frac{14}{5},3.618} \boxtimes 2^{5,720}_{\frac{26}{5},3.618}$
   The holo-equivalence class of one symmetry.
   Lagrangian condensible algebra A_i:
   $(1,0,0,1) \to \widetilde{\mathcal{R}}_{\text{Fib}}$-category symmetry (beyond algebraic symmetry of Ref. [9])

The holo-equivalent class of symmetries described by the symTO $4^{4,375}_{0,4.}$ (the double-semion topological order)) contains one symmetry, since the symTO has only one condensable algebra $A_i = (1,1,0,0)$. The condensable algebra gives rise to a fusion category $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}_{\text{semion}} \leftarrow 2^{4,437}_{1,2.}$, the fusion category of a single semion[10], whose Drinfeld center gives rise to the symTO $4^{4,375}_{0,4.}$. Such a symTO describes a single symmetry, the anomalous $\mathbb{Z}_2$ symmetry [147, 148].

Although the symTO $4^{2,750}_{0,4.}$ (the $\mathbb{Z}_2$ topological order) has two condensable algebra $A_i = (1,1,0,0)$ and $A_i = (1,0,1,0)$, the holo-equivalent class of symmetries described by the symTO contains only one symmetry. The condensable algebra $A_i = (1,1,0,0)$ gives rise to a boundary described by fusion category $\mathcal{V}ec_{\mathbb{Z}_2}$: $4^{2,750}_{0,4.} = \mathcal{Z}(\mathcal{V}ec_{\mathbb{Z}_2})$ [149, 150]. Thus, this condensable algebra gives rise to the $\mathcal{V}ec_{\mathbb{Z}_2}$-category symmetry, which is nothing but the usual $\mathbb{Z}_2$ symmetry. The condensable algebra $A_i = (1,0,1,0)$ gives rise to a boundary described by fusion category $\mathcal{R}ep_{\mathbb{Z}_2}$: $4^{2,750}_{0,4.} = \mathcal{Z}(\mathcal{R}ep_{\mathbb{Z}_2})$, and gives rise to $\mathcal{R}ep_{\mathbb{Z}_2}$-category symmetry, which corresponds to the dual of the $\mathbb{Z}_2$ symmetry [59, 151]. Since $\mathbb{Z}_2$ is an Abelian group, the $\mathbb{Z}_2$ symmetry is isomorphic to the dual $\mathbb{Z}_2$ symmetry. Therefore, the holo-equivalent class $\mathcal{M} = 4^{2,750}_{0,4.}$ contains only one symmetry.

The holo-equivalent class of symmetries described by the symTO $4^{5,872}_{0,13.09}$ (the double-Fibonacci topological order) also contains one symmetry. The symTO is Drinfeld center of fusion category $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}_{\text{Fib}} \leftarrow 2^{5,395}_{\frac{14}{5},3.618}$, the fusion category of a single Fibonacci anyon. Such a symTO corresponds to an non-invertible symmetry in 1-dimensional space that is beyond group theory description. It is also beyond the algebraic symmetry introduced in [9], since it is anomalous. This anomalous non-invertible symmetry can appear as a low energy emergent symmetry [152].

9.2 Rank 8

1. $8^{6,213}_{0,36.}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$
   $D^2 = 36.0 = 36$
   $T = (0, 0, 0, 0, \frac{1}{3}, \frac{2}{3}, 0, \frac{1}{2})$,
   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ 4,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ 4,\ -2,\ -2,\ 0,\ 0;\ \ -2,\ 4,\ 0,\ 0;\ \ -2,\ 0,\ 0;\ \ 3,\ -3;\ \ 3)$
   The holo-equivalence class of two symmetries.
   Lagrangian condensible algebra A_i:
   $(1,1,2,0,0,0,0,0) \to \mathcal{V}ec_{S_3}$-category symmetry $= S_3$ symmetry
   $(1,1,0,2,0,0,0,0) \to$ a symmetry isomorphic to the one above.
   $(1,0,0,1,0,0,1,0) \to \mathcal{R}ep_{S_3}$-category symmetry $=$ dual S_3 symmetry

[10] $\widetilde{\mathcal{R}} \leftarrow \mathcal{M}$ means that the fusion category $\widetilde{\mathcal{R}}$ is the fusion category formed by the objects in the braided fusion category $\mathcal{M}$, *i.e.* $\leftarrow$ is the forgetful functor which ignores the braiding. In this case the center of $\widetilde{\mathcal{R}}$ is given by $\mathcal{Z}(\widetilde{\mathcal{R}}) = \mathcal{M} \boxtimes \overline{\mathcal{M}}$.

$(1,0,1,0,0,0,1,0) \to$ a symmetry isomorphic to the one above.

2. $8^{12,101}_{0,36.}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$
   $D^2 = 36.0 = 36$
   $T = (0,0,0,0,\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4})$,
   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ 4,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ 4,\ -2,\ -2,\ 0,\ 0;\ \ -2,\ 4,\ 0,\ 0;\ \ -2,\ 0,\ 0;\ \ -3,\ 3;\ \ -3)$
   The holo-equivalence class of one symmetry
   Lagrangian condensible algebra A_i:
   $(1,1,2,0,0,0,0,0) \to$ anomalous $S_3^{(3)}$ symmetry
   $(1,1,0,2,0,0,0,0) \to$ a symmetry isomorphic to the one above.

3. $8^{18,162}_{0,36.}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$
   $D^2 = 36.0 = 36$
   $T = (0,0,0,\frac{1}{9},\frac{4}{9},\frac{7}{9},0,\frac{1}{2})$,
   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ 4,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ 2c_9^2,\ 2c_9^4,\ 2c_9^1,\ 0,\ 0;\ \ 2c_9^1,\ 2c_9^2,\ 0,\ 0;\ \ 2c_9^4,\ 0,\ 0;\ \ 3,\ -3;\ \ 3)$
   The holo-equivalence class of two symmetries
   Lagrangian condensible algebra A_i:
   $(1,1,2,0,0,0,0,0) \to$ anomalous $S_3^{(4)}$ symmetry
   $(1,0,1,0,0,0,1,0) \to$ an anomalous non-invertible symmetry

4. $8^{18,953}_{0,36.}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$
   $D^2 = 36.0 = 36$
   $T = (0,0,0,\frac{2}{9},\frac{5}{9},\frac{8}{9},0,\frac{1}{2})$,
   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ 4,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ 2c_9^4,\ 2c_9^2,\ 2c_9^1,\ 0,\ 0;\ \ 2c_9^1,\ 2c_9^4,\ 0,\ 0;\ \ 2c_9^2,\ 0,\ 0;\ \ 3,\ -3;\ \ 3)$
   The holo-equivalence class of two symmetries
   Lagrangian condensible algebra A_i:
   $(1,1,2,0,0,0,0,0) \to$ anomalous $S_3^{(2)}$ symmetry
   $(1,0,1,0,0,0,1,0) \to$ an anomalous non-invertible symmetry

5. $8^{36,495}_{0,36.}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$
   $D^2 = 36.0 = 36$
   $T = (0,0,0,\frac{1}{9},\frac{4}{9},\frac{7}{9},\frac{1}{4},\frac{3}{4})$,
   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ 4,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ 2c_9^2,\ 2c_9^4,\ 2c_9^1,\ 0,\ 0;\ \ 2c_9^1,\ 2c_9^2,\ 0,\ 0;\ \ 2c_9^4,\ 0,\ 0;\ \ -3,\ 3;\ \ -3)$
   The holo-equivalence class of one symmetry
   Lagrangian condensible algebra A_i:
   $(1,1,2,0,0,0,0,0) \to$ anomalous $S_3^{(1)}$ symmetry

6. $8^{36,171}_{0,36.}$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 2.0,\ 3.0,\ 3.0)$
   $D^2 = 36.0 = 36$
   $T = (0,0,0,\frac{2}{9},\frac{5}{9},\frac{8}{9},\frac{1}{4},\frac{3}{4})$,
   $S = (1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3;\ \ 1,\ 2,\ 2,\ 2,\ 2,\ -3,\ -3;\ \ 4,\ -2,\ -2,\ -2,\ 0,\ 0;\ \ 2c_9^4,\ 2c_9^2,\ 2c_9^1,\ 0,\ 0;\ \ 2c_9^1,\ 2c_9^4,\ 0,\ 0;\ \ 2c_9^2,\ 0,\ 0;\ \ -3,\ 3;\ \ -3)$
   The holo-equivalence class of one symmetry
   Lagrangian condensible algebra A_i:

$(1,1,2,0,0,0,0,0) \to$ anomalous $S_3^{(5)}$ symmetry

There are 6 symTOs at rank 8, giving rise to 6 holo-equivalent classes of symmetries. Each class contains an anomalous S_3-symmetry denoted as S_3^ω. In 1-dimensional space, the S_3-symmetry can have 6 types of anomalies $\omega \in H^3(S_3;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_6$ [39] which correspond to the 6 symTOs.

The first symTO $8^{6,213}_{0,36.}$ is the S_3-gauge theory in 2-dimensional space. One of its condensable algebra $A_i = (1,1,2,0,0,0,0,0)$ gives rise to a fusion category $\mathcal{V}\mathrm{ec}_{S_3}$. Thus the condensable algebra $A_i = (1,1,2,0,0,0,0,0)$ gives rise to a $\mathcal{V}\mathrm{ec}_{S_3}$-category symmetry, which is nothing but the usual S_3 symmetry.

The condensable algebra $A_i = (1,0,0,1,0,0,1,0)$ gives rise to a fusion category $\mathcal{R}\mathrm{ep}_{S_3}$. Thus the condensable algebra $A_i = (1,0,0,1,0,0,1,0)$ gives rise to a holo-equivalent symmetry – $\mathcal{R}\mathrm{ep}_{S_3}$-category symmetry, which was called dual S_3 symmetry in Ref. [9, 72], and is a non-invertible algebraic symmetry. It is interesting to see an ordinary group-like symmetry is holo-equivalent to a non-invertible algebraic symmetry.

We note that the symTO $8^{6,213}_{0,36.}$ has an automorphism of exchanging simple objects $i = 3$ and $i = 4$. The other two Lagrangian condensable algebras are related to the previous two by this automorphism. Thus, the other two condensable algebras also give rise to S_3 symmetry and dual S_3 symmetry, respectively. The symTO $8^{12,101}_{0,36.}$ also has an automorphism of exchanging simple objects $i = 3$ and $i = 4$. Its two Lagrangian condensable algebras are related by this automorphism. Such two Lagrangian condensable algebras give rise to the same fusion category $\widetilde{\mathcal{R}}$ and the same $\widetilde{\mathcal{R}}$-category symmetry.

Other rank-8 symTOs are Dijkgraaf-Witten S_3-gauge theories in 2-dimensional space. The corresponding holo-equivalence classes contain anomalous S_3^ω symmetry and anomalous non-invertible symmetry. For more discussions, see Ref. [152].

9.3 Rank 9

1. $9^{3,113}_{0,9.}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0)$
   $D^2 = 9.0 = 9$
   $T = (0,0,0,0,0,\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3})$,
   $S = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1;\ \ 1,\ 1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1;\ \ 1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1;\ \ 1,\ 1,\ -\zeta_6^1,\ \zeta_3^1,\ \zeta_3^1,\ -\zeta_6^1;\ \ 1,\ \zeta_3^1,\ -\zeta_6^1,\ -\zeta_6^1,\ \zeta_3^1;\ \ \zeta_3^1,\ -\zeta_6^1,\ 1,\ 1;\ \ \zeta_3^1,\ 1,\ 1;\ \ -\zeta_6^1,\ \zeta_3^1;\ -\zeta_6^1)$
   Factors $= 3^{3,527}_{2,3.} \boxtimes 3^{3,138}_{6,3.}$
   The holo-equivalence class of one symmetry
   Lagrangian condensible algebra A_i:
   $(1,1,1,0,0,0,0,0,0) \to \mathcal{V}\mathrm{ec}_{\mathbb{Z}_3}$-category symmetry $= \mathbb{Z}_3$ symmetry
   $(1,0,0,1,1,0,0,0,0) \to \mathcal{R}\mathrm{ep}_{\mathbb{Z}_3}$-category symmetry. Isomorphic to the above symmetry.

