MTC$[M_3, G]$: 3d Topological Order
Labeled by Seifert Manifolds

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We propose a correspondence between topological order in 2+1d and Seifert three-manifolds together with a choice of ADE gauge group G. Topological order in 2+1d is known to be characterized in terms of modular tensor categories (MTCs), and we thus propose a relation between MTCs and Seifert three-manifolds. The correspondence defines for every Seifert manifold and choice of G a fusion category, which we conjecture to be modular whenever the Seifert manifold has trivial first homology group with coefficients in the center of G. The construction determines the spins of anyons and their S-matrix, and provides a constructive way to determine the R- and F-symbols from simple building blocks. We explore the possibility that this correspondence provides an alternative classification of MTCs, which is put to the test by realizing all MTCs (unitary or non-unitary) with rank $r \leq 5$ in terms of Seifert manifolds and a choice of Lie group G.

1 Introduction

A tantalizing feature of three spacetime dimensions is the existence of non-trivial topological order. Mathematically 3d topological order is characterized by a modular tensor category (MTC) [1], for reviews see e.g. [2, 3, 4, 5]. In particular, this includes objects known as anyons, which are topological lines, along with data about their spins, braiding, and fusion. A particularly important piece of information is the modular data, encoded in the S- and T-matrices, which in turn specify the braiding and spins of anyons. Unfortunately it was recently shown that this modular data alone is insufficient to uniquely determine an MTC [6][1]. To pin down a 3d topological order uniquely, requires specifying the F- and R-symbols as well.

Nevertheless, a first order classification of topological order includes determining all the modular data. A known set of necessary conditions on the S- and T-matrices exist, which ensure that they correspond to the modular data of an MTC. Solutions to these (not necessarily sufficient) conditions were recently determined until rank $r = 11$ [7], with earlier results for lower rank in [8, 9, 10, 3, 11, 12]. But a conceptual and completely explicit classification is still very much wanting.

In this paper, we shall not find a solution to this formidable problem. However we will propose an alternative approach to studying and potentially classifying MTCs, by relating them to so-called Seifert three-manifold and a choice of gauge group G (usually chosen to be a Lie group of ADE type). In short, we propose a map

$$(\text{Seifert } M_3, G_{ADE}) \rightarrow (S, T) \text{ of MTC} [M_3, G_{ADE}] , \tag{1.1}$$

i.e. we extract the modular S- and T-matrices, though we will not address the ambiguity in terms of the modular data. However, we will show that this framework surprisingly maps out huge swaths of the set of known MTCs, and we shown that at least to rank $r = 5$ there is a realization of all known (unitary and non-unitary) MTCs within this framework.

The proposal is first of all quite ad hoc. Some initial inspiration may have come from the 3d-3d correspondence [13] and subsequently the conjectures in [14, 15]. However ultimately we make no claim of a connection at this point to the 3d-3d correspondence.

The conjectures of the present work can be formulated simply as a direct characterization of topological order (including non-unitary TQFTs), by a map from the data of the Seifert manifold and a choice of ADE Lie group, to a set of anyons and their T- and S-matrices. The correspondence passes some simple checks, e.g. that similarity transformations of Seifert mani-

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[1] Although the first such ambiguity occurs at a relatively high rank, 49.

folds map to the same modular data, and furthermore, we provide a (conjectured) criterion for when a three-manifold gives rise to modular data in terms of the condition $H_1(M_3, \mathcal{Z}_G) = 0$, where $\mathcal{Z}_G$ is the center of the gauge group G.

To briefly summarize the correspondence, the main ingredient is the data of a Seifert manifold, which is comprised of n pairs of coprime integers (p_i, q_i), each of which characterizes a singular fiber. We consider flat $G_{\mathbb{C}}$ connections on such a three-manifold, satisfying certain conditions, called **anyonic flat connections**. These will play the role of the anyons in the dual MTC. For each fiber parametrized by (p_i, q_i) we associate a pre-modular category $\mathcal{C}(\mathfrak{sl}(N), p_i, q_i)$. The fusion category associated to M_3 is obtained by taking a (graded) Deligne product of $\mathcal{C}(\mathfrak{sl}(N), p_i, q_i)$. We provide strong evidence that that this is modular if and only if $H_1(M_3, \mathcal{Z}_G) = 0$. Restricting to this set of three-manifolds we map out the low rank MTCs and find a complete list up to rank $r \leq 5$.

The proposal of this work may superficially have similarities with the one in [14, 15], which was further developed in [16], where also a curious connection between three-manifolds and 3d topological order was conjectured. In short the proposal there is that compactifying the 6d $(2,0)$ supersymmetric, super-conformal field theory with gauge algebra $\mathfrak{su}(2)$ on a three-fibered Seifert manifold M_3, gives rise to a UV $\mathcal{N} = 2$ supersymmetric 3d gauge theory $T[M_3]$, which conjecturally flows in the IR to a gapped phase. This approach yields all MTCs up to rank $r = 4$, but fails to produce all MTCs at higher rank (it fails to reproduce $r = 5$ models, concretely the models 1., 2., 21.-26. in table 4 do not have a realization in the proposal [14, 15]). The connection between three-manifolds M_3 and MTCs is made – in the spirit of the 3d-3d-correspondence [13] – via the Chern-Simons invariant and Reidemeister Torsion of flat $SL(2, \mathbb{C})$ connections on M_3. E.g. in determining all rank 4 models, it is crucial for [14] to include models with non-trivial $H_1(M_3, \mathcal{Z}_G)$ (which are thus not automatically modular), and require a gauging of a 1-form symmetry given by $H_1(M_3, \mathcal{Z}_G)$, and completion to a modular tensor category. Although the current work was motivated by this connection, a naive extension seems unclear (in particular the question of computing the torsion for higher number of fibers or higher rank groups is at best ill-defined). In addition, for a generic M_3, the 3d $\mathcal{N} = 2$ SQFT is not gapped in the infrared. It is clear that, without a deeper understanding of the 3d-3d correspondence and the RG flow from the UV 3d $\mathcal{N} = 2$ SQFT to the IR phase, a precise map between $T[M_3]$ and MTCs is currently out of reach.

The connection between Seifert manifolds and MTCs that we propose here is a priori ***distinct*** from the above, and in particular does not hinge on the 3d-3d correspondence. We will at this point simply provide a dictionary between the data Seifert manifolds and Lie groups G, and MTCs (not necessarily unitary). Crucially we relate triviality of $H_1(M_3, \mathcal{Z}_G)$

with modularity, and only consider Seifert manifolds with $H_1(M_3, \mathcal{Z}_G) = 0$. In particular this means that even for those MTCs that have a realization within the framework of [14], many will have non-trivial H_1, and thus our proposed realization in terms of Seifert manifolds is quite different, as exemplified in these concrete examples.

Finally, one might ask how our proposal differs from the realization of MTCs in terms of Chern-Simons (CS) theories (and related cosets) [1, 3]. The construction of $\text{MTC}[M_3, G]$ is based on building blocks, which are pre-modular categories. These are associated to building blocks (fibers) of the Seifert manifold M_3. The MTC is obtained as a graded Deligne product of these building blocks. Although some of the building blocks may be of CS-type, the graded Deligne product is not.

Furthermore, the construction we propose also provides a systematic way to compute the R- and F-symbols (using the graded Deligne product on the pre-modular categories). As the building blocks have well-known R- and F-symbols, we view this as an advantage of this construction, as it allows bootstrapping this data for MTCs which are otherwise notoriously hard to compute [10].

Summary. Let us summarize the construction: A Seifert manifold M_3 is characterized in terms of n fibers (see below). We associate to each fiber a pre-modular tensor category. This is graded by the center of G: $\mathcal{Z}_G$. If $H_1(M_3, \mathcal{Z}_G) = 0$, the $\mathcal{Z}_G$-graded Deligne product of these pre-modular categories is modular and defines our $\text{MTC}[M_3, G]$:

- Modularity of this is equivalent to vanishing of $H_1(M_3, \mathcal{Z}_G)$ where $\mathcal{Z}_G$ is the center of the gauge group G.

- The choice of building blocks and the gluing is determined by the data of the Seifert three-manifold M_3.

- Equivalence between different presentations of the three-manifold give equivalent MTCs.

- Beyond providing the T- and S-matrices from the Seifert data and information of G, the construction also provides a concrete way to compute R- and F-symbols.

- This results in all MTCs (not necessarily unitary!) for rank $r \leq 5$ for which we list (M_3, G) in appendix D, and provide a model for each of the MTCs in appendix E of [7].

The structure of the paper is as follows: we provide background on Seifert manifolds and flat connections in section 2, and we introduce the notion of an anyonic flat connection, that will be central to our correspondence. A summary of MTCs can be found in section 3, where we also introduce a graded Deligne product, which will be the key tool for the contruction based on Seifert data. The main proposal relating these two things is explained in section 4.

We put the conjecture that this may provide a comprehensive list of MTCs to test in section 5, with tables of all MTCs up to rank 5 in the appendix D. The precise map between anyonic flat connections and anyons is detailed in appendix B.

2 Seifert Manifolds and Flat Connections

2.1 Seifert 3-manifolds

A Seifert manifold M_3 is a closed, connected, smooth 3-manifold that admits a smooth circle action, see [17, 18]. The base of the fibration is a two-dimensional orbifold, given by a genus g Riemann surface $\Sigma_{g,n}$ with n distinct marked points. In this work we are only interested in orientable Seifert fibered spaces with orientable base. A Seifert fibration is specified by the data b, g $\{(p_k, q_k)\}_{k=1}^n$, where g, b, p_k, q_k are integers, $g \geq 0$, $p_k \geq 2$, $\gcd(p_k, q_k) = 1$. The order of the pairs (p_k, q_k) is immaterial and there may be repeated pairs. We will restrict in this paper to $g = b = 0$ and denote the associated 3-manifold by

$$M_3 = [\{(p_k, q_k)\}_{k=1}^n] \,. \tag{2.1}$$

The Seifert data in this paper is not normalized. The fundamental group of the Seifert manifold $M_3 = [\{(p_k, q_k)\}_{k=1}^n]$ has generators $\{x_k\}_{k=1}^n$ and h subject to relations

$$x_k^{p_k} h^{q_k} = 1 \,, \quad x_1 x_2 \ldots x_n = 1 \,, \quad h \text{ is central} \,. \tag{2.2}$$

Hurewicz theorem implies that

$$H_1(M_3; \mathbb{Z}) \cong \operatorname{coker} M \,, \tag{2.3}$$

where M is the following $(n+1) \times (n+1)$ matrix,

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ p_1 & 0 & \cdots & 0 & q_1 \\ 0 & p_2 & \cdots & 0 & q_2 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & p_n & q_n \end{pmatrix} \,. \tag{2.4}$$

When $n = 3$ this simplifies to $H_1(M; \mathbb{Z}) \cong \mathbb{Z}_K \oplus \mathbb{Z}_L$, where

$$K = \gcd(p_1, p_2, p_3) \,, \quad L = \frac{p_1 p_2 p_3}{K} \left| \sum_{k=1}^3 \frac{q_k}{p_k} \right| \,. \tag{2.5}$$

The main focus of this paper will be on M_3 with trivial $H_1(M_3, \mathcal{Z}_G)$, where $\mathcal{Z}_G$ will be the center of the gauge group of ADE type.

2.2 Flat Connections

Fix the Seifert manifold $M_3 = [\{(p_k, q_k)\}_{k=1}^n]$ and the complex Lie group $SL(N, \mathbb{C})$. A flat $SL(N, \mathbb{C})$ connection on M_3 is the same as a group homomorphism

$$\rho : \pi_1(M_3) \to SL(N, \mathbb{C}) \,, \tag{2.6}$$

up to conjugation. A flat $SL(N, \mathbb{C})$ connection on M_3 is said to be **irreducible** if it has a finite stabilizer group

$$\mathrm{Stab}(\rho) = \{g \in SL(N, \mathbb{C}) | g\rho(x) = \rho(x)g, \forall x \in \pi_1(M_3)\} \,. \tag{2.7}$$

Assume ρ is an irreducible flat $SL(N, \mathbb{C})$ connection on M_3. Then, the following properties hold. Firstly, the matrix $\rho(h)$ must be an element in the center of $SL(N, \mathbb{C})$,

$$\rho(h) = e^{2\pi i \ell/N} \mathbb{I}_N \,, \quad \ell \in \{0, \dots, N-1\} \,. \tag{2.8}$$

Secondly, the matrix $\rho(x_k)$ is diagonalizable for each $k = 1, \dots, n$. Finally, at most $n-3$ out of the n matrices $\rho(x_k)$ can be a multiple of the identity matrix. These properties of irreducible flat connections are established in [19], where we count them systematically for various values of N, n.

2.3 Anyonic Flat Connections

We consider a particular type of irreducible flat $SL(N, \mathbb{C})$ connection, which we call **anyonic**, if for each $k = 1, \dots, n$, the eigenvalues of the matrix $\rho(x_k)$ are all distinct. It is this set of flat connections that will be relevant in formulating a correspondence to MTCs.

The moduli space of anyonic $SL(N, \mathbb{C})$ connections on $M_3 = [\{(p_k, q_k)\}_{k=1}^n]$ generically consists of multiple connected components. The matrix $\rho(h)$ and the eigenvalues of the matrices $\rho(x_k)$ are the same for all connections ρ in the same connected component. We have already parametrized $\rho(h)$ in (2.8). Next, we seek a convenient parametrization of the eigenvalues of $\rho(x_k)$ compatible with the relation $\rho(x_k)^{p_k}\rho(h)^{q_k} = \mathbb{I}_N$ and with the requirement that all eigenvalues of $\rho(x_k)$ be distinct. To this end, we write

$$\rho(x_k) \sim \mathrm{diag}(e^{2\pi i a_k^{(1)}}, \dots, e^{2\pi i a_k^{(N)}}) \,, \tag{2.9}$$

where $\sim$ is equality up to conjugation in $SL(N, \mathbb{C})$ and the rational numbers $a_k^{(I)}$, $I = 1, \dots, N$, satisfy

$$a_k^{(1)} < a_k^{(2)} < \dots < a_k^{(N)} < a_k^{(1)} + 1 \,, \quad \sum_{I=1}^N a_k^{(I)} = 0 \,. \tag{2.10}$$

This is up to conjugation the most general diagonal matrix with distinct entries that are roots of unity. The relation $\rho(x_k)^{p_k}\rho(h)^{q_k} = \mathbb{I}_N$ implies that the $a_k^{(I)}$ take the form

$$a_k^{(I)} = \frac{1}{p_k}\left(\nu_k^{(I)} - \frac{\ell q_k}{N}\right) \,, \quad I = 1, \dots, N \,, \tag{2.11}$$

where $\nu_k^{(I)} \in \mathbb{Z}$ with $\sum_{I=1}^N \nu_k^{(I)} = \ell q_k$. Here the integers ν are unique once we demand $\nu_k^{(1)} < \dots < \nu_k^{(N)} < \nu_k^{(1)} + p_k$ together with $\sum_{I=1}^N \nu_k^{(I)} = \ell q_k$.

Our discussion above can be summarized by stating that, to each connected component in the space of anyonic $SL(N, \mathbb{C})$ flat connections on M_3, we can associate an element in the finite set

$$\mathcal{R}^N_{\{(p_k, q_k)\}^n_{k=1}} := \bigsqcup_{\ell=0}^{N-1} \left\{ \begin{array}{l} \nu_k^{(I)} \in \mathbb{Z} \ , \ 1 \le I \le N \ , \ 1 \le k \le n \\ \nu_k^{(1)} < \cdots < \nu_k^{(N)} < \nu_k^{(1)} + p_k \\ \sum_{I=1}^N \nu_k^{(I)} = \ell q_k \end{array} \right\} \ . \tag{2.12}$$

On the other hand, it is a highly nontrivial question which element in $\mathcal{R}^N_{\{(p_k, q_k)\}^n_{k=1}}$ defines an anyonic flat connection.

For given N and Seifert data it is possible to set up an explicit numerical counting problem of anyonic $SL(N, \mathbb{C})$ flat connections. This will be reported in [19]. The findings of such numerical investigations indicate that every point in the set (2.12) is associated to a connected component in the space of anyonic flat connections. We then propose the following:

Conjecture 1. *The connected components of the anyonic $SL(N, \mathbb{C})$ flat connections on the Seifert manifold $M_3 = [\{(p_k, q_k)\}^n_{k=1}]$ are in 1-1 correspondence with points in the finite set $\mathcal{R}^N_{\{(p_k, q_k)\}^n_{k=1}}$ in (2.12).*

If ρ is a flat $SU(N)$ connection on M_3, the value of the classical Chern-Simons (CS) invariant $CS(\rho)$ of ρ is [20]

$$CS(\rho) = \frac{1}{2} \sum_{k=1}^n \left[p_k r_k \mathrm{Tr} X_k^2 - q_k s_k \mathrm{Tr} H^2 \right] \mathrm{mod} \ \mathbb{Z} \, , \tag{2.13}$$

where the integers r_k, s_k satisfy $p_k s_k - q_k r_k = 1$ and the traceless matrices X_k, H are given by

$$\begin{aligned} X_k &= \mathrm{diag}(a_k^{(1)}, \ldots, a_k^{(N)}) \ , \\ H &= \mathrm{diag}\left(\frac{\ell}{N}, \ldots, \frac{\ell}{N}, -\frac{(N-1)\ell}{N} \right) \ . \end{aligned} \tag{2.14}$$

Since $CS(\rho)$ depends on the eigenvalues of $\rho(h)$, $\rho(x_k)$, it is constant in each connected component of the space of flat connections. Crucially, the group we consider is the complexification $SL(N, \mathbb{C})$ and there is an important question that we will not address, whether the CS invariants on Seifert manifolds differ when the group is complexified. For $SU(2)$ and $SL(2, \mathbb{C})$ we know that the invariants agree. It would be interesting to analyze this question in general.

