Inhomogeneous SSH models and the doubling of orthogonal polynomials

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Abstract

We analyze Su–Schrieffer–Heeger (SSH) models using the doubling method for orthogonal polynomial sequences. This approach yields the analytical spectrum and exact eigenstates of the models. We demonstrate that the standard SSH model is associated with the doubling of Chebyshev polynomials. Extending this technique to the doubling of other finite sequences enables the construction of Hamiltonians for inhomogeneous SSH models which are exactly solvable. We detail the specific cases associated with Krawtchouk and q-Racah polynomials. This work highlights the utility of polynomial-doubling techniques in obtaining exact solutions for physical models.

Contents

1 Introduction

Exactly-solvable models of quantum-many body systems play a significant role in our understanding of emergent phenomena, such as long-range order [1] and quantum phase transitions [2–5], entanglement measures [6–8], relaxation dynamics out of equilibrium [9–11], or the impact of spatial inhomogeneities on physical properties [12, 13].

The Su–Schrieffer–Heeger (SSH) model is a prominent example of such models, and was originally introduced to describe polyacetylene [14], where carbon atoms form a dimerized chain with alternating strong and weak bonds. Owing to this alternating bonding pattern, the model hosts symmetry-protected topological excitations and represents the simplest quantum many-body system exhibiting such features [15]. The SSH model is exactly solvable with free fermion techniques, but nonetheless describes experimentally realized topological phases [16], making it a paradigmatic framework for studying topological phase transitions in condensed matter systems.

In this paper, we use the mathematical properties of orthogonal polynomials and their doubling to revisit the solution of the standard SSH model and construct different solvable inhomogeneous generalizations. A strong relationship exists between orthogonal polynomials of the Askey scheme and free-fermion models. This connection has already yielded exact results for inhomogeneous XX or XY spin chains [17, 18], and free-fermion models on various graphs [19–22] or higher-dimensional lattices [23]. Moreover, it allows for the analytical investigation of physical properties, such as perfect state transfer or entanglement measures [24–28].

The doubling procedure for orthogonal polynomials consists in combining two families of orthogonal polynomials in an appropriate way to produce a new one [29, 30]. This approach allows one to construct and diagonalize a symmetric tridiagonal matrix whose eigenvectors are expressible in terms of the original polynomial families. The resulting tridiagonal matrix has particularly nice features: the elements on the sub- and super-diagonals alternate between two sequences, t_n^+ and t_n^-, describing the alternating strength of the bonds and their inhomogeneity along the chain. We show that this construction provides a new perspective on the standard solution of the SSH model and enables the definition of a broader class of exactly solvable inhomogeneous SSH-like models.

Outline of the paper. The paper is organized as follows: in Sec. 2 we show that the eigenvalues and eigenvectors of the SSH model can be related to the well-known Chebyshev polynomials and to their doubling. Section 3 extends the doubling procedure to any finite sequence of orthogonal polynomials, leading to new models that generalize the SSH model while remaining exactly solvable. Finally, in

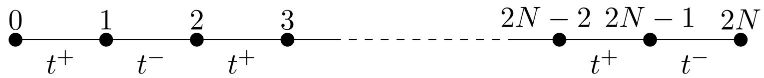

Figure 1: Representation of the usual SSH model with open boundary conditions.

Secs. 4 and 5, we illustrate the general construction using two specific families of discrete polynomials from the (q-)Askey scheme: the Krawtchouk and q-Racah polynomials.

Notations. We recall here the definitions of the generalized (q-)hypergeometric functions used in this paper. The generalized hypergeometric functions [31,32] are defined as

$$ {}_{r+1}F_r\left(\begin{matrix}-n,\ a_1,\ a_2,\ \cdots,\ a_r\\ b_1,\ b_2,\ \cdots,\ b_r\end{matrix}\ \middle|\ z\right)=\sum_{k=0}^{n}\frac{(-n,a_1,a_2,\cdots,a_r)_k}{k!(b_1,b_2,\cdots,b_r)_k}z^k\,, \tag{1.1}$$

for r,n non-negative integers, and where the Pochhammer symbols are

$$ (b_1,b_2,\cdots,b_r)_k=(b_1)_k(b_2)_k\cdots(b_r)_k\,,\qquad (b)_k=b(b+1)\cdots(b+k-1)\,. \tag{1.2}$$

The q-hypergeometric functions [31,32] are defined by

$$ {}_{r+1}\phi_r\left(\begin{matrix}q^{-n},\ a_1,\ a_2,\ \cdots,\ a_r\\ b_1,\ b_2,\ \cdots,\ b_r\end{matrix}\ \middle|\ q;z\right)=\sum_{k=0}^{n}\frac{(q^{-n},a_1,a_2,\cdots,a_r;q)_k}{(b_1,b_2,\cdots,b_r,q;q)_k}z^k\,, \tag{1.3}$$

for r,n non-negative integers, with the q-Pochhammer symbols given by

$$ (b_1,b_2,\cdots,b_r;q)_k=(b_1;q)_k(b_2;q)_k\cdots(b_r;q)_k\,,\quad (b;q)_k=(1-b)(1-qb)\cdots(1-q^{k-1}b)\,. \tag{1.4}$$

2 SSH model and doubling of Chebyshev polynomials

The SSH model is a one-dimensional free-fermion model with dimerized nearest-neighbor interactions. We consider a chain with an odd number of sites, labeled from 0 to $2N$ and endowed with open boundary conditions. The Hamiltonian $\mathcal{H}$ reads [14]

$$\begin{aligned}\mathcal{H}&=\sum_{x=1}^{N}\left(t^+c^\dagger_{2x-2}c_{2x-1}+t^-c^\dagger_{2x-1}c_{2x}+\text{h.c.}\right)\\ &=\sum_{x,y=1}^{2N+1}H_{x,y}c^\dagger_{x-1}c_{y-1},\end{aligned} \tag{2.1}$$

where $c_x^{(\dagger)}$ are the fermion operators satisfying the canonical anticommutation relations

$$ \{c^\dagger_x,c_y\}=\delta_{x,y},\ \ \{c_x,c_y\}=0=\{c^\dagger_x,c^\dagger_y\}, \tag{2.2}$$

and $t^\pm$ are the dimerized interactions. For the standard SSH model, the hoping parameters read

$$ t^\pm=\frac{1\pm\delta}{2} \tag{2.3}$$

with $-1\leqslant\delta\leqslant 1$. We represent the chain described by the Hamiltonian in Figure 1, where the alternating bounds are easily visualized.