2. $9^{9,620}_{0,9.}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0)$
   $D^2 = 9.0 = 9$
   $T = (0,0,0,\frac{1}{9},\frac{1}{9},\frac{4}{9},\frac{4}{9},\frac{7}{9},\frac{7}{9})$,
   $S = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1;\ \ 1,\ 1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1;\ \ 1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1;\ \ -\zeta_{18}^5,\ \zeta_9^2,\ \zeta_9^4,\ -\zeta_{18}^1,\ \zeta_9^1,\ -\zeta_{18}^7;\ \ -\zeta_{18}^5,\ -\zeta_{18}^1,\ \zeta_9^4,\ -\zeta_{18}^7,\ \zeta_9^1;\ \ \zeta_9^1,\ -\zeta_{18}^7,\ -\zeta_{18}^5,\ \zeta_9^2;\ \ \zeta_9^1,\ \zeta_9^2,\ -\zeta_{18}^5;\ \ \zeta_9^4,\ -\zeta_{18}^1;\ \ \zeta_9^4)$
   The holo-equivalence class of one symmetry
   Lagrangian condensible algebra A_i:
   $(1,1,1,0,0,0,0,0,0) \to$ anomalous $\mathbb{Z}_3^{(1)}$ symmetry

3. $9^{9,462}_{0,9.}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.0)$

$D^2 = 9.0 = 9$
$T = (0, 0, 0, \frac{2}{9}, \frac{2}{9}, \frac{5}{9}, \frac{5}{9}, \frac{8}{9}, \frac{8}{9})$,
$S = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1;\ \ 1,\ 1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1;\ \ 1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1,\ \zeta_3^1,\ -\zeta_6^1;\ \ -\zeta_{18}^1,\ \zeta_9^4,\ \zeta_9^2,\ -\zeta_{18}^5,\ -\zeta_{18}^7,\ \zeta_9^1;\ \ -\zeta_{18}^1,\ -\zeta_{18}^5,\ \zeta_9^2,\ \zeta_9^1,\ -\zeta_{18}^7;\ \ -\zeta_{18}^7,\ \zeta_9^1,\ -\zeta_{18}^1,\ \zeta_9^4;\ \ -\zeta_{18}^7,\ \zeta_9^4,\ -\zeta_{18}^1;\ \ \zeta_9^2,\ -\zeta_{18}^5;\ \ \zeta_9^2)$
The holo-equivalence class of one symmetry
Lagrangian condensible algebra A_i:
$(1, 1, 1, 0, 0, 0, 0, 0, 0) \to$ anomalous $\mathbb{Z}_3^{(2)}$ symmetry

4. $9^{16,447}_{0,16.}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.414,\ 1.414,\ 1.414,\ 1.414,\ 2.0)$
$D^2 = 16.0 = 16$
$T = (0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{16}, \frac{7}{16}, \frac{9}{16}, \frac{15}{16}, 0)$,
$S = (1,\ 1,\ 1,\ 1,\ \sqrt{2},\ \sqrt{2},\ \sqrt{2},\ \sqrt{2},\ 2;\ \ 1,\ 1,\ 1,\ -\sqrt{2},\ -\sqrt{2},\ -\sqrt{2},\ -\sqrt{2},\ 2;\ \ 1,\ 1,\ -\sqrt{2},\ \sqrt{2},\ -\sqrt{2},\ \sqrt{2},\ -2;\ \ 1,\ \sqrt{2},\ -\sqrt{2},\ \sqrt{2},\ -\sqrt{2},\ -2;\ \ 0,\ -2,\ 0,\ 2,\ 0;\ \ 0,\ 2,\ 0,\ 0;\ \ 0,\ -2,\ 0;\ \ 0,\ 0;\ \ 0)$
Factors $= 3^{16,598}_{\frac{1}{2},4.} \boxtimes 3^{16,639}_{\frac{15}{2},4.}$
The holo-equivalence class of one symmetry
Lagrangian condensible algebra A_i:
$(1, 1, 0, 0, 0, 0, 0, 0, 1) \to$ anomalous non-invertible $\widetilde{\mathcal{R}}_{\text{Ising}}$-category symmetry

5. $9^{16,624}_{0,16.}$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.414,\ 1.414,\ 1.414,\ 1.414,\ 2.0)$
$D^2 = 16.0 = 16$
$T = (0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{16}, \frac{5}{16}, \frac{11}{16}, \frac{13}{16}, 0)$,
$S = (1,\ 1,\ 1,\ 1,\ \sqrt{2},\ \sqrt{2},\ \sqrt{2},\ \sqrt{2},\ 2;\ \ 1,\ 1,\ 1,\ -\sqrt{2},\ -\sqrt{2},\ -\sqrt{2},\ -\sqrt{2},\ 2;\ \ 1,\ 1,\ -\sqrt{2},\ \sqrt{2},\ -\sqrt{2},\ \sqrt{2},\ -2;\ \ 1,\ \sqrt{2},\ -\sqrt{2},\ \sqrt{2},\ -\sqrt{2},\ -2;\ \ 0,\ -2,\ 0,\ 2,\ 0;\ \ 0,\ 2,\ 0,\ 0;\ \ 0,\ -2,\ 0;\ \ 0,\ 0;\ \ 0)$
Factors $= 3^{16,553}_{\frac{3}{2},4.} \boxtimes 3^{16,330}_{\frac{13}{2},4.}$
The holo-equivalence class of one symmetry
Lagrangian condensible algebra A_i:
$(1, 1, 0, 0, 0, 0, 0, 0, 1) \to$ anomalous non-invertible $\widetilde{\mathcal{R}}_{\text{twIsing}}$-category symmetry

6. $9^{7,161}_{0,86.41}$: $d_i = (1.0,\ 1.801,\ 1.801,\ 2.246,\ 2.246,\ 3.246,\ 4.48,\ 4.48,\ 5.48)$
$D^2 = 86.413 = 49 + 35c_7^1 + 14c_7^2$
$T = (0, \frac{1}{7}, \frac{6}{7}, \frac{2}{7}, \frac{5}{7}, 0, \frac{3}{7}, \frac{4}{7}, 0)$,
$S = (1,\ -c_7^3,\ -c_7^3,\ \xi_7^3,\ \xi_7^3,\ 2+c_7^1,\ 2+2c_7^1+c_7^2,\ 2+2c_7^1+c_7^2,\ 3+2c_7^1+c_7^2;\ \ -\xi_7^3,\ 2+c_7^1,\ 2+2c_7^1+c_7^2,\ 1,\ -2-2c_7^1-c_7^2,\ -3-2c_7^1-c_7^2,\ -c_7^3,\ \xi_7^3;\ \ -\xi_7^3,\ 1,\ 2+2c_7^1+c_7^2,\ -2-2c_7^1-c_7^2,\ -c_7^3,\ -3-2c_7^1-c_7^2,\ \xi_7^3;\ \ c_7^3,\ 3+2c_7^1+c_7^2,\ -c_7^3,\ -2-c_7^1,\ \xi_7^3,\ -2-2c_7^1-c_7^2;\ \ c_7^3,\ -c_7^3,\ \xi_7^3,\ -2-c_7^1,\ -2-2c_7^1-c_7^2;\ \ 3+2c_7^1+c_7^2,\ -\xi_7^3,\ -\xi_7^3,\ 1;\ \ 2+2c_7^1+c_7^2,\ 1,\ c_7^3;\ \ 2+2c_7^1+c_7^2,\ c_7^3;\ \ 2+c_7^1)$
Factors $= 3^{7,790}_{\frac{48}{7},9.295} \boxtimes 3^{7,245}_{\frac{8}{7},9.295}$
The holo-equivalence class of one symmetry
Lagrangian condensible algebra A_i:
$(1, 0, 0, 0, 0, 1, 0, 0, 1) \to$ anomalous non-invertible $\widetilde{\mathcal{R}}_{PSU(2)_5}$-category symmetry

At rank 9, the first Abelian symTO $9^{3,113}_{0,9.}$ is the Drinfeld center of $\mathcal{V}\text{ec}_{\mathbb{Z}_3}$. The corresponding holo-equivalent class of symmetries contains $\mathbb{Z}_3$ symmetry. The other two Abelian symTOs correspond to two holo-equivalent classes of symmetries which contain anomalous $\mathbb{Z}_3$ symmetries.

The symTO $9^{16,447}_{0,16.}$ is the Drinfeld center of Ising fusion category $\widetilde{\mathcal{R}}_{\text{Ising}} \leftarrow 3^{16,598}_{\frac{1}{2},4.}$ of rank 3: $9^{16,447}_{0,16.} = \mathcal{Z}(\widetilde{\mathcal{R}}_{\text{Ising}})$. Therefore, $9^{16,447}_{0,16.}$ describes the holo-equivalent class of $\widetilde{\mathcal{R}}_{\text{Ising}}$-category symmetry, which is a non-invertible symmetry beyond the algebraic symmetry of Ref. [9]. Such a symmetry was referred to as $\mathbb{Z}_2^e \vee \mathbb{Z}_2^m \vee \mathbb{Z}_2^{em}$ symmetry in Ref. [33]. It contains a $\mathbb{Z}_2^m$, and a dual of $\mathbb{Z}_2^m$ symmetry which is the $\mathbb{Z}_2^e$ symmetry. It also contains a $\mathbb{Z}_2^{em}$ duality symmetry that exchanges $\mathbb{Z}_2^e$ and $\mathbb{Z}_2^m$. This non-invertible $\widetilde{\mathcal{R}}_{\text{Ising}}$-category symmetry is important since it is the maximal emergent symmetry at the Ising critical point. In fact the self dual Ising model on 1-dimensional lattice

$$H = -\sum_i Z_i Z_{i+1} + X_i \tag{156}$$

has this $\widetilde{\mathcal{R}}_{\text{Ising}}$-category symmetry realized exactly, after projecting into the $\mathbb{Z}_2$ symmetric sub-Hilbert space [72].

In fact, the Ising fusion category $\widetilde{\mathcal{R}}_{\text{Ising}}$ is the fusion category that describes the fusion of the symmetry defects of the $\widetilde{\mathcal{R}}_{\text{Ising}}$-category symmetry [9,58]. With this understating, we can conclude that such a $\widetilde{\mathcal{R}}_{\text{Ising}}$-category symmetry is not anomaly-free. This is because the fusion category that describes the fusion of the symmetry defects of an anomaly-free symmetry must a local fusion category [9, 61], whose objects must all have integral quantum dimension. Since, $\widetilde{\mathcal{R}}_{\text{Ising}}$ has non-integral quantum dimensions, it cannot be an anomaly-free symmetry. In fact, there is no local fusion category whose Drinfeld center is the symTO $9^{16,447}_{0,16.}$. Thus the holo-equivalent class of symmetries for this symTO $9^{16,447}_{0,16.}$ contains no anomaly-free symmetries.

The symTOs $9^{16,624}_{0,16.}$ and $9^{7,161}_{0,86.41}$ also have a property that the holo-equivalent class of symmetries for the symTOs contain no anomaly-free symmetries. The holo-equivalent class of symTO $9^{16,624}_{0,16.}$ contains a $\widetilde{\mathcal{R}}_{\text{twIsing}}$-category symmetry, where the twisted-Ising fusion category $\widetilde{\mathcal{R}}_{\text{twIsing}}$ is given by $\widetilde{\mathcal{R}}_{\text{twIsing}} \leftarrow 3^{16,553}_{\frac{3}{2},4.}$. The holo-equivalent class of symTO $9^{7,161}_{0,86.41}$ contains a $\widetilde{\mathcal{R}}_{PSU(2)_5}$-category symmetry, where the fusion category $\widetilde{\mathcal{R}}_{PSU(2)_5}$ is given by $\widetilde{\mathcal{R}}_{PSU(2)_5} \leftarrow 3^{7,245}_{\frac{8}{7},9.295}$. Here $3^{7,245}_{\frac{8}{7},9.295} = PSU(2)_5$ is the non-pointed Deligne factor of braided fusion category $SU(2)_5$: $SU(2)_5 = PSU(2)_5 \boxtimes \mathcal{M}_{\text{pointed}}$.

9.4 Rank 10

1. $10^{12,155}_{0,89.56}$: $d_i = (1.0, 1.0, 2.732, 2.732, 2.732, 2.732, 2.732, 3.732, 3.732, 4.732)$

   $D^2 = 89.569 = 48 + 24\sqrt{3}$

   $T = (0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{5}{6}, \frac{7}{12}, \frac{7}{12}, 0, \frac{1}{2}, 0)$,

   $S = (1,\ 1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 3+\sqrt{3};\ \ 1,\ 1+\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -3-\sqrt{3};\ \ 1+\sqrt{3},\ -2-2\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ 0,\ 2+2\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ \ (3+\sqrt{3})\mathrm{i},\ (-3-\sqrt{3})\mathrm{i},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ (3+\sqrt{3})\mathrm{i},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ \ 1,\ 1,\ 3+\sqrt{3};\ \ 1,\ -3-\sqrt{3};\ \ 0)$

   The holo-equivalence class of $\widetilde{\mathcal{R}}_{\frac{1}{2}E(6)}$-category symmetry

   Lagrangian condensible algebra A_i:

   $(1, 0, 0, 0, 0, 0, 0, 1, 0, 1) \to \widetilde{\mathcal{R}}_{\frac{1}{2}E(6)}$-category symmetry

2. $10^{12,200}_{0,89.56}$: $d_i = (1.0, 1.0, 2.732, 2.732, 2.732, 2.732, 2.732, 3.732, 3.732, 4.732)$

   $D^2 = 89.569 = 48 + 24\sqrt{3}$

   $T = (0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{6}, \frac{5}{12}, \frac{5}{12}, 0, \frac{1}{2}, 0)$,

$S = (1,\ 1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ 3+\sqrt{3};\ 1,\ 1+\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 2+\sqrt{3},\ 2+\sqrt{3},\ -3-\sqrt{3};\ 1+\sqrt{3},\ -2-2\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ 0,\ 2+2\sqrt{3},\ 0,\ 0,\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ 1+\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ -1-\sqrt{3},\ 0;\ (-3-\sqrt{3})\mathrm{i},\ (3+\sqrt{3})\mathrm{i},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ (-3-\sqrt{3})\mathrm{i},\ -1-\sqrt{3},\ 1+\sqrt{3},\ 0;\ 1,\ 1,\ 3+\sqrt{3};\ 1,\ -3-\sqrt{3};\ 0)$

The holo-equivalence class of $\widetilde{\mathcal{R}}_{\frac{1}{2}\overline{E(6)}}$-category symmetry
Lagrangian condensible algebra A_i:
$(1,0,0,0,0,0,0,1,0,1) \to \widetilde{\mathcal{R}}_{\frac{1}{2}\overline{E(6)}}$-category symmetry

3. $10^{20,676}_{0,1435.}$: $d_i = (1.0,\ 9.472,\ 9.472,\ 9.472,\ 9.472,\ 9.472,\ 9.472,\ 16.944,\ 16.944,\ 17.944)$

$D^2 = 1435.541 = 720 + 320\sqrt{5}$

$T = (0,0,0,\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{2}{5},\frac{3}{5},0)$,

$S = (1,\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 8+4\sqrt{5},\ 8+4\sqrt{5},\ 9+4\sqrt{5};\ 15+6\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ 15+6\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ -5-2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ -3-6s^1_{20}-4c^2_{20}+14s^3_{20},\ -3+6s^1_{20}-4c^2_{20}-14s^3_{20},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ -3-6s^1_{20}-4c^2_{20}+14s^3_{20},\ 5+2\sqrt{5},\ 5+2\sqrt{5},\ 0,\ 0,\ 5+2\sqrt{5};\ -3+6s^1_{20}-4c^2_{20}-14s^3_{20},\ -3-6s^1_{20}-4c^2_{20}+14s^3_{20},\ 0,\ 0,\ 5+2\sqrt{5};\ -3+6s^1_{20}-4c^2_{20}-14s^3_{20},\ 0,\ 0,\ 5+2\sqrt{5};\ -6-2\sqrt{5},\ 14+6\sqrt{5},\ -8-4\sqrt{5};\ -6-2\sqrt{5},\ -8-4\sqrt{5};\ 1)$

The holo-equivalence class of two symmetries
Lagrangian condensible algebra A_i:
$(1,2,0,0,0,0,0,0,0,1) \to$ an anomalous non-invertible symmetry
$(1,0,2,0,0,0,0,0,0,1) \to$ a symmetry isomorphic to the one above.
$(1,1,1,0,0,0,0,0,0,1) \to$ an anomalous non-invertible symmetry

The first rank 10 symTO $10^{12,155}_{0,89.56}$ has only one condensable algebra $A_i = (1,0,0,0,0,0,0,1,0,1)$. Thus the symTO is the Drinfeld center of only one fusion category, the so called $\frac{1}{2}E(6)$ fusion category $\widetilde{\mathcal{R}}_{\frac{1}{2}E(6)}$ [153]. The holo-equivalent class of symmetries for this symTO contains only one symmetry which is the anomalous non-invertible $\widetilde{\mathcal{R}}_{\frac{1}{2}E(6)}$-category symmetry and is beyond the algebraic symmetry of Ref. [9]. The second rank 10 symTO $10^{12,200}_{0,89.56}$ is the complex conjugation of the first, which contains $\widetilde{\mathcal{R}}_{\frac{1}{2}\overline{E(6)}}$-category symmetry.

