Special Kähler geometries of $\mathcal{N} = 4$ superYang-Mills

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ABSTRACT: The low energy effective theory on the moduli space of vacua of 4d superYang-Mills (sYM) theory defines a special Kähler geometry. For simple sYM gauge algebras, $\mathfrak{g}$, we classify all compatible special Kähler structures by showing that they are in one-to-one correspondence with certain equivalence classes of integral symplectic representations of the Weyl group of $\mathfrak{g}$. We further demonstrate that, for principal Dirac pairing, these equivalence classes are in one-to-one correspondence with the S-duality orbits of the global structures of the corresponding $\mathfrak{g}$ sYM gauge theory, after a mistake in the field theory literature is corrected. This provides a low-energy test of S-duality. We also discuss twisted product geometries made from factors with special Kähler structures with non-principal Dirac pairings.

Contents

1 Introduction and summary

Information-rich and often exactly computable observables of a supersymmetric quantum field theory are the leading terms in the low-energy effective action of the massless degrees of freedom on $\mathcal{M}$, its moduli space of vacua. In the case of $4d$ theories with $\mathcal{N}{=}2$ supersymmetry, this moduli space is a continuum of states, and the low-energy effective action is encoded in a singular complex geometry – the moduli space geometry – that incorporates the moduli space of vacua. With more supersymmetry, the moduli space geometries become more constrained, and therefore more amenable to classification.

In this paper, we classify all possible moduli space geometries (with principal polarization) of theories with $\mathcal{N}{=}4$ superconformal symmetry and a weak coupling limit described by a superYang-Mills (sYM) theory with simple gauge algebra, $\mathfrak{g}$. The answer to this problem was seemingly given long ago [1] by the orbifold describing the moduli space of vacua,

$$\mathcal{M}_{\mathfrak{g}} \doteq \mathbb{C}^{3r}/(W(\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C}^3), \tag{1.1}$$

where $W(\mathfrak{g})$ is the Weyl group of the gauge group, and $r \doteq \mathrm{rk}(\mathfrak{g})$ is its rank. $W(\mathfrak{g})$ is a finite group acting via its defining real reflection representation on $\mathbb{R}^r$, "triply complexified" by tensoring with $\mathbb{C}^3$. $\mathcal{M}_{\mathfrak{g}}$ is a Kähler manifold with Kähler metric given by one inherited from a flat metric on $\mathbb{C}^{3r}$. But the moduli space geometry has more structure than just the Kähler orbifold geometry describing $\mathcal{M}_{\mathfrak{g}}$. In particular, it has a *special Kähler (SK) structure*,[1] whose definition and relation to the low-energy effective action is reviewed in appendix A. Our main result is the classification of SK structures that are compatible with the Kähler geometry of the $\mathcal{N}{=}4$ sYM moduli space of vacua (1.1).[2]

We show (see appendix A) that an $\mathcal{N}{=}4$ SK structure on $\mathcal{M}_{\mathfrak{g}}$ with polarization J (representing the Dirac pairing on the charge lattice of the low-energy theory) is defined by the equivalence class of a pair $(S, \boldsymbol{\tau})$ where $S : W_{\mathbb{C}} \to \mathrm{GL}(2r, \mathbb{Z})$ is an integral representation of the complexified Weyl group that is symplectic with respect to J, and $\boldsymbol{\tau}$ is a symmetric $r \times r$ matrix of low-energy $\mathfrak{u}(1)^r$ gauge couplings that is compatible with S. We show how to construct all such triples $(J, S, \boldsymbol{\tau})$ and compute the resulting equivalence classes, which thereby classifies all possible $\mathcal{N}{=}4$ sYM SK structures on $\mathcal{M}_{\mathfrak{g}}$. While the irreducible representations of Weyl groups into $\mathrm{GL}(r, \mathbb{Z})$ are well-studied, our classification of SK structures requires the classification of Weyl group representations into $\mathrm{GL}(2r, \mathbb{Z})$ that are

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[1]In the $\mathcal{N}{=}4$ class, it is a generalization of an $\mathcal{N}{=}2$ SK structure, called a "triple-SK structure" in [2], where it is described in detail.

[2]By assuming the existence of a weak-coupling limit described by an $\mathcal{N}{=}4$ sYM theory, the underlying Kähler geometry that describes the moduli space of vacua is determined to take the form (1.1). In particular, this assumption tells us that the Weyl group must act on $\mathbb{C}^{3r}$ through its reflection representation that defines it as a real reflection group. Beyond fixing the Kähler geometry, we do not take into account any additional, dynamical data of an $\mathcal{N}{=}4$ sYM theory that could be deduced from this assumption and attached to the SK geometry, such as its BPS charge lattice of physical, finite energy states. This additional data can be combined with an $\mathcal{N}{=}4$ SK structure and subsequently classified. We expect this would lead to some slight refinements of our classification results, but we leave these interesting considerations for future work.

reducible but not necessarily decomposable.[3] This is a module extension problem whose solution is summarized in table 3 and explained in 2.5.

For fixed polarization J, $\boldsymbol{\tau}$ is compatible with S if it resides in the fixed point set of S acting on the Seigel upper half-space of degree r, $\mathscr{H}_r$. This set, denoted $\mathrm{Fix}(S)$, forms a 1-parameter family of compatible $\boldsymbol{\tau}(\tau)$ matrices in $\mathscr{H}_r$ where the parameter τ is valued in $\mathscr{H}_1$. Thus, the inequivalent triples $(J, S, \boldsymbol{\tau}(\tau))$ organize into the equivalence classes of fixed-point set triples $(J, S, \mathrm{Fix}(S))$ which are labeled by the equivalence class of the representation S. We call such classes *SK orbits* and use them to present the classification of $\mathcal{N}{=}4$ SK structures in table 1. When J is *principal*,[4] and for a given Weyl group W, each row in the second column of table 1 represents an inequivalent SK orbit defined by the equivalence class of some representative $\mathrm{GL}(2r, \mathbb{Z})$, symplectic representation S of W. The 3rd column of the table is a certain modular group, called the *self-duality group of the SK orbit*, which are reviewed in appendix C. It is defined by (the equivalence class of) S, acts on τ through fractional linear transformations, and encodes the geometry of the 1-dimensional parameter space whose points are the inequivalent $\mathcal{N}{=}4$ SK structures $(J, S, \boldsymbol{\tau}(\tau))$ within the fixed-point set triple $(J, S, \mathrm{Fix}(S))$. This geometry can be realized as the modular curve formed by quotienting $\mathscr{H}_1$ by this self-duality group, and so is parameterized by $\tau \in \mathscr{H}_1$ relative to the identifications determined by the self-duality group. For this reason, we refer to this geometry as the *conformal manifold of the SK orbit*.

How does the classification of principal $\mathcal{N}{=}4$ SK structures compare with the predicted classification of the corresponding $\mathcal{N}{=}4$ sYM SCFT? As $4d$ gauge theories, the local space-time dynamics are specified by the gauge algebra $\mathfrak{g}$ and the complex gauge coupling $\tau_{\mathrm{YM}} \in \mathscr{H}_1$,[5] but they also carry discrete "global structure" data that is sensitive to the global topology of space-time [4]. This data can be expressed in terms of a maximal spectrum of Wilson-'t Hooft line operators that are mutually local, i.e. they are genuine line operators [5, 6]. Such a spectrum is specified by a rank-$2r$ Weyl-invariant lattice with a principal Dirac pairing, and points in this lattice correspond to the $\mathfrak{g} \times \mathfrak{g}^{\mathrm{L}}$ representations of the Wilson-'t Hooft operators.[6] Once this global structure data is specified, the gauge theory defines an absolute QFT, i.e. one that can be consistently placed on space-time manifolds with non-trivial topology. However, the set of consistent line operator spectra is not unique since there are distinct global structures that each define an absolute variant of the sYM gauge theory — these different choices are often called the *global variants* of the gauge theory. Furthermore, the conjectured S-duality of $\mathcal{N}{=}4$ sYM acts on these global variants to

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[3]The irreducible representations of the Weyl group that are $\mathbb{Q}$-equivalent to its defining reflection representation have been classified [3], and our required use of this result is a consequence of the sYM assumption; see appendix A.

[4]Principal J means the charge lattice is comprised of a maximal set of massive states that are mutually local. Correspondingly, there always exists an associated symplectic basis of J in which its matrix form is the ordinary symplectic form defining the symplectic matrix group $\mathrm{Sp}(2r, \mathbb{Z})$.

[5]Not to be confused with the boldface matrix $\boldsymbol{\tau}(\tau)$ of low-energy couplings on the moduli space or its complex parameter τ!

[6]Here, $\mathfrak{g}^{\mathrm{L}}$ is the Langlands dual Lie algebra to $\mathfrak{g}$. For simply laced $\mathfrak{g} = A_r, D_r, E_{6,7,8}$, $\mathfrak{g}^{\mathrm{L}} \cong \mathfrak{g}$, while for non-simply laced $\mathfrak{g}$ we have $B_r^{\mathrm{L}} \cong C_r$, $C_r^{\mathrm{L}} \cong B_r$, and the duals of G_2 and F_4 each have a rotated root system that leaves the algebra invariant [7].

combine them into various orbits – *S-duality orbits* – with an associated *self-duality group*. An absolute $\mathcal{N}=4$ theory is thus labeled by one of these orbits along with the marginal coupling $\tau_{\mathrm{YM}} \in \mathscr{H}_1$. But because S-duality also acts on τ_{YM}, the inequivalent absolute theories are labeled by a global variant orbit with the equivalence class of $\tau_{\mathrm{YM}} \in \mathscr{H}_1$ relative to the action of the self-duality group of that orbit. In other words, absolute theories are labeled by a global variant S-duality orbit and a point in a 1-dimensional geometry that is formed from the quotient of $\mathscr{H}_1$ by the self-duality group of the orbit; this geometry is the *conformal manifold of the S-duality orbit.* In summary, the inequivalent absolute $\mathcal{N}=4$ sYM SCFTs are in 1-to-1 correspondence with the families of global variant S-duality orbits that are parameterized by the *conformal manifold of that orbit.* These S-duality orbits and their self-duality groups have been worked out in detail for most simple $\mathfrak{g}$ in [4, 8], and we summarize these results in table 6.

The correspondence between the global variants of a $\mathcal{N}=4$ sYM theory with gauge algebra $\mathfrak{g}$; the global variants of the low-energy $\mathfrak{u}(1)^r$ gauge theory on its Coulomb branch in terms of *line lattices*; and the *principal*, $\mathfrak{g}$-SK geometries such line lattices canonically define,[7] all suggest there is a relation between principal $\mathcal{N}=4$ SK geometries with algebra $\mathfrak{g}$ and the absolute variants of a $\mathcal{N}=4$ sYM theory with gauge algebra $\mathfrak{g}$. Thus, we are led to compare principal, $\mathfrak{g}$-SK orbits and their self-duality groups with the S-duality orbits and self-duality groups of absolute, $\mathfrak{g}$ $\mathcal{N}=4$ sYM theories. In particular, we want to compare the number of inequivalent orbits and their self-duality groups, since this data is gauge-invariant and independent of the conformal manifold parameters τ and τ_{YM} of an SK orbit and S-duality orbit, respectively.

For a given Weyl group W, table 1 shows the number of distinct SK orbits (describing the inequivalent moduli space geometries) and their self-duality groups. These agree with the number of S-duality orbits and their self-duality groups except in those cases marked with ✗'s in the table, c.f. table 6. These disagreements come in two types that are correlated with the column the ✗ is in. The first type (second column ✗) is a disagreement in the number of SK orbits, and the second type (third column ✗) is a disagreement between the self-duality groups. For all the disagreements that occur in the $\mathfrak{g} = D_{2k}$ cases, they are due to a mistake in the field theory literature of not properly taking into account the (outer) automorphism interchanging spinor and conjugate-spinor representations. Doing so leads to additional equivalences between absolute $\mathcal{N}=4$ theories that either reduces the number of distinct S-duality orbits or increases the size of a self-duality orbit. For the A_1 and BC_r cases, although the number of orbits agree, one SK orbit always has a larger self-duality group.

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[7]The distinct sets of Wilson-'t Hooft line operators defining the global variants of the sYM theory can be put in 1-to-1 correspondence with the distinct sets of Wilson-'t Hooft line operators defining the global variants of the low-energy $u(1)^r$ gauge theory on its Coulomb branch [9]. The latter are specified by *line lattices*, which are refinements of the sYM BPS charge lattice to lattices that are maximal and mutually local, i.e. that have principal Dirac pairing. If the BPS charge lattice is non-principal, as is true for most simple $\mathfrak{g}$, the additional charges in the refined lattice represent genuine probe line operators. A line lattice canonically defines a principal SK geometry by identifying it with the homology lattice of the fiber; see [9] for more discussion on this correspondence.