3 3d Topological Order and MTCs

MTC and RFC. The second important ingredient in our setup are the so-called **modular tensor categories** (MTCs), characterizing 3d bosonic topological field theories, or bosonic topological order. Mathematically, an MTC $\mathcal{C}$ is a fusion category, whose objects are topological lines, or anyons. MTCs have finitely many simple objects X_j with $j \in \mathcal{I}$ for an index set $\mathcal{I}$, so that any object is a direct sum of X_i. The rank is defined as the cardinality $|\mathcal{I}|$ of the set $\mathcal{I}$. The fusion coefficients $N_{ij}^k \in \mathbb{N}$ appear in the tensor product

$$X_i \otimes X_j = \bigoplus_k N_{ij}^k X_k \tag{3.1}$$

and satisfy various consistency conditions. We will label an MTC $\mathcal{C}$ (tt is known that the modular data (S, T) does not uniquely determine an MTC [21]) in terms of the following data:

- Rank $r = |\mathcal{I}| \equiv |\mathcal{C}|$, which is the number of simple topological lines, i.e. simple anyons.

- The spin h_X, or equivalently the twist $\theta_X = \exp(2\pi i h_X)$ of the line $X \in \mathcal{C}$, which is a root of unity. The twists form a diagonal matrix $T_{ij} = \delta_{ij}\theta_i$, $i, j = 1, \ldots, r$. We follow the convention that the spin of the identity line is 0, or equivalently $T_{11} = 1$.

- The S-matrix S_{ij} is a unitary *non-degenerate* symmetric matrix that encodes the braidings of two anyons X_i and X_j.

If we drop the requirement of having a non-degenerate S-matrix, one instead has a ribbon fusion category (RFC), also known as a pre-modular category. For an MTC to be *unitary*, one simple necessary condition is to require all quantum dimensions $d_i \equiv S_{i0}/S_{00}$ to be real numbers greater or equal to 1.

The MTC $\mathcal{C}(\mathfrak{sl}(N), p, \mathfrak{q})$. One of the most important approaches to MTCs is through the representations of quantum groups at roots of unity. In particular, the semisimple quotient of the category of finite dimensional modules of quantum group $U_{\mathfrak{q}}(\mathfrak{sl}(N))$ at roots of unity gives rise to an (not necessarily modular) RFC, denoted as $\mathcal{C}(\mathfrak{sl}(N), p, \mathfrak{q})$ where $\mathfrak{q}^2$ is a root of unity of order p [22, 23, 24, 25].

We recall that the fundamental affine Weyl alcove $\Delta_{N,p}$ of the affine Lie algebra $\mathfrak{sl}(N)$ at level $p - N$ consists of those weights with integral Dynkin labels $[\lambda^{(1)}, \ldots, \lambda^{(N-1)}]$ in the set

$$\Delta_{N,p} := \left\{ 0 \leq \lambda^{(I)} \leq p - h^{\vee}, \ I = 0, \ldots, N - 1 \right\}, \tag{3.2}$$

where $\lambda^{(0)} := (p - h^{\vee}) - \sum_{I=1}^{N-1} \lambda^{(I)}$, $h^{\vee} = N$. Each $\lambda \in \Delta_{N,p}$ has an N-ality defined by the map

$$\phi(\lambda) := \sum_{I=1}^{N-1} I \lambda^{(I)} \mod N . \tag{3.3}$$

We depict $\Delta_{N,p}$ for $N = 3$, $p = 7$ in figure 1, where the colors represent the values of $\phi(\lambda)$. Fundamental alcoves for general simply-laced $\mathfrak{g}$ will be discussed in appendix C.

After these preliminaries, let us summarize some salient features of the RFC $\mathcal{C}(\mathfrak{sl}(N), p, \mathfrak{q})$:

- Simple objects are labelled by the weights in the fundamental affine Weyl alcove $\Delta_{N,p}$ of the affine Lie algebra $\mathfrak{sl}(N)$ at level $p - N$, defined in (3.2). One can see immediately that the rank $|\mathcal{C}(\mathfrak{sl}(N), p, \mathfrak{q})|$ is independent of $\mathfrak{q}$ and can be found from the coefficient of x^{p-N} in the power series expansion of $(1 - x)^{-N}$.

- The fusion rules are given by the familiar fusion coefficients of affine representations [26].

- For any two dominant weights $\lambda, \mu \in \Delta_{N,p}$, the S-matrix element is [25]

$$S_{\lambda\mu} = \frac{\sum_{\sigma \in W} \epsilon(\sigma) \mathfrak{q}^{2\langle \sigma(\lambda+\rho), \mu+\rho \rangle}}{\sum_{\sigma \in W} \epsilon(\sigma) \mathfrak{q}^{2\langle \sigma(\rho), \rho \rangle}} , \tag{3.4}$$

where ρ is the half sum of the positive roots and $\epsilon(\sigma)$ denotes the sign of the Weyl group element $\sigma \in W$ defined by the Bruhat order.
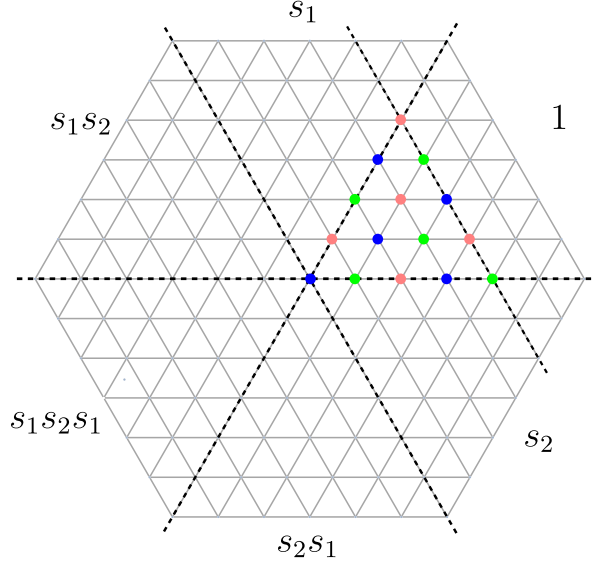
Figure 1: Fundamental affine Weyl alcove $\Delta_{3,7}$ for $\mathfrak{sl}(3)$ with $\phi(\lambda) = 0, 1, 2$ for blue, green and orange respectively. Here s_i are Weyl reflections, which label all the different alcoves. We will restrict to the fundamental one (labeled by 1).

- The twists are

$$\theta_\lambda = \mathfrak{q}^{\langle \lambda, \lambda + 2\rho \rangle} \ . \tag{3.5}$$

In addition, to fully characterize an MTC, one needs to provide F- and R-symbols subject to pentagon and hexagon identities. For $\mathfrak{g} = \mathfrak{sl}(2)$, this can be found in [27].

We observe that the S- and T-matrices contain fractional powers of $\mathfrak{q}$. This is because the inner product $\langle \lambda, \mu \rangle$ is quantized in units of $1/N$ if λ, μ are integral. In order to write unambiguous expressions for S and T, we parametrize $\mathfrak{q}$ as

$$\mathfrak{q} = \mathfrak{s}^N \ . \tag{3.6}$$

Then, $S_{\lambda\mu}$ is a ratio of polynomials in the variable $\mathfrak{s}^2$, while $\theta_\lambda = \mathfrak{s}^{M(\lambda)}$ with $M(\lambda)$ a nonnegative integer. If N is even, $M(\lambda)$ can be even or odd, depending on λ. If N is odd, $M(\lambda)$ is even for any λ. It follows that, for N odd, all entries of the S- and T-matrix are functions of $\mathfrak{s}^2$. In all cases, we require that $\mathfrak{s}^{2N}$ has order p to ensure that $\mathfrak{q}^2$ has order p.

We emphasize that the existence of S- and T-matrices does not necessarily imply that the category is modular, i.e. for a generic root of unity $\mathfrak{q}$, $\mathcal{C}(\mathfrak{sl}(N), p, \mathfrak{q})$ is not necessarily modular. Bruguieres [28] shows that $\mathcal{C}(\mathfrak{sl}(N), p, \mathfrak{q})$ for $\mathfrak{q} = e^{z\pi i/p}$ is modular if and only if $\gcd(z, Np) = 1$. In particular, when

$$\mathfrak{q}(k, N) = \exp\left(\frac{\pi i}{k+N}\right) :$$
$$\mathcal{C}(\mathfrak{sl}(N), k+N, \mathfrak{q}(k, N)) = \operatorname{Rep}\left(\widehat{\mathfrak{sl}}(N)_k\right) \tag{3.7}$$

is known to be an MTC given by the representation category $\operatorname{Rep}(\widehat{\mathfrak{sl}}(N)_k)$ of the affine Lie algebra, or equivalently the category of Wilson lines in the $\mathfrak{sl}(N)_k$ Chern Simons theory.

Graded Deligne Product. The final building block is to define a graded Deligne product $\boxtimes_G$ for RFCs as in [15]. Suppose A is an abelian group and $\mathcal{C} = \oplus_{g \in A} \mathcal{C}_g$ and $\mathcal{D} = \oplus_{g \in A} \mathcal{D}_g$ are two A-graded RFC's. The A-graded product $\boxtimes_A$ of $\mathcal{C}$ and $\mathcal{D}$ is defined to be $\mathcal{C} \boxtimes_A \mathcal{D} = \oplus_{g \in A} \mathcal{C}_g \boxtimes \mathcal{D}_g$, which is a subcategory of the usual Deligne product $\mathcal{C} \boxtimes \mathcal{D}$.

$\mathcal{C} \boxtimes_A \mathcal{D}$ is again a A-graded ribbon fusion category, where all of the category data are defined by a straightforward element-wise multiplication. If we write simple objects in a A-graded RFC $\mathcal{C}$ as $c_{g,i} \in \mathcal{C}_g$, $i = 1, \ldots, |\mathcal{C}_g|$, $\mathcal{C} \boxtimes_A \mathcal{D}$ has the data

- Simple objects: $c_{g,i} \boxtimes d_{g,j}$, for all $c_{g,i} \in \mathcal{C}_g$, $d_{g,j} \in \mathcal{D}_g$ and $g \in A$.

- Rank: $r = \sum_g |\mathcal{C}_g||\mathcal{D}_g|$.

- S-matrix:

$$S_{c_{g,i} \boxtimes d_{g,j}, c_{h,m} \boxtimes d_{h,n}} \equiv S_{c_{g,i}, c_{h,m}} S_{d_{g,j}, d_{h,n}} \,. \tag{3.8}$$

- Twists:

$$\theta_{c_{g,i} \boxtimes d_{h,j}} \equiv \theta_{c_{g,i}} \theta_{d_{h,j}} \,. \tag{3.9}$$

F- and R-symbols are defined similarly by a element-wise multiplication of the ones of $\mathcal{C}$ and $\mathcal{D}$ respectively.

More generally, the A-graded Deligne product of multiple A-graded RFCs can be defined in a similar manner

$$\mathcal{C}^1 \boxtimes_A \mathcal{C}^2 \boxtimes_A \cdots \boxtimes_A \mathcal{C}^n \equiv \oplus_{g \in A} \mathcal{C}_g^1 \boxtimes \mathcal{C}_g^2 \boxtimes \cdots \boxtimes \mathcal{C}_g^n \,, \tag{3.10}$$

where the ordering does not matter since the A-graded product is clearly commutative and associative.

However, it has been observed in [15] that A-graded Deligne product of A-graded MTCs may not be an MTC, namely its S-matrix may not be necessarily non-degenerate. It is therefore an interesting question when A-graded Deligne products of MTCs define MTCs. On the other hand, it is possible that the graded Deligne product of non-modular ribbon fusion categories (RFCs) turns out to be modular.

4 MTCs from Seifert Manifolds

We are now in a position to discuss the proposed map from Seifert manifolds to RFCs. We will present our two main conjectures, discuss the relation with anyonic flat connections, and present some examples.

Defining $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ and $\mathsf{MTC}(\mathfrak{sl}(N), M_3)$. Let us fix the Seifert manifold

$$M_3 = [\{(p_k, q_k)\}_{k=1}^n] \,, \tag{4.1}$$

and the complex Lie algebra $\mathfrak{g} = \mathfrak{sl}(N)$. Assume

$$p_k \geq N \,, \quad \gcd(N, q_k) = 1 \,, \quad k = 1, \ldots n \,. \tag{4.2}$$

To the pair $(\mathfrak{sl}(N), M_3)$ we associate the RFC denoted $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ and defined as the $\mathbb{Z}_N$-graded product of n factors,

$$\mathsf{RFC}(\mathfrak{sl}(N), M_3) := \mathcal{C}^1 \boxtimes_{\mathbb{Z}_N} \cdots \boxtimes_{\mathbb{Z}_N} \mathcal{C}^n \ , \tag{4.3}$$

where the factor $\mathcal{C}^k$ associated to the kth exceptional fiber of M_3 is the $\mathbb{Z}_N$-graded RFC

$$\mathcal{C}^k = \mathcal{C}(\mathfrak{sl}(N), p_k, \mathfrak{q}_k) \ , \tag{4.4}$$

with $\mathfrak{q}_k$ specified as

$$\mathfrak{q}_k = \mathfrak{s}_k^N \ , \quad \begin{cases} \mathfrak{s}_k = e^{2\pi i \frac{x_k}{2 p_k N}} & \text{for } N \text{ even,} \\ \mathfrak{s}_k^2 = e^{2\pi i \frac{y_k}{p_k N}} & \text{for } N \text{ odd,} \end{cases} \tag{4.5}$$

where x_k, y_k are integers satisfying

$$x_k = q_k^{-1} \bmod 2 p_k N \ , \quad y_k = q_k^{-1} \bmod p_k N \ . \tag{4.6}$$

In performing the graded Deligne product (4.3) we equip $\mathcal{C}^k$ with the $\mathbb{Z}_N$ grading defined by

$$g_k(\lambda) = q_k^{-1} \phi(\lambda) \bmod N \ . \tag{4.7}$$

Thus, simple objects in $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ are labeled by tuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ in the set

$$\mathcal{S}_{\{(p_k, q_k)\}_{k=1}^n}^N := \bigsqcup_{g=0}^{N-1} \left\{ \begin{matrix} \boldsymbol{\lambda} \in \Delta_{N, p_1} \times \cdots \times \Delta_{N, p_n} \text{ s.t.} \\ g_k(\lambda_k) = g \bmod N \end{matrix} \right\} \ , \tag{4.8}$$

with $g_k(\lambda)$ as defined in (4.7).

We observe that $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ may or may not be modular, depending on N and the Seifert data. When it is modular, we also use the notation $\mathsf{MTC}(\mathfrak{sl}(N), M_3)$.

Since the building blocks $\mathcal{C}^k$ are not necessarily unitary, the resulting MTCs are also not necessarily unitary. We will see that at least at low rank this proposal yields all unitary and non-unitary modular data.

Main conjectures. We formulate two main conjectures on $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$. The first conjecture asserts that, if we start from distinct Seifert data that yield diffeomorphic 3-manifolds, the associated RFCs have the same modular data. More precisely:

Conjecture 2. *Fix $N \geq 2$. Let $\{(p_k, q_k)_{k=1}^n\}$, $\{(p_k, q_k')_{k=1}^n\}$ be Seifert data with $p_k \geq N$, $\gcd(N, q_k) = 1$, $\gcd(N, q_k') = 1$, $k = 1, \ldots, n$, such that there exist integers $\{m_k\}_{k=1}^n$ with*

$$q_k' = q_k + p_k m_k \ , \quad k = 1, \ldots, n, \quad \sum_{k=1}^n m_k = 0 \ . \tag{4.9}$$

Write

$$M_3 = [\{(p_k, q_k)\}_{k=1}^n] \ , \quad M_3' = [\{(p_k, q_k')\}_{k=1}^n] \ . \tag{4.10}$$

Then $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ and $\mathsf{RFC}(\mathfrak{sl}(N), M_3')$ have the same modular data.

We have tested this conjecture in several examples, by verifying that $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ and $\mathsf{RFC}(\mathfrak{sl}(N), M_3')$ have the same T- and S-matrices up to a permutation of simple object labels.

Note that a generic Seifert 'gauge' transformation (4.9) does not preserve the assumption $\gcd(N, q_i) = 1$, while keeping $b = 0$. For more details see appendix A. The assumption $\gcd(N, q_i) = 1$ and $\gcd(N, q_i') = 1$ will be relaxed later.

Our second main conjecture relates the topological properties of M_3 to the non-degeneracy of the S-matrix of $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$. More precisely:

Conjecture 3. *Fix $N \geq 2$. Let $\{(p_k, q_k)_{k=1}^n\}$ be Seifert data with $p_k \geq N$, $\gcd(N, q_k) = 1$. Assume that at most one of the p_k is equal to N. Set $M_3 = [\{(p_k, q_k)\}_{k=1}^n]$. Then $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ is modular if and only if $H_1(M_3; \mathbb{Z}_N) = 0$.*

In fact for $n = 3$ (three-fibers) and $N = 2$ ($SL(2, \mathbb{C})$) this is proven in [15]. We have tested this conjecture is numerous examples including all the rank $r = 2, 3, 4, 5$ that we tabulate in the appendices, going beyond $n = 3$ and $N = 2$.

If we relax the hypothesis that at most one of the p_k is equal to N, the conjecture is no longer valid. For example, if $N = 2$, $M_3 = [(2, 1), (2, 1), (5, 1)]$ then $H_1(M_3; \mathbb{Z}_2) \cong \mathbb{Z}_2$ but $\mathsf{RFC}(\mathfrak{sl}(2), M_3)$ is modular. On the other hand, for H_1 non-trivial and only one $p_k = N$, we have confirmed in numerous examples that the models are not modular.

Defining $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ for N prime relaxing $\gcd(N, q_k) = 1$**.** When N is prime, we can extend the definition of $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ by allowing Seifert data in which we do not necessarily have $\gcd(N, q_k) = 1$ for every $k = 1, \ldots, n$. The form of $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ is the same as in (4.3), with building blocks as in (4.4). For each building block (4.4) we have to specify: (i) the value of $\mathfrak{q}_k$; (ii) the grading function $g_k(\lambda)$, which might differ from (4.7). We now describe these assignments in the cases $N = 2$ and N an odd prime. The case $N = 2$ has been first discussed in [15].

If $N = 2$, the value of $\mathfrak{q}_k$ is $\mathfrak{q}_k = \mathfrak{s}_k^2$ with

$$
\mathfrak{s}_k = \begin{cases} e^{2\pi i \frac{x_k}{4p_k}} & \text{if } q_k \text{ odd} , \\ e^{2\pi i \frac{\xi_k}{4p_k}} & \text{if } q_k \text{ even} , \end{cases}
\tag{4.11}
$$

where the integers x_k and ξ_k are determined mod $4p_k$ by

$$
\begin{aligned}
x_k &= q_k^{-1} && \mod 4p_k , \\
\xi_k &= (p_k + q_k)^{-2} q_k && \mod 4p_k .
\end{aligned}
\tag{4.12}
$$

For $N = 2$ there is only one choice of $\mathbb{Z}_2$ grading on $\mathcal{C}^k$: $g(\lambda) = \phi(\lambda) \mod 2$.