The third rank 10 symTO $10^{20,676}_{0,1435.}$ is a condensation reduction of $\mathcal{Z}(\mathcal{C})$ for some $\mathcal{C} \in \mathcal{NG}(\mathbb{Z}_4 \times \mathbb{Z}_4, 16)$, as described in Section 8, see [118]. Such a symTO has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group generated by the following exchanges of simple objects $i = 4 \leftrightarrow i = 5$) and $i = 6 \leftrightarrow i = 7$). The symTO appears to have three gapped boundaries, that give rise to only two non-invertible $\widetilde{\mathcal{R}}$-category symmetries, due to the automorphisms.

9.5 Rank 12

1. $12^{39,560}_{0,1276.}$: $d_i = (1.0,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 10.908,\ 11.908,\ 11.908,\ 11.908,\ 11.908)$

$D^2 = 1276.274 = \frac{1287+351\sqrt{13}}{2}$

$T = (0,\frac{2}{13},\frac{5}{13},\frac{6}{13},\frac{7}{13},\frac{8}{13},\frac{11}{13},0,0,0,\frac{1}{3},\frac{2}{3})$,

$S = (1,\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{11+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2};\ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2}c^2_{13},\ -\frac{9+3\sqrt{13}}{2}c^5_{13},\ -\frac{9+3\sqrt{13}}{2}c^6_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,0,0,0;\ -\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2}c^6_{13},\ -\frac{9+3\sqrt{13}}{2}c^2_{13},\ -\frac{9+3\sqrt{13}}{2}c^5_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,0,0,0;\ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2}c^5_{13},\ -\frac{9+3\sqrt{13}}{2}c^6_{13},\ -\frac{9+3\sqrt{13}}{2}c^2_{13},$

$-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;$
$-\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ 1,$
$\frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2};\ \ 13+3\sqrt{13},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2};$
$13+3\sqrt{13},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2};\ \ -\frac{13+3\sqrt{13}}{2},\ 13+3\sqrt{13};\ \ -\frac{13+3\sqrt{13}}{2})$

The holo-equivalent class of two symmetries.
Lagrangian condensible algebra A_i:
$(1,0,0,0,0,0,0,1,2,0,0,0) \to$ an anomalous non-invertible symmetry
$(1,0,0,0,0,0,0,1,0,2,0,0) \to$ a symmetry isomorphic to the one above
$(1,0,0,0,0,0,0,1,1,1,0,0) \to$ an anomalous non-invertible symmetry

2. $12^{117,251}_{0,1276.}$: $d_i = (1.0,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 10.908,\ 11.908,\ 11.908,\ 11.908,\ 11.908)$

$D^2 = 1276.274 = \frac{1287+351\sqrt{13}}{2}$

$T = (0, \frac{2}{13}, \frac{5}{13}, \frac{6}{13}, \frac{7}{13}, \frac{8}{13}, \frac{11}{13}, 0, 0, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}),$

$S = (1,\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{11+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},$
$\frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2};\ \ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2}c^2_{13},\ -\frac{9+3\sqrt{13}}{2}c^5_{13},$
$-\frac{9+3\sqrt{13}}{2}c^6_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,0,0,0;\ -\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2}c^6_{13},\ -\frac{9+3\sqrt{13}}{2}c^2_{13},$
$-\frac{9+3\sqrt{13}}{2}c^5_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,0,0,0;\ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2}c^5_{13},\ -\frac{9+3\sqrt{13}}{2}c^6_{13},\ -\frac{9+3\sqrt{13}}{2}c^2_{13},$
$-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;$
$-\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ 1,$
$\frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2};\ \ 13+3\sqrt{13},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2};$
$\frac{13+3\sqrt{13}}{2}c^2_{9},\ \frac{13+3\sqrt{13}}{2}c^4_{9},\ \frac{13+3\sqrt{13}}{2}c^1_{9};\ \ \frac{13+3\sqrt{13}}{2}c^1_{9},\ \frac{13+3\sqrt{13}}{2}c^2_{9};\ \ \frac{13+3\sqrt{13}}{2}c^4_{9})$

The holo-equivalent class of one symmetry.
Lagrangian condensible algebra A_i:
$(1,0,0,0,0,0,0,1,2,0,0,0) \to$ an anomalous non-invertible symmetry

3. $12^{117,145}_{0,1276.}$: $d_i = (1.0,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 9.908,\ 10.908,\ 11.908,\ 11.908,\ 11.908,\ 11.908)$

$D^2 = 1276.274 = \frac{1287+351\sqrt{13}}{2}$

$T = (0, \frac{2}{13}, \frac{5}{13}, \frac{6}{13}, \frac{7}{13}, \frac{8}{13}, \frac{11}{13}, 0, 0, \frac{2}{9}, \frac{5}{9}, \frac{8}{9}),$

$S = (1,\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{9+3\sqrt{13}}{2},\ \frac{11+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},$
$\frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2};\ \ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2}c^2_{13},\ -\frac{9+3\sqrt{13}}{2}c^5_{13},$
$-\frac{9+3\sqrt{13}}{2}c^6_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,0,0,0;\ -\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2}c^6_{13},\ -\frac{9+3\sqrt{13}}{2}c^2_{13},$
$-\frac{9+3\sqrt{13}}{2}c^5_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,0,0,0;\ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2}c^5_{13},\ -\frac{9+3\sqrt{13}}{2}c^6_{13},\ -\frac{9+3\sqrt{13}}{2}c^2_{13},$
$-\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;$
$-\frac{9+3\sqrt{13}}{2}c^3_{13},\ -\frac{9+3\sqrt{13}}{2}c^1_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ \ -\frac{9+3\sqrt{13}}{2}c^4_{13},\ -\frac{9+3\sqrt{13}}{2},\ 0,\ 0,\ 0,\ 0;\ 1,$
$\frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2},\ \frac{13+3\sqrt{13}}{2};\ \ 13+3\sqrt{13},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2},\ -\frac{13+3\sqrt{13}}{2};$
$\frac{13+3\sqrt{13}}{2}c^4_{9},\ \frac{13+3\sqrt{13}}{2}c^2_{9},\ \frac{13+3\sqrt{13}}{2}c^1_{9};\ \ \frac{13+3\sqrt{13}}{2}c^1_{9},\ \frac{13+3\sqrt{13}}{2}c^4_{9};\ \ \frac{13+3\sqrt{13}}{2}c^2_{9})$

The holo-equivalent class of one symmetry.
Lagrangian condensible algebra A_i:
$(1,0,0,0,0,0,0,1,2,0,0,0) \to$ an anomalous non-invertible symmetry

The first rank 12 symTO $12^{39,560}_{0,1276.}$ is the Haagerup-Izumi MTC $\mathrm{Haag}(1)_0$, which has three condensable algebras. Its holo-equivalent class contains only two symmetries, due to an automorphism that exchange two simple objects $i = 9 \leftrightarrow i = 10$. The other two rank

12 symTOs are the Haagerup-Izumi MTC $\text{Haag}(1)_1$ and $\text{Haag}(1)_{-1}$. Their corresponding holo-equivalent classes each contains one anomalous non-invertible symmetry.

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A d-number and $1/\rho_{\text{pMD}}(\mathfrak{s})_{ui}$

The unit row of $\rho_{\text{pMD}}(\mathfrak{s})$ has an useful property that $1/\rho_{\text{pMD}}(\mathfrak{s})_{ui}$ are the so called d-numbers.

Definition A.1. *Let $K/\mathbb{Q}$ be a Galois extension and O_K the ring of algebraic integers in K. An element $x \in Q_K$ is called a d-number if for any $\sigma \in \text{Gal}(K/\mathbb{Q})$, $\sigma(x) = xu$ for some unit $u \in O_K$.*

Theorem A.2. *Let K be a subfield of $\mathbb{Q}(\zeta_{p^\ell})$ such that $[K : \mathbb{Q}] = q$ is a prime number. Let O_K be the ring of algebraic integers in K. Then $x \in O_K$ is a d-number if and only if $x = nu^a v$ for some integer n, unit $u \in O_K$, $a = 0, \dots, q-1$ where $v = Norm_K^{\mathbb{Q}(\zeta_{p^\ell})}(1-\zeta_{p^\ell})$. If σ is a generator of $\text{Gal}(\mathbb{Q}(\zeta_{p^\ell})/\mathbb{Q})$, then*

$$v = (1-\zeta_{p^\ell})\sigma^q(1-\zeta_{p^\ell})\sigma^{2q}(1-\zeta_{p^\ell})\cdots\sigma^{p^{\ell-1}(p-1)-q}(1-\zeta_{p^\ell}).$$

Proof. Note that $\sigma(1-\zeta_{p^\ell}) = (1-\zeta_{p^\ell})u'$ for some unit u' in $\mathbb{Z}[\zeta_{p^\ell}]$, and K is the fixed field of $\langle\sigma^q\rangle$ by Galois correspondence. So $v \in K$ and $\sigma(v) = vu$ for some unit $u \in O_K$. Direct verification shows that $x = nv^a u$ for some integer n, unit $u \in O_K$, $a = 0, \dots, q-1$ is a d-number.

Conversely, let $x \in O_K$ be a d-number. Then $\sigma(x) = xu'$ for some unit $u' \in O_K$. Thus $x^q = mu''$ for some unit $u'' \in O_K$ where $m = |Norm(x)|$. Let $m = p_0^{k_0} p_1^{k_1} \cdots p_l^{k_l}$ be the prime factorization of m, where $p_0 = p$ and $k_0 \geq 0$. Since O_K is a Dedekind domain, for each $i > 0$,

$$(p_i) = \mathfrak{P}_{i,1}^{e_i} \cdots \mathfrak{P}_{i,g_i}^{e_i}$$

for some conjugate distinct prime ideals $\mathfrak{P}_{i,1}, \dots, \mathfrak{P}_{i,g_i}$ of O_K and $e_i f_i g_i = q$ where $f_i = \log_{p_i} |O_K/\mathfrak{P}_{i,1}|$. Since q is prime and p is the only totally ramified prime, $e_i < q$ for $i > 0$. Since q is a prime number, $e_i = 1$ and hence $(p_i) = \mathfrak{P}_{i,1} \cdots \mathfrak{P}_{i,g_i}$ for $i > 0$. The prime ideal factorization of (p) in O_K is $(p) = \mathfrak{P}_0^q$ where $\mathfrak{P}_0 = (u)$. Thus, we have

$$(x^q) = (x)^q = (m) = \mathfrak{P}_0^{qk_0}\mathfrak{P}_{1,1}^{k_1}\cdots\mathfrak{P}_{1,g_1}^{k_1}\cdots\mathfrak{P}_{l,1}^{k_l}\cdots\mathfrak{P}_{l,g_l}^{k_l}$$

and hence

$$(x) = \mathfrak{P}_0^{\alpha_0}\mathfrak{P}_{1,1}^{\alpha_{1,1}}\cdots\mathfrak{P}_{1,g_1}^{\alpha_{1,g_1}}\cdots\mathfrak{P}_{l,1}^{\alpha_{l,1}}\cdots\mathfrak{P}_{l,g_l}^{\alpha_{l,g_l}}$$

for some nonnegative integers $\alpha_0, \alpha_{i,j}$. By the unique factorizations of primes ideals, we have

$$q\alpha_0 = qk_0, \quad q\alpha_{i,j} = k_i \text{ for } i = 1, \dots, l.$$

Thus, $\alpha_0 = k_0$ and $\alpha_{i,j} = \alpha_{i,1}$ for $i > 0$. Hence,

$$(x) = \mathfrak{P}_0^{k_0}(p_1)^{\alpha_{1,1}} \cdots (p_l)^{\alpha_{l,1}}$$

and so

$$x = v^{k_0} p_1^{\alpha_{1,1}} \cdots p_l^{\alpha_{l,1}} u'''$$

for some unit $u''' \in O_K$. Since $(v^q) = (p)$, $x = nv^a u$ for some integer n, u an unit in O_K and $0 \le a \le q-1$. □

Corollary A.3. *Let $K = \mathbb{Q}(\sqrt{5})$. Then $x \in O_K$ is a d-number if and only if b $x = nu$ or $x = n\sqrt{5}u$ for some integer n and an unit $u \in O_K$.*

Proof. One can apply the preceding theorem as $[K : \mathbb{Q}] = 2$ and K is a subfield of $\mathbb{Q}(\zeta_5)$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$. Then σ^2 is the complex conjugation and so

$$v = (1-\zeta_5)\sigma^2(1-\zeta_5) = 2 - 2\cos(2\pi/5) = \sqrt{5}u'$$

for some unit $u' \in Q_K$. Now, the result follows immediately from Theorem A.2. □

Corollary A.4. *Let $K = \mathbb{Q}(\cos(2\pi/7))$. Then $x \in O_K$ is a d-number if and only if $x = nv^a u$ for some integer n, unit $u \in O_K$, $a = 0, 1, 2$, where $v = 2 - 2\cos(2\pi/7)$.*