W	distinct $\mathbf{j} = \mathbf{1}$ SK orbits	self-duality group of SK orbit
A_1	$\{(\mathbb{Z}_1, 0)\}$	$\mathrm{PSL}(2,\mathbb{Z})$ ✗
A_r $(r \geq 2)$	one SK structure orbit $\{(\mathbb{Z}_s, 0), \cdots\}$ for each $s^2 \mid (r+1)$ (see section 3.1)	$\Gamma_0((r+1)/s^2)$
BC_2	$\circlearrowright \{(\mathbb{Z}_2, 1), (\mathbb{Z}_2^\vee, 1)\}$ $\{(\mathbb{Z}_2, 0)\} \leftrightarrow \{(\mathbb{Z}_2^\vee, 0)\}$	H_4 $\mathrm{PSL}(2,\mathbb{Z})$ ✗
BC_{2k} $(k \geq 2)$	$\{(\mathbb{Z}_1, 0), (\mathbb{Z}_2, 0)\}$ $\{(\mathbb{Z}_2, 1), (\mathbb{Z}_2^\vee, 1)\}$ $\{(\mathbb{Z}_2^\vee, 0)\}$	$\Gamma_0(2)$ $\Gamma_0(2)$ $\mathrm{PSL}(2,\mathbb{Z})$ ✗
BC_{2k+1}	$\{(\mathbb{Z}_1, 0), (\mathbb{Z}_2, 0), (\mathbb{Z}_2^\vee, 1)\}$ $\{(\mathbb{Z}_2^\vee, 0)\}$	$\Gamma_0(4)$ $\mathrm{PSL}(2,\mathbb{Z})$ ✗
D_4	$\{(\mathbb{Z}_1, 0), (\mathbb{Z}_2^2, 00)\}$ ✗ $\{(\mathbb{Z}_2^V, 1), (\mathbb{Z}_2^2, 01)\} \leftrightarrow \{(\mathbb{Z}_2^S, 1), (\mathbb{Z}_2^2, 10)\} \leftrightarrow \{(\mathbb{Z}_2^C, 1), (\mathbb{Z}_2^2, 11)\}$ ✗ $\{(\mathbb{Z}_2^V, 0)\} \leftrightarrow \{(\mathbb{Z}_2^S, 0)\} \leftrightarrow \{(\mathbb{Z}_2^C, 0)\}$	$\Gamma_0(2)$ $\Gamma_0(2)$ $\mathrm{PSL}(2,\mathbb{Z})$
D_{4k} $(k \geq 2)$	$\{(\mathbb{Z}_1, 0), (\mathbb{Z}_2^2, 00)\}$ $\{(\mathbb{Z}_2^V, 1), (\mathbb{Z}_2^2, 01)\}$ ✗ $\{(\mathbb{Z}_2^S, 1), (\mathbb{Z}_2^2, 10)\} \leftrightarrow \{(\mathbb{Z}_2^C, 1), (\mathbb{Z}_2^2, 11)\}$ $\{(\mathbb{Z}_2^V, 0)\}$ ✗ $\{(\mathbb{Z}_2^S, 0)\} \leftrightarrow \{(\mathbb{Z}_2^C, 0)\}$	$\Gamma_0(2)$ $\Gamma_0(2)$ $\Gamma_0(2)$ $\mathrm{PSL}(2,\mathbb{Z})$ $\mathrm{PSL}(2,\mathbb{Z})$
D_{4k+2}	$\{(\mathbb{Z}_1, 0), (\mathbb{Z}_2^2, 00)\}$ $\{(\mathbb{Z}_2^V, 1), (\mathbb{Z}_2^2, 01)\}$ $\circlearrowright \{(\mathbb{Z}_2^S, 0), (\mathbb{Z}_2^C, 0), (\mathbb{Z}_2^2, 10)\}$ $\{(\mathbb{Z}_2^2, 11)\}$ $\{(\mathbb{Z}_2^V, 0)\}$	$\Gamma_0(2)$ $\Gamma_0(2)$ $\Gamma_0(2)$ ✗ Δ $\mathrm{PSL}(2,\mathbb{Z})$
D_{2k+1}	$\{(\mathbb{Z}_1, 0), (\mathbb{Z}_2, 1), (\mathbb{Z}_4, 0)\}$ $\{(\mathbb{Z}_2, 0)\}$	$\Gamma_0(4)$ $\mathrm{PSL}(2,\mathbb{Z})$
E_6	$\{(\mathbb{Z}_1, 0), (\mathbb{Z}_3, 0)\}$	$\Gamma_0(3)$
E_7	$\{(\mathbb{Z}_1, 0), (\mathbb{Z}_2, 0)\}$	$\Gamma_0(2)$
E_8	$\{(\mathbb{Z}_1, 0)\}$	$\mathrm{PSL}(2,\mathbb{Z})$
F_4	$\circlearrowright \{(\mathbb{Z}_1, 0), (\mathbb{Z}_1^\vee, 0)\}$	H_4
G_2	$\circlearrowright \{(\mathbb{Z}_1, 0), (\mathbb{Z}_1^\vee, 0)\}$	H_6

Table 1. Summary of classification results for $\mathcal{N}=4$ sYM SK structures. In the second column, each row containing (a set of) curly brackets represent an inequivalent SK orbit for the Weyl group, W, in the first column. As explained in 2.6, an SK orbit is determined by the $\mathrm{Sp}(2r, \mathbb{Z})$ orbit of an integral, symplectic representation of W. We label such representations by a pair, $(\mathbb{Z}_s^X, n)$, where first entry refers to subgroups of the center of a Lie algebra with Weyl group W, and the second entry refers to an element of their corresponding Ext group, which is summarized in table 3. The set of representations enclosed in curly brackets are all $\mathbb{Z}$-equivalent. The teal arrows between curly brackets denote further $\mathbb{Z}$-equivalences that incorporate Weyl group (outer) automorphisms; $\circlearrowright \{\cdots\}$ indicates self-equivalences within an SK orbit while $\{\cdots\} \leftrightarrow \{\cdots\}$ indicates equivalences between SK orbits that are not $\mathbb{Z}$-equivalent while using the identity automorphism of W. The 3rd column gives the self-duality group of each SK orbit as a modular subgroup (table C.5). Since circular arrows $\circlearrowright$ are self-equivalences of an SK orbit, they contribute to the self-duality group of that orbit, so we color the group in teal to reflect this fact. SK orbits or self-duality groups with ✗'s indicate those cases where either the number of orbits or the self-duality group, respectively, does not agree with the field theory S-duality conjectures as stated in [4, 5, 8], c.f. table 6.

We interpret this latter disagreement as resulting from the moduli space geometry simply failing to distinguish between inequivalent absolute $\mathcal{N}$=4 field theories because they lack sensitivity to do so as an IR observable. This is not a contradiction: there is no *a priori* reason that a low-energy observable such as the moduli space geometry should be different for all microscopically inequivalent field theories, i.e. the mapping from distinct SCFTs to a distinct (scale-invariant) moduli space geometry is allowed to be many-to-one.[8] Presumably, if we include additional data from the field theory beyond the Kähler geometry of the moduli space specified by (1.1), such as the BPS charge lattice of massive states out on the moduli space or the mass deformations of the SK geometry, then this discrepancy between self-duality groups would be resolved because such data would provide a non-trivial compatibility constraint on the self-duality transformations of the SK orbit.[9] Conversely, when classifying SK geometries, there is no *a priori* reason that all of them should be realized as the moduli space of a physical QFT. This is because we might lack knowledge of a sufficient set of physical consistency conditions to impose on the geometries to render them physical. Therefore, in our present classification, we can conclude that we are not lacking such knowledge if we find that each distinct SK geometry occurs as the moduli space of an absolute $\mathcal{N}$=4 sYM theory. Despite the enlarged SK orbit self-duality groups for the A_1 and BC_r Weyl groups, evidence for this conclusion comes from the fact that we always find the same number of SK orbits as there are S-duality orbits.

With the understanding that the S-dualities of $\mathcal{N}$=4 sYM theories are (formally) conjectures, our results can also be seen as a test of these equivalences because we compute the duality structure of one of their low-energy observable without direct reference to the duality structure of the field theory. We provide a more detailed comparison between the duality structure of SK orbits and S-duality orbits of absolute sYM theories in section 4. We find complete agreement with the field theory results apart from the discrepancies mentioned above, and we further elaborate on their explanations. How the moduli space geometry of an SCFT is reflected in and derived from the operator content of the field theory is a vexed question. Both $\mathcal{N}$=4 sYM theories and their moduli space geometries are well-studied, so the classification problem we address in this paper provides a concrete setting where this relationship can be studied. In table 2, we describe the relationship between some conformal field theory (CFT) observables and geometric quantities associated with the Kähler geometry of its moduli space of vacua (1.1) that are motivated by our

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[8]For example, it is known that there are distinct 4d $\mathcal{N}$=2 SCFTs with the same scale-invariant CB geometry. This already occurs at rank-1: the $\mathcal{N}$=4 $\mathfrak{su}(2)$ superYang-Mills theory (which has a hypermultiplet in the adjoint) and the $\mathfrak{su}(2)$ gauge theory with four fundamental hypermultiplets. Both theories have the same scale-invariant CB geometry (with an I_0^* singularity). They're distinguished by their mass deformations which is reflected in their flavor symmetries: the former has $SU(2)$ while the latter has $SO(8)$ flavor symmetry [10].

[9]Note that it is not consistent for the opposite disagreement between self-duality groups to occur: the self-duality group of an SK orbit cannot be smaller than the self-duality group of the SCFT it is the moduli space geometry of. This would imply there are self-identifications within the marginal coupling space of the SCFT that are not respected by their corresponding moduli space geometries. This is a contradiction because any identifications among SCFTs implies their low-energy observables, such as their moduli space geometries, must also be the same. We find no such discrepancies in our classification.

results; it also serves to introduce some terminology that we use in the rest of the paper.

$\mathcal{N}$=4 $\mathfrak{g}$ sYM CFT	relation	$\mathcal{N}$=4 $\mathfrak{g}$ sYM moduli space of vacua
Chiral ring of scalar BPS operators	1-to-1	Coordinate ring of moduli space Kähler geometry
Global structures of sYM gauge theory	many-to-1	Ext classes of pairs $(R_A, R_A^\vee)$; $R_A \doteq \mathrm{GL}(r, \mathbb{Z})$ irrep of $W(\mathfrak{g})$
S-duality orbit of absolute $\mathcal{N}$=4 sYM theory *	1-to-1	principal SK orbit $\doteq$ equivalence class of principal, $\mathrm{GL}(2r, \mathbb{Z})$ rep of $W(\mathfrak{g})$
S-duality group of absolute $\mathcal{N}$=4 sYM theory	subgroup of	Self-duality group of principal SK orbit
Conformal manifold of absolute $\mathcal{N}$=4 sYM theory	covers	Conformal manifold of principal SK orbit

Table 2. Relationship between some observables in the sYM conformal field theory with geometric quantities associated with its moduli space of vacua. The terminology in the right-most column is defined in section 2 of the text. The asterisk * is to indicate that the S-duality orbits and SK orbits are in 1-to-1 correspondence only after a correction to the list of global structure S-duality orbits, described in section 3, is incorporated.

Outline of the paper. Appendix A derives the properties the SK geometry of the moduli space of an $\mathcal{N}$=4 sYM theory must obey. This content is meant to serve as background material for the paper, but much of the discussion is probably well-known to experts.

Section 2 describes our procedure for classifying all possible $\mathcal{N}$=4 sYM SK structures. We keep the discussion general by not assuming that the Dirac pairing, J, is principal. Some of the mathematics we need concerning representations of finite groups over the integers is summarized in appendix B.

Section 3 is devoted to explaining our classification results for principal J. Table 3 summarizes the intermediate computation of Ext groups while table 1 summarizes our final results in terms of those Ext groups. A Mathematica notebook automating these calculations for every Weyl group is available from the authors upon request. Among the infinite series of Weyl groups corresponding to simple Lie algebras, we prove our results uniformly in r for the most intricate case in section 3.1: the Weyl groups $W(A_r)$ of the $A_r = \mathfrak{su}(r+1)$) series. We do not actually prove the classification results for all r of the BC_r and D_r series of Weyl groups. We instead report the pattern found by brute force computation, though a proof along the lines of section 3.1 should not be too hard, just lengthy. We detail the full classification computations for $\mathfrak{g} = \mathfrak{so}(12)$ in section 3.2 and $\mathfrak{g} = \mathfrak{so}(5) = \mathfrak{sp}(4)$ in section 3.3; these examples were chosen to illustrate the peculiar features of the most involved computations that occur for the cases in which outer automorphisms of the Weyl group must be taken into account.

Section 4 compares our SK orbit classification results to the field theory classification results of absolute sYM orbits that are predicted by S-duality, and discusses the relationship between global structures of the sYM gauge theory and SK geometries.

Section 5 raises the question of whether there are "exotic" principally polarized CB geometries corresponding to $\mathcal{N}$=4 sYM with non-simple gauge algebras which are not just the product of the simple ones. This hinges on finding reducible symplectic representations which are indecomposable, and involves the use of non-principally polarized symplectic representations as building blocks. We show how $\mathfrak{g} = \mathfrak{u}(N) = \mathfrak{u}(1) \oplus \mathfrak{su}(N)$ gives an example of this, albeit with a free factor.

Further directions. While considering the possible SK geometries that are consistent with unbroken $\mathcal{N}$=4 supersymmetry but not necessarily a weak-coupling limit corresponding to an $\mathcal{N}$=4 sYM theory, some obvious questions are:

▸ For simple $\mathfrak{g}$, the relative field theories of all $\mathcal{N}$=4 sYM theories have non-principal Dirac pairing on their BPS charge lattice besides $\mathfrak{g} = G_2$ and F_4 [9].[10] For each $\mathfrak{g}$, is there always a unique SK geometry with the corresponding non-principal polarization of the BPS charge lattice that could correspond to this minimal/"relative" version of the field theory? More generally, do all non-principally polarized SK geometries compatible with (1.1) have physical interpretations?

▸ Besides the $\mathfrak{u}(N)$ examples of section 5, are there other example of "exotic" principally polarized geometries built on reducible (non-principal) symplectic representations of Weyl groups?

▸ How are generalized global symmetries, such as 1- and 2-form symmetries [11] and non-invertible duality defects [12], reflected in the SK geometry, and how do the geometries change upon the (discrete) gauging of these symmetries? (See [13–15] for a related discussion.)

▸ If we drop the requirement of a weak-coupling limit, so we are not forced to assume the Kähler geometry of the moduli space of vacua described by the orbifold (1.1), are there other non-Weyl group isotrivial SK orbifold geometries compatible with $\mathcal{N}$=4 supersymmetry? More generally, are there non-orbifold isotrivial geometries? (See [16] for a related discussion.)

The project of classifying and constructing isotrivial moduli space geometries can also be extended to the more difficult cases that possess unbroken $\mathcal{N}$=3 or $\mathcal{N}$=2 supersymmetry. Many of the above questions also apply to these theories, or, in some cases, must be addressed. As the results of the present paper demonstrate, such making progress on these constructions promises to shed light on defining the analog of global structures in sYM theories to non-lagrangian theories where there is no notion of gauge group.

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[10]By relative field theory we mean the version of an $\mathcal{N}$=4 sYM theory that is defined by $\mathfrak{g}$ and its marginal coupling τ_{YM}, which determine its BPS charge lattice, but is not equipped with a choice of global structure that completes it to an absolute theory.

Note added: After the completion of this work, the paper [17] appeared, and we received a preliminary copy of [18], both of which have overlap with this paper. Where our results overlap, they agree.

2 Integer representations of Weyl and complexified Weyl groups

2.1 Ingredients of an $\mathcal{N}$=4 SK structure

We collect and briefly describe some ingredients of SK structures of $\mathcal{N}$=4 CB geometries that are needed to describe their classification in the rest of this section.

For a given Weyl group W = Weyl($\mathfrak{g}$) for a simple Lie algebra $\mathfrak{g}$ (with Cartan subalgebra $\mathfrak{h}$), and for a given integral symplectic form J on $\mathfrak{h} \oplus \mathfrak{h} \cong \mathbb{R}^{2r}$, we consider pairs $(S, \boldsymbol{\tau})$ where

- $S : W \to \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ is an integral, symplectic representation that is $\mathbb{R}$-equivent to two copies of the fundamental reflection representation of W,

- $\boldsymbol{\tau}$ is an $r \times r$ matrix in the Siegel upper half-space such that $\boldsymbol{\tau} \in \mathrm{Fix}(S)$, i.e., for all $w \in W$, $S(w) \circ \boldsymbol{\tau} = \boldsymbol{\tau}$, given in (A.14). This presupposes that we work in a basis where J takes the skew-block form (2.4).

We then define the concept of $\mathcal{N}$=4 sYM SK structures associated with $\mathfrak{g}$ and fixed choice of J as follows:

- An $\mathcal{N}$=4 *SK structure* for Lie algebra $\mathfrak{g}$ and integral symplectic form J is an equivalence class of pairs $(S, \boldsymbol{\tau})$, defined as above, under the equivalence relation

$$(S, \boldsymbol{\tau}) \sim (MSM^{-1}, M \circ \boldsymbol{\tau}), \qquad \text{for all} \qquad M \in \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z}). \tag{2.1}$$

- An *SK structure orbit* is an equivalence class of just the integral symplectic representation S, i.e., without specifying a compatible value of $\boldsymbol{\tau}$.