Let us now discuss the case of N an odd prime. The value of $\mathfrak{q}_k$ is $\mathfrak{q}_k = \mathfrak{s}_k^N$ with

$$
\mathfrak{s}_k^2 = \begin{cases} e^{2\pi i \frac{y_k}{p_k N}} & \text{if } N \nmid q_k , \\ e^{2\pi i \frac{\eta_k}{p_k N}} & \text{if } N \mid q_k , \end{cases}
\tag{4.13}
$$

where y_k is as in (4.6) while η_k is defined by

$$
\eta_k = (p_k + q_k)^{-2} q_k \mod p_k N .
\tag{4.14}
$$

Moreover, the grading function is

$$g_k(\lambda) = \begin{cases} q_k^{-1}\phi(\lambda) \bmod N \, , & \text{if } N \nmid q_k \, , \\ p_k^{-1}\phi(\lambda) \bmod N \, , & \text{if } N \mid q_k \, . \end{cases} \tag{4.15}$$

We notice that, since N is prime and $\gcd(p_k, q_k) = 1$, if $N|q_k$ then $\gcd(N, p_k) = 1$.

Match with CS-invariants of anyonic flat connections. We set $M_3 = [\{(p_k, q_k)\}_{k=1}^n]$ throughout this section. Our definition of $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ implies the following properties, which relate this RFC to flat connections on M_3:

- The simple objects of $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ are in 1-to-1 correspondence with (connected components of) anyonic flat $SL(N, \mathbb{C})$ connections on M_3.

- Simple objects are partitoned into sectors of definite $\mathbb{Z}_N$ charge g, and (connected components of) anyonic flat connections are partitioned into sectors with definite ℓ, see (2.8). The correspondence between simple objects and anyonic flat connections preserves these decompositions, with possible values of g in bijection with possible values of ℓ.

- If the simple object $\boldsymbol{\lambda}$ corresponds to the (connected component of the) anyonic flat $SL(N, \mathbb{C})$ connection ρ, the twist of $\boldsymbol{\lambda}$ matches the CS invariant of ρ,

$$\theta_{\boldsymbol{\lambda}} = K e^{-2\pi i \mathrm{CS}(\rho)} \, , \tag{4.16}$$

where the constant K depends of N and the Seifert data, but not on $\boldsymbol{\lambda}$.

A proof of these properties is given in appendix B. We discuss here some key aspects of the derivation.

To start with, we consider a generic $N \geq 2$ and impose the restriction $\gcd(N, q_k) = 1$ on all Seifert fibers. We want to define a 1-to-1 map between the set $\mathcal{R}^N_{\{(p_k, q_k)\}_{k=1}^n}$ in (2.12) (anyonic flat connections) and the set $\mathcal{S}^N_{\{(p_k, q_k)\}_{k=1}^n}$ in (4.8) (simple objects in the RFC). We map $\nu_k^{(I)}$ to the following $\lambda_k^{(I)}$,

$$\lambda_k^{(I)} = \nu_k^{(I+1)} - \nu_k^{(I)} - 1 \, , \quad I = 1, \ldots, N-1 \, . \tag{4.17}$$

One verifies that this map defines a 1-to-1 correspondence between $\mathcal{R}^N_{\{(p_k, q_k)\}_{k=1}^n}$ and $\mathcal{S}^N_{\{(p_k, q_k)\}_{k=1}^n}$. Here, we need to use the prescription for the grading function (4.7) as well as the assumption $\gcd(N, q_k) = 1$. See appendix B for more details.

As far as property (4.16) is concerned, it is established by direct computation using (2.8), (2.9), (2.11), (2.13), (3.5), (3.9), and (4.17), working separately in each Seifert fiber, as explained in appendix B. Here we simply point out that our assignent (4.5) for $\mathfrak{q}_k$ in terms of N, p_k, q_k is precisely engineered in such a way that (4.16) holds.

Next, let us now consider the case of N prime and relax the requirement $\gcd(N, q_k) = 1$. Crucially, the map (4.17) from $\nu_k^{(I)}$ to $\lambda_k^{(I)}$ needs to be modified in this case (cfr. [15] for $N = 2$). As a preliminary, we define

$$\nu_k^{(0)} := \nu_k^{(N)} - p_k \, . \tag{4.18}$$

Figure 2: Fundamental Weyl alcove for $\mathfrak{sl}(2)$ with $p = 3, 3, 3, 4$ respectively. $\mathbb{Z}_2$-grading is depicted by the color blue (even) and green (odd).

Now we generalize (4.17) by setting

$$\lambda_k^{(I)} = \begin{cases} \nu_k^{(I+1)} - \nu_k^{(I)} - 1 & \text{if } \gcd(N, q_k) = 1 \ , \\ \nu_k^{(\sigma[\ell](I)+1)} - \nu_k^{(\sigma[\ell](I))} - 1 & \text{if } \gcd(N, q_k) \neq 1 \ . \end{cases} \tag{4.19}$$

In the second line, $\sigma[\ell]$ is the cyclic permutation of $(0, 1, 2, \ldots, N - 1)$ that sends 0 to ℓ, i.e. $\sigma[\ell](I) = I + \ell \mod N$. In appendix B we demonstrate that this is the correct definition in order to have flat connections with definite ℓ to be mapped to weights λ_k with a definite $\mathbb{Z}_N$ charge. For N an odd prime, it is essential here to use the prescription (4.15) for the $\mathbb{Z}_N$ grading function.

The property (4.16) also holds in the case of N prime with $\gcd(N, q_k) = 1$ relaxed. As before, it is established working fiberwise. For those fibers with $\gcd(N, q_k) = 1$ there is nothing new to prove. For those fibers with $\gcd(N, q_k) \neq 1$, one uses (2.8), (2.9), (2.11), (2.13), (3.5), and the new map (4.19). We emphasise that the assignments (4.11), (4.13) for $\mathfrak{q}_k$ in the $N = 2$ and N odd prime cases are precisely engineered in such a way that (4.16) holds. We refer the reader to appendix B for more details.

Observation on fibers with $p_k = N$. We have the

Proposition 1. *Fix $N \geq 2$ and consider the Seifert manifolds*

$$M_3 = [(p_1, q_1), \ldots, (p_t, q_t), (N, q_{t+1}), \ldots, (N, q_n)] \ ,$$
$$\widetilde{M_3} = [(p_1, q_1), \ldots, (p_t, q_t), (N, 1)] \ , \tag{4.20}$$

where $\gcd(N, q_k) = 1$ for $k = 1, \ldots, n$, and $1 \leq t < n$, with $p_k > N$ for $k = 1, \ldots, t$. Then $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ and $\mathsf{RFC}(\mathfrak{sl}(N), \widetilde{M_3})$ have the same modular data.

Indeed, if the kth fiber has $p_k = N$, then Δ_{N,p_k} consists of a single element $\lambda_k = 0$. No matter the value of q_k, this element is assigned grading $g = 0$. Also, this fiber contributes $\theta_{\lambda=0} = 1$ and $S_{\lambda=0, \mu=0} = 1$ irrespective of the value of $\mathfrak{q}_k$. In the graded $\boxtimes$ construction we effectively project onto the $g = 0$ sector in each fiber. We see that, as soon as we have one fiber $(p_k = N, q_k)$, we can add any number of other fibers of the same kind, we arbitrary q's, without changing the T- and S-matrices.

In light of the above proposition, there is no loss in generality in restricting to Seifert data in which at most one of the p_k equals N, and setting $q_k = 1$ for that fiber.

Example. To illustrate this we now work out a concrete example, in preparation for the comprehensive list of models up to rank 5 in the next section. Consider the Seifert manifold with data

$$M_3 = [\{(3, 1), (3, 1), (3, 1), (4, 1)\}] \ , \tag{4.21}$$

and consider $G_{\mathbb{C}} = SL(2,\mathbb{C})$. Here $\mathcal{Z}_G = \mathbb{Z}_2$. We claim that this constructs the MTC of rank 3, given by (using the nomenclature of [7], this is model 6. at rank $r = 3$ in appendix E of that paper)

$$3^{16,639}_{\frac{15}{2},4.} \, . \tag{4.22}$$

This model has spins and S-matrix given by

$$h = \left\{ 0, \frac{1}{2}, \frac{15}{16} \right\} \, ,$$

$$S = \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \, . \tag{4.23}$$

We obtain this MTC as the graded Deligne product $\mathcal{C}^1 \boxtimes_{\mathbb{Z}_2} \mathcal{C}^2 \boxtimes_{\mathbb{Z}_2} \mathcal{C}^3 \boxtimes_{\mathbb{Z}_2} \mathcal{C}^4$, where

$$\mathcal{C}^1 \cong \mathcal{C}^2 \cong \mathcal{C}^3 \cong \mathcal{C}(\mathfrak{sl}(2), p = 3, \mathfrak{q}) \, , \quad \mathfrak{q} = \mathfrak{s}^2 \, , \quad \mathfrak{s} = e^{2\pi i \frac{1}{12}} \, ,$$

$$\mathcal{C}^4 \cong \mathcal{C}(\mathfrak{sl}(2), p = 4, \mathfrak{q}), \quad \mathfrak{q} = \mathfrak{s}^2, \quad \mathfrak{s} = e^{2\pi i \frac{1}{16}} \, , \tag{4.24}$$

where we have made use of (4.5). In figure 2 we depict the fundamental Weyl alcoves associated to $\mathcal{C}^1$, $\mathcal{C}^2$, $\mathcal{C}^3$, $\mathcal{C}^4$ together with their $\mathbb{Z}_2$ gradings. They have simple objects labelled by $\lambda_k^{(1)} \in \{0, 1\}$ with $k = 1, 2, 3$ and $\lambda_4^{(1)} \in \{0, 1, 2\}$, which are the weights inside the Weyl alcove. The spins and S-matrix of $\mathcal{C}(\mathfrak{sl}(2), p, \mathfrak{q})$ where $\mathfrak{q} = \mathfrak{s}^2$ as specified in (4.5), are given by the specialization of (3.4) to be

$$S_{\lambda_1, \lambda_2} = \frac{\mathfrak{q}^{(\lambda_1 + 1)(\lambda_2 + 1)} - \mathfrak{q}^{-(\lambda_1 + 1)(\lambda_2 + 1)}}{\mathfrak{q} - \mathfrak{q}^{-1}} \, , \tag{4.25}$$

and (recall $\theta_\lambda = e^{2\pi i h_\lambda}$)

$$\theta_\lambda = \mathfrak{q}^{\lambda(\lambda + 2)/2} = \mathfrak{s}^{\lambda(\lambda + 2)} \, . \tag{4.26}$$

Therefore for $\mathcal{C}^1$ they are given by

$$\mathcal{C}^1 \ : \quad h = \left\{ 0, \frac{1}{4} \right\} \, , \quad S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \, . \tag{4.27}$$

Those of $\mathcal{C}^2$, $\mathcal{C}^3$ are the same, while those of $\mathcal{C}^4$ read

$$\mathcal{C}^4 \ : \quad h = \left\{ 0, \frac{3}{16}, \frac{1}{2} \right\} \, , \quad S = \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \, . \tag{4.28}$$

The resulting MTC $\mathcal{C}^1 \boxtimes_{\mathbb{Z}_2} \cdots \boxtimes_{\mathbb{Z}_2} \mathcal{C}^4$ is again $\mathbb{Z}_2$-graded, with simple objects labeled by a tuple $\boldsymbol{\lambda} = (\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}, \lambda_4^{(1)})$. More precisely:

$$\begin{array}{lll} \text{sector with } \mathbb{Z}_2\text{-charge } g = 0: & & (0,0,0,0), \ (0,0,0,2) \, , \\ \text{sector with } \mathbb{Z}_2\text{-charge } g = 1: & & (1,1,1,1) \, . \end{array} \tag{4.29}$$

The spins are obtained by adding the spins in each of these sectors (mod 1), and the S-matrices by applying the graded Deligne product.

Note that $H_1(M_3, \mathbb{Z}_2) = 0$ for (4.21), and the model is indeed modular. Note that the same rank 3 model in [14] is realized with a Seifert manifold that has non-trivial $H_1(M_3, \mathbb{Z}_2)$, requiring them to gauge the $\mathbb{Z}_2$ 1-form symmetry, and rendering the model modular thereby.

5   All low rank MTCs from MTC(M_3, G)

Given the correspondence between Seifert manifolds and MTCs described above, it is natural to ask whether this provides a comprehensive list of MTCs. We do not have an answer to this question, but very strong evidence confirmed by low rank studies up to rank 5, that this may provide a complete classification.

In appendix D we provide a list of all MTCs from rank $r = 1, \cdots, 5$ and a realization in terms of Seifert data of an M_3 data and a choice of G_{ADE}. Note that this is not unique, but the point here is that each MTC has such a realization. We compare this with the recent classification in [7], and provide realizations for **unitary and non-unitary** MTCs. The comparison is done to appendix E of [7]. The tables 1, 2, 3, 4 provide the list of models for rank including $r = 5$. Note that many models will be reducible, as in be a Deligne product of lower rank MTCs, this is indicated by "(factors)". We provide a realization for all **prime models**, in particular.

We also explored rank 6 and in that case we find almost all prime models, and these are listed in appendix E. It would be interesting to extend the scan, and find realizations for all rank 6 models, and in fact complete the comparison to [7] until rank 11. Note that this would also provide clarity as to whether the models that do not have a CS-type realization (denoted "exotic models") may have a realization in terms of Seifert data.

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A More on Seifert Manifolds

We denote the Seifert 3-manifold M_3 associated to the Seifert data b, g, $\{(p_k, q_k)\}_{k=1}^n$ as

$$M_3 = [b; g; \{(p_k, q_k)\}_{k=1}^n] \; . \tag{A.1}$$

Orientation reversal is implemented on Seifert data as $b \mapsto -b$, $q_k \mapsto -q_k$. If two Seifert data b, g, $\{(p_k, q_k)\}_{k=1}^n$ and b', g, $\{(p_k, q_k')\}_{k=1}^n$ are related as

$$b' = b - \sum_{k=1}^n m_k \; , \; q_k' = q_k + p_k m_k \; , \; k = 1, \ldots, n \; , \tag{A.2}$$

with m_k integers, the associated 3-manifolds are diffeomorphic (via an orientation-preserving map).

Note that the simplifying assumption $\gcd(N, q_i) = 1$ is not preserved under the Seifert 'gauge' transformation (A.2), while keeping $b = 0$. Given a three manifold with Seifert data $M_3 = [0; 0; \{(p_k, q_k)\}_{k=1}^n]$, if certain q_i is not coprime to N, we can often find a 'gauge' where $\gcd(N, q_i) = 1$ for all i by means of (A.2). There are however cases where such 'gauge' does not exist. The simplest example is $(\{(3, 1), (3, 1), (3, 2)\})$ for $N = 2$ and it is easy to see that it is impossible to choose all three q_i' to be odd by using (A.2).

Let's start with $N = 2$ and any $n \geq 3$. Suppose two of the fibers have the Seifert data $\{(\text{even}, \text{odd}), (\text{even}, \text{odd})\}$, the parity will always be preserved under (A.2). Similarly for $\{(\text{odd}, \text{odd}), (\text{odd}, \text{even})\}$. On the other hand, $\{(\text{odd}, \text{even}), (\text{odd}, \text{even})\}$ can be changed to $\{(\text{odd}, \text{odd}), (\text{odd}, \text{odd})\}$ and $\{(\text{even}, \text{odd}), (\text{odd}, \text{even})\}$ can be changed to $\{(\text{odd}, \text{odd}), (\text{even}, \text{odd})\}$. Based on this observation, for any $n \geq 3$ fibers, we can always reduce it to the following $n + 2$ types.

We can have

$$\{(\text{odd}, \text{odd}), \ldots, (\text{odd}, \text{odd}), (\text{even}, \text{odd}), \ldots, (\text{even}, \text{odd})\} \; , \tag{A.3}$$

where the number of $(\text{even}, \text{odd})$ fiber can be any number from 0 to n. In addition, we have

$$\{(\text{odd}, \text{odd}), \ldots, (\text{odd}, \text{odd}), (\text{odd}, \text{even})\} \; , \tag{A.4}$$

which is only case that has $\gcd(2, q_i) \neq 0$. Furthermore it has trivial $H_1(M_3, \mathbb{Z}_2)$ if and only if

$$KL = \left(\prod p_i \right) \left| \sum_i \frac{q_i}{p_i} \right| \tag{A.5}$$

is odd, namely if and only if n is even.

For $N = 3$ and any $n \geq 3$, it is easy to see that one can always find q_i' with $\gcd(N, q_i') = 1$. So we don't lose any generality by assuming $\gcd(N, q_i) = 1$ for all i.

For $N = 4$, we are interested in whether it is possible to choose all q_i' to be odd. Let r_i be any integer with remainder i divided by 4, i.e. $r_i = i \mod 4$, for $i = 0, 1, 2, 3$. If there are two fibers with the Seifert data $\{(\text{odd}, r_0), (\text{odd}, r_0)\}$, one can always go to a gauge where both q's are odd. Similarly for $\{(\text{odd}, r_0), (\text{odd}, r_2)\}$ and $\{(\text{odd}, r_2), (\text{odd}, r_2)\}$. Therefore it suffices to consider the Seifert data with at most one q_i being r_0 or r_2. In addition, any pair $\{(\text{odd}, \text{odd}), (\text{odd}, r_2)\}$ can be made into $\{(\text{odd}, \text{odd}), (\text{odd}, r_0)\}$ and any pair

$\{(\text{even}, \text{odd}), (\text{odd}, r_0)\}$ can be made into $\{(\text{even}, \text{odd}), (\text{odd}, \text{odd})\}$. Therefore the only case with $\gcd(q_i, 4) \neq 1$ that can't be removed by using (A.2) is

$$\{(\text{odd}, \text{odd}), \dots, (\text{odd}, \text{odd}), (\text{odd}, r_0)\} , \tag{A.6}$$

where again r_0 is any multiple of 4. We leave the discussion of general N to future work.

B Mapping Anyons to Anyonic Flat Connections

In this appendix we verify the 1-to-1 correspondence between simple objects of $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ and anyonic $\mathfrak{sl}(N)$ flat connections on M_3. Throughout this appendix M_3 is the Seifert manifold $M_3 = [\{(p_k, q_k)\}_{k=1}^n]$.