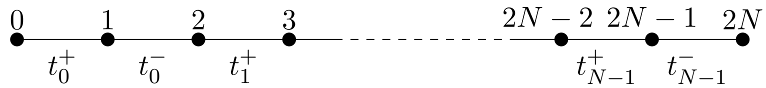

Figure 2: Representation of an inhomogeneous SSH model.

Because the full Hamiltonian $\mathcal{H}$ is quadratic in terms of the fermion operators, it suffices to diagonalize the $(2N+1)\times(2N+1)$ matrix H to obtain the eigenvalues and eigenvectors of $\mathcal{H}$. This matrix H is the single-particle Hamiltonian, and it reads

$$H = \begin{pmatrix} 0 & t^+ & & & & & \\ t^+ & 0 & t^- & & & & \\ 0 & t^- & 0 & t^+ & & & \\ & & & \ddots & & & \\ & & & t^+ & 0 & t^- \\ & & & & t^- & 0 \end{pmatrix}. \tag{2.4}$$

In the following, we consider solvable models with dimerized couplings $t_n^{\pm}$ which depend on the position in the chain, thereby defining inhomogeneous SSH-like models. We represent a schematic inhomogeneous SSH model in Fig. 2. To begin this study, we show in the following subsection that the eigenvalues and the eigenvectors of the homogeneous SSH model can be obtained with the doubling [29] of Chebyshev polynomials.

2.1 Diagonalisation by Chebyshev polynomials

In this section we show that the Chebyshev polynomials can be used to diagonalize the matrix H given in Eq. (2.4). To do this, let us recall some basic properties of the Chebyshev polynomials of the second kind, denoted as U_n $(n = 0, 1, 2, \dots)$. They can be expressed in terms of hypergeometric functions as

$$U_n(x) = (n+1)\ {}_2F_1\left(\begin{matrix} -n,\ n+2 \\ \frac{3}{2} \end{matrix}\middle|\,\frac{1-x}{2}\right). \tag{2.5}$$

Equivalently, using the substitution $x = \cos\theta$, they can be rewritten in trigonometric form,

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{2.6}$$

With the trigonometric form, it is immediate to obtain the roots x_k of U_n,

$$x_k = \cos\left(\frac{k\pi}{n+1}\right), \quad (k = 1, \dots, n). \tag{2.7}$$

Another well-known expression of U_n is

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}. \tag{2.8}$$

The Chebyshev polynomials of the second kind satisfy the three-term recurrence relation

$$U_0(x) = 1\,, \quad 2xU_n(x) = U_{n+1}(x) + U_{n-1}(x)\,, \quad (n = 0, 1, 2\dots)\,, \tag{2.9}$$

with the convention $U_{-1} = 0$. We define a new sequence of polynomials $(Q_n)_n$ as

$$Q_{2n}(x) = \frac{1+\delta}{2}U_n(\pi_x) + \frac{1-\delta}{2}U_{n-1}(\pi_x)\,, \tag{2.10}$$

$$Q_{2n+1}(x) = xU_n(\pi_x)\,, \tag{2.11}$$

where π_x is given as

$$\pi_x = \frac{2}{1-\delta^2}x^2 - \frac{1+\delta^2}{1-\delta^2}\,. \tag{2.12}$$

Note that Q_n is a polynomial of degree n in x. From the definition of Q_n, it follows that

$$xQ_{2n}(x) = \frac{1+\delta}{2}Q_{2n+1}(x) + \frac{1-\delta}{2}Q_{2n-1}(x)\,, \tag{2.13}$$

and

$$\begin{aligned} xQ_{2n+1}(x) = x^2U_n(\pi_x) &= \left(\frac{1-\delta^2}{2}\pi_x + \frac{1+\delta^2}{2}\right)U_n(\pi_x) \\ &= \frac{1-\delta^2}{4}(U_{n+1}(\pi_x) + U_{n-1}(\pi_x)) + \frac{1+\delta^2}{2}U_n(\pi_x) \\ &= \frac{1-\delta}{2}Q_{2n+2}(x) + \frac{1+\delta}{2}Q_{2n}(x)\,. \end{aligned} \tag{2.14}$$

To obtain the second line in the above relation, we have used the recurrence relation (2.9) satisfied by the Chebyshev polynomials $U_n(\pi(x))$. From Eqs. (2.13) and (2.14), we verify that the vector $\mathbf{Q}(x)$ given by

$$\mathbf{Q}(x) = \big(Q_0(x), Q_1(x), \dots, Q_{2N}(x)\big)^t\,, \tag{2.15}$$

is an eigenvector of H corresponding to the eigenvalue x, where x is a root of the characteristic polynomial:

$$Q_{2N+1}(x) = xU_N(\pi_x) = 0\,. \tag{2.16}$$

Therefore, knowing the roots of U_N in Eq. (2.7), the eigenvalues are given by

$$0,\ x_k^{\pm} = \pm\sqrt{\frac{1-\delta^2}{2}\cos\left(\frac{k\pi}{N+1}\right) + \frac{1+\delta^2}{2}}\,, \quad (k = 1, 2, \dots, N)\,. \tag{2.17}$$

It is important to emphasize that the parity of the number of sites plays a crucial role in obtainig analytical results. Indeed, when number of sites is equal to $2N$, the corresponding equation to determine the spectrum would be $Q_{2N}(x) = 0$. However the roots of this polynomial are not known explicitly in general.

Remark 2.1. *The construction of the sequence $(Q_n)_n$ can be seen as the doubling procedure introduced in [29]. Indeed, let us introduce the sequence of polynomials $(V_n)_n$ defined by*

$$V_n(x) = \frac{1+\delta}{2}U_n(x) + \frac{1-\delta}{2}U_{n-1}(x)\,. \tag{2.18}$$

One can show that the sequence $(Q_n)_n$ is constructed from $(V_n)_n$ as

$$Q_{2n}(x) = V_n(\pi_x)\,, \tag{2.19}$$

$$Q_{2n+1}(x) = \frac{x}{(1+\delta)(\pi_x - c)}\left(V_{n+1}(\pi_x) - \frac{V_{n+1}(c)}{V_n(c)}V_n(\pi_x)\right)\,, \tag{2.20}$$

with $c = -\frac{1+\delta^2}{1-\delta^2}$. This result follows from the relation

$$\frac{V_{n+1}(c)}{V_n(c)} = \frac{\delta+1}{\delta-1}\,, \tag{2.21}$$

which can be derived using (2.8). We conclude that the sequence $(Q_n)_n$ is obtained by doubling the sequence $(V_n)_n$.