These definitions and their motivation are described in detail in appendix A. A brief summary is that the SK structure of a rank-r $\mathcal{N}$=4 sYM moduli space, is specified by an r-dimensional complex orbifold

$$\mathcal{C} = \mathbb{C}^r / W_\mathbb{C}, \tag{2.2}$$

that carries a canonical Kähler structure but is also equipped with some extra structure that comprises a special Kähler structure. Here $W_\mathbb{C} \subset \mathrm{GL}(r, \mathbb{C})$ is the complexification of the Weyl group, a finite group acting linearly, holomorphically, and faithfully on $\mathbb{C}^r$. Because $W_\mathbb{C}$ is a reflection group, it is generated by reflections, elements $r_I \in W_\mathbb{C}$ of order 2 which fix a codimension 1 hyperplane in $\mathbb{C}^r$. The collection of these hyperplanes is the fixed point set of the orbifold action, and $\mathcal{C}^*$ is the smooth, non-simply connected component of $\mathcal{C}$ that is obtained by removing this fixed point set from $\mathcal{C}$.

Part of the extra structure is the rank-$2r$ lattice, Λ, of electric and magnetic charges of states and probes under the low energy $\mathrm{U}(1)^r$ gauge group. This lattice carries an integral symplectic form, J, giving the Dirac pairing between charges. The charge lattice may suffer

a monodromy $S_{\mathbb{Z}}(\gamma) \in \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ upon being dragged around the fixed point hyperplanes of $W_{\mathbb{C}}$ along some closed path γ.[11] These are basis changes of the lattice which preserve the symplectic pairing $J^\vee$ according to the matrix relation $S_{\mathbb{Z}}^t(\gamma) J^\vee S_{\mathbb{Z}}(\gamma) = J^\vee$. Such monodromies define an integral symplectic representation $S_{\mathbb{Z}}$ of $\pi_1(\mathcal{C}^*)$ into $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ that we call the *monodromy map*. The image $\mathrm{Im}S_{\mathbb{Z}} \subset \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ of the monodromy map forms a subgroup we call the *monodromy group*. As shown in appendix A, the monodromy group is isomorphic to the orbifold group $W_{\mathbb{C}}$ as abstract groups, so it is generated by the reflection elements, $r_I \in W_{\mathbb{C}}$. Through this isomorphism, we obtain an integral symplectic representation of $W_{\mathbb{C}}$ that we simply denote by S as used in the above definition of an $\mathcal{N}{=}4$ SK structure (orbit). Since $W_{\mathbb{C}}$ is the complexification of the real reflection group $W_{\mathbb{R}} \subset \mathrm{GL}(r, \mathbb{R})$, relative to a real basis of $\mathbb{C}^r$, $W_{\mathbb{C}}$ is the reducible $2r$-real-dimensional representation $W_{\mathbb{R}} \oplus W_{\mathbb{R}}$. Thus, for an $\mathcal{N}{=}4$ sYM CB, we have

$$\mathrm{Im}S_{\mathbb{Z}} \cong_{\mathbb{R}} W_{\mathbb{R}} \oplus W_{\mathbb{R}}, \tag{2.3}$$

or, in words, the monodromy group $\mathrm{Im}S_{\mathbb{Z}}$ defined by the integral symplectic representation $S_{\mathbb{Z}}$ must be equivalent over the reals to two copies of the real reflection representation of the orbifold group.

A *symplectic basis* of Λ is a choice of splitting into lagrangian sublattices ("magnetic" and "electric") with respect to J, $\Lambda = \Lambda_m \oplus \Lambda_e$, with respect to which J is skew block off-diagonal,

$$J = \begin{pmatrix} 0 & \mathbf{j} \\ -\mathbf{j}^t & 0 \end{pmatrix}, \tag{2.4}$$

with $\mathbf{j}$ an integral $r \times r$ matrix. Even with the matrix form of J fixed, there is still a large group of elements $M \in \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z}) \subset \mathrm{GL}(2r, \mathbb{Z})$ generating basis changes of Λ that preserve $J^\vee$ under the action $J^\vee \mapsto M^t J^\vee M = J^\vee$ which implies $J \mapsto MJM^t = J$. We define this group to be the *EM duality group* of the low-energy theory on $\mathcal{C}$. Two integral symplectic representations, S and S', satisfying (2.3) and with respect to Dirac pairings J and J' taking the same matrix form (so they have the same invariant factors) in a choice of their symplectic bases, are *integrally equivalent*,

$$S \cong_{\mathbb{Z}} S', \tag{2.5}$$

iff there is an $M \in \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ such that $S'(w) = MS(w)M^{-1}$ for all $w \in W$. We call M an *integral intertwiner* of the representations S and S', each of which define SK structure orbits. Therefore, two integrally equivalent symplectic representations of the Weyl group describe the same SK structure orbit, since they differ only by a symplectic change of charge lattice basis which is an EM duality transformation of the low-energy effective action.

We must actually use a slightly more general notion of integral equivalence of symplectic representations than the one described above and in (2.1). For any *reflection automorphisms* of the Weyl group, $\phi, \psi \in \mathrm{Aut}_{\mathrm{refl.}}(W)$, then $S \cong_{\mathbb{Z}} S'$ if there is an $M^{(\phi,\psi)} \in \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ such that $S'(\phi \circ w)M^{(\phi,\psi)} = M^{(\phi,\psi)}S(\psi \circ w)$ for all $w \in W$. In this case we

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[11]Here $J^\vee \doteq J^{-t}$ denotes the inverse transpose; the reason for its occurrence is explained in appendix A.

say $M^{(\phi,\psi)}$ is an integral *twisted intertwiner*. Denote the set of these twisted intertwiners which intertwine a representation S with itself by

$$\widetilde{\mathscr{S}_S} \doteq \bigcup_{\phi,\psi\in\mathrm{Aut}_{\mathrm{refl.}}(W)} \left\{ M \in \mathrm{Sp}_{J^\vee}(2r,\mathbb{Z}) \mid (S\circ\phi)\, M = M\,(S\circ\psi) \right\}. \tag{2.6}$$

The structure of this set is described in more detail in section 2.7, where we show that it is an image in $\mathrm{Sp}_{J^\vee}(2r,\mathbb{Z})$ of elements of the *SK structure orbit self-duality group* under corresponding self-intertwiner maps, which we also define there.

Relative to a choice of symplectic basis, (2.4), the low-energy effective action also specifies at each point on the moduli space an $r \times r$ complex matrix of gauge couplings,

$$\boldsymbol{\tau} \in \mathscr{H}_r \doteq \left\{ \boldsymbol{\tau} \in \mathrm{GL}(r,\mathbb{C}),\ \boldsymbol{\tau} = \boldsymbol{\tau}^t,\ \mathrm{Im}\,\boldsymbol{\tau} > 0 \right\}. \tag{2.7}$$

$\mathscr{H}_r$ is the Siegel upper half-space of degree r. The EM duality group acts on $\mathscr{H}_r$ via a fractional linear action (see appendix A) which we denote $\boldsymbol{\tau} \mapsto M \circ \boldsymbol{\tau}$ for $M \in \mathrm{Sp}_{J^\vee}(2r,\mathbb{Z})$. Two $\boldsymbol{\tau}$ related in this way are equivalent since they differ only by a change of basis of the charge lattice. In the case of $\mathcal{N}{=}4$ sYM moduli spaces, the SK structure is *isotrivial*, which means that $\boldsymbol{\tau}$ is constant over the moduli space. Thus the action $\boldsymbol{\tau} \mapsto S \circ \boldsymbol{\tau}$ of the monodromy group must leave $\boldsymbol{\tau}$ invariant. Call the fixed point set of this S action $\mathrm{Fix}(S)$. Therefore, for $\mathcal{C}$ isotrivial we must have

$$\boldsymbol{\tau} \in \mathrm{Fix}(S) \subset \mathscr{H}_r. \tag{2.8}$$

Fixed point sets of integrally equivalent representations are $\mathrm{Sp}_{J^\vee}(2r,\mathbb{Z})$ conjugate under the corresponding intertwiner. The twisted self-intertwiners, $\widetilde{\mathscr{S}_S}$, induce identifications on $\mathrm{Fix}(S)$. We call $\mathrm{Fix}(S)$ modulo these identifications the *conformal manifold of S*, denoted $\mathscr{C}(S)$, in light of its expected connection to SCFTs. It is the moduli space describing the inequivalent SK structures within a given SK orbit S, i.e., a given symplectic representation equivalence class. The group of identifications on $\mathrm{Fix}(S)$ is the *self-duality group of the SK orbit S*; we will see that for $\mathcal{N}{=}4$ moduli spaces, they are parameterized by $\mathrm{PSL}(2,\mathbb{R})$ matrices which form modular groups as discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

$\mathcal{N}{=}4$ sYM theories with simple gauge algebra have one exactly marginal complex coupling. Thus we expect $\mathscr{C}(S)$ to be one-dimensional.[12] The conformal manifold of the set of SK structures with a given $\mathfrak{g}$ and J is the disjoint union $\coprod_i \mathscr{C}(S_i)$ of all integrally inequivalent SK orbits S_i.

In summary, an $\mathcal{N}{=}4$ moduli space SK structure is determined by an equivalence class of pairs $(S, \boldsymbol{\tau})$ under EM duality transformations as claimed in (2.1). All other properties of the moduli space geometry (its special coordinates and its metric) are determined by the $(S, \boldsymbol{\tau})$ equivalence class, as explained in appendix A.

Thus, to classify all $\mathcal{N}{=}4$ sYM moduli spaces, for each gauge Lie algebra $\mathfrak{g}$ of rank r we need to compute all integrally inequivalent $\mathrm{Sp}_{J^\vee}(2r,\mathbb{Z})$ representations S of $W_\mathbb{C} = \mathbb{C}\otimes W(\mathfrak{g})$,

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[12]There is an equivalent realization of the geometry of the SK orbit conformal manifold as the modular curve formed by the quotient of $\mathscr{H}_1$ by the self-duality group of the SK orbit. We describe this perspective in section 2.7.

the complexified Weyl group, and also compute all $\tau \in \text{Fix}(S)$. We show how to carry this out in the rest of this section. For most of this section we keep our discussion general by not making any assumption on J. But, along the way, we will see that, modulo the overall normalization of J which cannot be determined in a scale-invariant isotrivial SK geometry,[13] there are only a finite number of allowed inequivalent polarizations.

2.2 Classifying integral symplectic representations of a Weyl group

In this section, we provide an explicit algorithm which, given a Weyl group W and a symplectic pairing J, fully classifies the corresponding $\mathcal{N}{=}4$ SK structures. We summarize here the steps and the main results, referring to the subsections below for the details.

▸ **Step 1: W-invariant lattices.** We construct a finite number of W-invariant lattices Γ_A for $A \in \mathcal{I}$ an index set. These are in one-to-one correspondence with $\mathbb{Z}$-equivalence classes of $\text{GL}(r, \mathbb{Z})$ representations of W which are $\mathbb{Q}$-equivalent to the reflection representation.

▸ **Step 2: A sufficient class of symplectic integral representations.** Given two such representations $A, B \in \mathcal{I}$ we construct a family of $\text{Sp}_{J^\vee}(2r, \mathbb{Z})$ representations $S_{(A,B;D)}$ of W, which depend on a *binding* D, and show that every SK structure contains at least one (S, τ) where $S \sim_{\mathbb{Z}} S_{(A,B;D)}$.

▸ **Step 3: Equivalence class of bindings.**

It is then enough to classify $\mathbb{Z}$-equivalence classes of these $S_{(A,B;D)}$ representations. We proceed as follows. Given $A, B \in \mathcal{I}$, we find for which D, D' are $S_{(A,B;D)}$ and $S_{(A,B;D')}$ $\mathbb{Z}$-equivalent. We show that the set of equivalence classes is parametrized by an extension group $\text{Ext}^1_{\mathbb{Z}W}(R_A, R_B^\vee)$ that we can explicitly compute. Hence we introduce a new finite index set

$$\mathfrak{I} \doteq \bigsqcup_{A,B \in \mathcal{I}} \text{Ext}^1_{\mathbb{Z}W}(R_A, R_B^\vee). \tag{2.9}$$

Elements of $\mathfrak{I}$ are denoted by

$$\text{labels of Weyl reps } R_A, \ R_B^\vee \ \rule{0pt}{0pt} \boxed{\mathcal{A} \doteq (A, B; D) \in \mathfrak{I},} \ \rule{0pt}{0pt} \text{element of } \text{Ext}^1_{\mathbb{Z}W}(R_A, R_B^\vee) \tag{2.10}$$

where we are now taking D to stand for an equivalence class of bindings, which is an element of the corresponding $\text{Ext}^1_{\mathbb{Z}W}$ group. (We have boxed and annotated this definition because it is a notation we employ extensively in the rest of the paper.) For each $\mathcal{A}$, its fixed point set in the rank-r Seigel half space, $\text{Fix}(S_\mathcal{A}) \subset \mathscr{H}_r$, is 1-dimensional (as expected).

▸ **Step 4: Integral equivalences of symplectic representations.**

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[13]The normalization of J can be determined by turning on mass deformations, as was shown, for example, in rank 1 SCFTs [10, 19]. Also, an induction argument in the rank [20] constrains the allowed J to a finite list at each rank for theories with characteristic dimension [21] not equal to 1 or 2.

We construct explicitly a family of maps labeled by a pair $(\mathcal{A}_i, \mathcal{A}_j) \in \mathfrak{I} \times \mathfrak{I}$

$$\mathcal{M}_{\mathcal{A}_i \mathcal{A}_j} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(2r,\mathbb{R}) \tag{2.11}$$

such that for all $w \in W$, $\mathcal{A}_i \in \mathfrak{I}$, and $\gamma_a \in \mathrm{SL}(2,\mathbb{R})$, they satisfy intertwiner and morphism properties,

$$\mathcal{M}_{\mathcal{A}_1 \mathcal{A}_2}(\gamma) S_{\mathcal{A}_2}(w) = S_{\mathcal{A}_1}(w) \mathcal{M}_{\mathcal{A}_1 \mathcal{A}_2}(\gamma), \tag{2.12}$$
$$\mathcal{M}_{\mathcal{A}_1 \mathcal{A}_2}(\gamma_1) \mathcal{M}_{\mathcal{A}_2 \mathcal{A}_3}(\gamma_2) = \mathcal{M}_{\mathcal{A}_1 \mathcal{A}_3}(\gamma_1 \gamma_2).$$

$\mathrm{SL}(2,\mathbb{R})$ acts by fractional linear transformations on the natural coordinate, $\tau \in \mathscr{H}_1$, parameterizing $\mathrm{Fix}(S_{\mathcal{A}})$. These maps and (2.12) thus encapsulate the algebra of equivalences among SK structures $(S_{\mathcal{A}}, \boldsymbol{\tau})$.