Proof. One can apply the preceding theorem as $[K : \mathbb{Q}] = 3$ and K is the real subfield of $\mathbb{Q}(\zeta_7)$. The assignment $\sigma : \zeta_7 \mapsto \zeta_7^3$ defines a generator $\mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$. Since $7 - 1 = 6$,

$$v = (1-\zeta_7)\sigma^3(1-\zeta_7) = (1-\zeta_7)(1-\zeta_7^{-1}) = 2 - 2\cos(2\pi/7).$$

Now, the result follows from Theorem A.2. □

Using the above results, we find that a d-number has the following explicit expression for simple conductors. When conductor $cnd = 1$, all the d-numbers are given by

$$\delta = n, \quad n \in \mathbb{Z}. \tag{157}$$

When conductor $cnd = 3$, all the d-numbers are given by

$$\delta = n\zeta_3^m, \quad n \in \mathbb{Z}, \quad m = 0, 1, 2. \tag{158}$$

When conductor $cnd = 4$, all the d-numbers are given by

$$\delta = n\zeta_4^m, \quad n \in \mathbb{Z}, \quad m = 0, 1, 2, 3. \tag{159}$$

When conductor $cnd = 5$, all the d-numbers are given by

$$\begin{aligned} &\delta = n(\zeta_{10} - 1)^k \zeta_{10}^m v^a, \quad n, k \in \mathbb{Z}, \quad m = 0,1,2,3,4,5,6,7,8,9 \\ &v = \zeta_{10} - \zeta_{10}^{-1}, \quad a = 0, 1, 2, 3. \end{aligned} \tag{160}$$

All real d-numbers of conductor 5 are given by

$$\delta = n(\frac{1+\sqrt{5}}{2})^k v^{2a}, \quad n, k \in \mathbb{Z}, \; a = 0, 1 \tag{161}$$

We note that the generic complex δ has the following form

$$\delta = n(\zeta_{10} - 1)^k \zeta_{10}^m v^a = n(\zeta_{20} - \zeta_{20}^{-1})^k \zeta_{20}^{2m+k} v^a \tag{162}$$

where $\zeta_{20} - \zeta_{20}^{-1}$ and $v = \zeta_{10} - \zeta_{10}^{-1}$ are imaginary. ζ_{20}^{2m+k} is real if $2m + k \bmod 10 = 0$ and it is imaginary if $2m + k \bmod 10 = 5$. In order for δ to be imaginary, $(\zeta_{20} - \zeta_{20}^{-1})^k \zeta_{20}^{2m+k}$ must be real or imaginary, since v^a is real or imaginary. When $(\zeta_{20} - \zeta_{20}^{-1})^k \zeta_{20}^{2m+k}$ is real or imaginary, it turns out that it can only be real. Thus v^a must be imaginary and a must be odd. This leads to the following result: all imaginary d-numbers of conductor 5 are given by

$$\delta = \begin{cases} n(\zeta_{20} - \zeta_{20}^{-1})^{2k} v^{2a+1} \\ n\,\mathrm{i}(\zeta_{20} - \zeta_{20}^{-1})^{2k+1} v^{2a+1} \end{cases} \tag{163}$$

We can rewrite (163) as

$$\delta = n(\mathrm{i}\zeta_{20} - \mathrm{i}\zeta_{20}^{-1})^k v^{2a+1} \quad \text{or} \quad \delta = n(\frac{1+\sqrt{5}}{2})^k (\sqrt{5})^a v, \quad a = 0, 1 \tag{164}$$

Thus all the imaginary d-numbers can be obtain from the real one by multiply a factor v.

When conductor $cnd = 7$, all the d-numbers are given by

$$\begin{aligned} &\delta = n(\zeta_{14} - 1)^k (\zeta_{14}^3 - 1)^l \zeta_{14}^m v^a, \qquad n, k, l \in \mathbb{Z}, \quad m = 0,1,2,3,4,5,6,7,8,9,10,11,12,13 \\ &v = \zeta_{14} - \zeta_{14}^{-1}, \quad a = 0,1,2,3,4,5. \end{aligned} \tag{165}$$

If δ is real, then

$$\begin{aligned} \delta &= n(2\cos(4\pi/7)^k (2\cos(4\pi/7) + 1)^l u^a, \qquad n, k, l \in \mathbb{Z} \\ u &= (1 - \zeta_7)(1 - \zeta_7^{-1}) = 2 - 2\cos(2\pi/7), \quad a = 0,1,2,3,4,5. \end{aligned} \tag{166}$$

We will use those results in next a few sections.

B Rank-12 representation-347

Among 5288 $\mathrm{SL}_2(\mathbb{Z})$ representations that we considered at rank 12, the 347^{th} representation is a difficult one for our GAP code to handle. The 347^{th} representation is given by: $\tilde{s} = (0, 0, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5})$. $\tilde{\rho}(\mathfrak{s}) =$

$$\begin{pmatrix}
-\frac{1}{5}s_5^1, & \frac{1}{5}s_5^2, & \frac{1+\sqrt{5}}{10}s_5^1, & 0, & \frac{3-\sqrt{5}}{10}s_5^1, & 0, & 0, & \frac{\sqrt{2}}{5}s_5^1, & 0, & 0, & \frac{\sqrt{2}}{5}s_{10}^1, & 0 \\
\frac{1}{5}s_5^2, & \frac{1}{5}s_5^1, & \frac{3-\sqrt{5}}{10}s_5^1, & 0, & -\frac{1+\sqrt{5}}{10}s_5^1, & 0, & 0, & -\frac{\sqrt{2}}{5}s_{10}^1, & 0, & 0, & \frac{\sqrt{2}}{5}s_5^1, & 0 \\
\frac{1+\sqrt{5}}{10}s_5^1, & \frac{3-\sqrt{5}}{10}s_5^1, & \frac{1}{5}s_5^1, & 0, & -\frac{1}{5}s_5^2, & 0, & 0, & \frac{\sqrt{2}}{5}s_{10}^1, & 0, & 0, & -\frac{\sqrt{2}}{5}s_5^1, & 0 \\
0, & 0, & 0, & -\frac{3-\sqrt{5}}{10}s_5^1, & 0, & \frac{1+\sqrt{5}}{10}s_5^1, & 0, & 0, & \frac{\sqrt{3}}{5}s_5^1, & \frac{\sqrt{3}}{5}s_{10}^1, & 0, & 0 \\
\frac{3-\sqrt{5}}{10}s_5^1, & -\frac{1+\sqrt{5}}{10}s_5^1, & -\frac{1}{5}s_5^2, & 0, & -\frac{1}{5}s_5^1, & 0, & 0, & \frac{\sqrt{2}}{5}s_5^1, & 0, & 0, & \frac{\sqrt{2}}{5}s_{10}^1, & 0 \\
0, & 0, & 0, & \frac{1+\sqrt{5}}{10}s_5^1, & 0, & \frac{3-\sqrt{5}}{10}s_5^1, & 0, & 0, & -\frac{\sqrt{3}}{5}s_{10}^1, & \frac{\sqrt{3}}{5}s_5^1, & 0, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & -\frac{5-\sqrt{5}}{10}s_5^1, & 0, & 0, & 0, & 0, & \frac{\sqrt{5}}{5}s_5^1 \\
\frac{\sqrt{2}}{5}s_5^1, & -\frac{\sqrt{2}}{5}s_{10}^1, & \frac{\sqrt{2}}{5}s_{10}^1, & 0, & \frac{\sqrt{2}}{5}s_5^1, & 0, & 0, & -\frac{1}{5}s_5^2, & 0, & 0, & \frac{1}{5}s_5^1, & 0 \\
0, & 0, & 0, & \frac{\sqrt{3}}{5}s_5^1, & 0, & -\frac{\sqrt{3}}{5}s_{10}^1, & 0, & 0, & \frac{1+\sqrt{5}}{10}s_5^1, & -\frac{3-\sqrt{5}}{10}s_5^1, & 0, & 0 \\
0, & 0, & 0, & \frac{\sqrt{3}}{5}s_{10}^1, & 0, & \frac{\sqrt{3}}{5}s_5^1, & 0, & 0, & -\frac{3-\sqrt{5}}{10}s_5^1, & -\frac{1+\sqrt{5}}{10}s_5^1, & 0, & 0 \\
\frac{\sqrt{2}}{5}s_{10}^1, & \frac{\sqrt{2}}{5}s_5^1, & -\frac{\sqrt{2}}{5}s_5^1, & 0, & \frac{\sqrt{2}}{5}s_{10}^1, & 0, & 0, & \frac{1}{5}s_5^1, & 0, & 0, & \frac{1}{5}s_5^2, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & \frac{\sqrt{5}}{5}s_5^1, & 0, & 0, & 0, & 0, & \frac{5-\sqrt{5}}{10}s_5^1
\end{pmatrix} \tag{167}$$

We use an orthogonal matrix U to transform the representation $\widetilde{\rho}$ into pMD representation $\rho_{\mathrm{pMD}} = U\widetilde{\rho}U^\dagger$. We start by finding all possible $D_{\rho_{\mathrm{pMD}}(\mathfrak{s})}(\sigma)$'s. We find that there is only one possible $D_{\rho_{\text{pMD}}(\mathfrak{s})}(\sigma)$ (up to equivalence), generated by

$$D_{\rho_{\text{pMD}}}(\sigma=2) = \begin{pmatrix} 0, & -1, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 1, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & -1, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & -1, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 1 \\ 0, & 0, & 0, & 0, & 0, & 0, & -1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & -1, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & -1, & 0, & 0, & 0 \end{pmatrix} \tag{168}$$

There are six possible choices of unit row $u = 1, 3, 5, 7, 8, 9$ (up to equivalence). We will only describe two cases, $u = 1$ and $u = 9$ here. The $u = 3, 5$ cases are similar to $u = 1$ case. The $u = 7, 8$ cases are similar to $u = 9$ case.

For the $u = 1$ case and at r-stage, we have $\rho_{\text{pMD}}(\mathfrak{s}) =$

$$\begin{pmatrix}
\boldsymbol{-\frac{3-\sqrt{5}}{4}s^1_5 r_{168} - \frac{1+\sqrt{5}}{4}s^1_5 r_{167}}, & \boldsymbol{\frac{1+\sqrt{5}}{4}s^1_5 r_{168} - \frac{3-\sqrt{5}}{4}s^1_5 r_{167}}, & \boldsymbol{\frac{3-\sqrt{5}}{4}s^1_5 r_{136} + \frac{1+\sqrt{5}}{4}s^1_5 r_{135}}, & \boldsymbol{\frac{3-\sqrt{5}}{4}s^1_5 r_{128} + \frac{1+\sqrt{5}}{4}s^1_5 r_{127}}, & \cdots \\
\boldsymbol{\frac{1+\sqrt{5}}{4}s^1_5 r_{168} - \frac{3-\sqrt{5}}{4}s^1_5 r_{167}}, & \frac{3-\sqrt{5}}{4}s^1_5 r_{168} + \frac{1+\sqrt{5}}{4}s^1_5 r_{167}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{136} + \frac{3-\sqrt{5}}{4}s^1_5 r_{135}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{128} + \frac{3-\sqrt{5}}{4}s^1_5 r_{127}, & \cdots \\
\boldsymbol{\frac{3-\sqrt{5}}{4}s^1_5 r_{136} + \frac{1+\sqrt{5}}{4}s^1_5 r_{135}}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{136} + \frac{3-\sqrt{5}}{4}s^1_5 r_{135}, & -\frac{2-\sqrt{5}}{15}s^1_5 + \frac{5-\sqrt{5}}{6}s^1_5 r_{120}, & -\frac{5-\sqrt{5}}{2}s^1_5 r_{104}, & \cdots \\
\boldsymbol{\frac{3-\sqrt{5}}{4}s^1_5 r_{128} + \frac{1+\sqrt{5}}{4}s^1_5 r_{127}}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{128} + \frac{3-\sqrt{5}}{4}s^1_5 r_{127}, & -\frac{5-\sqrt{5}}{2}s^1_5 r_{104}, & \frac{1+\sqrt{5}}{30}s^1_5 - \frac{5-\sqrt{5}}{6}s^1_5 r_{120}, & \cdots \\
\boldsymbol{\frac{1+\sqrt{5}}{4}s^1_5 r_{136} - \frac{3-\sqrt{5}}{4}s^1_5 r_{135}}, & \frac{3-\sqrt{5}}{4}s^1_5 r_{136} + \frac{1+\sqrt{5}}{4}s^1_5 r_{135}, & -\frac{3+\sqrt{5}}{30}s^1_5 + \frac{\sqrt{5}}{3}s^1_5 r_{120}, & -\sqrt{5}s^1_5 r_{104}, & \cdots \\
\boldsymbol{\frac{1+\sqrt{5}}{4}s^1_5 r_{128} - \frac{3-\sqrt{5}}{4}s^1_5 r_{127}}, & \frac{3-\sqrt{5}}{4}s^1_5 r_{128} + \frac{1+\sqrt{5}}{4}s^1_5 r_{127}, & -\sqrt{5}s^1_5 r_{104}, & -\frac{3-\sqrt{5}}{30}s^1_5 - \frac{\sqrt{5}}{3}s^1_5 r_{120}, & \cdots \\
\boldsymbol{\frac{3-\sqrt{5}}{4}s^1_5 r_{144} + \frac{1+\sqrt{5}}{4}s^1_5 r_{143}}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{144} + \frac{3-\sqrt{5}}{4}s^1_5 r_{143}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{16} + \frac{3-\sqrt{5}}{4}s^1_5 r_{15}, & -\frac{3-\sqrt{5}}{4}s^1_5 r_{24} - \frac{1+\sqrt{5}}{4}s^1_5 r_{23}, & \cdots \\
\boldsymbol{\frac{3-\sqrt{5}}{4}s^1_5 r_{152} + \frac{1+\sqrt{5}}{4}s^1_5 r_{151}}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{152} + \frac{3-\sqrt{5}}{4}s^1_5 r_{151}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{32} + \frac{3-\sqrt{5}}{4}s^1_5 r_{31}, & -\frac{3-\sqrt{5}}{4}s^1_5 r_{40} - \frac{1+\sqrt{5}}{4}s^1_5 r_{39}, & \cdots \\
\boldsymbol{\frac{3-\sqrt{5}}{4}s^1_5 r_{160} + \frac{1+\sqrt{5}}{4}s^1_5 r_{159}}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{160} + \frac{3-\sqrt{5}}{4}s^1_5 r_{159}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{48} + \frac{3-\sqrt{5}}{4}s^1_5 r_{47}, & -\frac{3-\sqrt{5}}{4}s^1_5 r_{56} - \frac{1+\sqrt{5}}{4}s^1_5 r_{55}, & \cdots \\
\boldsymbol{\frac{1+\sqrt{5}}{4}s^1_5 r_{144} - \frac{3-\sqrt{5}}{4}s^1_5 r_{143}}, & \frac{3-\sqrt{5}}{4}s^1_5 r_{144} + \frac{1+\sqrt{5}}{4}s^1_5 r_{143}, & \frac{3-\sqrt{5}}{4}s^1_5 r_{16} + \frac{1+\sqrt{5}}{4}s^1_5 r_{15}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{24} + \frac{3-\sqrt{5}}{4}s^1_5 r_{23}, & \cdots \\
\boldsymbol{\frac{1+\sqrt{5}}{4}s^1_5 r_{152} - \frac{3-\sqrt{5}}{4}s^1_5 r_{151}}, & \frac{3-\sqrt{5}}{4}s^1_5 r_{152} + \frac{1+\sqrt{5}}{4}s^1_5 r_{151}, & \frac{3-\sqrt{5}}{4}s^1_5 r_{32} + \frac{1+\sqrt{5}}{4}s^1_5 r_{31}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{40} + \frac{3-\sqrt{5}}{4}s^1_5 r_{39}, & \cdots \\
\boldsymbol{\frac{1+\sqrt{5}}{4}s^1_5 r_{160} - \frac{3-\sqrt{5}}{4}s^1_5 r_{159}}, & \frac{3-\sqrt{5}}{4}s^1_5 r_{160} + \frac{1+\sqrt{5}}{4}s^1_5 r_{159}, & \frac{3-\sqrt{5}}{4}s^1_5 r_{48} + \frac{1+\sqrt{5}}{4}s^1_5 r_{47}, & -\frac{1+\sqrt{5}}{4}s^1_5 r_{56} + \frac{3-\sqrt{5}}{4}s^1_5 r_{55}, & \cdots
\end{pmatrix} \tag{169}$$