We distinguish two cases: (1) $N \geq 2$ and $\gcd(N, q_k) = 1$ for all $k = 1, \dots, n$; (2) N prime but without any restriction on $\gcd(N, q_k)$. The most general case without any restrictions on N and $\gcd(N, q_k)$ will be left for future work.

B.1 The case where $\gcd(N, q_k) = 1$ for all Seifert fibers

Recall that a connected component in the moduli space of anyonic flat connections is determined by the data $(\ell, \nu_k^{(I)})$, $I = 1, \dots, N$, satisfying $\sum_{I=1}^N \nu_k^{(I)} = \ell q_k$, see (2.8), (2.9), (2.11), (2.12).

Our first task is to exhibit a 1-to-1 correspondence between simple objects in $\mathsf{RFC}(\mathfrak{sl}(N), M_3)$ and anyonic $SL(N, \mathbb{C})$ flat connections on M_3. To this end, we use the map from flat connection data $\nu_k^{(I)}$ to weight vectors $\lambda_k^{(I)}$ in (4.17), repeated here for convenience,

$$\lambda_k^{(I)} = \nu_k^{(I+1)} - \nu_k^{(I)} - 1 , \quad I = 1, \dots, N - 1 . \tag{B.1}$$

First, it is easy to verify that $\lambda_k^{(I)} \in \Delta_{N, p_k}$, for each $k = 1, \dots, n$. Next, we show that the $\mathbb{Z}_N$ charge g of $\lambda_k^{(I)}$, defined as $g = g_k(\lambda_k) = q_k^{-1} \phi(\lambda_k) \bmod N$, is independent of k and determined by ℓ. Indeed, using the definition of ϕ (3.3), the map (B.1), and the property $\sum_{I=1}^N \nu_k^{(I)} = \ell q_k$, one computes

$$g = q_k^{-1} \phi(\lambda_k) = q_k^{-1} \left(- q_k \ell - \frac{N(N-1)}{2} \right) = -\ell - q_k^{-1} \frac{N(N-1)}{2} \quad \bmod N . \tag{B.2}$$

To proceed, we prove the following relation,

$$q_k^{-1} \frac{N(N-1)}{2} = \frac{N(N-1)}{2} \quad \bmod N . \tag{B.3}$$

If N is odd, $(N-1)/2$ is integer and both sides vanish mod N. If N is even, q_k is odd because $\gcd(N, q_k) = 1$. Hence q_k^{-1} is odd and $q_k^{-1} - 1$ is even. As a result $(q_k^{-1} - 1)\frac{N(N-1)}{2} = 0 \bmod N$, giving (B.3). In summary, we have

$$g = -\ell - \frac{N(N-1)}{2} \quad \bmod N , \tag{B.4}$$

independent of k and determined by ℓ, as claimed. Using (B.4) it is straightforward to verify that (B.1) defines a 1-to-1 map from $\mathcal{R}_{\{(p_k, q_k)\}_{k=1}^n}^N$ to $\mathcal{S}_{\{(p_k, q_k)\}_{k=1}^n}^N$.

Next, we want to establish the match between CS invariants of anyonic flat connections and twists in the RFC. More precisely, let us start with an anyonic flat connection with data $(\ell, \nu_k^{(I)})$. Its CS invariant is given as a sum of terms, one for each Seifert fiber, see (2.13),

$$\text{CS} = \sum_{k=1}^{n} \text{CS}_k \mod \mathbb{Z} , \qquad \text{CS}_k := \frac{1}{2} p_k r_k \text{Tr} X_k^2 - \frac{1}{2} q_k s_k \text{Tr} H^2 \mod \mathbb{Z} . \tag{B.5}$$

Using (2.8), (2.9), (2.11), (2.13), and (2.14), we compute

$$\text{CS}_k = \frac{r_k}{2p_k} \sum_{I=1}^{N} (\nu_k^{(I)})^2 - \frac{\ell^2 q_k^2}{2N p_k} + \frac{(N-1)\ell^2}{N} \mod \mathbb{Z} , \tag{B.6}$$

where we have made use of $\sum_{I=1}^{N} \nu_k^{(I)} = \ell q_k$. We want to compare $e^{2\pi i \text{CS}_k}$ with the the quantity

$$\theta_{\lambda_k} = \mathfrak{q}_k^{\langle \lambda_k, \lambda_k + 2\rho \rangle} , \tag{B.7}$$

see (3.5), which is the contribution of the kth factor in the graded $\boxtimes$ presentation of the RFC to the total twist of the simple object labeled by $\boldsymbol{\lambda} = (\lambda_k)_{k=1}^{n}$. We write $\mathfrak{q}_k = e^{2\pi i \kappa_k/(2p_k N)}$ and therefore

$$\theta_k = \exp 2\pi i \left[\frac{\kappa_k}{2p_k N} \sum_{I,J=1}^{N-1} B_{IJ}^{(N)} \lambda_k^{(I)} \left(\lambda_k^{(J)} + 2 \right) \right] . \tag{B.8}$$

We have introduced the symmetric $(N-1) \times (N-1)$ matrix $B_{IJ}^{(N)}$ that represents N times the interior product $\langle \cdot, \cdot \rangle$ on the weight lattice. More explicitly,

$$B_{IJ}^{(N)} = N(A^{(N)-1})_{IJ} = N \min(I, J) - IJ , \tag{B.9}$$

where $A^{(N)}$ is the Cartan matrix of $\mathfrak{sl}(N)$. We notice that the entries of B are integers. We are now in a position to plug (B.1) into (B.8), and compare with (B.6). Our goal is to obtain a relation of the form

$$\theta_{\lambda_k} = K_k e^{-2\pi i \text{CS}_k} , \tag{B.10}$$

where K_k is a non-zero constant, depending on N and the Seifert data p_k, q_k, but independent of ℓ and $\nu_k^{(I)}$. We verify that (B.10) can indeed be achieved (independently for each k) precisely by choosing κ_k in such a way that $\mathfrak{q}_k$ is given as in (4.5) in terms of N and the Seifert data p_k, q_k.

Having established (B.10), by taking a product over $k = 1, \ldots, n$ we arrive at the desired relation between the twist and the CS invariant,

$$\theta_{\boldsymbol{\lambda}} = K e^{-2\pi i \text{CS}} , \tag{B.11}$$

where $K = \prod_{k=1}^{n} K_k$.

B.2   The case N prime with some Seifert fibers with $\gcd(N, q_k) \neq 1$

We now restrict N to be prime, but we allow Seifert fibers for which $\gcd(N, q_k) \neq 1$.

Let us focus on a fiber with label k for which $\gcd(N, q_k) \neq 1$. Since N is prime, it follows that $N | q_k$. Firstly, let us argue why we must modify the map (4.17) from eigenvalue data

$(\ell, \nu_k^{(I)})$ to weights $\lambda_k^{(I)}$. If we were to make use of (4.17), a direct computation shows that the N-ality function $\phi(\lambda)$ in (3.3) can be evaluated in terms of the eigenvalue data as

$$\text{using (4.17):} \quad \phi(\lambda_k) = -\ell q_k - \frac{N(N-1)}{2} + N\nu_k^{(N)}$$
$$= \begin{cases} -\ell q_k - 1 & \mod 2 \quad \text{if } N = 2 , \\ -\ell q_k & \mod N \quad \text{if } N \text{ is an odd prime} . \end{cases} \tag{B.12}$$

Since $N|q_k$, we see that the function $\phi(\lambda)$ assigns the same value to *all* flat connections, irrespectively of their ℓ. On the other hand, we know that the possible $\mathbb{Z}_N$ gradings on $\mathcal{C}(\mathfrak{sl}(N), p, \mathfrak{q})$ are all proportional to the function $\phi(\lambda)$, see (4.7). It is thus impossible to define a grading on $\mathcal{C}(\mathfrak{sl}(N), p, \mathfrak{q})$ such that weights λ with different gradings $g \in \mathbb{Z}_N$ correspond to anyonic flat connections with different values of ℓ.

Building on [15], we overcome this obstacle by modifying the map (4.17) from eigenvalue data $(\ell, \nu_k^{(I)})$ to weights $\lambda_k^{(I)}$. We proceed as follows. Define

$$\nu_k^{(0)} = \nu_k^{(N)} - p_k . \tag{B.13}$$

Let $\sigma[\ell]$ denote the cyclic permutation of $(0, 1, 2, \ldots, N-1)$ that assigns the value ℓ to 0, more explicitly,

$$\sigma[\ell] = \begin{pmatrix} 0 & 1 & 2 & 3 & \ldots & N-1 \\ \ell & \ell+1 & \ell+2 & \ell+3 & \ldots & \ell+N-1 \end{pmatrix} , \tag{B.14}$$

where all integers in the second row are understood to be reduced mod N to lie in the range $\{0, \ldots, N-1\}$. With this notation, the proposed modification of (4.17) is as follows,

$$\lambda_k^{(I)} = \begin{cases} \nu_k^{(I+1)} - \nu_k^{(I)} - 1 & \text{if } \gcd(N, q_k) = 1 , \\ \nu_k^{(\sigma[\ell](I)+1)} - \nu_k^{(\sigma[\ell](I))} - 1 & \text{if } \gcd(N, q_k) \neq 1 . \end{cases} \tag{B.15}$$

In other words, when $\gcd(N, q_k) \neq 1$ we use a different cyclic permutation for each value of ℓ. This definition is motivated by the fact that it ensures

$$\text{if } N = 2: \qquad \phi(\lambda_k) = \begin{cases} -\ell q_k - 1 & \mod 2 , \quad \text{if } q_k \text{ is odd} , \\ -\ell p_k - 1 & \mod 2 , \quad \text{if } q_k \text{ is even} , \end{cases}$$
$$\text{hence} \quad \phi(\lambda_k) = \ell + 1 \mod 2 . \tag{B.16}$$

$$\text{if } N \text{ is an odd prime:} \quad \phi(\lambda_k) = \begin{cases} -\ell q_k & \mod N , \quad \text{if } \gcd(N, q_k) = 1 , \\ -\ell p_k & \mod N , \quad \text{if } \gcd(N, q_k) \neq 1 . \end{cases}$$

For $N = 2$, we see that we have achieved our goal of having a 1-to-1 correspondence between λ's with definite ϕ mod 2, and anyonic flat connections with definite ℓ. In the case of N odd prime, we can also achieve this goal by choosing the $\mathbb{Z}_N$ grading function as in (4.15),

$$\text{if } N \text{ is an odd prime:} \quad g_k(\lambda) = \begin{cases} q_k^{-1}\phi(\lambda) \mod N , \quad \text{if } \gcd(N, q_k) = 1 , \\ p_k^{-1}\phi(\lambda) \mod N , \quad \text{if } \gcd(N, q_k) \neq 1 , \end{cases} \tag{B.17}$$
$$\text{hence} \quad g_k(\lambda_k) = -\ell \mod N .$$

We see that our definitions are engineered in such a way that

$$g = g_k(\lambda_k) = -\ell - \frac{N(N-1)}{2} \mod N , \tag{B.18}$$

for every prime N and for every fiber k (including those fibers with $\gcd(N, q_k) \neq 1$). But this is exactly the same result as in the case of general N with the constraint $\gcd(N, q_k) = 1$, see (B.4). By reasoning as in that previous case, one shows that the map (B.15) defines a 1-to-1 correspondence between anyonic flat connections and simple objects of the RFC.

Finally, we have to show that the twist of simple objects in the RFC matches with the CS invariant of anyonic flat connections. We proceed as in the previous subsection. It is convenient to analyze each Seifert fiber separately: we have to prove (B.10) for each fiber. For those fibers with $\gcd(N, q_k) = 1$ we can recycle the proof of the previous subsection verbatim. For those fibers with $\gcd(N, q_k) \neq 1$, the proof has to be revised, but the strategy is the same. On the CS side, we again make use of (2.8), (2.9), (2.11), (2.13), and (2.14). On the twist side, the only modification compared to the previous case is how we express $\lambda_k^{(I)}$ in terms of $\nu_k^{(I)}$: we have to make use of (B.15). A direct computation shows that we can indeed achieve (B.10) provided we fix $\mathfrak{q}_k$ according to the prescriptions (4.11) and (4.13) in the main text, for $N = 2$ and N an odd prime, respectively.

C   Fundamental Weyl Alcove for Simply-laced $\mathfrak{g}$

Let $\mathfrak{g}$ be an ADE Lie algebra with $\mathrm{rank}(\mathfrak{g}) = r$. An element λ in the weight lattice is expanded onto fundamental weights $\{\omega_I\}_{I=1}^r$ as $\lambda = \sum_{I=1}^r \lambda^{(I)} \omega_I$, where $\lambda^{(I)}$ are the Dynkin labels of λ.

The fundamental affine Weyl alcove $\Delta_{\mathfrak{g},p}$ is defined as

$$\Delta_{\mathfrak{g},p} := \{0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \,, \alpha \in \Phi^+\} \,, \tag{C.1}$$

where Φ^+ is the set of positive roots, $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ (which equals α for ADE algebras), and ρ is the Weyl vector $\rho = \sum_{I=1}^r \omega_I$. We can equivalently write

$$\Delta_{\mathfrak{g},p} = \{\langle \lambda + \rho, \alpha_I^\vee \rangle > 0 \,, \ \ \langle \lambda + \rho, \theta \rangle < p\} \,, \tag{C.2}$$

where α_I are the simple roots of $\mathfrak{g}$ and θ is the highest root,

$$\theta = \sum_{i=1}^r a_I \alpha_I = \sum_{i=1}^r a_I^\vee \alpha_I^\vee \,, \tag{C.3}$$

where the expansion coefficients a_I, $a_I^\vee$ are the marks, comarks of $\mathfrak{g}$ ($a_I = a_I^\vee$ for simply laced algebras). In terms of the Dynkin labels $\lambda^{(I)}$ we can write more explicitly

$$\Delta_{\mathfrak{g},p} = \left\{\lambda^{(I)} \geq 0 \,, \ \ \sum_{I=1}^r a_I^\vee \lambda^{(I)} \leq p - h^\vee\right\} \,, \tag{C.4}$$

where we have introduced the dual Coxeter number $h^\vee$ of $\mathfrak{g}$,

$$h^\vee = 1 + \sum_{I=1}^r a_i^\vee \,. \tag{C.5}$$

From (C.4), we see that $\Delta_{\mathfrak{g},p}$ is precisely the affine Weyl alcove, i.e. the set of integrable highest weight, for the affine algebra of $\mathfrak{g}$ at level $p - h^\vee$. Below we collect the comarks and

dual Coxeter numbers for all ADE Lie algebras (see e.g. [29]),

$$
\begin{aligned}
A_r : &\quad \left(a_I^\vee\right)_{I=1}^r = (1, 1, 1, \ldots, 1) \,, &\quad h^\vee = r + 1 \\
D_r : &\quad \left(a_I^\vee\right)_{I=1}^r = (1, 2^{r-3}, 1, 1) \,, &\quad h^\vee = 2r - 2 \\
E_6 : &\quad \left(a_I^\vee\right)_{I=1}^r = (1, 2, 3, 2, 1, 2) \,, &\quad h^\vee = 12 \\
E_7 : &\quad \left(a_I^\vee\right)_{I=1}^r = (2, 3, 4, 3, 2, 1, 2) \,, &\quad h^\vee = 18 \\
E_8 : &\quad \left(a_I^\vee\right)_{I=1}^r = (2, 3, 4, 5, 6, 4, 2, 3) \,, &\quad h^\vee = 30 \,.
\end{aligned}
\tag{C.6}
$$

where an exponent n indicates that an entry is repeated n times. We are using the standard ordering of simple roots, corresponding to the standard form of the Cartan matrices for ADE Lie algebras (as in [29]). For $\mathfrak{g} = A_{N-1}$, (C.4) reduces to (3.2).

D   Tables of MTCs from M_3 for Rank $r \leq 5$

We now identify the low rank MTCs, including non-unitary ones, with Seifert manifolds and a choice of $SL(N, \mathbb{C})$. To label the low rank MTCs we use the notation in [7]. In particular, each modular data in the list is labeled by $r_{c,D^2}^{\mathrm{ord}(T),\mathrm{fp}}$, where r is the rank of the MTC, i.e. the number of the simple objects, c is the chiral central charge, $D^2 \equiv \sum_i d_i^2$ is the total quantum dimension. $\mathrm{ord}(T)$ is the order of the T-matrix, i.e. the smallest positive integer n such that $T^n = \mathrm{id}$. fp is the "finger print", defined by the first three digits of $\left|\sum_i \left(s_i^2 - \frac{1}{4}\right) d_i\right|$.

 We list the rank and spins, or equivalently the T-matrix, but for all the models we tabulate we also checked agreement of the S-matrix with [7]. Labels are as in [7] and # indicates the model number within the set of MTCs of that rank in [7].

Summary of results:

1. Rank 1: trivially realized.

2. Rank 2: all models in [7].

3. Rank 3: all models in [7] including pseudo-unitary ones are realized, using $N = 2$, $n = 3$ and $n = 4$, and for the final two models (number 1. and 2.) we need $N = 3$ and $n = 3$.