In particular, this allows us to parameterize the set of intertwiners implementing integral equivalences between $S_{\mathcal{A}_1}$ and $S_{\mathcal{A}_2}$ by

$$\mathscr{S}_{\mathcal{A}_1 \mathcal{A}_2} \doteq \mathcal{M}_{\mathcal{A}_1 \mathcal{A}_2}^{-1}(\mathrm{Sp}_{J^\vee}(2r,\mathbb{Z})) \subset \mathrm{SL}(2,\mathbb{R}), \tag{2.13}$$

where $\mathcal{M}_{\mathcal{A}_1 \mathcal{A}_2}^{-1}$ denotes the preimage in $\mathrm{SL}(2,\mathbb{R})$. It is fully characterized in terms of divisibility conditions on the $\mathrm{SL}(2,\mathbb{R})$ matrix elements. We show that the set of intertwiners is exactly

$$\{M \in \mathrm{Sp}_{J^\vee}(2r,\mathbb{Z}) | M S_{\mathcal{A}_1} = S_{\mathcal{A}_2} M\} = \mathcal{M}_{\mathcal{A}_1 \mathcal{A}_2}(\mathscr{S}_{\mathcal{A}_1 \mathcal{A}_2}), \tag{2.14}$$

which is to say, $M \in \mathrm{Sp}_{J^\vee}(2r,\mathbb{Z})$ intertwines $S_{\mathcal{A}_1}$ and $S_{\mathcal{A}_2}$ iff there exists a $\gamma \in \mathrm{SL}(2,\mathbb{R})$ such that $M = \mathcal{M}_{\mathcal{A}_1 \mathcal{A}_2}(\gamma)$. Thus the maps $\mathcal{M}_{\mathcal{A}_1 \mathcal{A}_2}(\cdot)$ capture all purely inner intertwiners between $S_{\mathcal{A}_1}$ and $S_{\mathcal{A}_2}$,

So we define an equivalence relation on $\mathfrak{I}$ by

$$\mathcal{A}_1 \sim \mathcal{A}_2 \qquad \text{if and only if} \qquad \mathscr{S}_{\mathcal{A}_1 \mathcal{A}_2} \neq \varnothing. \tag{2.15}$$

Then the set of SK structure orbits associated with $\mathfrak{g}$ and J are in bijection with $\mathfrak{I}/\sim$. So, in particular, for fixed $\mathfrak{g}$ and J,

$$\# \,\text{SK structure orbits} \quad \equiv \quad \begin{array}{c} \#\text{connected components of} \\ \text{the conformal manifold of SK} \\ \text{structures with given } \mathfrak{g} \text{ and } J \end{array} \quad = \quad |\mathfrak{I}/\sim|. \tag{2.16}$$

▸ **Step 5: Self-duality groups of SK structure orbits.**

For a given SK structure corresponding to $\mathcal{A} \in \mathfrak{I}$, the set of integer self-intertwiners implementing self-equivalences,

$$\mathscr{S}_{\mathcal{A}} \doteq \mathcal{M}_{\mathcal{A}\mathcal{A}}^{-1}(\mathrm{Sp}_{J^\vee}(2r,\mathbb{Z})) \subset \mathrm{SL}(2,\mathbb{R}), \tag{2.17}$$

is a *group*. In particular, it is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ that parameterizes the similarity transformations of an SK structure which act trivially on $S_{\mathcal{A}}$. Thus, it is

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by definition the *self-duality group of the SK structure orbit* $S_{\mathcal{A}}$, or, for short, the *S-duality group of* $S_{\mathcal{A}}$. This definition of $\mathscr{S}_{\mathcal{A}}$, as well as the definition (2.13), will have to be modified to include intertwiners twisted by reflection automorphisms as in (2.6). This modification is explained in appendix A.3 and in section 2.7.

This notion of self-duality group of an SK structure orbit is related to, but in principle distinct from, the self-duality group of the associated $\mathcal{N} = 4$ sYM theory. As explained in section 4, we find the two do not always agree, and such cases are marked by red ✗'s in the summary table 1.

The results of the computation of the SK structure orbits and their self-duality groups are summarized for simple $\mathfrak{g}$ and principally polarized J in table 1. Section 3 discusses the details of these computations. In particular, for an explicit example of the calculations, the reader is urged to look at the $\mathfrak{g} = \mathfrak{so}(12)$ example worked out in section 3.2.

2.3 Weyl group invariant lattices

Our first task is to list $\mathbb{Z}$-equivalence classes of $\mathrm{GL}(r, \mathbb{Z})$ representations of W which are $\mathbb{Q}$-equivalent to the reflection representation of W. These are in one-to-one correspondence with lattice representations, i.e. representations formed by restricting the reflection representation of W on $\mathbb{R}^r$ to W-invariant lattices in $\mathbb{R}^r$, as a change of basis for a lattice is given by a matrix in $\mathrm{GL}(r, \mathbb{Z})$, and have been classified by Feit [3].[14] Hence, **Step 1** amounts to reviewing the classification of the $\mathbb{Z}$-equivalence classes of rank-r W-invariant lattices from which we can construct the corresponding lattice representations. We first introduce some notation.

▸ Consider a simple real Lie algebra $\mathfrak{g}$ of rank r, $\mathfrak{h}$ a Cartan subalgebra, and $\mathfrak{h}^*$ its linear dual. We denote by $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ the Killing form, which is a non-degenerate symmetric bilinear form. K will also denote the induced inner products on $\mathfrak{h}$ and $\mathfrak{h}^*$.

▸ We pick a basis $(\alpha_i)_{i=1,\dots,r}$ of simple roots, and denote by $\Gamma_{\mathrm{root}} \subset \mathfrak{h}^*$ the lattice they generate. The long roots are normalized such that they have norm squared equal to 2, i.e., $K(\alpha_{\mathrm{long}}, \alpha_{\mathrm{long}}) = 2$. Let $(\varpi_i)_{i=1,\dots,r}$ be the corresponding basis of fundamental weights,[15] and $\Gamma_{\mathrm{weight}} \subset \mathfrak{h}^*$ the lattice they generate. $Z \doteq \Gamma_{\mathrm{weight}}/\Gamma_{\mathrm{root}}$ is the center of the simply connected Lie group with Lie algebra $\mathfrak{g}$.

▸ The Weyl group W is the subgroup of $\mathrm{O}(\mathfrak{h}^*) \cong \mathrm{O}(r, \mathbb{R})$ generated by reflections about hyperplanes perpendicular to the simple roots. We call w_i the (simple) reflection about the simple root α_i. This defines an action of W on $\mathfrak{h}^*$ that renders $\mathfrak{h}^*$ an irreducible representation of W that we call the *reflection representation* and denote by R. It is a real orthogonal representation with respect to the positive definite scalar product K.

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[14]Note that [22] corrects an error in the classification of [3].

[15]Recall that to each root α is associated the one dimensional eigenspace $\mathfrak{g}_\alpha \subset \mathfrak{g}$, and there is a unique element $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(H_\alpha) = 2$. The fundamental weights are defined uniquely by the relation $\varpi_i(H_{\alpha_j}) = \delta_{ij}$.

Weyl($\mathfrak{g}$)	$A \subset Z(\mathfrak{g})$	Lattice Γ_A	$\mathrm{Ext}^1_{\mathbb{Z}W}(R_A, R_A^\vee)$
A_1	$\mathbb{Z}_1$	Γ_{root}	$\mathbb{Z}_1 = \{0\}$
A_r $(r \geq 2)$	$\mathbb{Z}_d$ $d\,\vert\,(r+1)$	Γ_d	$\mathbb{Z}_{\gcd(d,\,(r+1)/d)}$
BC_2	$\mathbb{Z}_2 \simeq \mathbb{Z}_2^\vee$	$\Gamma_{\mathrm{co-wt}} = \Gamma_{\mathrm{root}} \simeq \Gamma_{\mathrm{coroot}} = \Gamma_{\mathrm{wt}}$	$\mathbb{Z}_2 = \{0, 1\}$
BC_{2k} $(k \geq 2)$	$\mathbb{Z}_2$	$\Gamma^C_{\mathrm{root}} = \Gamma^B_{\mathrm{coroot}}$	$\mathbb{Z}_2 = \{0, 1\}$
	$\mathbb{Z}_2^\vee$	$\Gamma^B_{\mathrm{root}} = \Gamma^C_{\mathrm{coroot}}$	$\mathbb{Z}_2 = \{0, 1\}$
	$\mathbb{Z}_1$	$\Gamma^B_{\mathrm{wt}} = \Gamma^C_{\mathrm{co-wt}} = \Gamma^C_{\mathrm{wt}} = \Gamma^B_{\mathrm{co-wt}}$	$\mathbb{Z}_1 = \{0\}$
BC_{2k+1}	$\mathbb{Z}_2$	$\Gamma^C_{\mathrm{root}} = \Gamma^B_{\mathrm{coroot}}$	$\mathbb{Z}_1 = \{0\}$
	$\mathbb{Z}_2^\vee$	$\Gamma^B_{\mathrm{root}} = \Gamma^C_{\mathrm{coroot}}$	$\mathbb{Z}_2 = \{0, 1\}$
	$\mathbb{Z}_1$	$\Gamma^B_{\mathrm{wt}} = \Gamma^C_{\mathrm{co-wt}} = \Gamma^C_{\mathrm{wt}} = \Gamma^B_{\mathrm{co-wt}}$	$\mathbb{Z}_1 = \{0\}$
D_4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Γ_{root}	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 01, 10, 11\}$
	$\mathbb{Z}_{2V} \simeq \mathbb{Z}_{2S} \simeq \mathbb{Z}_{2C}$	$\Gamma_V \simeq \Gamma_S \simeq \Gamma_C$	$\mathbb{Z}_2 = \{0, 1\}$
	$\mathbb{Z}_1$	Γ_{weight}	$\mathbb{Z}_2 = \{0, 1\}$
D_{4k} $(k \geq 2)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Γ_{root}	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 01, 10, 11\}$
	$\mathbb{Z}_{2V}$	Γ_V	$\mathbb{Z}_2 = \{0, 1\}$
	$\mathbb{Z}_{2S} \simeq \mathbb{Z}_{2C}$	$\Gamma_S \simeq \Gamma_C$	$\mathbb{Z}_2 = \{0, 1\}$
	$\mathbb{Z}_1$	Γ_{weight}	$\mathbb{Z}_2 = \{0, 1\}$
D_{4k+2}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Γ_{root}	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 01, 10, 11\}$
	$\mathbb{Z}_{2V}$	Γ_V	$\mathbb{Z}_2 = \{0, 1\}$
	$\mathbb{Z}_{2S} \simeq \mathbb{Z}_{2C}$	$\Gamma_S \simeq \Gamma_C$	$\mathbb{Z}_1 = \{0\}$
	$\mathbb{Z}_1$	Γ_{weight}	$\mathbb{Z}_1 = \{0\}$
D_{2k+1}	$\mathbb{Z}_4$	Γ_{root}	$\mathbb{Z}_1 = \{0\}$
	$\mathbb{Z}_2$	Γ_V	$\mathbb{Z}_2 = \{0, 1\}$
	$\mathbb{Z}_1$	Γ_{weight}	$\mathbb{Z}_1 = \{0\}$
E_6	$\mathbb{Z}_3$	Γ_{root}	$\mathbb{Z}_1 = \{0\}$
	$\mathbb{Z}_1$	Γ_{weight}	$\mathbb{Z}_1 = \{0\}$
E_7	$\mathbb{Z}_2$	Γ_{root}	$\mathbb{Z}_1 = \{0\}$
	$\mathbb{Z}_1$	Γ_{weight}	$\mathbb{Z}_1 = \{0\}$
E_8	$\mathbb{Z}_1$	Γ_{root}	$\mathbb{Z}_1 = \{0\}$
F_4	$\mathbb{Z}_1 \simeq \mathbb{Z}_1^\vee$	$\Gamma_{\mathrm{root}} \simeq \Gamma_{\mathrm{coroot}}$	$\mathbb{Z}_1 = \{0\}$
G_2	$\mathbb{Z}_1 \simeq \mathbb{Z}_1^\vee$	$\Gamma_{\mathrm{root}} \simeq \Gamma_{\mathrm{coroot}}$	$\mathbb{Z}_1 = \{0\}$

Table 3. For each Weyl group (1st column, using Killing-Cartan notation) are listed the subgroups of the center of the corresponding Lie algebra(s) (2nd column). The 3rd column gives the names from Lie algebra theory of the corresponding Weyl-invariant lattices. The 4th column computes the Ext group (2.35) of bindings between R_A and $R_A^\vee$, and labels each Ext group element for use in table 1. Some subgroups A appear twice in the 2nd column as A and $A^\vee$ when the dual lattice $\Gamma_A^\vee$ is not integrally equivalent to $\Gamma_{A'}$ for any subgroup A'. Also, equivalences between subgroups/lattices that are due to outer reflection automorphisms of the Weyl groups are shown using $\simeq$. The 2nd column should be seen as attributing a name/label to each distinct lattice given by its corresponding subgroup with additional notation to distinguish lattices with identical subgroups.

- For any subgroup $H \subseteq Z$, there is a lattice $\Gamma_H \subset \mathfrak{h}^*$ such that $H = \Gamma_{\text{weight}}/\Gamma_H$. There is also a corresponding dual lattice $\Gamma_H^\vee \subset \mathfrak{h}^*$, where by dual we mean with respect to the scalar product K. In particular,

$$\begin{aligned}
\Gamma_{\text{weight}} &\doteq \Gamma_{\{1\}} \supseteq \Gamma_H \supseteq \Gamma_Z \doteq \Gamma_{\text{root}} \\
\Gamma_{\text{coroot}} &\doteq \Gamma_{\{1\}}^\vee \subseteq \Gamma_H^\vee \subseteq \Gamma_Z^\vee \doteq \Gamma_{\text{coweight}}
\end{aligned} \tag{2.18}$$

For simply laced algebras the dual lattices do not provide new lattices; in particular, $\Gamma_{\text{weight}} \cong \Gamma_{\text{coweight}}$ and $\Gamma_{\text{root}} \cong \Gamma_{\text{coroot}}$. For non-simply-laced algebras, however, the dual lattices can give non-isomorphic representations.

- The restrictions of the representation R to the lattices Γ_H and $\Gamma_H^\vee$ constructed above, for subgroup $H \subseteq Z$, provide *integral* representations of W that we call respectively R_H and $R_H^\vee$. These representations are all equivalent over $\mathbb{R}$ but they may or may not be equivalent over $\mathbb{Z}$.

- If we pick a basis $\boldsymbol{\gamma}^H = (\gamma_i^H)_{i=1,\dots,r}$ of Γ_H, we can associate to it the dual basis $\boldsymbol{\gamma}_H = (\gamma_H^i)_{i=1,\dots,r}$ defined by $K(\gamma_i^H, \gamma_H^j) = \delta_i^j$. When expressed in these bases, the representations R_H and $R_H^\vee$ induce explicit group morphisms $W \to \text{GL}(r, \mathbb{Z})$ that we denote $R_{H,\boldsymbol{\gamma}^H}$ and $R_{H,\boldsymbol{\gamma}_H}^\vee$,

$$R_{H,\boldsymbol{\gamma}^H} : w \mapsto \text{Mat}_{\boldsymbol{\gamma}^H}(R(w)), \qquad R_{H,\boldsymbol{\gamma}_H}^\vee : w \mapsto \text{Mat}_{\boldsymbol{\gamma}_H}(R(w)). \tag{2.19}$$

Then we have $R_{H,\boldsymbol{\gamma}_H}^\vee(w) = \left(R_{H,\boldsymbol{\gamma}^H}(w)\right)^{-t}$.