2.2 SSH model with non-vanishing chemical potential

The construction explained in Remark 2.1 is more general than the result used previously. Indeed, instead of using (2.12), one may define

$$\pi_x = \frac{2}{1-\delta^2}(x-\mu^+)(x-\mu^-) - \frac{1+\delta^2}{1-\delta^2}\,, \tag{2.22}$$

and define the sequence $(Q_n)_n$ by

$$Q_{2n}(x) = \frac{1+\delta}{2}U_n(\pi_x) + \frac{1-\delta}{2}U_{n-1}(\pi_x)\,, \tag{2.23}$$
$$Q_{2n+1}(x) = (x-\mu^+)U_n(\pi_x)\,. \tag{2.24}$$

Following the same computations as in the previous subsection, we find that the vector $\mathbf{Q}(x)$, whose components are $Q_0(x), Q_1(x), \dots, Q_{2N}(x)$ is an eigenvector of

$$H = \begin{pmatrix} \mu^+ & t^+ & & & & & \\ t^+ & \mu^- & t^- & & & & \\ 0 & t^- & \mu^+ & t^+ & & & \\ & & & \ddots & & & \\ & & & t^+ & \mu^- & t^- \\ & & & & t^- & \mu^+ \end{pmatrix}, \tag{2.25}$$

with x an eigenvalue chosen among

$$0,\ x_k^\pm = \frac{\mu^+ + \mu^- \pm \sqrt{(\mu^+-\mu^-)^2 + 2(1+\delta^2) + 2(1-\delta^2)\cos\left(\frac{k\pi}{N+1}\right)}}{2}\,, \qquad (k=1,2,\dots,N)\,. \tag{2.26}$$

The parameters $\mu^\pm$ can be interpreted as a chemical potential in the SSH model, which can depend on the parity of the sites.

3 Inhomogeneous SSH model from the doubling procedure

Inspired by the fact that the doubling procedure described in Remark 2.1 applies to any sequence of orthogonal polynomials, we can use the previous construction to obtain new models that generalize the SSH model and remain exactly solvable.

Instead of the Chebyshev polynomials U_n, let us consider a finite sequence of orthogonal polynomials $(P_n)_{n=0}^{N-1}$ satisfying a three-term recurrence relation

$$xP_n(x) = A_nP_{n+1}(x) - (A_n+C_n)P_n(x) + C_nP_{n-1}(x)\,, \tag{3.1}$$

with the initial conditions $C_0 = 0$, $A_{N-1} = 0$, $P_0(x) = 1$. Assuming $A_nC_{n+1} > 0$ (which we assume in the following), Favard's theorem (see e.g. [33]) guarantees that the sequence $(P_n)_n$ is orthogonal,

$$\sum_{x=0}^{N-1} \Omega(x)P_n(\lambda(x))P_m(\lambda(x)) = \delta_{n,m}\mathcal{N}_n\,, \tag{3.2}$$

with a certain weight function $\Omega(x)$ and norm $\mathcal{N}_n$. The sequence $(\lambda(x))_{x=0}^{N-1}$, appearing in the orthogonality relation, is called the grid associated to the polynomials P_n. The recurrence relation for $n = N-1$ is only valid for $x = \lambda(k)$ $(k = 0, 1, \dots, N-1)$. We can change the normalization of the sequence P_n to obtain a more convenient recurrence relation. Defining the sequence $(R_n)_n$ as

$$R_n = \epsilon^n \prod_{k=0}^{n-1} \frac{A_k}{\sqrt{A_kC_{k+1}}}P_n\,, \qquad (\epsilon = \pm 1)\,, \tag{3.3}$$

the three-term recurrence relation can be rewritten in the symmetric form

$$xR_n(x) = \epsilon\sqrt{A_nC_{n+1}}R_{n+1}(x) - (A_n + C_n)R_n(x) + \epsilon\sqrt{A_{n-1}C_n}R_{n-1}(x)\,. \tag{3.4}$$

Following the construction used for the Chebyshev polynomials (see Eqs. (2.10) and (2.11)), we define a new sequence of polynomials $(Q_n)_{n=0}^{2N}$ from $(R_n)_{n=0}^{N-1}$ as follows:

$$Q_{2n}(x) = t_n^+R_n(\pi_x) + t_{n-1}^-R_{n-1}(\pi_x)\,, \qquad (n = 0,\dots,N)\,, \tag{3.5a}$$

$$Q_{2n+1}(x) = xR_n(\pi_x)\,, \qquad (n = 0,\dots,N-1)\,. \tag{3.5b}$$

Here, we use the convention, $t_N^+ = 0$, $t_{-1}^- = 0$, and

$$\pi_x = \tau_2x^2 + \tau_0\,, \tag{3.6}$$

where τ_0, τ_2 are parameters that need to satisfy certain constraints (see below). The $Q_n(x)$ polynomials allow one to diagonalize a generalization of the SSH model, as per the following proposition.

Proposition 3.1. *The vector* $\mathbf{Q}(x)$*, whose components* $Q_0(x), Q_1(x),\dots,Q_{2N}(x)$ *are defined by* (3.5)*, diagonalizes the* $(2N+1)\times(2N+1)$ *matrix*

$$\begin{pmatrix} 0 & t_0^+ & & & & & \\ t_0^+ & 0 & t_0^- & & & & \\ 0 & t_0^- & 0 & t_1^+ & & & \\ & & & \ddots & & & \\ & & & t_{N-1}^+ & 0 & t_{N-1}^- \\ & & & & t_{N-1}^- & 0 \end{pmatrix}\mathbf{Q}(x) = x\mathbf{Q}(x)\,, \tag{3.7}$$

with the eigenvalue x *chosen among*

$$0,\; x_k^\pm = \pm\sqrt{\frac{\lambda(k)-\tau_0}{\tau_2}}\,, \qquad (k = 0,1,\dots,N-1)\,, \tag{3.8}$$

if the following constraints are satisfied:

$$t_n^\pm \neq 0\,, \quad (n = 0,1,\dots,N-1), \tag{3.9a}$$

$$\epsilon\sqrt{A_nC_{n+1}} = \tau_2t_n^-t_{n+1}^+\,, \tag{3.9b}$$

$$A_n + C_n + \tau_0 = -\tau_2((t_n^+)^2 + (t_n^-)^2)\,. \tag{3.9c}$$

Proof. The components of the eigenvalue problem (3.7) read

$$t_{n-1}^-Q_{2n-1}(x) + t_n^+Q_{2n+1}(x) = xQ_{2n}(x)\,, \quad (n = 0,1,\dots,N)\,, \tag{3.10}$$

$$t_n^+Q_{2n}(x) + t_n^-Q_{2n+2}(x) = xQ_{2n+1}(x)\,, \quad (n = 0,1,\dots,N-1)\,. \tag{3.11}$$

Relation (3.10) is a direct consequence of the definitions (3.5a) and (3.5b).