where only column $1, 2, 3, 4$ are displayed. The Galois conjugation and the action of $D_{\rho_{\text{pMD}}}(\sigma)$ on $\rho_{\text{pMD}}(\mathfrak{s})$ simply $\rho_{\text{pMD}}(\mathfrak{s})$ to have the above form. The $SL_2(\mathbb{Z})$ conditions, the simple $SL_2(\mathbb{Z})$ character conditions (117), and the orthogonality conditions of $\rho_{\text{pMD}}(\mathfrak{s})$ lead to many zero conditions on the r-variables. But those zero conditions are not enough to determine the r-variables.

The make progress, we note that $1/\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ is a d-number with conductor 5. The number of d-numbers of conductor 5 is infinite. If $1/\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ contained only one r-variable, the number of d-numbers of such a single-r form could be finite (see Section 5.6.8). However, for our case, $1/\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ contains two r-variables, the number of d-numbers of such a two-r form turn out to be infinite.

To make the infinite possibility finite, we manage to isolate a zero condition $-\frac{2}{25} + r_{168}^2 + r_{167}^2 = 0$ involving the variables in $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$. With this additional zero condition, we find, through an extensive search, that in order for the two variables r_{168} and r_{167} to satisfy $-\frac{2}{25} + r_{168}^2 + r_{167}^2 = 0$ and to make $1/\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ a d-number, they can only take the following eight sets of possible values

$$\begin{aligned}(r_{168}, r_{167}) = \; & (-7/25,\; 1/25),\; (-1/5,\; -1/5),\; (-1/5,\; 1/5),\; (-1/25,\; -7/25), \\ & (1/25,\; 7/25),\; (1/5,\; -1/5),\; (1/5,\; 1/5),\; (7/25,\; -1/25).\end{aligned} \tag{170}$$

This allows us to determine $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$. We stress that such a result is a conjecture at this stage, because our extensive search is still a finte one. To see how extensive is our search, a complicated (r_{168}, r_{167}) in our search is given by

$$(r_{168}, r_{167}) = \Big(\frac{225851433717}{390625}, \frac{956722026041}{390625}\Big) \tag{171}$$

Once the value of $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ is known, we can construct many inverse-pairs of integer conditions as decribed in Section 5.6.6. We find that those inverse-pairs of integer conditions all lead to contradictions. So the above eight solutions are all rejected.

Next, we consider the $u = 9$ case. We have $\rho_{\text{pMD}}(\mathfrak{s}) =$

$$\begin{pmatrix}
\cdots & \frac{3-\sqrt{5}}{4}s_5^1r_{64}+\frac{1+\sqrt{5}}{4}s_5^1r_{63}, & \frac{3-\sqrt{5}}{4}s_5^1r_{56}+\frac{1+\sqrt{5}}{4}s_5^1r_{55}, & \cdots & \boldsymbol{\frac{3-\sqrt{5}}{4}s_5^1r_{160}+\frac{1+\sqrt{5}}{4}s_5^1r_{159},} & \cdots \\
\cdots & -\frac{1+\sqrt{5}}{4}s_5^1r_{64}+\frac{3-\sqrt{5}}{4}s_5^1r_{63}, & -\frac{1+\sqrt{5}}{4}s_5^1r_{56}+\frac{3-\sqrt{5}}{4}s_5^1r_{55}, & \cdots & \boldsymbol{-\frac{1+\sqrt{5}}{4}s_5^1r_{160}+\frac{3-\sqrt{5}}{4}s_5^1r_{159},} & \cdots \\
\cdots & -\frac{3-\sqrt{5}}{10}s_5^1-\frac{5-\sqrt{5}}{2}s_5^1r_{120}, & \frac{5-\sqrt{5}}{6}s_5^1r_{104}, & \cdots & \boldsymbol{-\frac{1+\sqrt{5}}{4}s_5^1r_{128}+\frac{3-\sqrt{5}}{4}s_5^1r_{127},} & \cdots \\
\cdots & \frac{5-\sqrt{5}}{6}s_5^1r_{104}, & \frac{1}{5}s_5^1+\frac{5-\sqrt{5}}{2}s_5^1r_{120}, & \cdots & \boldsymbol{-\frac{3-\sqrt{5}}{4}s_5^1r_{136}-\frac{1+\sqrt{5}}{4}s_5^1r_{135},} & \cdots \\
\cdots & -\frac{1+\sqrt{5}}{10}s_5^1-\sqrt{5}s_5^1r_{120}, & \frac{\sqrt{5}}{3}s_5^1r_{104}, & \cdots & \boldsymbol{\frac{3-\sqrt{5}}{4}s_5^1r_{128}+\frac{1+\sqrt{5}}{4}s_5^1r_{127},} & \cdots \\
\cdots & \frac{\sqrt{5}}{3}s_5^1r_{104}, & \frac{1}{5}s_5^2+\sqrt{5}s_5^1r_{120}, & \cdots & \boldsymbol{-\frac{1+\sqrt{5}}{4}s_5^1r_{136}+\frac{3-\sqrt{5}}{4}s_5^1r_{135},} & \cdots \\
\cdots & -\frac{1+\sqrt{5}}{4}s_5^1r_{24}+\frac{3-\sqrt{5}}{4}s_5^1r_{23}, & -\frac{1+\sqrt{5}}{4}s_5^1r_{28}+\frac{3-\sqrt{5}}{4}s_5^1r_{27}, & \cdots & \boldsymbol{-\frac{3-\sqrt{5}}{4}s_5^1r_{144}-\frac{1+\sqrt{5}}{4}s_5^1r_{143},} & \cdots \\
\cdots & -\frac{1+\sqrt{5}}{4}s_5^1r_{40}+\frac{3-\sqrt{5}}{4}s_5^1r_{39}, & -\frac{3-\sqrt{5}}{4}s_5^1r_{48}-\frac{1+\sqrt{5}}{4}s_5^1r_{47}, & \cdots & \boldsymbol{-\frac{3-\sqrt{5}}{4}s_5^1r_{152}-\frac{1+\sqrt{5}}{4}s_5^1r_{151},} & \cdots \\
\cdots & \boldsymbol{-\frac{1+\sqrt{5}}{4}s_5^1r_{128}+\frac{3-\sqrt{5}}{4}s_5^1r_{127},} & \boldsymbol{-\frac{3-\sqrt{5}}{4}s_5^1r_{136}-\frac{1+\sqrt{5}}{4}s_5^1r_{135},} & \cdots & \boldsymbol{-\frac{3-\sqrt{5}}{4}s_5^1r_{168}-\frac{1+\sqrt{5}}{4}s_5^1r_{167},} & \cdots \\
\cdots & \frac{3-\sqrt{5}}{4}s_5^1r_{24}+\frac{1+\sqrt{5}}{4}s_5^1r_{23}, & \frac{3-\sqrt{5}}{4}s_5^1r_{28}+\frac{1+\sqrt{5}}{4}s_5^1r_{27}, & \cdots & \boldsymbol{-\frac{1+\sqrt{5}}{4}s_5^1r_{144}+\frac{3-\sqrt{5}}{4}s_5^1r_{143},} & \cdots \\
\cdots & \frac{3-\sqrt{5}}{4}s_5^1r_{40}+\frac{1+\sqrt{5}}{4}s_5^1r_{39}, & -\frac{1+\sqrt{5}}{4}s_5^1r_{48}+\frac{3-\sqrt{5}}{4}s_5^1r_{47}, & \cdots & \boldsymbol{-\frac{1+\sqrt{5}}{4}s_5^1r_{152}+\frac{3-\sqrt{5}}{4}s_5^1r_{151},} & \cdots \\
\cdots & \frac{3-\sqrt{5}}{4}s_5^1r_{128}+\frac{1+\sqrt{5}}{4}s_5^1r_{127}, & -\frac{1+\sqrt{5}}{4}s_5^1r_{136}+\frac{3-\sqrt{5}}{4}s_5^1r_{135}, & \cdots & \boldsymbol{-\frac{1+\sqrt{5}}{4}s_5^1r_{168}+\frac{3-\sqrt{5}}{4}s_5^1r_{167},} & \cdots
\end{pmatrix} \tag{172}$$

where only column $3, 4, 9$ are displayed. $1/\rho_{\text{pMD}}(\mathfrak{s})_{ui}$ are d-number of conductor 5 for all i's. The number of possible values of $1/\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ is infinite. To make the number of possible values finite, we need to isolate zero condition for the variables r_{167} and r_{168} in $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$. But we fail to isolate such zero condition.

To make progress, we consider variables in $\rho_{\text{pMD}}(\mathfrak{s})_{ij}$ for $i, j \in \{3, 4, 9\}$, which are $(r_{104},\ r_{120},\ r_{127},\ r_{128},\ r_{135},\ r_{136},\ r_{167},\ r_{168})$. We isolate the following zero conditions for those variables:

$$\begin{aligned}
&\frac{9}{5}r_{120}+9r_{120}^2+r_{104}^2=0 \ \text{ and } \ -\frac{2}{25}+\frac{1}{10}r_{168}+\frac{1}{10}r_{167}+r_{135}^2+r_{128}^2=0 \ \text{ and} \\
&-\frac{2}{25}+\frac{1}{10}r_{168}+\frac{1}{10}r_{167}+r_{136}^2+r_{127}^2=0 \ \text{ and } \ \frac{3}{10}r_{128}+\frac{3}{10}r_{127}+3r_{120}r_{127}+r_{104}r_{136}=0 \ \text{ and} \\
&-\frac{3}{10}r_{128}-\frac{3}{10}r_{127}-3r_{120}r_{128}+r_{104}r_{135}=0 \ \text{ and } \ \frac{3}{10}r_{136}+\frac{3}{10}r_{135}+3r_{120}r_{135}+r_{104}r_{128}=0 \ \text{ and} \\
&-\frac{3}{10}r_{136}-\frac{3}{10}r_{135}-3r_{120}r_{136}+r_{104}r_{127}=0 \ \text{ and } \ \frac{1}{10}r_{168}-\frac{3}{10}r_{167}-r_{135}r_{136}+r_{127}r_{128}
\end{aligned} \tag{173}$$

Also the following expressions of those variables must be cyclotomic integers:

$$\begin{aligned}
&\frac{1}{-\frac{3-\sqrt{5}}{4}s_5^1r_{168}-\frac{1+\sqrt{5}}{4}s_5^1r_{167}}=\text{cyc-int}, && \frac{1}{-\frac{1+\sqrt{5}}{4}s_5^1r_{128}+\frac{3-\sqrt{5}}{4}s_5^1r_{127}}=\text{cyc-int}, \\
&\frac{-\frac{1+\sqrt{5}}{4}s_5^1r_{128}+\frac{3-\sqrt{5}}{4}s_5^1r_{127}}{-\frac{3-\sqrt{5}}{4}s_5^1r_{168}-\frac{1+\sqrt{5}}{4}s_5^1r_{167}}=\text{cyc-int}, && \frac{-\frac{3-\sqrt{5}}{10}s_5^1-\frac{5-\sqrt{5}}{2}s_5^1r_{120}}{-\frac{3-\sqrt{5}}{4}s_5^1r_{168}-\frac{1+\sqrt{5}}{4}s_5^1r_{167}}=\text{cyc-int}, \\
&\frac{1}{-\frac{3-\sqrt{5}}{4}s_5^1r_{136}-\frac{1+\sqrt{5}}{4}s_5^1r_{135}}=\text{cyc-int}, && \frac{-\frac{3-\sqrt{5}}{4}s_5^1r_{136}-\frac{1+\sqrt{5}}{4}s_5^1r_{135}}{-\frac{3-\sqrt{5}}{4}s_5^1r_{168}-\frac{1+\sqrt{5}}{4}s_5^1r_{167}}=\text{cyc-int}, \\
&\frac{\frac{1}{5}s_5^1+\frac{5-\sqrt{5}}{2}s_5^1r_{120}}{-\frac{3-\sqrt{5}}{4}s_5^1r_{168}-\frac{1+\sqrt{5}}{4}s_5^1r_{167}}=\text{cyc-int}, && \frac{\frac{5-\sqrt{5}}{6}s_5^1r_{104}}{-\frac{3-\sqrt{5}}{4}s_5^1r_{168}-\frac{1+\sqrt{5}}{4}s_5^1r_{167}}=\text{cyc-int}, \\
&\frac{\frac{5-\sqrt{5}}{6}s_5^1r_{104}}{-\frac{3-\sqrt{5}}{4}s_5^1r_{136}-\frac{1+\sqrt{5}}{4}s_5^1r_{135}}=\text{cyc-int}, && \frac{\frac{5-\sqrt{5}}{6}s_5^1r_{104}}{-\frac{1+\sqrt{5}}{4}s_5^1r_{128}+\frac{3-\sqrt{5}}{4}s_5^1r_{127}}=\text{cyc-int},
\end{aligned} \tag{174}$$