- To lighten the notations, we assume a choice of basis $\boldsymbol{\gamma}^H = (\gamma_i^H)_{i=1,\dots,r}$ of Γ_H has been made once and for all, and we suppress it from the notations. Furthermore, we index the set of inequivalent representations and invariant lattices by some label set $\mathcal{I}$, so R_A for $A \in \mathcal{I}$ runs over the inequivalent R_H and $R_H^\vee$ for all $H \subset Z$, and similarly for lattices Γ_A along with their chosen bases, $\boldsymbol{\gamma}^A$. We also denote the inverse transpose of a matrix by a $^\vee$ superscript,

$$R_A^\vee \doteq R_A^{-t}. \tag{2.20}$$

Thus $R^\vee$ denotes what is commonly called the dual or contragredient representation to R.

- Since the set $\{R_A, A \in \mathcal{I}\}$ has, by definition, all inequivalent representations, the contragredient set $\{R_A^\vee, A \in \mathcal{I}\}$ does too, though generally permuted and relative to different bases. In the case of simply-laced Lie algebras, we can (and do) choose representatives $A \leftrightarrow \Gamma_H$, i.e., without using any dual lattices $\Gamma_H^\vee$. But for non-simply-laced Lie algebras there are necessarily be inequivalent dual representations, $R_H^\vee \not\cong_{\mathbb{Z}} R_{H'}$ for any H'. Consequently in these cases there are distinct labels $A \leftrightarrow R_H$ and $B \leftrightarrow R_H^\vee$, and their associated representations are contragredient,

$$R_A^\vee \doteq R_A^{-t} = \left(R_{H,\boldsymbol{\gamma}^H}\right)^{-t} = R_{H,\boldsymbol{\gamma}_H}^\vee = R_B. \tag{2.21}$$

The full list of lattices is given in table 3 [3, 22].

▸ We call I_{AB} the matrix of the identity map acting on $\mathfrak{h}$, mapping from $\mathfrak{h}$ in basis $\boldsymbol{\gamma}^B$ to $\mathfrak{h}$ in basis $\boldsymbol{\gamma}^A$. Similarly, we call K_{AB} the matrix of the Killing form in bases $\boldsymbol{\gamma}^A$ and $\boldsymbol{\gamma}^B$. (In particular, the subscript AB labels these matrices and not their matrix components.) This means we have the matrix equations,

$$I_{AB}R_B(w) = R_A(w)I_{AB}\,, \qquad \text{for all } w \in W, \tag{2.22}$$

and

$$K_{AB}R_B(w) = R_A^\vee(w)K_{AB}\,, \qquad \text{for all } w \in W. \tag{2.23}$$

Thus I_{AB} and K_{AB} are intertwiners (a.k.a., W-equivariant maps) between R_B and R_A or $R_A^\vee$, respectively. We use the defining properties (2.22) and (2.23) of these intertwiners extensively in all computations to follow. Note that

$$I_{AA} = \mathbf{1}_r, \qquad\qquad I_{AB}^{-1} = I_{BA}, \qquad\qquad K_{AB}^t = K_{BA}, \tag{2.24}$$

and

$$I_{AB}I_{BC} = I_{AC}, \qquad K_{AB}I_{BC} = K_{AC}, \qquad K_{AB}^\vee K_{BC} = I_{AC}, \tag{2.25}$$

but, in general, $I_{AB}^t \neq I_{BA}$, $K_{AB}^{-1} \neq K_{BA}$ and $I_{AB}K_{BC} \neq K_{AC}$.

The identification of the integer representations, or, equivalently, invariant lattices are familiar from gauge theory. But in application to non-lagrangian theories a description in terms of gauge theory does not exist, so we are interested in their properties without regard to any chosen basis. The intertwiner matrices, I_{AB} and K_{AB}, defined above carry this information. By defining them as the matrices of the identity map and Killing form on $\mathfrak{h}$ with respect to our conventional lattice bases $\boldsymbol{\gamma}^A$ and $\boldsymbol{\gamma}^B$, we have chosen a particular normalization for I_{AB} and K_{AB}. Basis changes of the Γ_A and Γ_B lattices change I_{AB} and K_{AB} by multiplication on the left and right by arbitrary $\mathrm{GL}(r,\mathbb{Z})$ matrices. Thus invariant information in the individual intertwiner matrices can be encoded by normalizing them to integer matrices by dividing them by the rational gcd of their entries, i.e., $K_{AB}/\gcd(K_{AB})$ and $I_{AB}/\gcd(I_{AB})$. Then, by multiplication on the left and right by $\mathrm{GL}(r,\mathbb{Z})$ matrices they can be put in unique invariant factor form (a.k.a., Smith normal form), for example, $K_{AB}/\gcd(K_{AB}) \sim \mathrm{diag}\left\{1, f_1, f_1 f_2, \ldots, \prod_{i=1}^{r-1} f_i\right\}$, for positive integers f_i. (The first entry is 1 because we have factored out the gcd.) Note that this implies that $K_{AB}^\vee/\gcd(K_{AB}^\vee) \sim \mathrm{diag}\{1, f_{r-1}, f_{r-1}f_{r-2}, \ldots, \prod_{i=1}^{r-1} f_i\}$, and that

$$n_{AB} \doteq \gcd(K_{AB})^{-1} \cdot \gcd(K_{BA}^\vee)^{-1} = \prod_{i=1}^{r-1} f_i, \tag{2.26}$$

is an integer invariant. We will see below that this invariant governs the S-duality groups of many of the SK structures associated to the pair of representation $(R_A, R_B^\vee)$.

Similar definitions apply to the I intertwiners. Furthermore, since, by definition, $R_A \not\cong_\mathbb{Z} R_B$ for $A \neq B$, it follows that I_{AB} cannot be invertible over the integers for $A \neq B$, and thus their invariant factors cannot all be 1. Conversely, if $A = B$, then $I_{AA} = \mathbf{1}$, and its invariant factors are all obviously 1.

We emphasize, however, that the invariant factors of the I's and K's do not exhaust the invariant information about the representations that they encode. For example, the Γ_A and Γ_B basis changes which put, say, I_{AB} in invariant factor form need not put K_{AB} in that form, nor allow I_{BC} or K_{BC} to be put in that form by a basis change of Γ_C. Thus there is much more invariant information in the whole set of intertwiners than in their individual invariant factors.

2.4 Symplectic representations of Weyl groups and their fixed points

We now perform **Step 2** of the method outlined in section 2.2, that is, we construct a sufficient class of symplectic integral representations. Consider an integral symplectic form J of the block skew-diagonal form (2.4), where the skew block, **j**, is a non-degenerate $r \times r$ matrix with integer coefficients. The group $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ is the group of matrices $M \in \mathrm{Mat}(2r, \mathbb{Z})$ such that $M^t J^\vee M = J^\vee$.[16] The usual, or *principally polarized*, symplectic group is recovered by taking $\mathbf{j} = \mathbf{1}$, in which case $J = J^\vee$.

Introduce for $A, B \in \mathcal{I}$, following [23], a *binding* of R_A with $R_B^\vee$, which is a map $L_{(AB,D)} : W \to \mathrm{Mat}(r, \mathbb{Z})$ of the form

$$L_{(A,B;D)}(w) \doteq R_A(w)I_{AB}D - I_{AB}DR_B^\vee(w) \ \in \ \mathrm{Mat}(r, \mathbb{Z}) \tag{2.27}$$

for all $w \in W$ for some symmetric matrix

$$D = D^t \ \in \ \mathrm{Mat}(r, \mathbb{R}). \tag{2.28}$$

Note that a non-zero binding is not a group morphism. And to each binding we associate the symplectic representation

$$\boxed{\begin{aligned} S_{(A,B;D)} : W &\to \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z}) \\ w &\mapsto \begin{pmatrix} R_A(w) & L_{(A,B;D)}(w) \\ 0 & R_B^\vee(w) \end{pmatrix} \end{aligned}}, \tag{2.29}$$

with J given by (2.4) with

$$\mathbf{j} = I_{AB}. \tag{2.30}$$

Interpreted as the Dirac pairing in the low energy theory on the CB, **j** is integral (the Dirac quantization condition) and its invariant factors are physical observables. On the other hand, I_{AB} is generally not an integral matrix as we have defined it. But its normalization does not affect the SK geometry since J only enters linearly in the definition of the EM duality group. We are thus free to normalize it to be integral by multiplying it by $n/\gcd(I_{AB})$ for any non-zero integer n, though we cannot determine the value of n by our methods. So we ignore the physical normalization of **j** and just take it to be (2.30) from now on.

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[16]As reviewed in appendix A, it is $\mathrm{Sp}_{J^\vee}$, and not Sp_J, which governs how the special coordinates and $\boldsymbol{\tau}$ transform under EM duality transformations. The two matrix groups are related by the inverse transpose of their elements, i.e. $M \in \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ iff $M^{-t} \in \mathrm{Sp}_J(2r, \mathbb{Z})$.

The fact that (2.29) is a representation follows upon noticing that

$$S_{(A,B;D)} = \begin{pmatrix} 1 & I_{AB}D \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} R_A & 0 \\ 0 & R_B^\vee \end{pmatrix} \begin{pmatrix} 1 & I_{AB}D \\ 0 & 1 \end{pmatrix}. \tag{2.31}$$

That it is symplectic follows from direct calculation, where Schur's lemma implies that $S_{(A,B;D)}$ preserves $J^\vee$ iff $D - D^t \propto K_{BB}^\vee$. It then follows from (2.24) that the factor of proportionality vanishes and so D must be symmetric.

A theorem, reviewed in appendix B, implies that every SK structure has a representative of the form $S_{(A,B;D)}$ given by (2.29). More precisely, it says that any representation $W \to \mathrm{Sp}_{J^\vee}(2r, \mathbb{R})$ that is $\mathbb{R}$-equivalent to the direct sum of two copies of the fundamental reflection representation is $\mathbb{Z}$-equivalent to some $S_{(A,B;D)}$. So it is sufficient to determine when two $S_{(A,B;D)}$ are integrally equivalent to find all the inequivalent symplectic representations of W.

The condition that $\tau \in \mathrm{Fix}(S)$ is easily translated, using the action (A.14) of S on $\mathscr{H}_r$, to the condition that for all $w \in W$, $R_A(w)(\boldsymbol{\tau} + I_{AB}DI_{BA}^\vee) = (\boldsymbol{\tau} + I_{AB}DI_{BA}^\vee)R_A^\vee(w)$. So, by Schur's lemma and (2.23), there exists a complex number τ (not to be confused with the $r \times r$ complex matrix $\boldsymbol{\tau}$) such that $\boldsymbol{\tau} + I_{AB}DI_{BA}^\vee = \tau K_{AA}^\vee$, implying $\mathrm{Fix}(S)$ is the one-complex-dimensional set,

$$\boxed{\boldsymbol{\tau} = \tau K_{AA}^\vee - I_{AB}DI_{BA}^\vee, \qquad \tau \in \mathbb{C}.} \tag{2.32}$$

$\boldsymbol{\tau} \in \mathscr{H}_r$ iff $\boldsymbol{\tau}$ is symmetric, which is automatic for all $\boldsymbol{\tau}$ from (2.32), and if $\mathrm{Im}\,\boldsymbol{\tau} > 0$. This latter condition follows if τ is restricted to the upper half-plane, $\tau \in \mathscr{H}_1$, i.e.,

$$\mathrm{Im}\tau > 0, \tag{2.33}$$

because K_{AA} is positive definite since the Killing form on $\mathfrak{h}^*$ from which it is derived is. We will eventually identify τ with the exactly marginal gauge coupling of the $\mathcal{N}=4$ sYM theory.

We refer to the whole 1-dimensional family of SK structures associated to S, parameterized by $\tau \in \mathscr{H}_1$, as an *SK structure orbit*. Note that for any $M \in \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$, $(S_M, \boldsymbol{\tau}_M) \doteq (MSM^{-1}, M \circ \boldsymbol{\tau})$ is an equivalent SK structure, and S and S' define the same SK structure orbit. These equivalences are just the low energy EM duality frame equivalences, which identify $\mathrm{Fix}(S)$ and $\mathrm{Fix}(MSM^{-1})$ in the Siegel half-space. But if an $M \neq 1$ exists such that $MSM^{-1} = S$ then it implies an identification $\boldsymbol{\tau} \equiv M \circ \boldsymbol{\tau}$ on $\mathrm{Fix}(S)$, and thus an identification on $\tau \in \mathscr{H}_1$. We call its set of inequivalent $\tau \in \mathscr{H}_1$ the *conformal manifold* of the SK structure orbit represented by $S_{(A,B;D)}$. The group of such identifications on $\mathscr{H}_1$ form the *self-duality group of the SK structure orbit*, since the conformal manifold of the SK structure orbit is the quotient of the covering space, $\mathscr{H}_1$, by the action of this group on τ.

2.5 Classifying bindings by Ext groups

We now turn to **Step 3** of the method outlined in section 2.2, which is the first step in determining when two $S_{(A,B;D)}$ are integrally equivalent. This is to determine for given

$A, B \in \mathcal{I}$ which bindings D give integrally inequivalent symplectic representations $S_{(A,B;D)}$. A binding can be thought of as specifying an extension of the R_A representation (module) by $R_B^\vee$,

$$1 \to R_B^\vee \to S_{(A,B;D)} \to R_A \to 1. \tag{2.34}$$

Such extensions are classified by Ext groups. In particular, introduce the notion of *inner binding* as a binding of the form (2.27) but with $I_{AB}D \in \mathrm{Mat}(r, \mathbb{Z})$. In this case, by (2.31), $S_{(A,B;D)} \cong_\mathbb{Z} S_{(AB,0)}$. Let $\mathcal{B}$ be the group of bindings (under addition), and $\mathcal{B}_0$ be the subgroup of inner bindings. Then

$$\mathrm{Ext}^1_{\mathbb{Z}W}(R_A, R_B^\vee) = \mathcal{B}/\mathcal{B}_0 \tag{2.35}$$

classifies the inequivalent $S_{(A,B;D)}$ extensions. The largest denominator appearing in a rational entry in D is bounded by the order of W, thus giving an upper bound on the size of the $\mathrm{Ext}^1_{\mathbb{Z}W}(R_A, R_B^\vee)$ groups; however, for Weyl groups, the denominators are typically much smaller.[17] These results are reviewed in appendix B.

Note that if $D \propto K_{BB}^\vee$, then $L \equiv 0$ by (2.22) and (2.23). So the groups $\mathcal{B}$ and $\mathcal{B}_0$ can be computed as the groups of rational and integer $I_{AB}D$, respectively, satisfying (2.27), as long as the ideal in $\mathrm{Mat}(r, \mathbb{Q})$ generated by $K_{BB}^\vee$ is quotiented out in computing $\mathcal{B}$. Upon clearing denominators, (2.27) for all $w \in W$ becomes a set of linear diophantine equations for the numerators and common denominator of entries of D.