To prove Eq. (3.11), let us compute

$$\begin{aligned} xQ_{2n+1}(x) &= x^2R_n(\pi_x) = \frac{1}{\tau_2}(\tau_2x^2+\tau_0)R_n(\pi_x) - \frac{\tau_0}{\tau_2}R_n(\pi_x) \\ &= \frac{\epsilon}{\tau_2}\sqrt{A_nC_{n+1}}R_{n+1}(\pi_x) - \frac{1}{\tau_2}(A_n+C_n+\tau_0)R_n(\pi_x) + \frac{\epsilon}{\tau_2}\sqrt{A_{n-1}C_n}R_{n-1}(\pi_x)\,. \end{aligned} \tag{3.12}$$

The last equality is obtained using the three term recurrence relation for R_n. Using the constraints (3.9b) and (3.9c), one obtains

$$\begin{aligned} xQ_{2n+1}(x) &= t_n^-t_{n+1}^+R_{n+1}(\pi_x) + ((t_n^+)^2 + (t_n^-)^2)R_n(\pi_x) + t_{n-1}^-t_n^+R_{n-1}(x) \\ &= t_n^-Q_{2n+2}(x) + t_n^+Q_{2n}(x)\,, \end{aligned} \tag{3.13}$$

which proves (3.11). Note that the three term recurrence relation used for (3.12) is valid only if $\pi_x = \lambda(k)$ $(k = 0,1,\dots,N-1)$ when $n = N-1$. Observing that (3.11) holds for $x = 0$ and finding the roots x of $\pi_x = \lambda(k)$, we obtain the eigenvalues (3.8). This concludes the proof of the proposition. □

The matrix in (3.7) can be interpreted as the Hamiltonian of an inhomogeneous SSH model which can be represented as in Fig. 2. Let us remark that it is important that R_n be a finite sequence for $n = 0, 1, \ldots, N-1$ such that the spectrum is analytically computable.

At first glance, the following solution

$$\epsilon = \tau_2 = 1\,, \quad \tau_0 = 0\,, \quad t_n^+ = \sqrt{C_n}, \quad t_n^- = \sqrt{A_n}, \tag{3.14}$$

seems to be valid for any finite family of orthogonal polynomials. However, the constraint $t_n^\pm \neq 0$ is not satisfied since $t_{N-1}^- = A_{N-1} = 0 = C_0 = t_0^+$. Nevertheless, we can restrict ourselves to the subsystem made only of the sites $1, \ldots, 2N-1$ (we do not consider the uncoupled sites 0 and $2N$) and adapt the proof of the previous proposition to get the solution of this model. Fortunately, we do not need to study these restricted models since we can show that these can be obtained through the doubling of orthogonal polynomials.

To recover the SSH model described in Sec. 2.1, we set

$$\epsilon = +1\,, \quad \tau_0 = -2 - 2\frac{1+\delta^2}{1-\delta^2}\,, \quad \tau_2 = \frac{4}{1-\delta^2}\,, \quad t_n^\pm = \frac{1 \pm \delta}{2}\,, \quad A_n = C_n = 1\,, \tag{3.15}$$

and

$$R_n(x) = U_n\left(\frac{x+2}{2}\right)\,. \tag{3.16}$$

Let us point out that, due to the affine transformation in the parameter of (3.16), it is necessary to make the same affine transformation for the function π_x to compare with the one given by (2.12):

$$\frac{\tau_0 + \tau_2 x^2 + 2}{2} = \frac{2}{1-\delta^2}x^2 - \frac{1+\delta^2}{1-\delta^2}\,. \tag{3.17}$$

Remark 3.2. *The observation regarding the usual SSH model made in remark 2.1 remains valid for the general construction of the sequence $(Q_n)_n$ given in this section: $(Q_n)_n$ can be seen as the doubling procedure introduced in [29]. Indeed, let us introduce the sequence of polynomials $(V_n)_n$ defined by*

$$V_n(x) = t_n^+ R_n(x) + t_{n-1}^- R_{n-1}(x)\,. \tag{3.18}$$

It is straightforward to observe that $Q_{2n}(x) = V_n(\pi_x)$. Moreover, we verify that

$$Q_{2n+1}(x) = \frac{\tau_2 t_n^- x}{(\pi_x - \tau_0)}\left(V_{n+1}(\pi_x) + \frac{t_n^+}{t_n^-}V_n(\pi_x)\right)\,. \tag{3.19}$$

This follows from the definition of $V_n(x)$ and the constrains (3.9b), (3.9c). Indeed, the right hand side of (3.19) is equal to

$$\begin{aligned}
&\frac{\tau_2 t_n^- x}{(\pi_x - \tau_0)}\left(t_{n+1}^+ R_{n+1}(\pi_x) + \left(t_n^- + \frac{(t_n^+)^2}{t_n^-}\right)R_n(\pi_x) + \frac{t_n^+}{t_n^-}t_{n-1}^- R_{n-1}(\pi_x)\right)\\
&= \frac{\tau_2 t_n^- x}{(\pi_x - \tau_0)}\left(\frac{\epsilon\sqrt{A_n C_{n+1}}}{\tau_2 t_n^-}R_{n+1}(\pi_x) - \frac{(A_n + C_n + \tau_0)}{\tau_2 t_n^-}R_n(\pi_x) + \frac{\epsilon\sqrt{A_{n-1}C_n}}{\tau_2 t_n^-}R_{n+1}(\pi_x)\right)\\
&= \frac{\tau_2 t_n^- x}{(\pi_x - \tau_0)}\frac{(\pi_x - \tau_0)R_n(\pi_x)}{\tau_2 t_n^-} = Q_{2n+1}(x),
\end{aligned} \tag{3.20}$$

which is the left hand side of (3.19). In addition to that, we have the following property:

$$\begin{aligned}
V_n(\tau_0)t_n^+ + V_{n+1}(\tau_0)t_n^- &= (t_n^+)^2 R_n(\tau_0) + t_n^+ t_{n-1}^- R_{n-1}(\tau_0) + t_n^- t_{n+1}^+ R_{n+1}(\tau_0) + (t_n^-)^2 R_n(\tau_0)\\
&= \frac{1}{\tau_2}(\epsilon\sqrt{A_n C_{n+1}}R_{n+1}(\tau_0) - (A_n + C_n + \tau_0)R_n(\tau_0) + \epsilon\sqrt{A_{n-1}C_n}R_{n-1}(\tau_0))\\
&= \frac{1}{\tau_2}(\tau_0 R_n(\tau_0) - \tau_0 R_n(\tau_0)) = 0,
\end{aligned} \tag{3.21}$$

which is equivalent to $\frac{t_n^+}{t_n^-} = -\frac{V_{n+1}(\tau_0)}{V_n(\tau_0)}$. *Finally, relation* (3.19) *becomes*

$$Q_{2n+1}(x) = \frac{\tau_2 t_n^- x}{(\pi_x - \tau_0)} \left(V_{n+1}(\pi_x) - \frac{V_{n+1}(\tau_0)}{V_n(\tau_0)} V_n(\pi_x) \right) , \tag{3.22}$$

which is the Christoffel transformation used in [29].*

In the following, we use the above construction with well-known finite sequences of orthogonal polynomials from the (q-)Askey scheme (see e.g. [32]) to provide examples of inhomogeneous generalizations of the SSH model, and hence construct interesting solvable physical models.