4. Rank 4: all prime models in [7].

5. Rank 5: all models in [7].

6. Rank 6: See appendix E.

 Let us make a note on pseudo-unitary models. A pseudo-unitary model has anyons of negative quantum dimension $d_i = S_{0i}/S_{00}$. It is related to a unitary model, as follows. The T- and S-matrices of the unitary model are given in terms of those of the pseudo-unitary model by

$$
e^{2\pi i \theta_j'} = \mathrm{sign}(d_j) e^{2\pi i \theta_j} \,, \qquad S_{ij}' = \mathrm{sign}(d_i)\, \mathrm{sign}(d_j) S_{ij} \,.
\tag{D.1}
$$

N	n	$M_3(\{(p_i, q_i)\})$	Rank	Spins	Label	#
2	3	$\{(2,1),(3,1),(5,1)\}$	2	$\{0, \frac{2}{5}\}$	$2^{5,395}_{\frac{14}{5},3.618}$	3.(prime)
2	3	$\{(2,1),(3,1),(5,3)\}$	2	$\{0, \frac{4}{5}\}$	$2^{5,491}_{\frac{38}{5},1.381}$	6.(prime)
2	3	$\{(2,1),(3,1),(5,7)\}$	2	$\{0, \frac{1}{5}\}$	$2^{5,120}_{\frac{2}{5},1.381}$	5.(prime)
2	3	$\{(2,1),(3,1),(5,9)\}$	2	$\{0, \frac{3}{5}\}$	$2^{5,720}_{\frac{26}{5},3.618}$	4.(prime)
2	3	$\{(3,1),(3,1),(3,1)\}$	2	$\{0, \frac{3}{4}\}$	$2^{4,625}_{7,2.}$	2.(prime) $\sim$ 7.(prime, pseudo)
2	3	$\{(3,1),(3,1),(3,7)\}$	2	$\{0, \frac{1}{4}\}$	$2^{4,437}_{1,2.}$	1.(prime) $\sim$ 8.(prime, pseudo)
2	4	$\{(3,1),(3,1),(3,1),(3,2)\}$	2	$\{0, \frac{1}{4}\}$	$2^{4,625}_{1,2.}$	7.(prime, pseudo) $\sim$ 2.(prime)
2	4	$\{(3,1),(3,1),(3,1),(3,4)\}$	2	$\{0, \frac{3}{4}\}$	$2^{4,562}_{7,2.}$	8.(prime, pseudo) $\sim$ 1.(prime)

Table 1: MTCs from 3-manifolds: Rank 2. Note models 7. and 8. are pseudo-unitary and related to 2. and 1., respectively.

E   Rank 6

In this final appendix we explore the rank 6 models. The vast majority of these are factored into rank 2 and 3 models. However we find most of the prime models, bar 10 models (out of a total of 192 rank 6 models). Clearly one obvious place to search for these models is to generalize the dictionary between MTCs and M_3, $\mathfrak{sl}(N, \mathbb{C})$ data to include cases where N and q are not co-prime.

N	n	$M_3(\{(p_i,q_i)\})$	Rank	Spins	Label	#
2	3	$\{(2,1),(3,1),(7,1)\}$	3	$\left\{0,\frac{2}{7},\frac{6}{7}\right\}$	$3^{7,245}_{\frac{8}{7},9.295}$	20.(prime)
2	3	$\{(2,1),(3,1),(7,3)\}$	3	$\left\{0,\frac{2}{7},\frac{3}{7}\right\}$	$3^{7,768}_{\frac{12}{7},2.862}$	21.(prime, not-pseudo)
2	3	$\{(2,1),(3,1),(7,5)\}$	3	$\left\{0,\frac{4}{7},\frac{6}{7}\right\}$	$3^{7,604}_{\frac{52}{7},1.841}$	24.(prime, not-pseudo)
2	3	$\{(2,1),(3,1),(7,9)\}$	3	$\left\{0,\frac{1}{7},\frac{3}{7}\right\}$	$3^{7,953}_{\frac{4}{7},1.841}$	23.(prime, not-pseudo)
2	3	$\{(2,1),(3,1),(7,11)\}$	3	$\left\{0,\frac{4}{7},\frac{5}{7}\right\}$	$3^{7,531}_{\frac{44}{7},2.862}$	22.(prime, not-pseudo)
2	3	$\{(2,1),(3,1),(7,13)\}$	3	$\left\{0,\frac{1}{7},\frac{5}{7}\right\}$	$3^{7,790}_{\frac{48}{7},9.295}$	19.(prime)
2	3	$\{(3,1),(3,1),(4,1)\}$	3	$\left\{0,\frac{1}{2},\frac{11}{16}\right\}$	$3^{16,648}_{\frac{11}{2},4.}$	13.(prime)
2	3	$\{(3,1),(3,1),(4,3)\}$	3	$\left\{0,\frac{1}{2},\frac{9}{16}\right\}$	$3^{16,343}_{\frac{9}{2},4.}$	17.(pseudo) $\sim$ 3.(prime)
2	3	$\{(3,1),(3,1),(4,5)\}$	3	$\left\{0,\frac{1}{2},\frac{15}{16}\right\}$	$3^{16,113}_{\frac{15}{2},4.}$	18.(pseudo) $\sim$ 4.(prime)
2	3	$\{(3,1),(3,1),(4,7)\}$	3	$\left\{0,\frac{1}{2},\frac{13}{16}\right\}$	$3^{16,330}_{\frac{13}{2},4.}$	14.(prime)
2	3	$\{(3,1),(3,1),(4,9)\}$	3	$\left\{0,\frac{3}{16},\frac{1}{2}\right\}$	$3^{16,553}_{\frac{3}{2},4.}$	11.(prime)
2	3	$\{(3,1),(3,1),(4,11)\}$	3	$\left\{0,\frac{1}{16},\frac{1}{2}\right\}$	$3^{16,980}_{\frac{1}{2},4.}$	15.(pseudo) $\sim$ 5.(prime)
2	3	$\{(3,1),(3,1),(4,13)\}$	3	$\left\{0,\frac{7}{16},\frac{1}{2}\right\}$	$3^{16,167}_{\frac{7}{2},4.}$	16.(pseudo) $\sim$ 6.(prime)
2	3	$\{(3,1),(3,1),(4,15)\}$	3	$\left\{0,\frac{5}{16},\frac{1}{2}\right\}$	$3^{16,465}_{\frac{5}{2},4.}$	12.(prime)
2	4	$\{(3,1),(3,1),(3,1),(4,1)\}$	3	$\left\{0,\frac{1}{2},\frac{15}{16}\right\}$	$3^{16,639}_{\frac{15}{2},4.}$	6.(prime)
2	4	$\{(3,1),(3,1),(3,1),(4,3)\}$	3	$\left\{0,\frac{1}{2},\frac{13}{16}\right\}$	$3^{16,830}_{\frac{13}{2},4.}$	10.(pseudo) $\sim$ 12.(prime)
2	4	$\{(3,1),(3,1),(3,1),(4,5)\}$	3	$\left\{0,\frac{3}{16},\frac{1}{2}\right\}$	$3^{16,538}_{\frac{3}{2},4.}$	7.(pseudo) $\sim$ 13.(prime)
2	4	$\{(3,1),(3,1),(3,1),(4,7)\}$	3	$\left\{0,\frac{1}{16},\frac{1}{2}\right\}$	$3^{16,598}_{\frac{1}{2},4.}$	3.(prime)
2	4	$\{(3,1),(3,1),(3,1),(4,9)\}$	3	$\left\{0,\frac{7}{16},\frac{1}{2}\right\}$	$3^{16,332}_{\frac{7}{2},4.}$	4.(prime)
2	4	$\{(3,1),(3,1),(3,1),(4,11)\}$	3	$\left\{0,\frac{5}{16},\frac{1}{2}\right\}$	$3^{16,345}_{\frac{5}{2},4.}$	8.(pseudo) $\sim$ 14.(prime)
2	4	$\{(3,1),(3,1),(3,1),(4,13)\}$	3	$\left\{0,\frac{1}{2},\frac{11}{16}\right\}$	$3^{16,564}_{\frac{11}{2},4.}$	9.(pseudo) $\sim$ 11.(prime)
2	4	$\{(3,1),(3,1),(3,1),(4,15)\}$	3	$\left\{0,\frac{1}{2},\frac{9}{16}\right\}$	$3^{16,156}_{\frac{9}{2},4.}$	5.(prime)
3	3	$\{(4,1),(4,1),(4,5)\}$	3	$\left\{0,\frac{1}{3},\frac{1}{3}\right\}$	$3^{3,527}_{2,3}$	1.(prime)
3	3	$\{(4,1),(4,5),(4,5)\}$	3	$\left\{0,\frac{2}{3},\frac{2}{3}\right\}$	$3^{3,138}_{6,3}$	2.(prime)

Table 2: MTCs from 3-manifolds: Rank 3. The table shows the complete set of rank 3 models from $SL(2,\mathbb{C})$ and $n=3,4$-fibers: Rank 3 and $SL(3,\mathbb{C})$ with $n=3$ fibers.

N	n	$M_3(\{(p_i,q_i)\})$	Rank	Spins	Label	#
2	3	$\{(2,1),(3,1),(9,1)\}$	4	$\left\{0,\frac{2}{9},\frac{1}{3},\frac{2}{3}\right\}$	$4^{9,459}_{\frac{10}{3},19.23}$	24.(prime)
2	3	$\{(2,1),(3,1),(9,5)\}$	4	$\left\{0,\frac{1}{3},\frac{4}{9},\frac{2}{3}\right\}$	$4^{9,544}_{\frac{14}{3},5.445}$	26.(prime)
2	3	$\{(2,1),(3,1),(9,7)\}$	4	$\left\{0,\frac{1}{3},\frac{2}{3},\frac{8}{9}\right\}$	$4^{9,549}_{\frac{22}{3},2.319}$	29.(prime)
2	3	$\{(2,1),(3,1),(9,11)\}$	4	$\left\{0,\frac{1}{9},\frac{1}{3},\frac{2}{3}\right\}$	$4^{9,199}_{\frac{2}{3},2.319}$	28.(prime)
2	3	$\{(2,1),(3,1),(9,13)\}$	4	$\left\{0,\frac{1}{3},\frac{5}{9},\frac{2}{3}\right\}$	$4^{9,616}_{\frac{10}{3},5.445}$	27.(prime)
2	3	$\{(2,1),(3,1),(9,17)\}$	4	$\left\{0,\frac{1}{3},\frac{2}{3},\frac{7}{9}\right\}$	$4^{9,614}_{\frac{14}{3},19.23}$	25.(prime)
2	3	$\{(2,1),(5,1),(5,1)\}$	4	$\left\{0,\frac{2}{5},\frac{2}{5},\frac{4}{5}\right\}$	$4^{5,479}_{\frac{28}{5},13.09}$	18.(factors)
2	3	$\{(2,1),(5,1),(5,3)\}$	4	$\left\{0,\frac{1}{5},\frac{2}{5},\frac{4}{5}\right\}$	$4^{5,426}_{\frac{12}{5},5.}$	39.(factors)
2	3	$\{(2,1),(5,1),(5,7)\}$	4	$\left\{0,\frac{1}{5},\frac{2}{5},\frac{3}{5}\right\}$	$4^{5,375}_{\frac{16}{5},5.}$	38.(factors)
2	3	$\{(2,1),(5,1),(5,9)\}$	4	$\left\{0,0,\frac{2}{5},\frac{3}{5}\right\}$	$4^{5,872}_{0,13.09}$	22.(factors)
2	3	$\{(2,1),(5,3),(5,3)\}$	4	$\left\{0,\frac{3}{5},\frac{4}{5},\frac{4}{5}\right\}$	$4^{5,690}_{\frac{36}{5},1.909}$	21.(factors)
2	3	$\{(2,1),(5,3),(5,7)\}$	4	$\left\{0,0,\frac{1}{5},\frac{4}{5}\right\}$	$4^{5,456}_{0,1.909}$	23.(factors)
2	3	$\{(2,1),(5,3),(5,9)\}$	4	$\left\{0,\frac{2}{5},\frac{3}{5},\frac{4}{5}\right\}$	$4^{5,223}_{\frac{24}{5},5.}$	41.(factors)
2	3	$\{(2,1),(5,7),(5,7)\}$	4	$\left\{0,\frac{1}{5},\frac{1}{5},\frac{2}{5}\right\}$	$4^{5,248}_{\frac{4}{5},1.909}$	20.(factors)
2	3	$\{(2,1),(5,7),(5,9)\}$	4	$\left\{0,\frac{1}{5},\frac{3}{5},\frac{4}{5}\right\}$	$4^{5,332}_{\frac{28}{5},5.}$	40.(factors)
2	3	$\{(2,1),(5,9),(5,9)\}$	4	$\left\{0,\frac{1}{5},\frac{3}{5},\frac{3}{5}\right\}$	$4^{5,443}_{\frac{12}{5},13.09}$	19.(factors)
2	3	$\{(3,1),(3,1),(5,1)\}$	4	$\left\{0,\frac{1}{4},\frac{2}{5},\frac{13}{20}\right\}$	$4^{20,304}_{\frac{19}{5},7.236}$	10.(factors) $\sim$ 31.(pseudo)
2	3	$\{(3,1),(3,1),(5,3)\}$	4	$\left\{0,\frac{11}{20},\frac{3}{4},\frac{4}{5}\right\}$	$4^{20,210}_{\frac{33}{5},2.763}$	17.(factors)
2	3	$\{(3,1),(3,1),(5,7)\}$	4	$\left\{0,\frac{1}{5},\frac{3}{4},\frac{19}{20}\right\}$	$4^{20,210}_{\frac{37}{5},2.763}$	16.(factors)
2	3	$\{(3,1),(3,1),(5,9)\}$	4	$\left\{0,\frac{1}{4},\frac{3}{5},\frac{17}{20}\right\}$	$4^{20,505}_{\frac{31}{5},7.236}$	11.(factors) $\sim$ 33.(pseudo)
2	3	$\{(3,1),(3,1),(5,11)\}$	4	$\left\{0,\frac{3}{20},\frac{2}{5},\frac{3}{4}\right\}$	$4^{20,451}_{\frac{9}{5},7.236}$	12.(factors) $\sim$ 30.(pseudo)
2	3	$\{(3,1),(3,1),(5,13)\}$	4	$\left\{0,\frac{1}{20},\frac{1}{4},\frac{4}{5}\right\}$	$4^{20,525}_{\frac{3}{5},2.763}$	15.(factors)
2	3	$\{(3,1),(3,1),(5,17)\}$	4	$\left\{0,\frac{1}{5},\frac{1}{4},\frac{9}{20}\right\}$	$4^{20,278}_{\frac{7}{5},2.763}$	14.(factors)
2	3	$\{(3,1),(3,1),(5,19)\}$	4	$\left\{0,\frac{7}{20},\frac{3}{5},\frac{3}{4}\right\}$	$4^{20,341}_{\frac{21}{5},7.236}$	13.(factors) $\sim$ 32.(pseudo)
2	4	$\{(3,1),(3,1),(3,1),(5,18)\}$	4	$\left\{0,\frac{1}{5},\frac{4}{5},\frac{1}{4}\right\}$	$4^{20,456}_{\frac{3}{5},2.763}$	34.(factors) $= 2.6 \boxtimes 2.7$
2	4	$\{(3,1),(3,1),(3,1),(5,2)\}$	4	$\left\{0,\frac{9}{20},\frac{1}{5},\frac{1}{4}\right\}$	$4^{20,379}_{\frac{7}{5},2.763}$	35.(factors) $= 2.5 \boxtimes 2.7$
2	4	$\{(3,1),(3,1),(3,1),(5,8)\}$	4	$\left\{0,\frac{11}{20},\frac{4}{5},\frac{3}{4}\right\}$	$4^{20,771}_{\frac{33}{5},2.763}$	36.(factors) $= 2.6 \boxtimes 2.8$
2	4	$\{(3,1),(3,1),(3,1),(5,12)\}$	4	$\left\{0,\frac{19}{20},\frac{1}{5},\frac{3}{4}\right\}$	$4^{20,294}_{\frac{37}{5},2.763}$	37.(factors) $= 2.5 \boxtimes 2.8$
4	3	$\{(5,1),(5,1),(5,1)\}$	4	$\left\{0,\frac{1}{8},\frac{1}{8},\frac{1}{2}\right\}$	$4^{8,718}_{1,4.}$	6.(prime) $\sim$ 44.(pseudo)
4	3	$\{(5,1),(5,1),(5,3)\}$	4	$\left\{0,\frac{1}{2},\frac{7}{8},\frac{7}{8}\right\}$	$4^{8,781}_{7,4.}$	9.(prime) $\sim$ 43.(pseudo)
4	3	$\{(5,1),(5,1),(5,7)\}$	4	$\left\{0,\frac{3}{8},\frac{3}{8},\frac{1}{2}\right\}$	$4^{8,468}_{3,4.}$	7.(prime) $\sim$ 45.(pseudo)
4	3	$\{(5,1),(5,1),(5,13)\}$	4	$\left\{0,\frac{1}{2},\frac{5}{8},\frac{5}{8}\right\}$	$4^{8,312}_{5,4.}$	8.(prime) $\sim$ 42.(pseudo)
Spin(16)	3	$\{(15,1),(15,1),(15,1)\}$	4	$\left\{0,0,0,\frac{1}{2}\right\}$	$4^{2,750}_{0,4.}$	1.(prime, toric code)
Spin(8)	3	$\{(7,1),(7,1),(7,1)\}$	4	$\left\{0,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\}$	$4^{2,250}_{4,4.}$	2.(prime)
			4	$\left\{0,0,\frac{1}{4},\frac{3}{4}\right\}$	$4^{4,375}_{0,4.}$	3.(factors) $= 2.1 \boxtimes 2.2$
Spin(12)	3	$\{(11,1),(11,1),(11,1)\}$	4	$\left\{0,\frac{1}{2},\frac{1}{4},\frac{1}{4}\right\}$	$4^{4,625}_{2,4.}$	4.(factors) $= 2.1 \boxtimes 2.1$
Spin(20)	3	$\{(19,1),(19,1),(19,1)\}$	4	$\left\{0,\frac{1}{2},\frac{3}{4},\frac{3}{4}\right\}$	$4^{4,375}_{6,4.}$	5.(factors) $= 2.2 \boxtimes 2.2$

Table 3: MTCs from 3-manifolds: Rank 4. The table shows the complete set of rank 4 models. For all prime models we provide a realization.