Practical computation. Such equations can be solved by any of various efficient algorithms for computing the Hermite normal form of the coefficient matrix. The solutions depend on the relative divisibility properties of the matrix elements of the R_A and $R_B^\vee$ representations, and seems complicated in general. In particular, the solutions for D, and thus the $\mathrm{Ext}^1_{\mathbb{Z}W}(R_A, R_B^\vee)$ groups, do not seem to be expressed in terms of the K and I intertwiners in any simple way.

Instead, we list the results in the fourth column of table 3 in the case where $B = A$, which are the symplectic representations which preserve a principal polarization. We prove these results for the most intricate case, the A_r (i.e., $\mathfrak{su}(r+1)$) series of Weyl groups, in section 3.1. But we do not actually prove the results listed in table 3 for the BC_r and D_r series of Weyl groups; instead, we have reported the pattern found by brute force computation at low ranks. We implemented these computations for any Weyl group in a Mathematica notebook which is available from the authors upon request. A proof of the BC_r and D_r series results along the lines of that of the A_r series results of section 3.1 should not be too hard, just lengthy.

In slightly more detail, in order to compute the extension group, we first consider an arbitrary symmetric $r \times r$ matrix $D = (D_{\mu\nu})$, with $D_{\mu\nu} = D_{\nu\mu} \in \mathbb{Q}$. In order to impose condition (2.27), we compute the r matrices $R_A(w_i)D - DR_A^\vee(w_i)$, whose entries are r^3 linear forms of the $r(r+1)/2$ variables $D_{\mu\nu}$ with integer coefficients (as the matrices $R_A(w_i)$

---

[17]For Weyl groups of rank r, the largest denominator grows only as $\sim \sqrt{r}$ even though the order of the groups grow as $\sim r!$.

have integer entries). These linear forms can be repackaged as a linear map of lattices $\mathcal{Y} : \mathbb{Z}^{r(r+1)/2} \to \mathbb{Z}^{r^3}$, along with a chosen basis (corresponding to the parametrization of D). Hence $\mathcal{Y}$ can be seen as a $r^3 \times r(r+1)/2$ matrix with integer entries. Using the Hermite decomposition algorithm, one can find a *unimodular* map $\mathcal{U} : \mathbb{Z}^{r^3} \to \mathbb{Z}^{r^3}$ (equivalently, an $r^3 \times r^3$ matrix with integer entries and determinant 1) such that $\mathcal{U}\mathcal{Y}$ is *upper triangular*. It is then straightforward to compute $(\mathcal{U}\mathcal{Y})^{-1}(\mathbb{Z}^{r^3})$, as the upper-triangularity reduces the problem to solving r^3 linear equations in *one* variable. This allows to write down the generic solution to (2.27) as a linear combination

$$D = \beta I_{BA} K_{AB}^{\vee} + n_1 \mathfrak{D}_1 + \cdots + n_s \mathfrak{D}_s \tag{2.36}$$

with $\beta \in \mathbb{Q}$ and $n_1, \ldots, n_s \in \mathbb{Z}$. Using this form, it is then straightforward to compute the group of bindings, that of inner bindings, and thus the quotient. See section 3.2 for an explicit example of such a computation.

2.6 Integral equivalence of symplectic representations

Once we have found the inequivalent bindings D for a choice of integral Weyl reflection representations A and B which preserve a given $J^{\vee}$, it still remains to determine which $S_{\mathcal{A}}$ with $\mathcal{A} = (A, B; D)$ are integrally equivalent among all the different choices of $\mathcal{A}$. Mathematically, this amounts to classifying the following equivalence classes. We have seen that a pair (S, τ) determines an SK geometry, where $S : W \to \mathrm{Sp}_{J^{\vee}}(2r, \mathbb{Z})$ is a faithful representation of the Weyl group $W_{\mathbb{C}}(\mathfrak{g})$ that is $\mathbb{Q}$-equivalent to two copies of the reflection representation of W, and $\tau \in \mathrm{Fix}(S) \subset \mathscr{H}_r$. And, based on the EM duality of the IR effective action, two such descriptions define the same geometry, $(S, \tau) \simeq (S', \tau')$, if there exists an $M \in \mathrm{Sp}_{J^{\vee}}(2r, \mathbb{Z})$ such that

$$MS(w)M^{-1} = S'(w) \qquad \text{and} \qquad M \circ \tau = \tau' \qquad \forall w \in W. \tag{2.37}$$

Actually, this is not the most general notion of equivalence, and below we will correct it by allowing for equivalences between integral symplectic representations S and S' that are intertwined by some $M \in \mathrm{Sp}_{J^{\vee}}(2r, \mathbb{Z})$ after one of them has been composed with a reflection preserving automorphism of the Weyl group. But, as a first pass, we classify SK geometries with respect to this definition (2.37) of equivalence. This is **Step 4** of the classification strategy described in section 2.2.

The key to classifying these equivalence classes is the following technical lemma that allows us to characterize the $\mathrm{Sp}_{J^{\vee}}(2r, \mathbb{Z})$ matrices M that satisfy the intertwining condition in (2.37):

Lemma 1. *Any* $\mathrm{GL}(2r, \mathbb{R})$ *matrix that intertwines* $S_{\mathcal{A}_1}$ *and* $S_{\mathcal{A}_2}$ *(2.37) is of the form*

$$\boxed{\mathcal{M}_{\mathcal{A}_1 \mathcal{A}_2}(\gamma) \doteq \begin{pmatrix} 1 & -I_{A_1 B_1} D_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{a} I_{A_1 A_2} & \underline{b} K_{A_1 B_2}^{\vee} \\ \underline{c} K_{B_1 A_2} & \underline{d} I_{B_1 B_2}^{\vee} \end{pmatrix} \begin{pmatrix} 1 & I_{A_2 B_2} D_2 \\ 0 & 1 \end{pmatrix}}, \tag{2.38}$$

where

$$\gamma \doteq \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}), \tag{2.39}$$

and

$$\det\left[\mathcal{M}_{\mathcal{A}_1\mathcal{A}_2}(\gamma)\right] = (\det\gamma)^r \det I_{A_2A_1} \det I_{B_1B_2}. \tag{2.40}$$

Here $\mathcal{A}_i = (A_i, B_i; D_i)$ as usual, and we underline $\underline{a}, \underline{b}, \underline{c}$, and $\underline{d}$ only because we are running out of letters. The expression (2.38) comes from a direct computation and several uses of Schur's lemma. The determinant comes from the formula for a block matrix determinant, and from the identity $K_{B_2A_1} I_{A_1A_2} K^\vee_{A_2B_1} = I^\vee_{B_2B_1}$, which follows from (2.24) and (2.25).

The $\mathcal{M}_{\mathcal{A}_1\mathcal{A}_2}$ so defined are thus maps

$$\mathcal{M}_{\mathcal{A}_1\mathcal{A}_2} : \mathrm{GL}(2,\mathbb{R}) \to \mathrm{GL}(2r,\mathbb{R}), \tag{2.41}$$

whose image intertwines $S_{\mathcal{A}_1}$ and $S_{\mathcal{A}_2}$,

$$S_{\mathcal{A}_1}(w)\, \mathcal{M}_{\mathcal{A}_1\mathcal{A}_2}(\gamma) = \mathcal{M}_{\mathcal{A}_1\mathcal{A}_2}(\gamma)\, S_{\mathcal{A}_2}(w), \qquad \forall w \in W,\ \gamma \in \mathrm{GL}(2,\mathbb{R}). \tag{2.42}$$

These maps also obey the important composition property:

Lemma 2. *For any* $\gamma, \gamma' \in \mathrm{GL}(2,\mathbb{R})$ *and any* $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathfrak{I}$,

$$\mathcal{M}_{\mathcal{A}\mathcal{B}}(\gamma)\mathcal{M}_{\mathcal{B}\mathcal{C}}(\gamma') = \mathcal{M}_{\mathcal{A}\mathcal{C}}(\gamma\gamma'). \tag{2.43}$$

This follows from the identities (2.25).

Such matrices give an equivalence between the two symplectic representations if $\mathcal{M}_{\mathcal{A}_1\mathcal{A}_2}$ is integral and has determinant ± 1. By virtue of (2.42), they satisfy $\mathcal{M}^t_{\mathcal{A}_1\mathcal{A}_2} J^\vee_1 \mathcal{M}_{\mathcal{A}_1\mathcal{A}_2} \propto J^\vee_2$, where J_i are the symplectic forms preserved by the $S_{\mathcal{A}_i}$ representations. If J_1 and J_2 have different invariant factors then there does not exist an integral invertible $\mathcal{M}_{\mathcal{A}_1\mathcal{A}_2}$ intertwining them. Thus there can only be an equivalence between representations with equivalent symplectic forms. Furthermore, if J_1 and J_2 are equivalent symplectic forms, then any $\mathcal{M}_{\mathcal{A}_1\mathcal{A}_2}$ intertwining them has determinant $+1$. So we introduce the sets of 2×2 matrices

$$\mathscr{S}_{\mathcal{A}_1\mathcal{A}_2} \doteq \left\{\, \gamma \in \mathrm{GL}(2,\mathbb{R}) \mid \mathcal{M}_{\mathcal{A}_1\mathcal{A}_2}(\gamma) \in \mathrm{SL}(2r,\mathbb{Z}) \,\right\}, \tag{2.44}$$

that yield such integral intertwiners. Note that in general the composition property (2.43) does not require $\mathscr{S}_{\mathcal{A}_1\mathcal{A}_2}$ to form a group. The determinant condition can only have a solution if $(\det I_{A_2A_1} \det I_{B_1B_2})^{1/r}$ is rational. Since by (2.30) the normalized integral pairings preserved by the two representations are $\mathbf{j}_1 = I_{A_1B_1}/\gcd(I_{A_1B_1})$ and $\mathbf{j}_2 = I_{A_2B_2}/\gcd(I_{A_2B_2})$, it follows that

$$\det I_{A_2A_1} \det I_{B_1B_2} = \det I_{A_2B_2} \det I_{B_1A_1} = (\det\mathbf{j}_2/\det\mathbf{j}_1)\left[\gcd(I_{A_1B_1})/\gcd(I_{A_2B_2})\right]^r,$$

where we have used that $I_{B_2A_2}I_{A_2A_1} = I_{B_2B_1}I_{B_1A_1}$ by (2.25). $\mathbf{j}_1$ and $\mathbf{j}_2$ are equivalent iff they have the same invariant factors, in which case their determinants are the same. So, a refinement of **Lemma 1** is

Lemma 3. *Let $S_{\mathcal{A}_1}$ and $S_{\mathcal{A}_2}$ be symplectic representations with respect to equivalent symplectic forms. Then they are $\mathbb{Z}$-equivalent representations iff there is an <u>integral</u> $\mathcal{M}_{\mathcal{A}_1\mathcal{A}_2}(\gamma)$ of the form* (2.38) *with $\gamma \in \mathrm{GL}(2,\mathbb{R})$ and $\det\gamma = \gcd(I_{A_2B_2})/\gcd(I_{A_1B_1})$.*

This gives a computable condition for the pair of symplectic representations $S_{\mathcal{A}_1}$ and $S_{\mathcal{A}_2}$ to be $\mathbb{Z}$-equivalent. For, since we are assuming that the I's, K's, and D's are given (because we have solved the extension problem described in the previous subsection), integrality of $\mathcal{M}_{\mathcal{A}_1\mathcal{A}_2}(\gamma)$ amounts to some divisibility criteria on $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d}$ which can be solved algorithmically as for the Ext problem.

We have carried out these calculations for all pairs of representations symplectic with respect to a principally polarized symplectic form. As we discuss in section 2.8, these are the representations $S_{\mathcal{A}}$ with $\mathcal{A} = (A, A; D)$. The results are recorded in the 2nd column of table 1 which lists the $\mathcal{A}$ equivalence classes in terms of the naming system for the invariant lattices and Ext group elements summarized in table 3. The identifications of $\mathcal{A}$ equivalence classes indicated by blue arrows in that table do not follow from the above lemma. We will discuss these additional equivalences shortly. These computations have also been implemented in a Mathematica notebook available from the authors on request.

2.7   SK structure orbit self-duality groups

We now turn to **Step 5** of the method described in section 2.2, where we characterize the group of self-equivalences of an SK structure.

For $\mathcal{A} = (A, B; D)$, the self-intertwiners $\mathcal{M}_{\mathcal{A}\mathcal{A}}$ take the form,

$$\mathcal{M}_{\mathcal{A}\mathcal{A}}(\gamma) = \begin{pmatrix} 1 & -I_{AB}D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{a} & \underline{b}K^{\vee}_{AB} \\ \underline{c}K_{BA} & \underline{d} \end{pmatrix} \begin{pmatrix} 1 & I_{AB}D \\ 0 & 1 \end{pmatrix}, \tag{2.45}$$

where, as above, $\gamma \doteq \left(\begin{smallmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$. From **Lemma 2**, $\mathcal{M}_{\mathcal{A}\mathcal{A}}$ is a group morphism; in fact, it is an isomorphism onto its image. We introduce the group $\mathscr{S}_{\mathcal{A}}$ of 2×2 real determinant 1 matrices which parameterize the self-intertwiners of the SK structure orbit $\mathcal{A}$. It is therefore defined as

$$\mathscr{S}_{\mathcal{A}} \doteq \mathscr{S}_{\mathcal{A}\mathcal{A}} = \mathcal{M}^{-1}_{\mathcal{A}\mathcal{A}}(\mathrm{Sp}_{J^{\vee}}(2r,\mathbb{Z})) \subset \mathrm{SL}(2,\mathbb{R}), \tag{2.46}$$

where $\mathcal{M}^{-1}_{\mathcal{A}\mathcal{A}}$ denotes the preimage in $\mathrm{SL}(2,\mathbb{R})$. It is some discrete subgroup defined by divisibility conditions. If there were no additional automorphism-twisted intertwiners, then $\mathscr{S}_{\mathcal{A}}$ would be the duality group of the SK orbit $\mathcal{A}$; we will refine the definition of S-duality group below when we incorporate twisted intertwiners.

$\widetilde{\mathscr{S}_{\mathcal{A}}} \doteq \mathcal{M}_{\mathcal{A}\mathcal{A}}(\mathscr{S}_{\mathcal{A}})$ is thus the group of intertwiners of the $\mathcal{A}$ SK structure orbit.[18] $\widetilde{\mathscr{S}_{\mathcal{A}}}$ acts on $\tau \in \mathscr{H}_r$ via the action $\tau \mapsto \mathcal{M}_{\mathcal{A}\mathcal{A}}(\gamma) \circ \tau$ given in (A.14). If this τ is in $\mathrm{Fix}(S)$ then a calculation shows that this maps the $\tau \in \mathscr{H}_1$ coordinate on $\mathrm{Fix}(S)$ via fractional linear transformations,

$$\gamma : \tau \mapsto \left(\begin{smallmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{smallmatrix}\right) \circ \tau = \frac{\underline{a}\tau + \underline{b}}{\underline{c}\tau + \underline{d}}. \tag{2.47}$$

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[18]It is only a subgroup of the $\widetilde{\mathscr{S}_{\mathcal{A}}}$ defined in (2.6) since it does not (yet) contain the automorphism twisted intertwiners.