4 Inhomogeneous SSH model associated to Krawtchouk polynomials

In this section, we give a solution to the constraints of Proposition 3.1 associated to the Krawtchouk polynomials. These polynomials are defined as follows in terms of the ${}_2F_1$ hypergeometric function, for $0 < p < 1$ and $0 \leqslant n \leqslant N-1$:

$$K_n(x; p, N-1) = {}_2F_1 \left(\begin{matrix} -n, \ -x \\ -N+1 \end{matrix} \middle| \frac{1}{p} \right) , \tag{4.1}$$

and satisfy the three-term recurrence relation

$$\begin{aligned} &- xK_n(x; p, N-1) = \\ &\quad A_n^{(K)} K_{n+1}(x; p, N-1) - (A_n^{(K)} + C_n^{(K)}) K_n(x; p, N-1) + C_n^{(K)} K_{n-1}(x; p, N-1) , \end{aligned} \tag{4.2}$$

with

$$A_n^{(K)} = p(N-n-1) , \quad C_n^{(K)} = n(1-p) . \tag{4.3}$$

Equivalently, they also satisfy the orthogonality relation

$$\sum_{x=0}^{N-1} \binom{N-1}{x} p^x (1-p)^{n-1-x} K_n(x; p, N-1) K_m(x; p, N-1) = \frac{(-1)^n n!}{(-N+1)_n} \left(\frac{1-p}{p} \right)^n \delta_{n,m}. \tag{4.4}$$

As discussed in the previous section, in order to obtain a more convenient recurrence relation, we define renormalized Krawtchouk polynomials, for $\epsilon = \pm 1$, as

$$R_n^{(K)}(x) = \epsilon^n \sqrt{\frac{p^n}{(1-p)^n} \binom{N-1}{n}} K_n(-x; p, N-1) , \tag{4.5}$$

which satisfy

$$xR_n^{(K)}(x) = \epsilon \sqrt{A_n^{(K)} C_{n+1}^{(K)}} R_{n+1}^{(K)}(x) - (A_n^{(K)} + C_n^{(K)}) R_n^{(K)}(x) + \epsilon \sqrt{A_{n-1}^{(K)} C_n^{(K)}} R_{n-1}^{(K)}(x) . \tag{4.6}$$

The grid associated to this sequence of polynomials is $\lambda(k) = -k$, $(k = 0, 1, \ldots, N-1)$. The constraints (3.9b) and (3.9c) read

$$\epsilon \sqrt{p(1-p)(N-n-1)(n+1)} = \tau_2 t_n^- t_{n+1}^+ , \tag{4.7a}$$
$$-(p(N-n-1) + n(1-p) + \tau_0) = \tau_2((t_n^+)^2 + (t_n^-)^2) . \tag{4.7b}$$

*There are small modifications with respect to [29] due to the fact that the polynomials considered here are not monic and that the coefficient of x^2 in π_x is not 1.

A solution of these constraints is

$$\epsilon = \tau_2 = -1\,, \quad \tau_0 = 1\,, \quad t_n^+ = \sqrt{p(N-n)}\,, \quad t_n^- = \sqrt{(1-p)(n+1)}\,. \tag{4.8}$$

With these choices, Proposition 3.1 allows us to diagonalize exactly the following $(2N+1)\times(2N+1)$ matrix,

$$H = \begin{pmatrix} 0 & u_0\sqrt{1+\delta} & & & & & \\ u_0\sqrt{1+\delta} & 0 & u_{N-1}\sqrt{1-\delta} & & & & \\ & u_{N-1}\sqrt{1-\delta} & 0 & u_1\sqrt{1+\delta} & & & \\ & & & \ddots & & & \\ & & & u_{N-1}\sqrt{1+\delta} & 0 & u_0\sqrt{1-\delta} \\ & & & & u_0\sqrt{1-\delta} & 0 \end{pmatrix}. \tag{4.9}$$

Here, $\delta = 2p-1$ and $u_i = \sqrt{\frac{N-i}{2}}$. The spectrum is given by

$$0, x_k^\pm = \pm\sqrt{k+1}, \qquad (k = 0, 1, \dots, N-1). \tag{4.10}$$

Using Proposition 3.1 and after manipulating the Krawtchouk polynomials, the components of the eigenvectors $\mathbf{Q}(x_i^\pm)$ $(i = 0, 1, \dots, N-1)$ are

$$Q_{2n}(x_k^\pm) = (-1)^n \sqrt{\frac{Np^{n+1}}{(1-p)^n}\binom{N}{n}}\, K_n(k+1; p, N)\,, \tag{4.11a}$$

$$Q_{2n+1}(x_k^\pm) = \pm(-1)^n \sqrt{\frac{(k+1)p^n}{(1-p)^n}\binom{N-1}{n}}\, K_n(k; p, N-1)\,. \tag{4.11b}$$

To prove relation (4.11a), we used the dual of the backward shift relation of the Krawtchouk polynomials [32] (or the contiguity relation (KI) of Ref. [34]), i.e.,

$$(1-p)nK_{n-1}(k; p, N) - p(N+1-n)K_n(k; p, N) = -p(N+1)K_n(k+1; p, N+1)\,. \tag{4.12}$$

The components of the eigenvector $\mathbf{Q}(0)$, associated to the vanishing eigenvalue, are:

$$Q_{2n}(0) = (-1)^n \sqrt{\frac{Np^{n+1}}{(1-p)^n}\binom{N}{n}}\,, \quad Q_{2n+1}(0) = 0\,. \tag{4.13}$$

The eigenvectors $\mathbf{Q}(x_k^\pm)$, $\mathbf{Q}(0)$ are orthogonal, with norms

$$\mathbf{Q}(x_k^\pm)^t\ \mathbf{Q}(x_k^\pm) = \frac{2(k+1)(1-p)^{i-N+1}}{p^i\binom{N-1}{i}}\,, \qquad \mathbf{Q}(0)^t\ \mathbf{Q}(0) = \frac{pN}{(1-p)^N}\,, \tag{4.14}$$

for $k = 0, 1, \dots, N-1$. To prove (4.14), we use the self-duality property of the Krawtchouk polynomials, namely $K_n(x; p, N) = K_x(n; p, N)$, which is obvious from their hypergeometric representation, together with the orthogonality relation (4.4).