Although the number of possible values for the three inverse d-numbers, $\rho_{\text{pMD}}(\mathfrak{s})_{ui}$, $i = 3, 4, 9$, is infinite, through an extensive research, we conjecture that none of the three combined d-numbers can satisfy the two sets of conditions (173) and (174). We note that, once we know r_{127}, r_{128}, r_{135}, r_{136}, r_{167}, r_{168}, we obtain a linear relation between r_{104} and r_{120}, from the zero conditions (173). We find that such a linear relation, together with the qudratic relation $\frac{9}{5}r_{120} + 9r_{120}^2 + r_{104}^2 = 0$, only leads to a few solutions of r_{104} and r_{120}. Those solutions are rejected by the cyclotomic-integer conditions (174). To see how extensive is our search, a complicated (r_{168}, r_{167}) in our search is given by

$$(r_{168}, r_{167}) = \Big(-\frac{3571}{78125}, -\frac{15127}{78125}\Big) \tag{175}$$

C Rank-12 representation-333

The rank 12, 333^{th} representation is also hard for our GAP code. The 333^{th} representation is given by: $\tilde{s} =(\frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{4}{5}, 0, 0, 0, 0)$ and $\tilde{\rho}(\mathfrak{s}) =$

$$\begin{pmatrix}
\frac{3-\sqrt{5}}{10}, & 0, & -\frac{1+\sqrt{5}}{5}, & 0, & -\frac{1-\sqrt{5}}{5}, & 0, & -\frac{3+\sqrt{5}}{10}, & 0, & \frac{\sqrt{6}}{5}, & 0, & 0, & 0 \\
0, & -\frac{5+\sqrt{5}}{10}, & 0, & 0, & 0, & 0, & 0, & -\frac{5-\sqrt{5}}{10}, & 0, & 0, & 0, & -\frac{\sqrt{10}}{5} \\
-\frac{1+\sqrt{5}}{5}, & 0, & \frac{3+\sqrt{5}}{10}, & 0, & \frac{3-\sqrt{5}}{10}, & 0, & \frac{1-\sqrt{5}}{5}, & 0, & \frac{\sqrt{6}}{5}, & 0, & 0, & 0 \\
0, & 0, & 0, & -\frac{5-\sqrt{5}}{10}, & 0, & -\frac{5+\sqrt{5}}{10}, & 0, & 0, & 0, & 0, & -\frac{\sqrt{10}}{5}, & 0 \\
-\frac{1-\sqrt{5}}{5}, & 0, & \frac{3-\sqrt{5}}{10}, & 0, & \frac{3+\sqrt{5}}{10}, & 0, & \frac{1+\sqrt{5}}{5}, & 0, & \frac{\sqrt{6}}{5}, & 0, & 0, & 0 \\
0, & 0, & 0, & -\frac{5+\sqrt{5}}{10}, & 0, & -\frac{5-\sqrt{5}}{10}, & 0, & 0, & 0, & 0, & \frac{\sqrt{10}}{5}, & 0 \\
-\frac{3+\sqrt{5}}{10}, & 0, & \frac{1-\sqrt{5}}{5}, & 0, & \frac{1+\sqrt{5}}{5}, & 0, & \frac{3-\sqrt{5}}{10}, & 0, & -\frac{\sqrt{6}}{5}, & 0, & 0, & 0 \\
0, & -\frac{5-\sqrt{5}}{10}, & 0, & 0, & 0, & 0, & 0, & -\frac{5+\sqrt{5}}{10}, & 0, & 0, & 0, & \frac{\sqrt{10}}{5} \\
\frac{\sqrt{6}}{5}, & 0, & \frac{\sqrt{6}}{5}, & 0, & \frac{\sqrt{6}}{5}, & 0, & -\frac{\sqrt{6}}{5}, & 0, & -\frac{1}{5}, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 1, & 0, & 0 \\
0, & 0, & 0, & -\frac{\sqrt{10}}{5}, & 0, & \frac{\sqrt{10}}{5}, & 0, & 0, & 0, & 0, & -\frac{\sqrt{5}}{5}, & 0 \\
0, & -\frac{\sqrt{10}}{5}, & 0, & 0, & 0, & 0, & 0, & \frac{\sqrt{10}}{5}, & 0, & 0, & 0, & \frac{\sqrt{5}}{5}
\end{pmatrix} \tag{176}$$

At r-stage, there are 117 cases from different possible $D_{\rho_{pMD}}(\sigma)$'s, different possible unit index u and different or-connected zero conditions. For some cases, $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ contain only one r variable. In this case, $1/\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ can only take a finite number of possible values in order for it to be a d-number of conductor 5 (see Section 5.6.8).

For some other cases, $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ contain two r variables. We will discuss one such case here. Other two-variable cases are similar. One of the two-variable case has $\rho_{\text{pMD}}(\mathfrak{s}) =$

$$\begin{pmatrix}
-\frac{3+\sqrt{5}}{10} + r_{109}, & -r_{57}, & \cdots & \mathbf{\frac{1-\sqrt{5}}{2} r_{144} - r_{141}}, & \frac{1+\sqrt{5}}{2} r_{144} - r_{141}, & \frac{1-\sqrt{5}}{2} r_8 - r_5, & \frac{1+\sqrt{5}}{2} r_8 - r_5 \\
-r_{57}, & \frac{1-\sqrt{5}}{10} - r_{109}, & \cdots & \mathbf{\frac{1+\sqrt{5}}{2} r_{136} - r_{133}}, & \frac{1-\sqrt{5}}{2} r_{136} - r_{133}, & -\frac{1+\sqrt{5}}{2} r_{32} + r_{29}, & -\frac{1-\sqrt{5}}{2} r_{32} + r_{29} \\
\frac{1+\sqrt{5}}{2} r_{56}, & -\frac{1+\sqrt{5}}{2} r_{80}, & \cdots & \mathbf{-\frac{1-\sqrt{5}}{2} r_{128} + r_{125}}, & -\frac{1+\sqrt{5}}{2} r_{128} + r_{125}, & \frac{1-\sqrt{5}}{2} r_{28} - r_{25}, & \frac{1+\sqrt{5}}{2} r_{28} - r_{25} \\
-\frac{1+\sqrt{5}}{2} r_{68}, & \frac{1+\sqrt{5}}{2} r_{76}, & \cdots & \mathbf{\frac{1+\sqrt{5}}{2} r_{120} - r_{117}}, & \frac{1-\sqrt{5}}{2} r_{120} - r_{117}, & -\frac{1+\sqrt{5}}{2} r_{20} + r_{17}, & -\frac{1-\sqrt{5}}{2} r_{20} + r_{17} \\
\frac{1-\sqrt{5}}{2} r_{56}, & -\frac{1-\sqrt{5}}{2} r_{80}, & \cdots & \mathbf{-\frac{1+\sqrt{5}}{2} r_{128} + r_{125}}, & -\frac{1-\sqrt{5}}{2} r_{128} + r_{125}, & \frac{1+\sqrt{5}}{2} r_{28} - r_{25}, & \frac{1-\sqrt{5}}{2} r_{28} - r_{25} \\
-\frac{1-\sqrt{5}}{2} r_{68}, & \frac{1-\sqrt{5}}{2} r_{76}, & \cdots & \mathbf{\frac{1-\sqrt{5}}{2} r_{120} - r_{117}}, & \frac{1+\sqrt{5}}{2} r_{120} - r_{117}, & -\frac{1-\sqrt{5}}{2} r_{20} + r_{17}, & -\frac{1+\sqrt{5}}{2} r_{20} + r_{17} \\
-\frac{3-\sqrt{5}}{10} + r_{109}, & -r_{57}, & \cdots & \mathbf{\frac{1+\sqrt{5}}{2} r_{144} - r_{141}}, & \frac{1-\sqrt{5}}{2} r_{144} - r_{141}, & \frac{1+\sqrt{5}}{2} r_8 - r_5, & \frac{1-\sqrt{5}}{2} r_8 - r_5 \\
r_{57}, & -\frac{1+\sqrt{5}}{10} + r_{109}, & \cdots & \mathbf{-\frac{1-\sqrt{5}}{2} r_{136} + r_{133}}, & -\frac{1+\sqrt{5}}{2} r_{136} + r_{133}, & \frac{1-\sqrt{5}}{2} r_{32} - r_{29}, & \frac{1+\sqrt{5}}{2} r_{32} - r_{29} \\
\mathbf{\frac{1-\sqrt{5}}{2} r_{144} - r_{141}}, & \mathbf{\frac{1+\sqrt{5}}{2} r_{136} - r_{133}}, & \cdots & \mathbf{\frac{1+\sqrt{5}}{2} r_{160} - r_{157}}, & \mathbf{\frac{1-\sqrt{5}}{2} r_{160} - r_{157}}, & \mathbf{-\frac{1-\sqrt{5}}{2} r_{152} + r_{149}}, & \mathbf{-\frac{1+\sqrt{5}}{2} r_{152} + r_{149}} \\
\frac{1+\sqrt{5}}{2} r_{144} - r_{141}, & \frac{1-\sqrt{5}}{2} r_{136} - r_{133}, & \cdots & \mathbf{\frac{1-\sqrt{5}}{2} r_{160} - r_{157}}, & \frac{1+\sqrt{5}}{2} r_{160} - r_{157}, & -\frac{1+\sqrt{5}}{2} r_{152} + r_{149}, & -\frac{1-\sqrt{5}}{2} r_{152} + r_{149} \\
\frac{1-\sqrt{5}}{2} r_8 - r_5, & -\frac{1+\sqrt{5}}{2} r_{32} + r_{29}, & \cdots & \mathbf{-\frac{1-\sqrt{5}}{2} r_{152} + r_{149}}, & -\frac{1+\sqrt{5}}{2} r_{152} + r_{149}, & \frac{2}{5} - \frac{1+\sqrt{5}}{2} r_{160} + r_{157}, & \frac{2}{5} - \frac{1-\sqrt{5}}{2} r_{160} + r_{157} \\
\frac{1+\sqrt{5}}{2} r_8 - r_5, & -\frac{1-\sqrt{5}}{2} r_{32} + r_{29}, & \cdots & \mathbf{-\frac{1+\sqrt{5}}{2} r_{152} + r_{149}}, & -\frac{1-\sqrt{5}}{2} r_{152} + r_{149}, & \frac{2}{5} - \frac{1-\sqrt{5}}{2} r_{160} + r_{157}, & \frac{2}{5} - \frac{1+\sqrt{5}}{2} r_{160} + r_{157}
\end{pmatrix} \tag{177}$$

where only column $1, 2, 9, 10, 11, 12$ are displayed. $1/\rho_{\text{pMD}}(\mathfrak{s})_{ui}$ are d-number of conductor 5 for all i's. The number of possible values of $1/\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ is infinite. To make the number of possible values finite, we need to isolate zero condition for the variables r_{157} and r_{160} in $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$. But we fail to isolate such zero condition.

To make progress, consider variables in $\rho_{\text{pMD}}(\mathfrak{s})_{ij}$ for $i, j \in \{1, 2, 9\}$, which are r_{57}, r_{109}, r_{133}, r_{136}, r_{141}, r_{144}, r_{157}, r_{160}. We isolate the following zero conditions for those variables:

$$
\begin{aligned}
&-\frac{3}{25}-\frac{2}{5}r_{109}+r_{109}^2+r_{57}^2=0 \quad\text{and}\quad -\frac{2}{25}-\frac{2}{5}r_{160}+r_{144}^2+r_{136}^2=0 \quad\text{and}\\
&-\frac{3}{25}-\frac{3}{5}r_{160}+\frac{1}{5}r_{109}+2r_{144}^2+r_{109}r_{160}=0 \quad\text{and}\quad \frac{3}{5}r_{136}-r_{109}r_{136}+r_{57}r_{144}=0 \quad\text{and}\\
&\frac{1}{5}r_{144}+r_{109}r_{144}+r_{57}r_{136}=0 \quad\text{and}\quad -\frac{3}{25}-\frac{1}{5}r_{157}+r_{144}^2-2r_{141}r_{144}+r_{141}^2+r_{133}^2=0 \quad\text{and}\\
&-\frac{3}{25}r_{136}-\frac{3}{5}r_{136}r_{160}+\frac{1}{5}r_{109}r_{136}+2r_{136}r_{144}^2+r_{109}r_{136}r_{160}=0 \quad\text{and}\quad =0 \quad\text{and}\\
&-\frac{1}{25}-\frac{1}{5}r_{160}+r_{144}^2-r_{141}r_{144}+r_{133}r_{136}=0 \quad\text{and}\\
&-\frac{1}{5}r_{144}+\frac{3}{5}r_{141}+r_{109}r_{144}-r_{109}r_{141}+r_{57}r_{133}=0 \quad\text{and}\\
&\frac{1}{5}r_{136}+\frac{1}{5}r_{133}-r_{109}r_{136}+r_{109}r_{133}+r_{57}r_{141}
\end{aligned}
\tag{178}
$$

Although the number of possible values for the three inverse d-numbers, $\rho_{\text{pMD}}(\mathfrak{s})_{ui}$, $i = 1, 2, 9$, is infinite, through an extensive research, we conjecture that there are only 12 sets of solutions can satisfy the zero conditions (178):

$$
\begin{aligned}
&(r[160], r[157], r[144], r[141], r[136], r[133], r[109], r[57]) \qquad (179)\\
= \;& (-\tfrac{1}{5}, -\tfrac{1}{5}, 0, -\tfrac{1}{5}, 0, -\tfrac{1}{5}, \tfrac{1}{5}, -\tfrac{2}{5}), \quad (-\tfrac{1}{5}, -\tfrac{1}{5}, 0, -\tfrac{1}{5}, 0, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5}), \quad (-\tfrac{1}{5}, -\tfrac{1}{5}, 0, \tfrac{1}{5}, 0, -\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5}),\\
& (-\tfrac{1}{5}, -\tfrac{1}{5}, 0, \tfrac{1}{5}, 0, \tfrac{1}{5}, \tfrac{1}{5}, -\tfrac{2}{5}), \quad (\tfrac{1}{5}, -\tfrac{1}{5}, -\tfrac{2}{5}, -\tfrac{1}{5}, 0, -\tfrac{1}{5}, 0), \quad (\tfrac{1}{5}, -\tfrac{1}{5}, -\tfrac{2}{5}, -\tfrac{1}{5}, \tfrac{1}{5}, -\tfrac{1}{5}, 0),\\
& (\tfrac{1}{5}, -\tfrac{1}{5}, 0, -\tfrac{1}{5}, -\tfrac{2}{5}, -\tfrac{1}{5}, \tfrac{3}{5}, 0), \quad (\tfrac{1}{5}, -\tfrac{1}{5}, 0, -\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{3}{5}, 0), \quad (\tfrac{1}{5}, -\tfrac{1}{5}, 0, \tfrac{1}{5}, -\tfrac{2}{5}, -\tfrac{1}{5}, \tfrac{3}{5}, 0),\\
& (\tfrac{1}{5}, -\tfrac{1}{5}, 0, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{3}{5}, 0), \quad (\tfrac{1}{5}, -\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, 0, -\tfrac{1}{5}, -\tfrac{1}{5}, 0), \quad (\tfrac{1}{5}, -\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, 0, \tfrac{1}{5}, -\tfrac{1}{5}, 0)
\end{aligned}
$$

Once the value of $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ is known, we can construct many inverse-pairs of integer conditions as decribed in Section 5.6.6. We find that those inverse-pairs of integer conditions all lead to contradictions. So the above 12 solutions are all rejected.