N	n	$M_3(\{(p_i,q_i)\})$	Rank	Spins	Label	#
2	3	$\{(2,1),(3,1),(11,1)\}$	5	$\left\{0,\frac{1}{11},\frac{2}{11},\frac{6}{11},\frac{9}{11}\right\}$	$5^{11,640}_{\frac{16}{11},34.64}$	12.(prime)
2	3	$\{(2,1),(3,1),(11,3)\}$	5	$\left\{0,\frac{2}{11},\frac{3}{11},\frac{4}{11},\frac{8}{11}\right\}$	$5^{11,189}_{\frac{20}{11},3.323}$	17.(prime)
2	3	$\{(2,1),(3,1),(11,5)\}$	5	$\left\{0,\frac{4}{11},\frac{7}{11},\frac{9}{11},\frac{10}{11}\right\}$	$5^{11,540}_{\frac{56}{11},9.408}$	14.(prime)
2	3	$\{(2,1),(3,1),(11,7)\}$	5	$\left\{0,\frac{4}{11},\frac{5}{11},\frac{6}{11},\frac{8}{11}\right\}$	$5^{11,181}_{\frac{40}{11},4.814}$	15.(prime)
2	3	$\{(2,1),(3,1),(11,9)\}$	5	$\left\{0,\frac{1}{11},\frac{5}{11},\frac{8}{11},\frac{10}{11}\right\}$	$5^{11,611}_{\frac{80}{11},2.806}$	19.(prime)
2	3	$\{(2,1),(3,1),(11,13)\}$	5	$\left\{0,\frac{1}{11},\frac{3}{11},\frac{6}{11},\frac{10}{11}\right\}$	$5^{11,238}_{\frac{8}{11},2.806}$	20.(prime)
2	3	$\{(2,1),(3,1),(11,15)\}$	5	$\left\{0,\frac{3}{11},\frac{5}{11},\frac{6}{11},\frac{7}{11}\right\}$	$5^{11,393}_{\frac{48}{11},4.814}$	16.(prime)
2	3	$\{(2,1),(3,1),(11,17)\}$	5	$\left\{0,\frac{1}{11},\frac{2}{11},\frac{4}{11},\frac{7}{11}\right\}$	$5^{11,549}_{\frac{32}{11},9.408}$	13.(prime)
2	3	$\{(2,1),(3,1),(11,19)\}$	5	$\left\{0,\frac{3}{11},\frac{7}{11},\frac{8}{11},\frac{9}{11}\right\}$	$5^{11,508}_{\frac{68}{11},3.323}$	18.(prime)
2	3	$\{(2,1),(3,1),(11,21)\}$	5	$\left\{0,\frac{2}{11},\frac{5}{11},\frac{9}{11},\frac{10}{11}\right\}$	$5^{11,216}_{\frac{72}{11},34.64}$	11.(prime)
2	3	$\{(3,1),(3,1),(6,1)\}$	5	$\left\{0,0,\frac{1}{8},\frac{1}{3},\frac{5}{8}\right\}$	$5^{24,940}_{2,12.}$	3.(prime)
2	3	$\{(3,1),(3,1),(6,5)\}$	5	$\left\{0,0,\frac{1}{8},\frac{5}{8},\frac{2}{3}\right\}$	$5^{24,512}_{6,12.}$	6.(prime)
2	3	$\{(3,1),(3,1),(6,7)\}$	5	$\left\{0,0,\frac{1}{3},\frac{3}{8},\frac{7}{8}\right\}$	$5^{24,148}_{2,12.}$	5.(prime)
2	3	$\{(3,1),(3,1),(6,11)\}$	5	$\left\{0,0,\frac{3}{8},\frac{2}{3},\frac{7}{8}\right\}$	$5^{24,592}_{6,12.}$	4.(prime)
2	3	$\{(3,1),(3,1),(6,5)\}$	5	$\left\{0,0,\frac{1}{8},\frac{5}{8},\frac{2}{3}\right\}$	$5^{24,273}_{6,12.}$	7.(prime) same M_3 as 6.
2	3	$\{(3,1),(3,1),(6,7)\}$	5	$\left\{0,0,\frac{1}{3},\frac{3}{8},\frac{7}{8}\right\}$	$5^{24,741}_{2,12.}$	8.(prime) same M_3 as 5.
2	3	$\{(3,1),(3,1),(6,1)\}$	5	$\left\{0,0,\frac{1}{8},\frac{1}{3},\frac{5}{8}\right\}$	$5^{24,615}_{2,12.}$	9.(prime) same M_3 as 3.
2	3	$\{(3,1),(3,1),(6,11)\}$	5	$\left\{0,0,\frac{3}{8},\frac{2}{3},\frac{7}{8}\right\}$	$5^{24,814}_{6,12.}$	10.(prime) same M_3 as 4.
3	3	$\{(3,1),(4,1),(7,1)\}$	5	$\left\{0,\frac{1}{7},\frac{3}{7},\frac{6}{7},\frac{6}{7}\right\}$	$5^{7,101}_{\frac{18}{7},35.34}$	22.(prime)
3	3	$\{(3,1),(4,1),(7,2)\}$	5	$\left\{0,\frac{3}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7}\right\}$	$5^{7,125}_{\frac{30}{7},4.501}$	24.(prime)
3	3	$\{(3,1),(4,1),(7,4)\}$	5	$\left\{0,\frac{2}{7},\frac{5}{7},\frac{5}{7},\frac{6}{7}\right\}$	$5^{7,255}_{\frac{50}{7},2.155}$	26.(prime)
3	3	$\{(3,1),(4,1),(7,5)\}$	5	$\left\{0,\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{4}{7}\right\}$	$5^{7,408}_{\frac{26}{7},4.501}$	23.(prime)
3	3	$\{(3,1),(4,1),(7,10)\}$	5	$\left\{0,\frac{1}{7},\frac{2}{7},\frac{2}{7},\frac{5}{7}\right\}$	$5^{7,342}_{\frac{6}{7},2.155}$	25.(prime)
3	3	$\{(3,1),(4,1),(7,13)\}$	5	$\left\{0,\frac{1}{7},\frac{1}{7},\frac{4}{7},\frac{6}{7}\right\}$	$5^{7,386}_{\frac{38}{7},35.34}$	21.(prime)
5	3	$\{(6,1),(6,1),(6,1)\}$	5	$\left\{0,\frac{1}{5},\frac{1}{5},\frac{4}{5},\frac{4}{5}\right\}$	$5^{5,110}_{0,5.}$	1.(prime)
5	3	$\{(6,1),(6,1),(6,7)\}$	5	$\left\{0,\frac{2}{5},\frac{2}{5},\frac{3}{5},\frac{3}{5}\right\}$	$5^{5,210}_{4,5}$	2.(prime)

Table 4: MTCs from 3-manifolds: Rank 5. 7-10 have same spins as 3-6. and are obtained by identifying different spin 0 line as vacuum.

N	n	$M_3(\{(p_i,q_i)\})$	Rank	Spins	Label	#
2	3	$\{(2,1),(3,1),(13,1)\}$	6	$\{0,\frac{2}{13},\frac{4}{13},\frac{6}{13},\frac{7}{13},\frac{12}{13}\}$	$6^{13,131}_{\frac{46}{13},56.74}$	137.(prime)
2	3	$\{(2,1),(3,1),(13,3)\}$	6	$\{0,\frac{2}{13},\frac{4}{13},\frac{5}{13},\frac{10}{13},\frac{11}{13}\}$	$6^{13,107}_{\frac{102}{13},4.798}$	143.(prime)
2	3	$\{(2,1),(3,1),(13,5)\}$	6	$\{0,\frac{3}{13},\frac{4}{13},\frac{5}{13},\frac{6}{13},\frac{9}{13}\}$	$6^{13,162}_{\frac{30}{13},3.717}$	145.(prime)
2	3	$\{(2,1),(3,1),(13,7)\}$	6	$\{0,\frac{1}{13},\frac{4}{13},\frac{8}{13},\frac{11}{13},\frac{12}{13}\}$	$6^{13,300}_{\frac{14}{13},15.04}$	140.(prime)
2	3	$\{(2,1),(3,1),(13,9)\}$	6	$\{0,\frac{5}{13},\frac{6}{13},\frac{8}{13},\frac{10}{13},\frac{12}{13}\}$	$6^{13,241}_{\frac{86}{13},7.390}$	142.(prime)
2	3	$\{(2,1),(3,1),(13,11)\}$	6	$\{0,\frac{3}{13},\frac{7}{13},\frac{10}{13},\frac{11}{13},\frac{12}{13}\}$	$6^{13,764}_{\frac{94}{13},3.297}$	147.(prime)
2	3	$\{(2,1),(3,1),(13,15)\}$	6	$\{0,\frac{1}{13},\frac{2}{13},\frac{3}{13},\frac{6}{13},\frac{10}{13}\}$	$6^{13,560}_{\frac{10}{13},3.297}$	148.(prime)
2	3	$\{(2,1),(3,1),(13,17)\}$	6	$\{0,\frac{1}{13},\frac{3}{13},\frac{5}{13},\frac{7}{13},\frac{8}{13}\}$	$6^{13,415}_{\frac{18}{13},7.390}$	141.(prime)
2	3	$\{(2,1),(3,1),(13,19)\}$	6	$\{0,\frac{1}{13},\frac{2}{13},\frac{5}{13},\frac{9}{13},\frac{12}{13}\}$	$6^{13,102}_{\frac{90}{13},15.04}$	139.(prime)
2	3	$\{(2,1),(3,1),(13,21)\}$	6	$\{0,\frac{4}{13},\frac{7}{13},\frac{8}{13},\frac{9}{13},\frac{10}{13}\}$	$6^{13,481}_{\frac{74}{13},3.717}$	146.(prime)
2	3	$\{(2,1),(3,1),(13,23)\}$	6	$\{0,\frac{2}{13},\frac{3}{13},\frac{8}{13},\frac{9}{13},\frac{11}{13}\}$	$6^{13,604}_{\frac{2}{13},4.798}$	144.(prime)
2	3	$\{(2,1),(3,1),(13,25)\}$	6	$\{0,\frac{1}{13},\frac{6}{13},\frac{7}{13},\frac{9}{13},\frac{11}{13}\}$	$6^{13,502}_{\frac{58}{13},56.74}$	138.(prime)
2	3	$\{(2,1),(5,1),(7,1)\}$	6	$\{0,\frac{9}{35},\frac{2}{7},\frac{2}{5},\frac{24}{35},\frac{6}{7}\}$	$6^{35,363}_{\frac{138}{35},33.63}$	114.(factors)
2	3	$\{(2,1),(5,1),(7,3)\}$	6	$\{0,\frac{2}{7},\frac{2}{5},\frac{3}{7},\frac{24}{35},\frac{29}{35}\}$	$6^{35,274}_{\frac{158}{35},10.35}$	122.(factors)
2	3	$\{(2,1),(5,1),(7,5)\}$	6	$\{0,\frac{9}{35},\frac{2}{5},\frac{4}{7},\frac{6}{7},\frac{34}{35}\}$	$6^{35,121}_{\frac{78}{35},6.661}$	128.(factors)
2	3	$\{(2,1),(5,1),(7,9)\}$	6	$\{0,\frac{1}{7},\frac{2}{5},\frac{3}{7},\frac{19}{35},\frac{29}{35}\}$	$6^{35,153}_{\frac{118}{35},6.661}$	126.(factors)
2	3	$\{(2,1),(5,1),(7,11)\}$	6	$\{0,\frac{4}{35},\frac{2}{5},\frac{4}{7},\frac{5}{7},\frac{34}{35}\}$	$6^{35,423}_{\frac{38}{35},10.35}$	124.(factors)
2	3	$\{(2,1),(5,1),(7,13)\}$	6	$\{0,\frac{4}{35},\frac{1}{7},\frac{2}{5},\frac{19}{35},\frac{5}{7}\}$	$6^{35,955}_{\frac{58}{35},33.63}$	113.(factors)
2	3	$\{(2,1),(5,3),(7,1)\}$	6	$\{0,\frac{3}{35},\frac{2}{7},\frac{23}{35},\frac{4}{5},\frac{6}{7}\}$	$6^{35,138}_{\frac{26}{35},12.84}$	120.(factors)
2	3	$\{(2,1),(5,3),(7,3)\}$	6	$\{0,\frac{3}{35},\frac{8}{35},\frac{2}{7},\frac{3}{7},\frac{4}{5}\}$	$6^{35,437}_{\frac{46}{35},3.956}$	129.(factors)
2	3	$\{(2,1),(5,3),(7,5)\}$	6	$\{0,\frac{13}{35},\frac{4}{7},\frac{23}{35},\frac{4}{5},\frac{6}{7}\}$	$6^{35,724}_{\frac{246}{35},2.544}$	136.(factors)
2	3	$\{(2,1),(5,3),(7,9)\}$	6	$\{0,\frac{1}{7},\frac{8}{35},\frac{3}{7},\frac{4}{5},\frac{33}{35}\}$	$6^{35,346}_{\frac{6}{35},2.544}$	134.(factors)
2	3	$\{(2,1),(5,3),(7,11)\}$	6	$\{0,\frac{13}{35},\frac{18}{35},\frac{4}{7},\frac{5}{7},\frac{4}{5}\}$	$6^{35,723}_{\frac{206}{35},3.956}$	131.(factors)
2	3	$\{(2,1),(5,3),(7,13)\}$	6	$\{0,\frac{1}{7},\frac{18}{35},\frac{5}{7},\frac{4}{5},\frac{33}{35}\}$	$6^{35,105}_{\frac{226}{35},12.84}$	118.(factors)
2	3	$\{(2,1),(5,7),(7,1)\}$	6	$\{0,\frac{2}{35},\frac{1}{5},\frac{2}{7},\frac{17}{35},\frac{6}{7}\}$	$6^{35,669}_{\frac{54}{35},12.84}$	119.(factors)
2	3	$\{(2,1),(5,7),(7,3)\}$	6	$\{0,\frac{1}{5},\frac{2}{7},\frac{3}{7},\frac{17}{35},\frac{22}{35}\}$	$6^{35,769}_{\frac{74}{35},3.956}$	130.(factors)
2	3	$\{(2,1),(5,7),(7,5)\}$	6	$\{0,\frac{2}{35},\frac{1}{5},\frac{4}{7},\frac{27}{35},\frac{6}{7}\}$	$6^{35,692}_{\frac{274}{35},2.544}$	135.(factors)
2	3	$\{(2,1),(5,7),(7,9)\}$	6	$\{0,\frac{1}{7},\frac{1}{5},\frac{12}{35},\frac{3}{7},\frac{22}{35}\}$	$6^{35,711}_{\frac{34}{35},2.544}$	133.(factors)
2	3	$\{(2,1),(5,7),(7,11)\}$	6	$\{0,\frac{1}{5},\frac{4}{7},\frac{5}{7},\frac{27}{35},\frac{32}{35}\}$	$6^{35,690}_{\frac{234}{35},3.956}$	132.(factors)
2	3	$\{(2,1),(5,7),(7,13)\}$	6	$\{0,\frac{1}{7},\frac{1}{5},\frac{12}{35},\frac{5}{7},\frac{32}{35}\}$	$6^{35,615}_{\frac{254}{35},12.84}$	117.(factors)
2	3	$\{(2,1),(5,9),(7,1)\}$	6	$\{0,\frac{2}{7},\frac{16}{35},\frac{3}{5},\frac{6}{7},\frac{31}{35}\}$	$6^{35,224}_{\frac{222}{35},33.63}$	116.(factors)
2	3	$\{(2,1),(5,9),(7,3)\}$	6	$\{0,\frac{1}{35},\frac{2}{7},\frac{3}{7},\frac{3}{5},\frac{31}{35}\}$	$6^{35,120}_{\frac{242}{35},10.35}$	121.(factors)
2	3	$\{(2,1),(5,9),(7,5)\}$	6	$\{0,\frac{6}{35},\frac{16}{35},\frac{4}{7},\frac{3}{5},\frac{6}{7}\}$	$6^{35,532}_{\frac{162}{35},6.661}$	127.(factors)
2	3	$\{(2,1),(5,9),(7,9)\}$	6	$\{0,\frac{1}{35},\frac{1}{7},\frac{3}{7},\frac{3}{5},\frac{26}{35}\}$	$6^{35,488}_{\frac{202}{35},6.661}$	125.(factors)
2	3	$\{(2,1),(5,9),(7,11)\}$	6	$\{0,\frac{6}{35},\frac{11}{35},\frac{4}{7},\frac{3}{5},\frac{5}{7}\}$	$6^{35,246}_{\frac{122}{35},10.35}$	123.(factors)
2	3	$\{(2,1),(5,9),(7,13)\}$	6	$\{0,\frac{1}{7},\frac{11}{35},\frac{3}{5},\frac{5}{7},\frac{26}{35}\}$	$6^{35,429}_{\frac{142}{35},33.63}$	115.(factors)
2	3	$\{(3,1),(3,1),(7,1)\}$	6	$\{0,\frac{1}{28},\frac{2}{7},\frac{17}{28},\frac{3}{4},\frac{6}{7}\}$	$6^{28,212}_{\frac{1}{7},18.59}$	96. $*$ (factors)
2	3	$\{(3,1),(3,1),(7,3)\}$	6	$\{0,\frac{1}{4},\frac{2}{7},\frac{3}{7},\frac{15}{28},\frac{19}{28}\}$	$6^{28,193}_{\frac{19}{7},5.725}$	98. $*$ (factors)
2	3	$\{(3,1),(3,1),(7,5)\}$	6	$\{0,\frac{9}{28},\frac{4}{7},\frac{17}{28},\frac{3}{4},\frac{6}{7}\}$	$6^{28,452}_{\frac{45}{7},3.682}$	103. $*$ (factors)
2	3	$\{(3,1),(3,1),(7,9)\}$	6	$\{0,\frac{1}{7},\frac{5}{28},\frac{3}{7},\frac{3}{4},\frac{25}{28}\}$	$6^{28,318}_{\frac{53}{7},3.682}$	101. $*$ (factors)
2	3	$\{(3,1),(3,1),(7,11)\}$	6	$\{0,\frac{1}{4},\frac{4}{7},\frac{5}{7},\frac{23}{28},\frac{27}{28}\}$	$6^{28,133}_{\frac{51}{7},5.725}$	100. $*$ (factors)
2	3	$\{(3,1),(3,1),(7,13)\}$	6	$\{0,\frac{1}{7},\frac{13}{28},\frac{5}{7},\frac{3}{4},\frac{25}{28}\}$	$6^{28,114}_{\frac{41}{7},18.59}$	95. $*$ (factors)

Table 5: MTCs from 3-manifolds: Rank 6.