5 Inhomogeneous SSH model associated to q-Racah polynomials

In the case of the Krawtchouk polynomials discussed in the previous section, a significant simplification arises in the computation (4.11a) of $Q_{2n}(x)$, since it can be expressed as a single Krawtchouk polynomial. The relations of type (4.12) used to achieve this simplification are called contiguity relations, and analogous relations exist for other finite sequences in the (q-)Askey scheme. This construction is now extended to the q-Racah polynomials, making use of the classification of contiguity relations provided in [34].

5.1 Properties of the q-Racah polynomials

The q-Racah polynomials $P_n^{(qR)}$ are defined in terms of q-hypergeometric functions as follows. For N a nonnegative integer and $0 \leqslant n \leqslant N$,

$$P_n^{(qR)}(\lambda_x; \boldsymbol{\rho}) = {}_4\phi_3 \left(\begin{matrix} q^{-n},\ \alpha\beta q^{n+1},\ q^{-x},\ \gamma\delta q^{x+1} \\ \alpha q,\ \beta\delta q,\ \gamma q \end{matrix} \,\middle|\, q; q \right) , \tag{5.1}$$

where $\lambda_x = q^{-x} + \gamma\delta q^{x+1}$, $\boldsymbol{\rho} = \alpha, \beta, \gamma, \delta$ and one of the following truncation conditions is imposed:

$$\alpha q = q^{-N}, \quad \beta\delta q = q^{-N}, \quad \text{or} \quad \gamma q = q^{-N}. \tag{5.2}$$

The q-Racah polynomials enjoy several remarkable properties, including orthogonality, bispectrality and duality [32]. In particular, they satisfy the orthogonality relation

$$\sum_{x=0}^{N} P_n^{(qR)}(\lambda_x; \boldsymbol{\rho}) P_m^{(qR)}(\lambda_x; \boldsymbol{\rho}) w(x; \boldsymbol{\rho}) = \delta_{n,m} h_{n,\boldsymbol{\rho}} \,, \tag{5.3}$$

where the weight function and the norm are given by

$$w(x; \boldsymbol{\rho}) = \frac{(\alpha q, \beta\delta q, \gamma q, \gamma\delta q; q)_x (1 - \gamma\delta q^{2x+1})}{(q, \delta\gamma\alpha^{-1} q, \beta^{-1}\gamma q, \delta q; q)_x (\alpha\beta q)^x (1 - \gamma\delta q)} \,, \tag{5.4}$$

$$h_{n,\boldsymbol{\rho}} = \frac{(\alpha^{-1}\beta^{-1}\gamma, \alpha^{-1}\delta, \beta^{-1}, \gamma\delta q^2; q)_\infty}{(\alpha^{-1}\beta^{-1}q^{-1}, \alpha^{-1}\gamma\delta q, \beta^{-1}\gamma q, \delta q; q)_\infty} \frac{(1 - \alpha\beta q)(\gamma\delta q)^n}{(1 - \alpha\beta q^{2n+1})} \frac{(q, \alpha\beta\gamma^{-1} q, \alpha\delta^{-1} q, \beta q; q)_n}{(\alpha q, \alpha\beta q, \beta\delta q, \gamma q; q)_n} \,. \tag{5.5}$$

The polynomials $P_n^{(qR)}(\lambda_x; \boldsymbol{\rho})$ satisfy the following three-term recurrence relation:

$$-(1 - q^{-x})(1 - \gamma\delta q^{x+1}) P_n^{(qR)}(\lambda_x; \boldsymbol{\rho}) = A_{n,\boldsymbol{\rho}}^{(qR)}\ P_{n+1}^{(qR)}(\lambda_x; \boldsymbol{\rho}) - (A_{n,\boldsymbol{\rho}}^{(qR)} + C_{n,\boldsymbol{\rho}}^{(qR)})\ P_n^{(qR)}(\lambda_x; \boldsymbol{\rho}) + C_{n,\boldsymbol{\rho}}^{(qR)}\ P_{n-1}^{(qR)}(\lambda_x; \boldsymbol{\rho}) \,, \tag{5.6}$$

where

$$A_{n,\boldsymbol{\rho}}^{(qR)} = \frac{(1 - \gamma q^{n+1})(1 - \alpha q^{n+1})(1 - \alpha\beta q^{n+1})(1 - \beta\delta q^{n+1})}{(1 - \alpha\beta q^{2n+1})(1 - \alpha\beta q^{2n+2})} \,, \tag{5.7a}$$

$$C_{n,\boldsymbol{\rho}}^{(qR)} = \frac{q(1 - q^n)(1 - \beta q^n)(\gamma - \alpha\beta q^n)(\delta - \alpha q^n)}{(1 - \alpha\beta q^{2n})(1 - \alpha\beta q^{2n+1})} \,. \tag{5.7b}$$

Moreover, the polynomials $P_n^{(qR)}(\lambda_x; \boldsymbol{\rho})$ satisfy the duality relation

$$P_n^{(qR)}(\lambda_x; \boldsymbol{\rho}) = P_x^{(qR)}(\lambda_n^d; \boldsymbol{\rho}^d) \tag{5.8}$$

for $0 \leqslant n, x \leqslant N$, where $\lambda_n^d = q^{-n} + \alpha\beta q^{n+1}$ and $\boldsymbol{\rho}^d = \gamma, \delta, \alpha, \beta$. Furthermore, the following identity holds

$$w(n; \boldsymbol{\rho}^d) = \prod_{k=0}^{n-1} \frac{A_{k,\boldsymbol{\rho}}^{(qR)}}{C_{k+1,\boldsymbol{\rho}}^{(qR)}} \,. \tag{5.9}$$

Consequently, the normalization appearing in (3.3) is related to the weight as

$$\epsilon^n \prod_{k=0}^{n-1} \frac{A_{k,\boldsymbol{\rho}}^{(qR)}}{\sqrt{A_{k,\boldsymbol{\rho}}^{(qR)} C_{k+1,\boldsymbol{\rho}}^{(qR)}}} = \epsilon^n (-1)^{\nu_n} \sqrt{w(n; \boldsymbol{\rho}^d)}, \tag{5.10}$$

where ν_n denotes the number of indices $k \in \{0, 1, \ldots, N\}$ such that $A_{k,\boldsymbol{\rho}}^{(qR)}$ is negative.