D Rank-12 representation-3246

The rank 12, 3246^{th} representation has $\tilde{s} = (0, 0, \frac{1}{2}, 0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 0, 0)$. At r-stage, there is only one case with unit row $u = 8$ and $\rho_{\text{pMD}}(\mathfrak{s}) =$

$$
\begin{pmatrix}
\cdots & \mathbf{r_{73}}, & r_{73}, & r_{73}, & r_{19}, & r_{25}\\
\cdots & \mathbf{-(\frac{1}{3}-c_7^1)r_{72}-(\frac{1}{3}-c_7^2)r_{71}}, & -(\frac{1}{3}-c_7^2)r_{72}+(\frac{2}{3}+c_7^1+c_7^2)r_{71}, & (\frac{2}{3}+c_7^1+c_7^2)r_{72}-(\frac{1}{3}-c_7^1)r_{71}, & 0, & 0\\
\cdots & \mathbf{(\frac{2}{3}+c_7^1+c_7^2)r_{72}-(\frac{1}{3}-c_7^1)r_{71}}, & -(\frac{1}{3}-c_7^1)r_{72}-(\frac{1}{3}-c_7^2)r_{71}, & -(\frac{1}{3}-c_7^2)r_{72}+(\frac{2}{3}+c_7^1+c_7^2)r_{71}, & 0, & 0\\
\cdots & \mathbf{(\frac{1}{3}+c_7^2)r_{72}-(\frac{2}{3}-c_7^1-c_7^2)r_{71}}, & -(\frac{2}{3}-c_7^1-c_7^2)r_{72}+(\frac{1}{3}+c_7^1)r_{71}, & (\frac{1}{3}+c_7^1)r_{72}+(\frac{1}{3}+c_7^2)r_{71}, & 0, & 0\\
\cdots & \mathbf{(\frac{1}{3}+c_7^2)r_{72}-(\frac{2}{3}-c_7^1-c_7^2)r_{71}}, & -(\frac{2}{3}-c_7^1-c_7^2)r_{72}+(\frac{1}{3}+c_7^1)r_{71}, & (\frac{1}{3}+c_7^1)r_{72}+(\frac{1}{3}+c_7^2)r_{71}, & 0, & 0\\
\cdots & \mathbf{-(\frac{2}{3}-c_7^1-c_7^2)r_{72}+(\frac{1}{3}+c_7^1)r_{71}}, & (\frac{1}{3}+c_7^1)r_{72}+(\frac{1}{3}+c_7^2)r_{71}, & (\frac{1}{3}+c_7^2)r_{72}-(\frac{2}{3}-c_7^1-c_7^2)r_{71}, & 0, & 0\\
\cdots & \mathbf{(\frac{1}{3}+c_7^1)r_{72}+(\frac{1}{3}+c_7^2)r_{71}}, & (\frac{1}{3}+c_7^2)r_{72}-(\frac{2}{3}-c_7^1-c_7^2)r_{71}, & -(\frac{2}{3}-c_7^1-c_7^2)r_{72}+(\frac{1}{3}+c_7^1)r_{71}, & 0, & 0\\
\cdots & \mathbf{c_7^2r_{84}+c_7^3r_{83}+r_{79}}, & \mathbf{c_7^3r_{84}+c_7^1r_{83}+r_{79}}, & \mathbf{c_7^1r_{84}+c_7^2r_{83}+r_{79}}, & \mathbf{-r_{43}}, & \mathbf{-r_{49}}\\
\cdots & \mathbf{c_7^3r_{84}+c_7^1r_{83}+r_{79}}, & c_7^1r_{84}+c_7^2r_{83}+r_{79}, & c_7^2r_{84}+c_7^3r_{83}+r_{79}, & -r_{43}, & -r_{49}\\
\cdots & \mathbf{c_7^1r_{84}+c_7^2r_{83}+r_{79}}, & c_7^2r_{84}+c_7^3r_{83}+r_{79}, & c_7^3r_{84}+c_7^1r_{83}+r_{79}, & -r_{43}, & -r_{49}\\
\cdots & \mathbf{-r_{43}}, & -r_{43}, & -r_{43}, & \frac{3}{2}+r_{84}+r_{83}-3r_{79}+r_{13}, & -r_1\\
\cdots & \mathbf{-r_{49}}, & -r_{49}, & -r_{49}, & -r_1, & -r_{13}
\end{pmatrix}
\tag{180}
$$

where only column $8, 9, 10, 11, 12$ are displayed. $1/\rho_{\text{pMD}}(\mathfrak{s})_{ui}$ are d-number of conductor 7 for all i's. The number of possible values of $1/\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ is infinite. To make the number of possible values finite, we need to isolate zero condition for the variables $r[79]$, r_{83}, and r_{84} in $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$. However, the isolated zero conditions are not enough to make number of possible values finite.

To make progress, we isolate zero condition for the variables r_{79}, r_{83}, r_{84}, plus one more variable. We obtain the following zero conditions:

$$-\frac{1}{49}+r_{84}^2-r_{83}r_{84}+r_{83}^2=0, \quad \text{and} \quad -\frac{1}{49}r_{84}-\frac{1}{49}r_{83}+r_{84}^3+r_{83}^3=0, \quad \text{and}$$
$$-\frac{1}{49}r_{84}+r_{84}^3-r_{83}r_{84}^2+r_{83}^2r_{84}=0, \quad \text{and} \quad -\frac{2}{49}-\frac{1}{7}r_{84}+\frac{2}{7}r_{83}+r_{72}^2=0, \quad \text{and}$$
$$-\frac{2}{49}-\frac{1}{7}r_{84}-\frac{1}{7}r_{83}+r_{71}^2=0, \quad \text{and} \quad -\frac{1}{6}-\frac{1}{6}r_{84}-\frac{1}{6}r_{83}+\frac{1}{2}r_{79}+r_{73}^2=0. \tag{181}$$

Although the number of possible values for the inverse d-number $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ is infinite, through an extensive search, we conjecture that there are only three sets of solutions can satisfy the zero conditions (181):

$$(r_{84}, r_{83}, r_{79}) = (-\frac{1}{7}, 0, -\frac{3}{14}), \quad (0, -\frac{1}{7}, -\frac{3}{14}), \quad (\frac{1}{7}, \frac{1}{7}, -\frac{1}{14}). \tag{182}$$

To see how extensive is our search, an complicated triple (r_{84}, r_{83}, r_{79}) in our search is given by

$$(r_{84}, r_{83}, r_{79}) = \Big(\frac{127603558175}{100352}, \frac{459867333397}{200704}, -\frac{16225407395}{28672}\Big). \tag{183}$$

We remark that we search for possible d-numbers $1/\rho_{\text{pMD}}(\mathfrak{s})_{uu}$, *i.e.* search for different values of r_{84}, r_{83}, r_{79}. But the zero condition with an extra variable, say, $-\frac{1}{6}-\frac{1}{6}r_{84}-\frac{1}{6}r_{83}+\frac{1}{2}r_{79}+r_{73}^2=0$ is still useful. When $(r_{84}, r_{83}, r_{79}) = (\frac{1}{7}, \frac{1}{7}, -\frac{1}{14})$, the zero condition $-\frac{1}{6}-\frac{1}{6}r_{84}-\frac{1}{6}r_{83}+\frac{1}{2}r_{79}+r_{73}^2 = -1/4+r_{73}^2$ has solutions for $r_{73} = \pm 1/2$. But when $(r_{84}, r_{83}, r_{79}) = (\frac{1}{7}, \frac{1}{7}, \frac{1}{14})$, the zero condition $-\frac{1}{6}-\frac{1}{6}r_{84}-\frac{1}{6}r_{83}+\frac{1}{2}r_{79}+r_{73}^2 = -5/28+r_{73}^2$ has no solution for a rational r_{73}. This is why the zero condition with one extra variable can still reject most possible values of the d-numbers.

Once the value of $\rho_{\text{pMD}}(\mathfrak{s})_{uu}$ is known, we can construct many inverse-pairs of integer conditions as descried in Section 5.6.6. We find that those inverse-pairs of integer conditions all lead to contradictions. So the above three solutions are all rejected.

The rank 12 representation-1372 and representation-3251 can be handled in a similar way.

E Condensable algebras, boundaries, and domain walls

Let us use a, b, c to label the simple objects in a modular tensor category (MTC) $\mathcal{M}$. $\mathcal{M}$ is characterized by modular matrices $\widetilde{S}_\mathcal{M} = (\widetilde{S}_\mathcal{M}^{ab})$ and $\widetilde{T}_\mathcal{M} = (\widetilde{T}_\mathcal{M}^{ab})$, whose indices are labeled by the simple objects. $\widetilde{S}_\mathcal{M}, \widetilde{T}_\mathcal{M}$ are unitary matrices that generate a representation of $SL(2, \mathbb{Z}_n)$, where n is the smallest integer that satisfy $\widetilde{T}_\mathcal{M}^n = \text{id}$. We call n as the order of $\widetilde{T}_\mathcal{M}$ and denote it as $n = \text{ord}(\widetilde{T}_\mathcal{M})$. $\widetilde{T}_\mathcal{M}$ is a diagonal matrix and $\widetilde{S}_\mathcal{M}$ is a symmetric matrix.

From $\widetilde{S}_\mathcal{M}$ and $\widetilde{T}_\mathcal{M}$, we define normalized S, T-matrices

$$S_\mathcal{M} = \widetilde{S}_\mathcal{M}/\widetilde{S}_\mathcal{M}^{\mathbf{11}}, \qquad T_\mathcal{M} = \widetilde{T}_\mathcal{M}/\widetilde{T}_\mathcal{M}^{\mathbf{11}}. \tag{184}$$

Let d_a be the quantum dimension of simple object a, which is given by $d_a = (S_\mathcal{M})^{a\mathbf{1}}$. Let s_a be the topological spin of simple object a, which is given by $e^{i2\pi s_a} = (T_\mathcal{M})^{aa}$. The total dimension of $\mathcal{M}$ is defined as $D_\mathcal{M}^2 \equiv \sum_{a\in\mathcal{M}} d_a^2$. Also let $d_\mathcal{A}$ be the quantum dimension of the condensable algebra $\mathcal{A}$, *i.e.* if

$$\mathcal{A} = \bigoplus_{a\in\mathcal{M}} A^a a \tag{185}$$

then $d_{\mathcal{A}} = \sum_a A^a d_a$. Similarly, we use i, j, k to label the simple object in $\mathcal{M}_{/\mathcal{A}}$, where $\mathcal{M}_{/\mathcal{A}}$ is the MTC obtained from $\mathcal{M}$ by condensing $\mathcal{A}$. Following the above, we can define $\widetilde{S}_{\mathcal{M}_{/\mathcal{A}}} = (\widetilde{S}^{ij}_{\mathcal{M}_{/\mathcal{A}}})$, $\widetilde{T}_{\mathcal{M}_{/\mathcal{A}}} = (\widetilde{T}^{ij}_{\mathcal{M}_{/\mathcal{A}}})$, $S_{\mathcal{M}_{/\mathcal{A}}}$, $T_{\mathcal{M}_{/\mathcal{A}}}$, as well as d_i, s_i, and $D^2_{\mathcal{M}_{/\mathcal{A}}}$. Then we have the following properties

- The distinct s_i's form a subset of $\{s_a \mid a \in \mathcal{M}\}$.
- $D_{\mathcal{M}} = D_{\mathcal{M}_{/\mathcal{A}}} d_{\mathcal{A}}$. $D_{\mathcal{M}}, D_{\mathcal{M}_{/\mathcal{A}}}, d_{\mathcal{A}}$ are cyclotomic integers.
- A^a in $\mathcal{A}$ are non-negative integers, $A^a = A^{\bar{a}}$, and $A^{\mathbf{1}} = 1$.
- For $a \in \mathcal{A}$ (*i.e.* for $A^a \neq 0$), the corresponding $s_a = 0 \bmod 1$.
- if $a, b \in \mathcal{A}$, then at least one of the fusion products in $a \otimes b$ must be contained in $\mathcal{A}$, *i.e.* $\exists c \in \mathcal{A}$ such that $a \otimes b = c \oplus \cdots$.