Thus $\mathscr{S}_{\mathcal{A}}$ acts as a group of identifications of the coupling τ in the upper half-plane, giving a conformal manifold of the SK structure orbit $\mathcal{A}$ as $\mathscr{H}_1/\mathscr{S}_{\mathcal{A}}$, the modular curve of $\mathscr{S}_{\mathcal{A}}$.[19] (More properly, $\mathscr{S}_{\mathcal{A}}$ should be defined as a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ because there is no difference between the action of $\gamma, -\gamma \in \mathscr{S}_{\mathcal{A}}$ on τ; for convenience, we stick with the realization of $\mathscr{S}_{\mathcal{A}}$ as $\mathrm{SL}(2,\mathbb{R})$ matrices where this $\mathbb{Z}_2$ identification is understood.)

In the special case where the binding vanishes, $D = 0$, we can be even more specific about what the group $\mathscr{S}_{\mathcal{A}}$ is. In this case $\mathcal{M}_{\mathcal{A}\mathcal{A}}(\gamma) \in \mathrm{Sp}_{J^\vee}(2r,\mathbb{Z})$ iff

$$\gamma = \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} = \begin{pmatrix} \underline{\alpha} & \underline{\beta}/\mathrm{gcd}(K_{AB}^\vee) \\ \underline{\gamma}/\mathrm{gcd}(K_{AB}) & \underline{\delta} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Q}), \qquad \underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta} \in \mathbb{Z}. \tag{2.48}$$

By conjugating $\gamma \mapsto N\gamma N^{-1}$ by $N = \begin{pmatrix} \sqrt{\mathrm{gcd}(K_{AB}^\vee)} & 0 \\ 0 & 1/\sqrt{\mathrm{gcd}(K_{AB}^\vee)} \end{pmatrix}$, we see that the self-duality group of the $\mathcal{A} = (AB, 0)$ orbit is equivalent to the modular congruence group

$$\mathscr{S}_{(AB,0)} \simeq \Gamma_0(n_{AB}) \doteq \left\{ \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \ \underline{c} = 0 \ (\mathrm{mod} \ n_{AB}) \right\}, \tag{2.49}$$

where n_{AB} is the integer invariant defined in (2.26).

Interlude on outer automorphisms of Weyl groups.

We now turn to including automorphism twisted intertwiners in the SK structure orbit self-duality groups. First, we summarize a few facts about automorphisms of Weyl groups, mostly taken from [24, 25]. For each Weyl group W, one can define:

- ► $\mathrm{Aut}(W)$ is the group of all group automorphisms of W.

- ► $\mathrm{Inn}(W)$ is the group of inner automorphisms, i.e., those which act on W by conjugation, $W \mapsto vWv^{-1}$ for some $v \in W$.

- ► $\mathrm{Out}(W) = \frac{\mathrm{Aut}(W)}{\mathrm{Inn}(W)}$ is the group of outer automorphisms. They are listed in table 4.

- ► $\mathrm{Aut}_{\mathrm{refl.}}(W)$ is the group of all $\phi \in \mathrm{Aut}(W)$ preserving reflections, i.e., such that $\phi(w)$ is a reflection whenever w is a reflection. All inner automorphisms are reflection automorphisms, $\mathrm{Inn}(W) \subset \mathrm{Aut}_{\mathrm{refl.}}(W)$.

- ► $\mathrm{Out}_{\mathrm{refl.}}(W) = \frac{\mathrm{Aut}_{\mathrm{refl.}}(W)}{\mathrm{Inn}(W)}$ is the group of outer automorphisms preserving reflections, or *reflection outer automorphisms* for short, also listed in table 4.

In the next paragraph, we will see that in the computation of the modifications of SK structure orbits and their self-duality groups that come from the inclusion of automorphism intertwiners, the relevant group is $\mathrm{Out}_{\mathrm{refl.}}(W)$.

Symmetries of the Coxeter diagram[20] for W correspond to reflection automorphisms of W, since these symmetries interchange nodes of the diagrams which corresponds to

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[19]Again, upon including automorphism-twisted identifications, as we do below, the S-duality group may be enlarged, and the conformal manifold correspondingly reduced.

[20]Recall that Coxeter diagrams of Weyl groups are undirected versions of the Dynkin diagrams of the corresponding Lie algebras, i.e., without the arrow or marking differentiating long from short roots.

interchanging the corresponding simple generating reflections which define W as a finite reflection group. These diagram symmetries are shown in table 4. Though they are reflection automorphisms, they are not necessarily outer automorphisms: only the diagram automorphisms of the BC_2, D_{2r}, F_4, and G_2 Weyl groups are outer. These turn out to be all the reflection outer automorphisms of Weyl groups.

Finally, as a quotient of $\mathrm{Aut}_{\mathrm{refl.}}(W)$, $\mathrm{Out}_{\mathrm{refl.}}(W)$ may fail to be a subgroup of $\mathrm{Aut}_{\mathrm{refl.}}(W)$. But in the Weyl group case, the quotient splits as $\mathrm{Aut}_{\mathrm{refl.}}(W) = \mathrm{Inn}(W) \rtimes \mathrm{Out}_{\mathrm{refl.}}(W)$, so we can (and do) realize $\mathrm{Out}_{\mathrm{refl.}}(W) \subset \mathrm{Aut}_{\mathrm{refl.}}(W)$.

W	$\mathrm{Out}(W) = \frac{\mathrm{Aut}(W)}{\mathrm{Inn}(W)}$	$\mathrm{Out}_{\mathrm{refl.}}(W) = \frac{\mathrm{Aut}_{\mathrm{refl.}}(W)}{\mathrm{Inn}(W)}$	$\mathrm{Aut}(\mathrm{Cox}(W))$
$A_{r \neq 5}$	1	1	$\mathbb{Z}_2$
A_5	$\mathbb{Z}_2$	1	$\mathbb{Z}_2$
BC_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$BC_{\mathrm{odd}>2}$	$\mathbb{Z}_2$	1	1
$BC_{\mathrm{even}>2}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	1	1
D_4	$S_3 \times \mathbb{Z}_2$	S_3	S_3
$D_{\mathrm{odd}>4}$	1	1	$\mathbb{Z}_2$
$D_{\mathrm{even}>4}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
E_6	1	1	$\mathbb{Z}_2$
E_7	1	1	1
E_8	$\mathbb{Z}_2$	1	1
F_4	D_8	$\mathbb{Z}_2$	$\mathbb{Z}_2$
G_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table 4. Weyl groups W (first column), their outer automorphisms $\mathrm{Out}(W)$ (second column), their outer automorphisms preserving reflections $\mathrm{Out}_{\mathrm{refl.}}(W)$ (third column), and automorphisms of the W Coxeter diagram $\mathrm{Aut}(\mathrm{Cox}(W))$ (fourth column). D_8 is the dihedral group of order 8.

Additional equivalences from reflection outer automorphisms of Weyl groups

The definition (2.46) of self-duality groups is not quite correct, since the definition (2.37) of $\mathbb{Z}$-equivalent representations missed the *automorphism twisted $\mathbb{Z}$-equivalences* discussed in appendix A.3. Briefly, a reflection automorphism maps a CB geometry to an isometric one with the same EM monodromies and low energy coupling τ simply because it gives an isomorphism $f : \mathcal{C} \to \mathcal{C}$ of the orbifold Kähler structure which maps the SK structure representation $S_\mathcal{A}$ on the CB to another one $S_\mathcal{B}$ on the isometric CB given by pulling back by the inverse map, $S_\mathcal{B} = (f^{-1})^* S_\mathcal{A}$. Said another way, a Weyl group automorphism induces a fiber-preserving symplectomorphism of the total space of the CB. Thus the two CB geometries are the same. Equivalently, there is no low energy way to physically distinguish the effective actions on two CBs related in this way.

So we generalize integral equivalence of reflection representations to automorphism twisted $\mathbb{Z}$-equivalences, defined in (A.28) and formalized in appendix B. It is useful to apply this both to the the integral irreducible Weyl group reflection representations, R_A, and to the integral reducible symplectic complexified Weyl group reflection representations,

$S_{\mathcal{A}}$. Written out for the irreducible reflection representations this is

$$R_A \cong_{\mathbb{Z},\phi} R_B \quad \text{if} \quad M R_A = (R_B \circ \phi) M \quad \text{for} \quad M \in \mathrm{GL}(r,\mathbb{Z}) \quad \text{and} \quad \phi \in \mathrm{Aut}_{\mathrm{refl.}}(W). \quad (2.50)$$

If $\phi \in \mathrm{Inn}(W)$ is an inner automorphism, so $\phi(w) = v w v^{-1}$ for some $v \in W$, then $\widetilde{M} \doteq R_B(v^{-1}) M$ gives an equivalence between R_A and R_B with $\phi = \mathrm{id}$. So setting $\phi = \mathrm{id}$ in the equivalence relation (2.50) automatically covers the case of $\phi \in \mathrm{Inn}(W)$, and (2.50) can be modified by restricting to outer automorphisms without loss of generality,

$$\phi \in \mathrm{Out}_{\mathrm{refl.}}(W). \quad (2.51)$$

If $\phi = \mathrm{id}$, $\cong_{\mathbb{Z},\mathrm{id}}$ is just the old $\cong_{\mathbb{Z}}$; call such an equivalence an *inner equivalence*.

Since the set $\{R_A\}$ with $A \in \mathcal{I}$ constructed in section 2.3 consists of all the $\mathbb{Z}$-inequivalent irreducible reflection representations of a given W, the only inner equivalences are $R_A \cong_{\mathbb{Z}} R_A$, by definition. The outer equivalences are $R_A \cong_{\mathbb{Z},\phi} R_{\phi(A)}$, which serves to define the action of the outer automorphism ϕ on the set $\mathcal{I}$ of reflection representations.

This action of $\mathrm{Out}_{\mathrm{refl.}}(W)$ on the representations is indicated in the second and third columns of table 3 by the blue arrows in the cases where it acts non-trivially in $\mathcal{I}$. These actions are easy to see from the description of the outer automorphism as a symmetry of the Coxeter diagram. This induces an action on the weight space of the associated Lie algebra, and thus on the weight lattices we used to define the representations. In the $D_{2\ell}$ cases the automorphism acts on the weight space as the reflection which interchanges the roots corresponding to the spinor and conjugate spinor, leaving the other basis vectors unchanged. Its action on the representation lattices then follows: Γ_{root} and Γ_{weight} are invariant almost by definition, Γ_V is also invariant by inspection, and the spinor and conjugate spinor lattices are interchanged, $\Gamma_S \leftrightarrow \Gamma_C$. In the BC_2, G_2, and F_4 cases the automorphism is the reflection which exchanges long root directions with short root directions, so results in interchanging the root and co-root lattices.

We apply the same expanded definition of equivalence to SK structures: $(S_{\mathcal{A}}, \boldsymbol{\tau}) \cong_{\mathbb{Z},\phi} (S_{\mathcal{A}'}, \boldsymbol{\tau}')$ if there exists an $M \in \mathrm{Sp}_{J^\vee}(2r,\mathbb{Z})$ and an $\phi \in \mathrm{Out}_{\mathrm{refl.}}(W)$ such that

$$M S_{\mathcal{A}}(w) M^{-1} = S_{\mathcal{A}'}(\phi(w)) \qquad \text{and} \qquad M \circ \boldsymbol{\tau} = \boldsymbol{\tau}' \qquad \forall w \in W. \quad (2.52)$$

The automorphism can be taken to be outer without loss of generality by the same argument as given above in the case of irreducible Weyl representations. Given this more general definition of equivalence, we formalize the notion of the self-duality group, $\mathscr{S}_{\mathcal{A}}$, of an SK structure orbit as the set of all equivalences $(S_{\mathcal{A}}, \boldsymbol{\tau}) \cong_{\mathbb{Z},\phi} (S_{\mathcal{A}}, \boldsymbol{\tau}')$ of $S_{\mathcal{A}}$ to itself. For each pair $\phi, \psi \in \mathrm{Out}_{\mathrm{refl.}}(W)$, define the sets

$$\widetilde{\mathscr{S}}_{\mathcal{A}}^{(\phi,\psi)} \doteq \{ M \in \mathrm{Sp}_{J^\vee}(2r,\mathbb{Z}) \mid M S_{\mathcal{A}}(\psi(w)) = S_{\mathcal{A}}(\phi(w)) M, \ \forall \ w \in W \}, \quad (2.53)$$

which are subsets of $\mathrm{Sp}_{J^\vee}(2r,\mathbb{Z})$, but are generally not subgroups when $\phi \neq \psi$. Instead, they satisfy the groupoid properties

$$\widetilde{\mathscr{S}}_{\mathcal{A}}^{(\phi,\psi)} \widetilde{\mathscr{S}}_{\mathcal{A}}^{(\psi,\chi)} = \widetilde{\mathscr{S}}_{\mathcal{A}}^{(\phi,\chi)}, \qquad\qquad \left(\widetilde{\mathscr{S}}_{\mathcal{A}}^{(\phi,\psi)} \right)^{-1} = \widetilde{\mathscr{S}}_{\mathcal{A}}^{(\psi,\phi)}, \quad (2.54)$$

in a notation where we multiply or invert the elements of each set. The set of all self-intertwiners of $\mathcal{A}$ is the union over all pairs $\phi, \psi \in \mathrm{Out}_{\mathrm{refl.}}(W)$ of $\widetilde{\mathscr{T}}_{\mathcal{A}}^{(\phi,\psi)}$, as in (2.6). Note that, from its definition, the sets $\widetilde{\mathscr{T}}_{\mathcal{A}}^{(\phi,\psi)}$ only depend on ϕ and ψ in the combination $\phi\psi^{-1}$. That is, $\widetilde{\mathscr{T}}_{\mathcal{A}}^{(\phi,\psi)} = \widetilde{\mathscr{T}}_{\mathcal{A}}^{(\phi',\psi')}$ if $\phi\psi^{-1} = \phi'(\psi')^{-1}$. Their preimages in $\mathrm{SL}(2,\mathbb{R})$ under the $\mathcal{M}_{\mathcal{A}\mathcal{A}}$ maps form the self-duality group of the SK structure orbit. The reason is that all automorphism images of $\mathcal{A}$ have the same fixed point set in $\mathscr{H}_r$, i.e., $\mathrm{Fix}(S_{\mathcal{A}\circ\phi}) = \mathrm{Fix}(S_{\mathcal{A}\circ\psi})$ for all $\phi, \psi \in \mathrm{Out}_{\mathrm{refl.}}(W)$, as shown in appendix A.3. Thus the induced identifications act on the same $\tau \in \mathscr{H}_1$ coordinate irrespective of the automorphism labels, so the groupoid multiplication law collapses to a group law.