5.2 First exactly solvable model

In this subsection, we consider the q-Racah polynomials with parameters $\boldsymbol{\rho} = \alpha, q\beta, q^{-N}, \delta/q$, and renormalize them as follows:

$$R_n^{(qR)}(x,\boldsymbol{\rho}) = \epsilon^n(-1)^{\nu_n}\sqrt{w(n;\boldsymbol{\rho}^d)}P_n^{(qR)}(\lambda_x,\boldsymbol{\rho})\,, \tag{5.11}$$

where $\epsilon = \pm 1$ and $\boldsymbol{\rho}^d = q^{-N}, \delta/q, \alpha, q\beta$. The orthogonality grid for this sequence of polynomials is

$$\lambda_{k,\boldsymbol{\rho}} = -(1-q^{-k})(1-\delta q^{k-N}), \qquad (k = 0,1,\dots,N-1). \tag{5.12}$$

Proposition 5.1. *A solution to the constraints* (3.9) *is given by*

$$\tau_0 = -\,(1-\delta)\,(1-q^{-N})\,, \quad \tau_2 = -1\,, \quad \epsilon = -1\,, \tag{5.13a}$$

$$t_n^+ = \sqrt{\frac{(1-q^{n-N})(1-\alpha\beta q^{n+1})(1-\beta q^{n+1})(\delta-\alpha q^{n+1})}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})}}\,, \tag{5.13b}$$

$$t_n^- = \sqrt{\frac{(1-\beta\delta q^{n+1})(1-q^{n+1})(1-\alpha\beta q^{n+N+2})(1-\alpha q^{n+1})}{q^N(1-\alpha\beta q^{2n+2})(1-\alpha\beta q^{2n+3})}}\,. \tag{5.13c}$$

Proof. The proof follows directly from a straightforward calculation using the expressions for τ_0, τ_2, t_n^+, and t_n^-. □

With these solutions, the results of Proposition 3.1 allow one to exactly diagonalize the corresponding $(2N+1)\times(2N+1)$ matrix given in (3.7), whose spectrum is

$$0,\; x_k^{\pm} = \pm\sqrt{(1-q^{k-N})(\delta-q^{-k})}, \qquad (k = 0,1,\dots,N-1). \tag{5.14}$$

By applying Proposition 3.1 and performing suitable manipulations of the q-Racah polynomials, the components of the eigenvectors $\boldsymbol{Q}(x_k^{\pm})$, for $k = 0,1,\dots,N-1$, are given by

$$Q_{2n}(x_k^{\pm}) = \epsilon^n(-1)^{\nu_n}\sqrt{\frac{(1-\beta q)(\delta-\alpha q)(1-q^{-N})w(n,\overline{\boldsymbol{\rho}}^d)}{(1-\alpha\beta q^2)}}P_n^{(qR)}(\lambda_k;\overline{\boldsymbol{\rho}})\,, \quad (n = 0,\dots,N)\,, \tag{5.15a}$$

$$Q_{2n+1}(x_k^{\pm}) = \pm\epsilon^n(-1)^{\nu_n}\sqrt{(1-q^{k-N})(\delta-q^{-k})w(n,\boldsymbol{\rho}^d)}P_n^{(qR)}(\lambda_k;\boldsymbol{\rho})\,, \quad (n = 0,\dots,N-1)\,. \tag{5.15b}$$

The parameters are specified as follows:

$$\boldsymbol{\rho} = \alpha, q\beta, q^{-N}, \delta/q, \quad \boldsymbol{\rho}^d = q^{-N}, \delta/q, \alpha, q\beta, \quad \overline{\boldsymbol{\rho}} = \alpha, \beta, q^{-N-1}, \delta \quad \text{and} \quad \overline{\boldsymbol{\rho}}^d = q^{-N-1}, \delta, \alpha, \beta. \tag{5.16}$$

To establish the relation (5.15a), we use the following contiguity relation of the q-Racah polynomials (relation (qRI) of Ref. [34]):

$$P_n^{(qR)}(\lambda_x;\overline{\boldsymbol{\rho}}) = \frac{(1-q^{n-N})(1-\alpha\beta q^{n+1})}{(1-q^{-N})(1-\alpha\beta q^{2n+1})}P_n^{(qR)}(\lambda_x,\boldsymbol{\rho}) + \frac{(1-q^n)(1-\alpha\beta q^{N+n+1})}{(1-q^N)(1-\alpha\beta q^{2n+1})}P_{n-1}^{(qR)}(\lambda_x,\boldsymbol{\rho}). \tag{5.17}$$

The components of the eigenvector $\mathbf{Q}(0)$, corresponding to the vanishing eigenvalue, are given by

$$Q_{2n}(0) = \epsilon^n(-1)^{\nu_n}\sqrt{\frac{(1-\beta q)(\delta-\alpha q)(1-q^{-N})w(n,\overline{\boldsymbol{\rho}}^d)}{(1-\alpha\beta q^2)}}P_n^{(qR)}(\lambda_N;\overline{\boldsymbol{\rho}})\,, \quad Q_{2n+1}(0) = 0\,. \tag{5.18}$$

The eigenvectors $\mathbf{Q}(x_k^{\pm})$, $\mathbf{Q}(0)$ are orthogonal, with norms, for $k = 0,1,\dots,N-1$:

$$\mathbf{Q}(x_k^{\pm})^t\,\mathbf{Q}(x_k^{\pm}) = 2(1-q^{k-N})(\delta-q^{-k})h_{k,\boldsymbol{\rho}^d}, \tag{5.19}$$

$$\mathbf{Q}(0)^t\,\mathbf{Q}(0) = \frac{(1-\beta q)(\delta-\alpha q)(1-q^{-N})}{(1-\alpha\beta q^2)}h_{N,\overline{\boldsymbol{\rho}}^d}\,. \tag{5.20}$$

The relations (5.19) and (5.20) follow from the duality property (5.8) of the q-Racah polynomials.

Finally, we remark that the contiguity relation $(qRIII)$ given in [34] yields the same result as above, up to the following reparameterization:

$$\beta \to \alpha, \qquad \delta \to \frac{1}{\delta}. \tag{5.21}$$

5.3 Second exactly solvable model

Additional contiguity relations for q-Racah polynomials were derived in [34]. Some of these relations do not alter the value of N, in contrast to those used previously (see (4.12) or (5.17)). By employing such relations, alternative solutions to the constraints can be obtained but they apply to a chain with an even number of sites and a slight modification of Proposition 3.1 is needed.