N	n	$M_3(\{(p_i,q_i)\})$	Rank	Spins	Label	#
2	3	$\{(3,1),(3,1),(7,15)\}$	6	$\{0,\frac{3}{28},\frac{1}{4},\frac{2}{7},\frac{15}{28},\frac{6}{7}\}$	$6^{28,289}_{\frac{15}{7},18.59}$	94. $*$ (factors)
2	3	$\{(3,1),(3,1),(7,17)\}$	6	$\{0,\frac{1}{28},\frac{5}{28},\frac{2}{7},\frac{3}{7},\frac{3}{4}\}$	$6^{28,424}_{\frac{5}{7},5.725}$	97. $*$ (factors)
2	3	$\{(3,1),(3,1),(7,19)\}$	6	$\{0,\frac{3}{28},\frac{1}{4},\frac{4}{7},\frac{23}{28},\frac{6}{7}\}$	$6^{28,411}_{\frac{3}{7},3.682}$	104. $*$ (factors)
2	3	$\{(3,1),(3,1),(7,23)\}$	6	$\{0,\frac{1}{7},\frac{1}{4},\frac{11}{28},\frac{3}{7},\frac{19}{28}\}$	$6^{28,112}_{\frac{11}{7},3.682}$	102. $*$ (factors)
2	3	$\{(3,1),(3,1),(7,25)\}$	6	$\{0,\frac{9}{28},\frac{13}{28},\frac{4}{7},\frac{5}{7},\frac{3}{4}\}$	$6^{28,257}_{\frac{37}{7},5.725}$	99. $*$ (factors)
2	3	$\{(3,1),(3,1),(7,27)\}$	6	$\{0,\frac{1}{7},\frac{1}{4},\frac{11}{28},\frac{5}{7},\frac{27}{28}\}$	$6^{28,108}_{\frac{55}{7},18.59}$	93. $*$ (factors)
2	3	$\{(3,1),(4,1),(5,1)\}$	6	$\{0,\frac{3}{16},\frac{2}{5},\frac{1}{2},\frac{47}{80},\frac{9}{10}\}$	$6^{80,424}_{\frac{43}{10},14.47}$	61. $*$ (factors)
2	3	$\{(3,1),(4,1),(5,3)\}$	6	$\{0,\frac{3}{10},\frac{39}{80},\frac{1}{2},\frac{11}{16},\frac{4}{5}\}$	$6^{80,664}_{\frac{51}{10},5.527}$	86. $*$ (factors)
2	3	$\{(3,1),(4,1),(5,7)\}$	6	$\{0,\frac{1}{5},\frac{1}{2},\frac{11}{16},\frac{7}{10},\frac{71}{80}\}$	$6^{80,423}_{\frac{59}{10},5.527}$	85. $*$ (factors)
2	3	$\{(3,1),(4,1),(5,9)\}$	6	$\{0,\frac{1}{10},\frac{3}{16},\frac{1}{2},\frac{3}{5},\frac{63}{80}\}$	$6^{80,828}_{\frac{67}{10},14.47}$	62. $*$ (factors)
2	3	$\{(3,1),(4,1),(5,11)\}$	6	$\{0,\frac{7}{80},\frac{2}{5},\frac{1}{2},\frac{11}{16},\frac{9}{10}\}$	$6^{80,270}_{\frac{3}{10},14.47}$	65. $*$ (factors)
2	3	$\{(3,1),(4,1),(5,13)\}$	6	$\{0,\frac{3}{16},\frac{3}{10},\frac{1}{2},\frac{4}{5},\frac{79}{80}\}$	$6^{80,132}_{\frac{11}{10},5.527}$	82. $*$ (factors)
2	3	$\{(3,1),(4,1),(5,17)\}$	6	$\{0,\frac{3}{16},\frac{1}{5},\frac{31}{80},\frac{1}{2},\frac{7}{10}\}$	$6^{80,485}_{\frac{19}{10},5.527}$	81. $*$ (factors)
2	3	$\{(3,1),(4,1),(5,19)\}$	6	$\{0,\frac{1}{10},\frac{23}{80},\frac{1}{2},\frac{3}{5},\frac{11}{16}\}$	$6^{80,528}_{\frac{27}{10},14.47}$	66. $*$ (factors)
2	3	$\{(3,1),(4,3),(5,1)\}$	6	$\{0,\frac{1}{16},\frac{2}{5},\frac{37}{80},\frac{1}{2},\frac{9}{10}\}$	$6^{80,941}_{\frac{33}{10},14.47}$	69. $*$ (factors)
2	3	$\{(3,1),(4,3),(5,3)\}$	6	$\{0,\frac{3}{10},\frac{29}{80},\frac{1}{2},\frac{9}{16},\frac{4}{5}\}$	$6^{80,589}_{\frac{41}{10},5.527}$	80. $*$ (factors)
2	3	$\{(3,1),(4,3),(5,7)\}$	6	$\{0,\frac{1}{5},\frac{1}{2},\frac{9}{16},\frac{7}{10},\frac{61}{80}\}$	$6^{80,727}_{\frac{49}{10},5.527}$	91. $*$ (factors)
2	3	$\{(3,1),(4,3),(5,9)\}$	6	$\{0,\frac{1}{16},\frac{1}{10},\frac{1}{2},\frac{3}{5},\frac{53}{80}\}$	$6^{80,544}_{\frac{57}{10},14.47}$	73. $*$ (factors)
2	3	$\{(3,1),(4,3),(5,11)\}$	6	$\{0,\frac{2}{5},\frac{1}{2},\frac{9}{16},\frac{9}{10},\frac{77}{80}\}$	$6^{80,113}_{\frac{73}{10},14.47}$	71. $*$ (factors)
2	3	$\{(3,1),(4,3),(5,13)\}$	6	$\{0,\frac{1}{16},\frac{3}{10},\frac{1}{2},\frac{4}{5},\frac{69}{80}\}$	$6^{80,387}_{\frac{1}{10},5.527}$	92. $*$ (factors)
2	3	$\{(3,1),(4,3),(5,17)\}$	6	$\{0,\frac{1}{16},\frac{1}{5},\frac{21}{80},\frac{1}{2},\frac{7}{10}\}$	$6^{80,787}_{\frac{9}{10},5.527}$	79. $*$ (factors)
2	3	$\{(3,1),(4,3),(5,19)\}$	6	$\{0,\frac{1}{10},\frac{13}{80},\frac{1}{2},\frac{9}{16},\frac{3}{5}\}$	$6^{80,426}_{\frac{17}{10},14.47}$	95. $*$ (factors)
2	3	$\{(3,1),(4,5),(5,1)\}$	6	$\{0,\frac{2}{5},\frac{7}{16},\frac{1}{2},\frac{67}{80},\frac{9}{10}\}$	$6^{80,439}_{\frac{63}{10},14.47}$	70. $*$ (factors)
2	3	$\{(3,1),(4,5),(5,3)\}$	6	$\{0,\frac{3}{10},\frac{1}{2},\frac{59}{80},\frac{4}{5},\frac{15}{16}\}$	$6^{80,102}_{\frac{71}{10},5.527}$	90. $*$ (factors)
2	3	$\{(3,1),(4,5),(5,7)\}$	6	$\{0,\frac{11}{80},\frac{1}{5},\frac{1}{2},\frac{7}{10},\frac{15}{16}\}$	$6^{80,135}_{\frac{79}{10},5.527}$	77. $*$ (factors)
2	3	$\{(3,1),(4,5),(5,9)\}$	6	$\{0,\frac{3}{80},\frac{1}{10},\frac{7}{16},\frac{1}{2},\frac{3}{5}\}$	$6^{80,191}_{\frac{7}{10},14.47}$	74. $*$ (factors)
2	3	$\{(3,1),(4,5),(5,11)\}$	6	$\{0,\frac{27}{80},\frac{2}{5},\frac{1}{2},\frac{9}{10},\frac{15}{16}\}$	$6^{80,675}_{\frac{23}{10},14.47}$	72. $*$ (factors)
2	3	$\{(3,1),(4,5),(5,13)\}$	6	$\{0,\frac{19}{80},\frac{3}{10},\frac{7}{16},\frac{1}{2},\frac{4}{5}\}$	$6^{80,478}_{\frac{31}{10},5.527}$	78. $*$ (factors)
2	3	$\{(3,1),(4,5),(5,17)\}$	6	$\{0,\frac{1}{5},\frac{7}{16},\frac{1}{2},\frac{51}{80},\frac{7}{10}\}$	$6^{80,489}_{\frac{39}{10},5.527}$	89. $*$ (factors)
2	3	$\{(3,1),(4,5),(5,19)\}$	6	$\{0,\frac{1}{10},\frac{1}{2},\frac{43}{80},\frac{3}{5},\frac{15}{16}\}$	$6^{80,143}_{\frac{47}{10},14.47}$	76. $*$ (factors)
2	3	$\{(3,1),(4,7),(5,1)\}$	6	$\{0,\frac{5}{16},\frac{2}{5},\frac{1}{2},\frac{57}{80},\frac{9}{10}\}$	$6^{80,884}_{\frac{53}{10},14.47}$	63. $*$ (factors)
2	3	$\{(3,1),(4,7),(5,3)\}$	6	$\{0,\frac{3}{10},\frac{1}{2},\frac{49}{80},\frac{4}{5},\frac{13}{16}\}$	$6^{80,785}_{\frac{61}{10},5.527}$	88. $*$ (factors)
2	3	$\{(3,1),(4,7),(5,7)\}$	6	$\{0,\frac{1}{80},\frac{1}{5},\frac{1}{2},\frac{7}{10},\frac{13}{16}\}$	$6^{80,529}_{\frac{69}{10},5.527}$	87. $*$ (factors)
2	3	$\{(3,1),(4,7),(5,9)\}$	6	$\{0,\frac{1}{10},\frac{5}{16},\frac{1}{2},\frac{3}{5},\frac{73}{80}\}$	$6^{80,657}_{\frac{77}{10},14.47}$	64. $*$ (factors)
2	3	$\{(3,1),(4,7),(5,11)\}$	6	$\{0,\frac{17}{80},\frac{2}{5},\frac{1}{2},\frac{13}{16},\frac{9}{10}\}$	$6^{80,621}_{\frac{13}{10},14.47}$	67. $*$ (factors)
2	3	$\{(3,1),(4,7),(5,13)\}$	6	$\{0,\frac{9}{80},\frac{3}{10},\frac{5}{16},\frac{1}{2},\frac{4}{5}\}$	$6^{80,400}_{\frac{21}{10},5.527}$	84. $*$ (factors)
2	3	$\{(3,1),(4,7),(5,17)\}$	6	$\{0,\frac{1}{5},\frac{5}{16},\frac{1}{2},\frac{41}{80},\frac{7}{10}\}$	$6^{80,495}_{\frac{29}{10},5.527}$	83. $*$ (factors)
2	3	$\{(3,1),(4,7),(5,19)\}$	6	$\{0,\frac{1}{10},\frac{33}{80},\frac{1}{2},\frac{3}{5},\frac{13}{16}\}$	$6^{80,629}_{\frac{37}{10},14.47}$	68. $*$ (factors)

Table 6: MTCs from 3-manifolds: Rank 6. (cont.)

N	n	$M_3(\{(p_i,q_i)\})$	Rank	Spins	Label	#
3	3	$\{(4,1),(4,1),(5,1)\}$	6	$\{0,\frac{1}{3},\frac{1}{3},\frac{3}{5},\frac{14}{15},\frac{14}{15}\}$	$6^{15,166}_{\frac{36}{5},10.85}$	22.(factors)
3	3	$\{(4,1),(4,1),(5,4)\}$	6	$\{0,\frac{1}{3},\frac{1}{3},\frac{2}{5},\frac{11}{15},\frac{11}{15}\}$	$6^{15,257}_{\frac{24}{5},10.85}$	21.(factors)
3	3	$\{(4,1),(4,1),(5,7)\}$	6	$\{0,\frac{2}{15},\frac{2}{15},\frac{1}{3},\frac{1}{3},\frac{4}{5}\}$	$6^{15,481}_{\frac{8}{5},4.145}$	26.(factors)
3	3	$\{(4,1),(4,1),(5,13)\}$	6	$\{0,\frac{1}{5},\frac{1}{3},\frac{1}{3},\frac{8}{15},\frac{8}{15}\}$	$6^{15,440}_{\frac{12}{5},4.145}$	25.(factors)
3	3	$\{(4,1),(4,5),(5,1)\}$	6	$\{0,\frac{4}{15},\frac{4}{15},\frac{3}{5},\frac{2}{3},\frac{2}{3}\}$	$6^{15,262}_{\frac{16}{5},10.85}$	24.(factors)
3	3	$\{(4,1),(4,5),(5,4)\}$	6	$\{0,\frac{1}{15},\frac{1}{15},\frac{2}{5},\frac{2}{3},\frac{2}{3}\}$	$6^{15,801}_{\frac{4}{5},10.85}$	23.(factors)
3	3	$\{(4,1),(4,5),(5,7)\}$	6	$\{0,\frac{7}{15},\frac{7}{15},\frac{2}{3},\frac{2}{3},\frac{4}{5}\}$	$6^{15,623}_{\frac{28}{5},4.145}$	28.(factors)
3	3	$\{(4,1),(4,5),(5,13)\}$	6	$\{0,\frac{1}{5},\frac{2}{3},\frac{2}{3},\frac{13}{15},\frac{13}{15}\}$	$6^{15,350}_{\frac{32}{5},4.145}$	27.(factors)
2	4	$\{(3,1),(3,1),(4,3),(5,6)\}$	6	$\{0,\frac{1}{16},\frac{2}{5},\frac{37}{80},\frac{1}{2},\frac{9}{10}\}$	$6^{80,798}_{\frac{33}{10},14.47}$	29.(factors)
2	4	$\{(3,1),(3,1),(4,3),(5,14)\}$	6	$\{0,\frac{1}{16},\frac{1}{10},\frac{1}{2},\frac{3}{5},\frac{53}{80}\}$	$6^{80,376}_{\frac{57}{10},14.47}$	30.(factors)
2	4	$\{(3,1),(3,1),(4,5),(5,6)\}$	6	$\{0,\frac{2}{5},\frac{7}{16},\frac{1}{2},\frac{67}{80},\frac{9}{10}\}$	$6^{80,146}_{\frac{63}{10},14.47}$	31.(factors)
2	4	$\{(3,1),(3,1),(4,5),(5,14)\}$	6	$\{0,\frac{3}{80},\frac{1}{10},\frac{7}{16},\frac{1}{2},\frac{3}{5}\}$	$6^{80,111}_{\frac{7}{10},14.47}$	32.(factors)
2	4	$\{(3,1),(3,1),(4,3),(5,16)\}$	6	$\{0,\frac{2}{5},\frac{1}{2},\frac{9}{16},\frac{9}{10},\frac{77}{80}\}$	$6^{80,215}_{\frac{73}{10},14.47}$	33.(factors)
2	4	$\{(3,1),(3,1),(4,3),(5,4)\}$	6	$\{0,\frac{1}{10},\frac{13}{80},\frac{1}{2},\frac{9}{16},\frac{3}{5}\}$	$6^{80,878}_{\frac{17}{10},14.47}$	34.(factors)
2	4	$\{(3,1),(3,1),(4,5),(5,16)\}$	6	$\{0,\frac{27}{80},\frac{2}{5},\frac{1}{2},\frac{9}{10},\frac{15}{16}\}$	$6^{80,108}_{\frac{23}{10},14.47}$	35.(factors)
2	4	$\{(3,1),(3,1),(4,5),(5,4)\}$	6	$\{0,\frac{1}{10},\frac{1}{2},\frac{43}{80},\frac{3}{5},\frac{15}{16}\}$	$6^{80,518}_{\frac{47}{10},14.47}$	36.(factors)
2	4	$\{(3,1),(3,1),(4,1),(5,6)\}$	6	$\{0,\frac{3}{16},\frac{2}{5},\frac{1}{2},\frac{47}{80},\frac{9}{10}\}$	$6^{80,596}_{\frac{43}{10},14.47}$	37.(factors)
2	4	$\{(3,1),(3,1),(4,7),(5,6)\}$	6	$\{0,\frac{5}{16},\frac{2}{5},\frac{1}{2},\frac{57}{80},\frac{9}{10}\}$	$6^{80,136}_{\frac{53}{10},14.47}$	38.(factors)
2	4	$\{(3,1),(3,1),(4,1),(5,16)\}$	6	$\{0,\frac{7}{80},\frac{2}{5},\frac{1}{2},\frac{11}{16},\frac{9}{10}\}$	$6^{80,750}_{\frac{3}{10},14.47}$	39.(factors)
2	4	$\{(3,1),(3,1),(4,7),(5,16)\}$	6	$\{0,\frac{17}{80},\frac{2}{5},\frac{1}{2},\frac{13}{16},\frac{9}{10}\}$	$6^{80,399}_{\frac{13}{10},14.47}$	40.(factors)
2	4	$\{(3,1),(3,1),(4,1),(5,14)\}$	6	$\{0,\frac{1}{10},\frac{3}{16},\frac{1}{2},\frac{3}{5},\frac{63}{80}\}$	$6^{80,100}_{\frac{67}{10},14.47}$	41.(factors)
2	4	$\{(3,1),(3,1),(4,7),(5,14)\}$	6	$\{0,\frac{1}{10},\frac{5}{16},\frac{1}{2},\frac{3}{5},\frac{73}{80}\}$	$6^{80,157}_{\frac{77}{10},14.47}$	42.(factors)
2	4	$\{(3,1),(3,1),(4,1),(5,4)\}$	6	$\{0,\frac{1}{10},\frac{23}{80},\frac{1}{2},\frac{3}{5},\frac{11}{16}\}$	$6^{80,392}_{\frac{27}{10},14.47}$	43.(factors)
2	4	$\{(3,1),(3,1),(4,7),(5,4)\}$	6	$\{0,\frac{1}{10},\frac{33}{80},\frac{1}{2},\frac{3}{5},\frac{13}{16}\}$	$6^{80,857}_{\frac{37}{10},14.47}$	44.(factors)
2	4	$\{(3,1),(3,1),(4,7),(5,12)\}$	6	$\{0,\frac{1}{80},\frac{1}{5},\frac{1}{2},\frac{7}{10},\frac{13}{16}\}$	$6^{80,106}_{\frac{69}{10},5.527}$	45.(factors)
2	4	$\{(3,1),(3,1),(4,7),(5,18)\}$	6	$\{0,\frac{9}{80},\frac{3}{10},\frac{5}{16},\frac{1}{2},\frac{4}{5}\}$	$6^{80,384}_{\frac{21}{10},5.527}$	46.(factors)
2	4	$\{(3,1),(3,1),(4,1),(5,2)\}$	6	$\{0,\frac{3}{16},\frac{1}{5},\frac{31}{80},\frac{1}{2},\frac{7}{10}\}$	$6^{80,519}_{\frac{19}{10},5.527}$	47.(factors)
2	4	$\{(3,1),(3,1),(4,1),(5,8)\}$	6	$\{0,\frac{3}{10},\frac{39}{80},\frac{1}{2},\frac{11}{16},\frac{4}{5}\}$	$6^{80,717}_{\frac{51}{10},5.527}$	48.(factors)
2	4	$\{(3,1),(3,1),(4,3),(5,2)\}$	6	$\{0,\frac{1}{16},\frac{1}{5},\frac{21}{80},\frac{1}{2},\frac{7}{10}\}$	$6^{80,458}_{\frac{9}{10},5.527}$	49.(factors)
2	4	$\{(3,1),(3,1),(4,3),(5,18)\}$	6	$\{0,\frac{1}{16},\frac{3}{10},\frac{1}{2},\frac{4}{5},\frac{69}{80}\}$	$6^{80,117}_{\frac{1}{10},5.527}$	50.(factors)
2	4	$\{(3,1),(3,1),(4,5),(5,2)\}$	6	$\{0,\frac{1}{5},\frac{7}{16},\frac{1}{2},\frac{51}{80},\frac{7}{10}\}$	$6^{80,488}_{\frac{39}{10},5.527}$	51.(factors)
2	4	$\{(3,1),(3,1),(4,5),(5,18)\}$	6	$\{0,\frac{19}{80},\frac{3}{10},\frac{7}{16},\frac{1}{2},\frac{4}{5}\}$	$6^{80,305}_{\frac{31}{10},5.527}$	52.(factors)
2	4	$\{(3,1),(3,1),(4,3),(5,12)\}$	6	$\{0,\frac{1}{5},\frac{1}{2},\frac{9}{16},\frac{7}{10},\frac{61}{80}\}$	$6^{80,464}_{\frac{49}{10},5.527}$	53.(factors)
2	4	$\{(3,1),(3,1),(4,3),(5,8)\}$	6	$\{0,\frac{3}{10},\frac{29}{80},\frac{1}{2},\frac{9}{16},\frac{4}{5}\}$	$6^{80,194}_{\frac{41}{10},5.527}$	54.(factors)
2	4	$\{(3,1),(3,1),(4,5),(5,12)\}$	6	$\{0,\frac{11}{80},\frac{1}{5},\frac{1}{2},\frac{7}{10},\frac{15}{16}\}$	$6^{80,822}_{\frac{79}{10},5.527}$	55.(factors)
2	4	$\{(3,1),(3,1),(4,5),(5,8)\}$	6	$\{0,\frac{3}{10},\frac{1}{2},\frac{59}{80},\frac{4}{5},\frac{15}{16}\}$	$6^{80,240}_{\frac{71}{10},5.527}$	56.(factors)
2	4	$\{(3,1),(3,1),(4,7),(5,2)\}$	6	$\{0,\frac{1}{5},\frac{5}{16},\frac{1}{2},\frac{41}{80},\frac{7}{10}\}$	$6^{80,420}_{\frac{29}{10},5.527}$	57.(factors)
2	4	$\{(3,1),(3,1),(4,7),(5,8)\}$	6	$\{0,\frac{3}{10},\frac{1}{2},\frac{49}{80},\frac{4}{5},\frac{13}{16}\}$	$6^{80,862}_{\frac{61}{10},5.527}$	58.(factors)
2	4	$\{(3,1),(3,1),(4,1),(5,12)\}$	6	$\{0,\frac{1}{5},\frac{1}{2},\frac{11}{16},\frac{7}{10},\frac{71}{80}\}$	$6^{80,113}_{\frac{59}{10},5.527}$	59.(factors)
2	4	$\{(3,1),(3,1),(4,1),(5,18)\}$	6	$\{0,\frac{3}{16},\frac{3}{10},\frac{1}{2},\frac{4}{5},\frac{79}{80}\}$	$6^{80,545}_{\frac{11}{10},5.527}$	60.(factors)