Now, let us assume $\mathcal{A}$ to be Lagrangian, then the $\mathcal{A}$-condensed boundary of $\mathcal{M}$ is gapped. Let us use x to label the (simple) excitations on the gapped boundary. If we bring a bulk excitation a to such a boundary and fuse it with a boundary excitation y, it will become a (composite) boundary excitation

$$a \otimes y = \bigoplus_x M_x^{ay} x, \quad M_x^{ay} \in \mathbb{N}. \tag{186}$$

Then A^a is given by $A^a = M^a_{\mathbf{1}}$, where $M^a_x \equiv M^{a\mathbf{1}}_x$. In other words, $A^a \neq 0$ means that a condenses on the boundary (*i.e.* the bulk a can become the null excitation $\mathbf{1}$ on the boundary).

The boundary excitations form a fusion category, denoted as $\mathcal{F}$. We can fuse a, b in bulk to c, and then fuse c to the boundary. Or we can fuse a, b in bulk to the boundary to get x, y, and then fuse x, y on boundary to z. The two ways of fusion should give rise to the same result. This requires M^a_x to satisfy

$$\sum_c N^{ab}_{\mathcal{M},c} M^c_x = \sum_{y,z} M^a_y M^b_z N^{yz}_{B,x} \tag{187}$$

where $N^{ab}_{\mathcal{M},c}$ describes the fusion ring of the bulk excitations in $\mathcal{M}$ and $N^{xy}_{B,z}$ describes the fusion ring of the boundary excitations.

Taking $x = \mathbf{1}$, (187) reduces to

$$\sum_c N^{ab}_{\mathcal{M},c} A^c = A^a A^b + \sum_{x \neq \mathbf{1}} M^a_x M^b_{\bar{x}} \tag{188}$$

Since $M^a_x \geq 0$, we obtain an additional condition on A^a

$$\sum_c N^{ab}_{\mathcal{M},c} A^c \geq A^a A^b. \tag{189}$$

From the conservation of quantum dimensions, we have

$$d_a = \sum_x M^a_x d_x = A^a + \sum_{x \neq \mathbf{1}} M^a_x d_x. \tag{190}$$

This implies that, if $d_a \neq$integer, $\sum_{x \neq \mathbf{1}} M^a_x d_x \geq 1$, or

$$A^a \leq d_a - \delta(d_a), \tag{191}$$

where $\delta(d_a)$ is defined as

$$\delta(x)=\begin{cases}0 & \text{if } x\in\mathbb{Z}\\ 1 & \text{if } x\notin\mathbb{Z}\end{cases} \tag{192}$$

By writing $\sum_y=\sum_{y=\mathbf{1}}+\sum_{y\neq\mathbf{1}}$ in (187), we find

$$\sum_c N^{ab}_{\mathcal{M},c}M^c_x = A^a M^b_x + \sum_{y\neq\mathbf{1},z} M^a_y M^b_z N^{yz}_{B,x}. \tag{193}$$

Noticing that $d_a - A^a = \sum_{x\neq\mathbf{1}} M^a_x d_x$. We multiply d_x and sum over $x\neq\mathbf{1}$ in the above

$$\begin{aligned}\sum_{c,x} N^{ab}_{\mathcal{M},c}(d_c - A^c) &= A^a(d_b - A^b) + \sum_{y\neq\mathbf{1},z,x\neq\mathbf{1}} M^a_y M^b_z N^{yz}_{B,x} d_x\\ &= A^a(d_b - A^b) + \sum_{x\neq\mathbf{1}} M^a_x A^b d_x + \sum_{y\neq\mathbf{1},z\neq\mathbf{1},x\neq\mathbf{1}} M^a_y M^b_z N^{yz}_{B,x} d_x\\ &\geq A^a(d_b - A^b) + A^b(d_a - A^a).\end{aligned} \tag{194}$$

Taking $b=\bar{a}$ in (188), we find

$$\sum_c N^{a\bar{a}}_{\mathcal{M},c}A^c \geq (A^a)^2 + \delta(d_a). \tag{195}$$

Summarizing the above discussions and adding a modular covariant condition, we have

$$\begin{aligned}&\boldsymbol{A} = D^{-1}_{\mathcal{M}} S_{\mathcal{M}}\boldsymbol{A}, && \boldsymbol{A} = T_{\mathcal{M}}\boldsymbol{A},\\ &A^a \leq d_a - \delta(d_a), && D_{\mathcal{M}} = d_{\mathcal{A}} = \sum_a d_a A^a,\\ &A^a A^b \leq \sum_c N^{ab}_{\mathcal{M},c}A^c - \delta_{a,\bar{b}}\delta(d_a), && \sum_{c,x} N^{ab}_{\mathcal{M},c}(d_c - A^c) \geq A^a(d_b - A^b) + A^b(d_a - A^a),\end{aligned} \tag{196}$$

where $\boldsymbol{A} = (A^{\mathbf{1}},\cdots,A^a,\cdots)^\top$.

Now, let us assume $\mathcal{A}$ not to be Lagrangian. In this case, the condensation of $\mathcal{A}$ will change $\mathcal{M}$ to a non-trivial $\mathcal{M}_{/\mathcal{A}}$. Let us consider the domain wall between $\mathcal{M}$ and $\mathcal{M}_{/\mathcal{A}}$. Such a domain wall can be viewed as a boundary of $\mathcal{M}\boxtimes\overline{\mathcal{M}}_{/\mathcal{A}}$ topological order formed by stacking $\mathcal{M}$ and the spatial reflection of $\mathcal{M}_{/\mathcal{A}}$. Since the domain wall, and hence the boundary, is gapped, there must be a Lagrangian condensable algebra $\mathcal{A}_{\mathcal{M}\boxtimes\overline{\mathcal{M}}_{/\mathcal{A}}}$ in $\mathcal{M}\boxtimes\overline{\mathcal{M}}_{/\mathcal{A}}$, whose condensation gives rise to the gapped boundary. Let

$$\mathcal{A}_{\mathcal{M}\boxtimes\overline{\mathcal{M}}_{/\mathcal{A}}} = \bigoplus_{a\in\mathcal{M},\ i\in\mathcal{M}_{/\mathcal{A}}} A^{ai}\ a\otimes i, \tag{197}$$

then the matrix $A=(A^{ai})$ satisfies

$$\begin{aligned}&D^{-1}_{\mathcal{M}}S_{\mathcal{M}}A = AD^{-1}_{\mathcal{M}_{/\mathcal{A}}}S_{\mathcal{M}_{/\mathcal{A}}}, \quad T_{\mathcal{M}}A = AT_{\mathcal{M}_{/\mathcal{A}}}, \quad A^{ai}\leq d_a d_i - \delta(d_a d_i),\\ &A^{ai}A^{bj} \leq \sum_{c,k} N^{ab}_{\mathcal{M},c}N^{ij}_{\mathcal{M}_{/\mathcal{A}},k}A^{ck} - \delta_{a,\bar{b}}\delta_{i,\bar{j}}\delta(d_a d_i)\\ &\sum_{c,x} N^{ab}_{\mathcal{M},c}N^{ij}_{\mathcal{M}_{/\mathcal{A}},k}(d_c d_k - A^{ck}) \geq A^{ai}(d_b d_j - A^{bj}) + A^{bj}(d_a d_i - A^{ai})\end{aligned} \tag{198}$$

The above conditions only require the domain wall between $\mathcal{M}$ and $\mathcal{M}_{/\mathcal{A}}$ to be gapped. However, since $\mathcal{M}$ and $\mathcal{M}_{/\mathcal{A}}$ are related by a condensation of $\mathcal{A}$, there is a special domain wall (called the canonical domain wall) that satisfies the following condition:

$$\text{For any } i\text{, there exists an } a \text{ such that } \quad A^{ai} \neq 0. \tag{199}$$

The above, together with $T_{\mathcal{M}}A = AT_{\mathcal{M}_{/\mathcal{A}}}$, impies that the eigenvalues of $T_{\mathcal{M}_{/\mathcal{A}}}$ are also eigenvalues of $T_{\mathcal{M}}$.

The canonical domain wall can be viewed as $\mathcal{A}_{\mathcal{M}\to\mathcal{M}_{/\mathcal{A}}}$-condensed boundary of $\mathcal{M}$ with

$$\mathcal{A}_{\mathcal{M}\to\mathcal{M}_{/\mathcal{A}}} = \bigoplus A^{a\mathbf{1}} a. \tag{200}$$

We note that anyon a in $\mathcal{M}$ condenses on the canonical domain wall between $\mathcal{M}$ and $\mathcal{M}_{/\mathcal{A}}$, if and only if $A^{a\mathbf{1}} \neq 0$. This implies that

$$\mathcal{A}_{\mathcal{M}\to\mathcal{M}_{/\mathcal{A}}} = \mathcal{A}, \qquad A^a = A^{a\mathbf{1}}. \tag{201}$$

The domain wall can also be viewed as $\mathcal{A}_{\mathcal{M}_{/\mathcal{A}}\to\mathcal{M}}$-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$ with

$$\mathcal{A}_{\mathcal{M}_{/\mathcal{A}}\to\mathcal{M}} = \bigoplus A^{\mathbf{1}i} i. \tag{202}$$

Since $\mathcal{M}_{/\mathcal{A}}$ comes from a condensation of $\mathcal{M}$, the canonical domain wall must be an $\mathbf{1}$-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$, *i.e.*

$$\mathcal{A}_{\mathcal{M}_{/\mathcal{A}}\to\mathcal{M}} = \mathbf{1}, \quad A^{\mathbf{1}i} = \delta_{\mathbf{1},i}. \tag{203}$$

We can obtain more conditions on A^a. From (198), we find

$$\frac{D_{\mathcal{M}_{/\mathcal{A}}}}{D_{\mathcal{M}}} \sum_{b\in\mathcal{M}} (S_{\mathcal{M}})^{ab} A^{bi} = \sum_{j\in\mathcal{M}_{/\mathcal{A}}} A^{aj} (S_{\mathcal{M}_{/\mathcal{A}}})^{ji}, \tag{204}$$

Setting $a = \mathbf{1}$ in the above, we find that

$$\frac{D_{\mathcal{M}_{/\mathcal{A}}}}{D_{\mathcal{M}}} \sum_{b\in\mathcal{M}} d_b A^{bi} = d_i, \tag{205}$$

or

$$D_{\mathcal{M}} = D_{\mathcal{M}_{/\mathcal{A}}} \sum_{b\in\mathcal{M}} d_b A^b. \tag{206}$$

Eq. (204) also implies

$$\frac{\sum_{b\in\mathcal{M}} (S_{\mathcal{M}})^{ab} A^{bi}}{\sum_{b\in\mathcal{M}} d_b A^b} = \sum_{j\in\mathcal{M}_{/\mathcal{A}}} A^{aj} (S_{\mathcal{M}_{/\mathcal{A}}})^{ji} = \text{cyclotomic integer}, \qquad \text{for all } a\in\mathcal{M},\quad i\in\mathcal{M}_{/\mathcal{A}} \tag{207}$$

Setting $i = \mathbf{1}$ in the above, we find that $\mathcal{A} = \bigoplus_a A^a a$ must satisfies

$$\frac{\sum_{b\in\mathcal{M}} (S_{\mathcal{M}})^{ab} A^{b}}{\sum_{b\in\mathcal{M}} d_b A^b} = \sum_{j\in\mathcal{M}_{/\mathcal{A}}} A^{aj} d_j = \text{cyclotomic integer for all } a\in\mathcal{M} \tag{208}$$

Setting $i = j = \mathbf{1}$ in (198), we also obtain

$$A^a \le d_a - \delta(d_a),$$
$$A^a A^b \le \sum_c N^{ab}_{\mathcal{M},c} A^c - \delta_{a,\bar a}\delta(d_a)$$
$$\sum_c N^{ab}_{\mathcal{M},c}(d_c - A^c) \ge A^a(d_b - A^b) + A^b(d_a - A^a) \tag{209}$$

We also have

$$D^{-1}_{\mathcal{M}}(S_{\mathcal{M}})_{ab}A^b = A^{ai}D^{-1}_{\mathcal{M}_{/\mathcal{A}}}(S_{\mathcal{M}_{/\mathcal{A}}})_{i\mathbf{1}}, \quad (T_{\mathcal{M}})_{ab}A^b = A^{ai}(T_{\mathcal{M}_{/\mathcal{A}}})_{i\mathbf{1}}, \tag{210}$$

which only leads to a weak condition on the vector $\boldsymbol{A} = (A^{\mathbf{1}}, \cdots, A^a, \cdots)^\top$

$$T_{\mathcal{M}}\boldsymbol{A} = \boldsymbol{A}. \tag{211}$$

To summarize, a condensable algebra A^a, which may not a Lagrangian, must satisfies

$$T_{\mathcal{M}}\boldsymbol{A} = \boldsymbol{A}, \quad \delta_{a\mathbf{1}} \le A^a \le d_a - \delta(d_a), \quad \frac{D_{\mathcal{M}}}{\sum_{b\in\mathcal{M}} d_b A^b} = D_{\mathcal{M}_{/\mathcal{A}}} = d\text{-number} \ge 1,$$
$$\frac{\sum_{b\in\mathcal{M}}(S_{\mathcal{M}})^{ab}A^b}{\sum_{b\in\mathcal{M}} d_b A^b} = \text{cyclotomic integer} > 0, \quad \text{for all } a \in \mathcal{M}, \tag{212}$$
$$\sum_c N^{ab}_{\mathcal{M},c}A^c - A^aA^b - \delta_{a,\bar b}\delta(d_a) \ge 0, \quad (d_a - A^a)(d_b - A^b) - (\sum_c N^{ab}_{\mathcal{M},c}A^c - A^aA^b) \ge 0.$$

The last condition comes from $\sum_c N^{ab}_{\mathcal{M},c}(d_c - A^c) \ge A^a(d_b - A^b) + A^b(d_a - A^a)$. The last two conditions in (212) imply that

$$A^ad_b + A^bd_a + 2\delta_{a,\bar b}\delta(d_a) \le \sum_c N^{ab}_{\mathcal{M},c}A^c + d_ad_b \quad \text{linear in } A^a$$
$$(d_a - A^a)(d_b - A^b) - \delta_{a,\bar b}\delta(d_a) \ge 0 \quad \text{independent of } N^{ab}_{\mathcal{M},c}. \tag{213}$$

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