Consider now q-Racah polynomials with parameters $\boldsymbol{\rho} = \alpha, q\beta, q^{-N}, \delta$, and renormalize them as follows:

$$R_n^{(qR)}(x, \boldsymbol{\rho}) = \epsilon^n \sqrt{w(n; \boldsymbol{\rho}^d)} P_n^{(qR)}(\lambda_x, \boldsymbol{\rho}), \tag{5.22}$$

where $\epsilon = \pm 1$ and $\boldsymbol{\rho}^d = q^{-N}, \delta, \alpha, q\beta$. The grid for this sequence of polynomials is

$$\lambda_{k,\boldsymbol{\rho}} = -(1 - q^{-k})(1 - \delta q^{k-N+1}), \qquad (k = 0, 1, \ldots, N-1). \tag{5.23}$$

Proposition 5.2. *A solution to the constraints* (3.9b) *and* (3.9c) *is given by*

$$\begin{aligned}
\tau_0 &= -\left(1 - \beta^{-1} q^{-N}\right)(1 - q\beta\delta), \quad \tau_2 = -\beta^{-1} q^{-N}, \quad \epsilon = 1, \\
t_n^+ &= \sqrt{\frac{(1 - \beta q^{n+1})(1 - \alpha\beta q^{n+1})(1 - \alpha\beta q^{n+N+1})(1 - \beta\delta q^{n+1})}{(1 - \alpha\beta q^{2n+1})(1 - \alpha\beta q^{2n+2})}}, \\
t_n^- &= -q\beta \sqrt{\frac{q^{N-1}(1 - q^{n-N+1})(1 - \alpha q^{n+1})(1 - q^{n+1})(\delta - \alpha q^{n+1})}{(1 - \alpha\beta q^{2n+2})(1 - \alpha\beta q^{2n+3})}}.
\end{aligned}$$

Proof. The proof follows directly from a straightforward calculation using the expressions for τ_0, τ_2, t_n^+, and t_n^-. □

In this case, $t_{N-1}^- = 0$. We can eliminate the last row and column of the $(2N+1) \times (2N+1)$ matrix and consider the resulting subsystem. By adapting the proof of Proposition 3.1, we can show that the spectrum of the corresponding $2N \times 2N$ matrix is given by

$$x_k^\pm = \pm\sqrt{(1 - \delta\beta q^{k+1})(1 - \beta q^{N-k})}, \qquad (k = 0, 1, \ldots, N-1), \tag{5.25}$$

and the components of the eigenvectors $\boldsymbol{Q}(x_k^\pm)$, for $k = 0, 1, \ldots, N-1$, are given by

$$Q_{2n}(x_k^\pm) = (-1)^n \sqrt{\frac{(1 - \beta q)(1 - \alpha\beta q^{N+1})(1 - \beta\delta q) w(n; \overline{\boldsymbol{\rho}}^d)}{1 - \alpha\beta q^2}} P_n^{(qR)}(\lambda_k; \overline{\boldsymbol{\rho}}), \tag{5.26a}$$

$$Q_{2n+1}(x_k^\pm) = \pm(-1)^n \sqrt{(1 - \delta\beta q^{k+1})(1 - \beta q^{N-k}) w(n; \boldsymbol{\rho}^d)} P_n^{(qR)}(\lambda_k; \boldsymbol{\rho}), \tag{5.26b}$$

for $n = 0, \ldots, N-1$. The parameters are specified as follows:

$$\boldsymbol{\rho} = \alpha, q\beta, q^{-N}, \delta, \quad \overline{\boldsymbol{\rho}} = \alpha, \beta, q^{-N}, \delta, \quad \boldsymbol{\rho}^d = q^{-N}, \delta, \alpha, q\beta \quad \text{and} \quad \overline{\boldsymbol{\rho}}^d = q^{-N}, \delta, \alpha, \beta. \tag{5.27}$$

To establish the relation (5.26a), we employ the following contiguity relation of the q-Racah polynomials (see $(qRII)$ in [34]):

$$P_n^{(qR)}(\lambda_x; \overline{\boldsymbol{\rho}}) = \frac{(1 - \alpha\beta q^{n+1})(1 - \beta\delta q^{n+1})}{(1 - \beta\delta)(1 - \alpha\beta q^{2n+1})} P_n^{(qR)}(\lambda_x, \boldsymbol{\rho}) - \frac{\beta(1 - q^n)(\delta - \alpha q^n)}{(1 - \beta\delta)(1 - \alpha\beta q^{2n+1})} P_{n-1}^{(qR)}(\lambda_x, \boldsymbol{\rho}). \tag{5.28}$$

We note that there is no shift in the parameter $N-1$ when comparing $\boldsymbol{\rho}$ and $\overline{\boldsymbol{\rho}}$. The eigenvectors $\mathbf{Q}(x_i^\pm)$ are orthogonal, with norms, for $k=0,1,\dots,N-1$:

$$\mathbf{Q}(x_k^\pm)^t\ \mathbf{Q}(x_k^\pm) = 2(1-\delta\beta q^{k+1})(1-\beta q^{N-k})h_{k,\boldsymbol{\rho}^d}\,. \tag{5.29}$$

The relations (5.29) follow from the duality property (5.8) of the q-Racah polynomials.

Finally, we remark that the contiguity relation $(qRIV)$ given in [34] yields the same result as above, up to the following reparametrization:

$$\beta \to \alpha, \qquad \gamma \to \frac{1}{\gamma}. \tag{5.30}$$

6 Conclusion

Summing up, we have shown that the eigenvalues and eigenvectors of the standard SSH model can be elegantly derived through a connection with Chebyshev polynomials and the doubling method for polynomial sequences. Exploiting this analytical framework, we introduced new exactly solvable models that extend the conventional SSH model by allowing for inhomogeneous dimerized couplings along the chain. This generalization preserves the model's tractability while significantly enriching its structure. In particular, we derived closed-form expressions for the full energy spectrum and the corresponding eigenstates, providing tools that are essential for detailed theoretical analysis.

A natural next step is to investigate how the topological properties of the SSH model, along with other physically relevant quantities such as entanglement measures and correlation functions, are influenced by the introduction of these inhomogeneities. Since the doubling procedure provides a clear analytical handle on both the energy spectrum and the eigenstates, we expect that the computation and interpretation of these more intricate quantities will remain accessible through analytical methods. We will explore these aspects in detail in a forthcoming paper.

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