Table 7: MTCs from 3-manifolds: Rank 6. (cont.)

N	n	$M_3(\{(p_i,q_i)\})$	Rank	Spins	Label	#
2	4	$\{(3,1),(3,1),(3,1),(7,6)\}$	6	$\{0,\frac{1}{7},\frac{1}{4},\frac{11}{28},\frac{5}{7},\frac{27}{28}\}$	$6^{28,124}_{\frac{55}{7},18.59}$	159.(pseudo)
2	4	$\{(3,1),(3,1),(3,1),(7,16)\}$	6	$\{0,\frac{1}{7},\frac{5}{28},\frac{3}{7},\frac{3}{4},\frac{25}{28}\}$	$6^{28,127}_{\frac{53}{7},3.682}$	168.(factors)
2	4	$\{(3,1),(3,1),(3,1),(7,8)\}$	6	$\{0,\frac{1}{28},\frac{2}{7},\frac{17}{28},\frac{3}{4},\frac{6}{7}\}$	$6^{28,277}_{\frac{1}{7},18.59}$	162.(factors)
2	4	$\{(3,1),(3,1),(3,1),(7,4)\}$	6	$\{0,\frac{9}{28},\frac{13}{28},\frac{4}{7},\frac{5}{7},\frac{3}{4}\}$	$6^{28,806}_{\frac{37}{7},5.725}$	165.(factors)
2	4	$\{(3,1),(3,1),(3,1),(7,2)\}$	6	$\{0,\frac{1}{7},\frac{1}{4},\frac{11}{28},\frac{3}{7},\frac{19}{28}\}$	$6^{28,782}_{\frac{11}{7},3.682}$	167.(factors)
2	4	$\{(3,1),(3,1),(3,1),(7,10)\}$	6	$\{0,\frac{1}{4},\frac{2}{7},\frac{3}{7},\frac{15}{28},\frac{19}{28}\}$	$6^{28,399}_{\frac{19}{7},5.725}$	164.(factors)
2	4	$\{(3,1),(3,1),(3,1),(7,12)\}$	6	$\{0,\frac{9}{28},\frac{4}{7},\frac{17}{28},\frac{3}{4},\frac{6}{7}\}$	$6^{28,756}_{\frac{45}{7},3.682}$	170.(factors)
2	4	$\{(3,1),(3,1),(3,1),(7,18)\}$	6	$\{0,\frac{1}{4},\frac{4}{7},\frac{5}{7},\frac{23}{28},\frac{27}{28}\}$	$6^{28,267}_{\frac{51}{7},5.725}$	166.(factors)
2	4	$\{(3,1),(3,1),(3,1),(7,20)\}$	6	$\{0,\frac{1}{7},\frac{13}{28},\frac{5}{7},\frac{3}{4},\frac{25}{28}\}$	$6^{28,130}_{\frac{41}{7},18.59}$	160.(factors)
2	4	$\{(3,1),(3,1),(3,1),(7,11)\}$	6	$\{0,\frac{3}{28},\frac{1}{4},\frac{2}{7},\frac{15}{28},\frac{6}{7}\}$	$6^{28,779}_{\frac{15}{7},18.59}$	161.(factors)
2	4	$\{(3,1),(3,1),(3,1),(7,24)\}$	6	$\{0,\frac{1}{28},\frac{5}{28},\frac{2}{7},\frac{3}{7},\frac{3}{4}\}$	$6^{28,578}_{\frac{5}{7},5.725}$	163.(factors)
2	4	$\{(3,1),(3,1),(3,1),(7,26)\}$	6	$\{0,\frac{3}{28},\frac{1}{4},\frac{4}{7},\frac{23}{28},\frac{6}{7}\}$	$6^{28,797}_{\frac{3}{7},3.682}$	169.(factors)
Spin(8)	3	$\{(6,1),(7,1),(9,17)\}$	4	$\{0,\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{3},\frac{2}{3}\}$	$6^{9,186}_{\frac{8}{3},74.61}$	149.(prime)
Spin(8)	3	$\{(6,1),(7,1),(9,1)\}$	4	$\{0,\frac{8}{9},\frac{8}{9},\frac{8}{9},\frac{2}{3},\frac{1}{3}\}$	$6^{9,452}_{\frac{16}{3},74.61}$	150.(prime)
Spin(8)	3	$\{(6,1),(7,1),(9,11)\}$	4	$\{0,\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{2}{3},\frac{1}{3}\}$	$6^{9,226}_{\frac{8}{3},3.834}$	151.(prime)
Spin(8)	3	$\{(6,1),(7,1),(9,7)\}$	4	$\{0,\frac{5}{9},\frac{5}{9},\frac{5}{9},\frac{2}{3},\frac{1}{3}\}$	$6^{9,342}_{\frac{16}{3},3.834}$	152.(prime)
Spin(8)	3	$\{(6,1),(7,1),(9,13)\}$	4	$\{0,\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{3},\frac{1}{3}\}$	$6^{9,789}_{\frac{4}{3},2.548}$	153.(prime)
Spin(8)	3	$\{(6,1),(7,1),(9,5)\}$	4	$\{0,\frac{7}{9},\frac{7}{9},\frac{7}{9},\frac{2}{3},\frac{1}{3}\}$	$6^{9,632}_{\frac{20}{3},2.548}$	154.(prime)
E_7	3	$\{(18,1),(19,1),(21,5)\}$	6	$\{0,0,\frac{1}{7},\frac{2}{7},\frac{4}{7},\frac{2}{3}\}$	$6^{21,154}_{6,100.6}$	155.(prime)
E_7	3	$\{(18,1),(19,1),(21,1)\}$	6	$\{0,0,\frac{3}{7},\frac{5}{7},\frac{6}{7},\frac{1}{3}\}$	$6^{21,320}_{2,100.6}$	156.(prime)
E_7	3	$\{(18,1),(19,1),(21,19)\}$	6	$\{0,0,\frac{1}{3},\frac{1}{7},\frac{2}{7},\frac{4}{7}\}$	$6^{21,215}_{2,4.382}$	157.(prime)
E_7	3	$\{(18,1),(19,1),(21,11)\}$	6	$\{0,0,\frac{2}{3},\frac{3}{7},\frac{5}{7},\frac{6}{7}\}$	$6^{21,604}_{6,4.382}$	158.(prime)

Table 8: MTCs from 3-manifolds: Rank 6. (cont.)

Rank	Spins	Label	#
6	$\{0,\frac{1}{9},\frac{2}{3},\frac{4}{9},\frac{1}{3},\frac{7}{9}\}$	$6^{9,746}_{\frac{16}{3},9.}$	171.(prime)
6	$\{0,\frac{8}{9},\frac{1}{3},\frac{5}{9},\frac{2}{3},\frac{2}{9}\}$	$6^{9,711}_{\frac{8}{3},9.}$	172.(prime)
6	$\{0,0,0,\frac{1}{5},\frac{4}{5},\frac{1}{2}\}$	$6^{10,699}_{0,20.}$	105.(prime)
6	$\{0,0,0,\frac{2}{5},\frac{3}{5},\frac{1}{2}\}$	$6^{10,990}_{4,20.}$	106.(prime)
6	$\{0,0,0,\frac{2}{5},\frac{3}{5},\frac{1}{2}\}$	$6^{10,101}_{4,20.}$	107.(prime), same spin as 106.
6	$\{0,0,0,\frac{1}{5},\frac{4}{5},\frac{1}{2}\}$	$6^{10,419}_{0,20.}$	108.(prime), same spin as 105.
6	$\{0,0,\frac{1}{5},\frac{4}{5},\frac{1}{4},\frac{3}{4}\}$	$6^{20,139}_{0,20.}$	109.(prime)
6	$\{0,0,\frac{2}{5},\frac{3}{5},\frac{1}{4},\frac{3}{4}\}$	$6^{20,739}_{4,20.}$	110.(prime)
6	$\{0,0,\frac{2}{5},\frac{3}{5},\frac{1}{4},\frac{3}{4}\}$	$6^{20,180}_{4,20.}$	111.(prime), same spin as 110.
6	$\{0,0,\frac{1}{5},\frac{4}{5},\frac{1}{4},\frac{3}{4}\}$	$6^{20,419}_{0,20.}$	112.(prime), same spins as 109.

Table 9: MTCs from 3-manifolds: Rank 6: missing Prime models

Rank	Spins	Label	#
6	$\{0, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{7}{12}, \frac{7}{12}\}$	$6^{12,534}_{3,6.}$	1.(factors)
6	$\{0, \frac{1}{3}, \frac{1}{3}, \frac{3}{4}, \frac{1}{12}, \frac{1}{12}\}$	$6^{12,701}_{1,6.}$	2.(factors)
6	$\{0, \frac{2}{3}, \frac{2}{3}, \frac{1}{4}, \frac{11}{12}, \frac{11}{12}\}$	$6^{12,113}_{7,6.}$	3.(factors)
6	$\{0, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{5}{12}, \frac{5}{12}\}$	$6^{12,298}_{5,6.}$	4.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{16}, \frac{5}{16}\}$	$6^{16,688}_{\frac{3}{2},8.}$	5.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{16}, \frac{7}{16}\}$	$6^{16,511}_{\frac{5}{2},8.}$	6.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{9}{16}, \frac{13}{16}\}$	$6^{16,548}_{\frac{11}{2},8.}$	7.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{11}{16}, \frac{15}{16}\}$	$6^{16,107}_{\frac{13}{2},8.}$	8.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{16}, \frac{13}{16}\}$	$6^{16,357}_{\frac{15}{2},8.}$	9.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{16}, \frac{15}{16}\}$	$6^{16,710}_{\frac{1}{2},8.}$	10.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{5}{16}, \frac{9}{16}\}$	$6^{16,346}_{\frac{7}{2},8.}$	11.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{7}{16}, \frac{11}{16}\}$	$6^{16,357}_{\frac{9}{2},8.}$	12.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{16}, \frac{13}{16}\}$	$6^{16,107}_{\frac{15}{2},8.}$	13.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{16}, \frac{15}{16}\}$	$6^{16,460}_{\frac{1}{2},8.}$	14.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{5}{16}, \frac{9}{16}\}$	$6^{16,246}_{\frac{7}{2},8.}$	15.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{7}{16}, \frac{11}{16}\}$	$6^{16,107}_{\frac{9}{2},8.}$	16.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{16}, \frac{5}{16}\}$	$6^{16,438}_{\frac{3}{2},8.}$	17.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{16}, \frac{7}{16}\}$	$6^{16,261}_{\frac{5}{2},8.}$	18.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{9}{16}, \frac{13}{16}\}$	$6^{16,798}_{\frac{11}{2},8.}$	19.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{11}{16}, \frac{15}{16}\}$	$6^{16,132}_{\frac{13}{2},8.}$	20.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{16}, \frac{1}{4}, \frac{3}{4}, \frac{5}{16}\}$	$6^{16,507}_{\frac{3}{2},8.}$	173.(factors)
6	$\{0, \frac{1}{2}, \frac{1}{16}, \frac{1}{4}, \frac{3}{4}, \frac{13}{16}\}$	$6^{16,130}_{\frac{15}{2},8.}$	174.(factors)
6	$\{0, \frac{1}{2}, \frac{7}{16}, \frac{1}{4}, \frac{3}{4}, \frac{3}{16}\}$	$6^{16,154}_{\frac{5}{2},8.}$	175.(factors)
6	$\{0, \frac{1}{2}, \frac{7}{16}, \frac{1}{4}, \frac{3}{4}, \frac{11}{16}\}$	$6^{16,772}_{\frac{9}{2},8.}$	176.(factors)
6	$\{0, \frac{1}{2}, \frac{9}{16}, \frac{1}{4}, \frac{3}{4}, \frac{5}{16}\}$	$6^{16,656}_{\frac{7}{2},8.}$	177.(factors)
6	$\{0, \frac{1}{2}, \frac{9}{16}, \frac{1}{4}, \frac{3}{4}, \frac{13}{16}\}$	$6^{16,861}_{\frac{11}{2},8.}$	178.(factors)
6	$\{0, \frac{1}{2}, \frac{15}{16}, \frac{1}{4}, \frac{3}{4}, \frac{3}{16}\}$	$6^{16,818}_{\frac{1}{2},8.}$	179.(factors)
6	$\{0, \frac{1}{2}, \frac{15}{16}, \frac{1}{4}, \frac{3}{4}, \frac{11}{16}\}$	$6^{16,199}_{\frac{13}{2},8.}$	180.(factors)
6	$\{0, \frac{1}{2}, \frac{3}{16}, \frac{1}{4}, \frac{3}{4}, \frac{7}{16}\}$	$6^{16,595}_{\frac{5}{2},8.}$	181.(factors)
6	$\{0, \frac{1}{2}, \frac{3}{16}, \frac{1}{4}, \frac{3}{4}, \frac{15}{16}\}$	$6^{16,156}_{\frac{1}{2},8.}$	182.(factors)
6	$\{0, \frac{1}{2}, \frac{5}{16}, \frac{1}{4}, \frac{3}{4}, \frac{1}{16}\}$	$6^{16,242}_{\frac{3}{2},8.}$	183.(factors)
6	$\{0, \frac{1}{2}, \frac{5}{16}, \frac{1}{4}, \frac{3}{4}, \frac{9}{16}\}$	$6^{16,684}_{\frac{7}{2},8.}$	184.(factors)
6	$\{0, \frac{1}{2}, \frac{11}{16}, \frac{1}{4}, \frac{3}{4}, \frac{7}{16}\}$	$6^{16,227}_{\frac{9}{2},8.}$	185.(factors)
6	$\{0, \frac{1}{2}, \frac{11}{16}, \frac{1}{4}, \frac{3}{4}, \frac{15}{16}\}$	$6^{16,949}_{\frac{13}{2},8.}$	186.(factors)
6	$\{0, \frac{1}{2}, \frac{13}{16}, \frac{1}{4}, \frac{3}{4}, \frac{1}{16}\}$	$6^{16,553}_{\frac{15}{2},8.}$	187.(factors)
6	$\{0, \frac{1}{2}, \frac{13}{16}, \frac{1}{4}, \frac{3}{4}, \frac{9}{16}\}$	$6^{16,111}_{\frac{11}{2},8.}$	188.(factors)
6	$\{0, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{7}{12}, \frac{7}{12}\}$	$6^{12,520}_{3,6.}$	189.(factors)
6	$\{0, \frac{1}{3}, \frac{1}{3}, \frac{3}{4}, \frac{1}{12}, \frac{1}{12}\}$	$6^{12,354}_{1,6.}$	190.(factors)
6	$\{0, \frac{2}{3}, \frac{2}{3}, \frac{1}{4}, \frac{11}{12}, \frac{11}{12}\}$	$6^{12,854}_{7,6.}$	191.(factors)
6	$\{0, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{5}{12}, \frac{5}{12}\}$	$6^{12,208}_{5,6.}$	192.(factors)

Table 10: MTCs from 3-manifolds: Rank 6. Missing factored models.

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