Note that the union of all the $\widetilde{\mathscr{T}}^{(\phi,\psi)}$ forms a subgroup of $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$. However, this group need not be isomorphic to the self-duality group of the SK structure orbit (c.f. (2.17)) because the $\mathcal{M}_{\mathcal{A}\mathcal{A}}$ maps (2.38) of **Lemma 1** are different for each $\widetilde{\mathscr{T}}_{\mathcal{A}}^{(\phi,\psi)}$ subset.

So, denoting intertwiner preimages in $\mathrm{PSL}(2,\mathbb{R})$ by

$$\mathscr{S}_{\mathcal{A}}^{(\phi,\psi)} \doteq \mathcal{M}_{\mathcal{A}\circ\phi,\mathcal{A}\circ\psi}^{-1}(\widetilde{\mathscr{T}}_{\mathcal{A}}^{(\phi,\psi)}), \tag{2.55}$$

the self-duality group of the $\mathcal{A}$ SK structure orbit is a union over $|\mathrm{Out}_{\mathrm{refl.}}(W)|$ many sets:

$$\mathscr{S}_{\mathcal{A}} \doteq \bigcup_{\substack{\phi,\psi \ \in \\ \mathrm{Out}_{\mathrm{refl.}}(W)}} \mathscr{S}_{\mathcal{A}}^{(\phi,\psi)} = \bigcup_{\substack{\psi \ \in \\ \mathrm{Out}_{\mathrm{refl.}}(W)}} \mathscr{S}_{\mathcal{A}}^{(1,\psi)}. \tag{2.56}$$

We illustrate this in the case $\mathrm{Out}_{\mathrm{refl.}}(W) = \mathbb{Z}_2 \doteq \{0, 1\}$. There are just two symplectic representations related to $S_{\mathcal{A}}$ by an outer automorphism: $S_{\mathcal{A}} \circ 0 = S_{\mathcal{A}}$ and $S_{\mathcal{A}} \circ 1$. Then their intertwiner groupoid and self-duality group can be depicted as

$$\tag{2.57}$$

where the blue arrows denote the $\widetilde{\mathscr{T}}_{\mathcal{A}}^{(i,j)}$ intertwiner actions. Note, furthermore, that by the groupoid properties and the equivalence of SK structures under the isometry induced by the reflection automorphism, it follows that $\widetilde{\mathscr{T}}_{\mathcal{A}}^{(0,0)} = \widetilde{\mathscr{T}}_{\mathcal{A}}^{(1,1)}$ and $\widetilde{\mathscr{T}}_{\mathcal{A}}^{(1,0)} = \left(\widetilde{\mathscr{T}}_{\mathcal{A}}^{(0,1)}\right)^{-1}$. In particular, if $\widetilde{\mathscr{T}}^{(1,0)} = \varnothing$ is empty, then the self-duality group $\mathscr{S}_{\mathcal{A}}$ does not change from its value (2.46) found just using $\mathbb{Z}$-equivalence (i.e., without taking account of automorphism equivalences). Conversely, if $\widetilde{\mathscr{T}}_{\mathcal{A}}^{(1,0)} \neq \varnothing$, then the self-duality group is enlarged from its naive value, and, correspondingly, the conformal manifold of the orbit is smaller.

The inclusion of twisted intertwiners can not only change the self-duality groups of SK structure orbits, but can also change the number of orbits. For simplicity of exposition, we specialize to the $\mathrm{Out}_{\mathrm{refl.}}(W) = \mathbb{Z}_2$ case; the other, $\mathfrak{g} = D_4$, case is similar. We have a set of $\mathbb{Z}$-inequivalent representations, $\{S_{\mathcal{A}}\}$, and we are seeking to understand their equivalences, a.k.a., SK structure orbits. Define sets of automorphism twisted intertwiners between

representations $\mathcal{A}$ and $\mathcal{B}$ by

$$\widetilde{\mathscr{T}}_{\mathcal{AB}}^{(\phi,\psi)} \doteq \{\, M \in \mathrm{Sp}_{J^\vee}(2r,\mathbb{Z}) \;\mid\; M\,(S_{\mathcal{A}} \circ \psi) = (S_{\mathcal{A}} \circ \phi)\,M \,\}, \tag{2.58}$$

thus generalizing the definition (2.53), $\widetilde{\mathscr{T}}_{\mathcal{A}}^{(\phi,\psi)} = \widetilde{\mathscr{T}}_{\mathcal{AA}}^{(\phi,\psi)}$. Assume $\widetilde{\mathscr{T}}_{\mathcal{AB}}^{(0,0)} = \varnothing$ for $\mathcal{A} \neq \mathcal{B}$: $\mathcal{A}$ and $\mathcal{B}$ label distinct SK structure orbits before the imposition of automorphism equivalences. If we have the situation that there is no non-trivial outer automorphism intertwiners of $S_{\mathcal{A}}$ with itself, $\widetilde{\mathscr{T}}_{\mathcal{A}}^{(1,0)} = \varnothing$, but there are automorphism intertwiners of $S_{\mathcal{A}}$ with another representation $S_{\mathcal{B}}$, $\mathcal{B} \neq \mathcal{A}$, then the two orbits are identified, but their (common) self-duality groups are not changed from their non-automorphism values:

$$\text{if} \quad \widetilde{\mathscr{T}}_{\mathcal{A}}^{(1,0)} = \varnothing \quad \text{and} \quad \widetilde{\mathscr{T}}_{\mathcal{AB}}^{(1,0)} \neq \varnothing, \quad \text{then} \quad \mathscr{S}_{\mathcal{A}} = \mathcal{M}_{\mathcal{A}\circ 0, \mathcal{A}\circ 0}^{-1}\big(\widetilde{\mathscr{T}}_{\mathcal{A}}^{(0,0)}\big) = \mathscr{S}_{\mathcal{B}}, \tag{2.59}$$

and $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ are in the same orbit. In other words, in this case the set $\widetilde{\mathscr{T}}_{\mathcal{AB}}^{(1,0)}$ of equivalences do not contribute to the self-duality group, but rather serve to identify what might have been thought to be inequivalent orbits. On the other hand,

$$\text{if} \quad \widetilde{\mathscr{T}}_{\mathcal{A}}^{(1,0)} \neq \varnothing, \quad \text{then} \quad \mathscr{S}_{S_{\mathcal{A}}} = \mathcal{M}_{\mathcal{A}\circ 0, \mathcal{A}\circ 0}^{-1}\big(\widetilde{\mathscr{T}}_{\mathcal{A}}^{(0,0)}\big) \cup \mathcal{M}_{\mathcal{A}\circ 1, \mathcal{A}\circ 0}^{-1}\big(\widetilde{\mathscr{T}}_{\mathcal{A}}^{(1,0)}\big). \tag{2.60}$$

That is to say, in this case the outer automorphism intertwiners do not enlarge the number of representations in an orbit, but they do impose further identifications within the $S_{\mathcal{A}}$ orbit, and so enlarge its self-duality group.

Then, one checks for the existence of outer automorphism intertwiners as in (2.58), finding the results shown in table 1. They are not hard to predict, following the rules (2.59) and (2.60) developed above. In particular, for the D_{4r} cases only case (2.59) occurs since the spinor and conjugate spinor lattices always appear in different SKS orbits, so the outer automorphism simply identifies the orbits as indicated by the blue double arrows $\leftrightarrow$ in table 1. This has the effect in these cases of simply reducing the number of inequivalent duality orbits relative to the naive (no outer automorphism) number. In the D_{4r+2} cases and in the non-simply-laced cases, on the other hand, the outer automorphism intertwiners are of the (2.60) variety, so impose identifications within an orbit, as indicated in table 1 by the blue curved arrows $\curvearrowright$. (BC_2 actually has both types of identification.) These imply a 2-fold identification on their conformal manifolds, thus enlarging their self-duality groups by a factor of 2 to the ones shown.

2.8 Principally polarized symplectic representations

The results presented in tables 1 and 3 are for the special case of SK structures with principally polarized symplectic forms. This is the case relevant for absolute $\mathcal{N}{=}4$ sYM theories with simple gauge algebras; though see the discussion in section 5 for the possible role of non-principal symplectic forms in absolute $\mathcal{N}{=}4$ theories with semi-simple gauge algebras.

Principally polarized symplectic representations of W are those for which the invariant factors of $\mathbf{j}$ are all 1. From (2.30) it follows that this occurs if and only if $A = B$ in (2.29), in which case $\mathbf{j} = \mathbf{1}$ and $J = J^\vee$ is the canonical symplectic form. We drop the $J^\vee$ subscript

from the symplectic group $\mathrm{Sp}(2r,\mathbb{Z})$ in this case. The results of the last subsection simplify somewhat upon setting $A = B$. Thus principally polarized representations are ones of the form

$$S_{(AA,D)} = \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} R_A & 0 \\ 0 & R_A^\vee \end{pmatrix} \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}(2r,\mathbb{Z}), \tag{2.61}$$

with

$$D = D^t \ \in \ \big(\mathrm{Mat}(r,\mathbb{Q})/\langle K_{AA}^\vee\rangle\big)/\mathrm{Mat}(r,\mathbb{Z}). \tag{2.62}$$

The last is the solution to the extension problem, described above, and classifies the inequivalent bindings of R_A with $R_A^\vee$.

Lemma 1 (2.38) simplifies to the statement that $M \in \mathrm{Sp}(2r,\mathbb{Z})$ intertwines two principal symplectic integral representations $S_{(A_1 A_1, D_1)}$ and $S_{(A_2 A_2, D_2)}$ iff M has the form

$$\mathcal{M}_{\mathcal{A}_1 \mathcal{A}_2}(\gamma) \doteq \begin{pmatrix} 1 & -D_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{a} I_{A_1 A_2} & \underline{b} K_{A_1 A_2}^\vee \\ \underline{c} K_{A_1 A_2} & \underline{d} I_{A_1 A_2}^\vee \end{pmatrix} \begin{pmatrix} 1 & D_2 \\ 0 & 1 \end{pmatrix}, \tag{2.63}$$

for $\gamma \doteq \left(\begin{smallmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Q})$. The structure of this set of matrices was computed, giving the results shown in black in column two of table 1. The additional outer automorphism intertwiners were also computed, giving the further identifications shown in blue, and described in the last subsection. Examples of these computations in the A_r, D_6, and BC_2 cases are given in the next section.

The resulting self-duality groups (2.56) were also computed, and are shown in column three of table 1, which gives the self-duality groups as certain modular subgroups or as Hecke groups. These are defined in appendix C.

3   Results and examples: SK structures with simple $\mathfrak{g}$ and principal J

We now illustrate the general arguments of the last section in the case of principally polarized $\mathcal{N}{=}4$ sYM CB geometries corresponding to simple gauge algebras $\mathfrak{g}$. In addition to the five exceptional Weyl groups corresponding to the five exceptional simple Lie algebras, there are three infinite series of corresponding Weyl groups: A_r corresponding to $\mathfrak{g} = \mathfrak{su}(r+1)$; BC_r corresponding to $\mathfrak{g} = \mathfrak{so}(2r+1)$ or $\mathfrak{sp}(2r)$; and D_r corresponding to $\mathfrak{g} = \mathfrak{so}(2r)$. In this section we construct the SK structures for the $\mathfrak{su}(N)$ series, and compute their orbits and self-duality groups, for all N. We highlight this case both because of its intricacy — it depends on divisibility properties of N — and because it can be treated with pleasing uniformity.

Similar arguments can be applied to the two other infinite series as well, but they separate into seven distinct special sub-cases, as indicated in table 1, so a general proof of their properties is tiresome. Instead, we have automated these calculations in a Mathematica notebook which is available from the authors upon request, from which the patterns shown in table 1 can be induced from examples. In sections 3.2 and 3.3 we illustrate these computations in two of the most intricate cases: the D_6 or $\mathfrak{g} = \mathfrak{so}(12)$ case, and the BC_2 or $\mathfrak{g} = \mathfrak{so}(5) = \mathfrak{sp}(4)$ case. For the $E_{6,7,8}$, F_4, and G_2 exceptional groups we simply calculated them explicitly.

3.1 $\mathfrak{su}(N)$ SK structures

As indicated in table 3, the integrally inequivalent $(N-1)$-dimensional irreducible representations of the Weyl group S_N are labeled by the divisors of N, when $N > 2$. The $N = 2$ special case is discussed at the end of this subsection (3.1.1). Thus we denote these representations R_d, with $d|N$. We use the basis

$$(\alpha_2, \ldots, \alpha_{N-1}, -d\varpi_{N-1}) \tag{3.1}$$

for the lattice Γ_d, where $(\alpha_1, \alpha_2, \ldots, \alpha_{N-1})$ is a basis of simple roots and $(\varpi_1, \ldots, \varpi_{N-1})$ the corresponding basis of fundamental weights. In these Γ_d bases, we find

$$I_{dd'} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & d/d' \end{pmatrix}, \qquad K_{dd'} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -d' \\ & & & -d & \frac{N-1}{N}dd' \end{pmatrix}, \tag{3.2}$$

and $\gcd(K_{dd'}) = \gcd(N, dd')/N$. If $dd' \neq N$, then it is easy to see that $K_{dd'}/\gcd(K_{dd'})$ is not invertible over the integers, while if $dd' = N$, then all its invariant factors are 1, and it is invertible over the integers. Thus

$$R_d \cong_{\mathbb{Z}} R_{d'}^{\vee} \quad \text{iff} \quad dd' = N. \tag{3.3}$$

This result also follows from the Lie algebra definitions of the Γ_d invariant lattices as those corresponding to subgroups $\mathbb{Z}_d \subset \mathbb{Z}_N$ of the center, together with the co-lattice equivalences (2.18).

Principally polarized symplectic representations are those $S_{(dd',D)}$ with $d = d'$, as in (2.61). Note that for $d = d'$, $\gcd(K_{dd}) = ds_d/N$, where we have defined the integer

$$s_d \doteq \gcd\left(\frac{N}{d}, d\right). \tag{3.4}$$

s_d plays an important role in what follows. Note that $s_d^2|N$; we call it the "square-divisor of N associated to d".

A somewhat laborious calculation using the explicit forms of the R_d representation matrices shows that the condition for a symmetric $D \in \mathrm{Mat}(r, \mathbb{R})$ to give an integral $S_{(dd,D)}$ representation is

$$D = \beta K_{dd}^{\vee} + \frac{d}{N}\begin{pmatrix} \mathbf{0}_{r-1} & \vec{0} \\ \vec{0} & z \end{pmatrix} \quad \mod 1, \qquad \beta \in \mathbb{R}, \qquad z \in \mathbb{Z}_{N/d}. \tag{3.5}$$

Here the matrix is all zeros except for the lower right entry, z, which is an integer, and the "mod 1" means that an arbitrary integer symmetric matrix can be added to D. These integer matrices give the inner bindings which should be quotiented out in, as in (2.62). This is what restricts z to be in in $\mathbb{Z}_{N/d}$. The ideal generated by $K_{dd}^{\vee}$ over the rationals should also be quotiented out since it gives an identically vanishing binding. Note that,

using the explicit form (3.2) for $K_{dd}^\vee$, if $z = nd$ for some integer n, then, by taking $q = n$ in (3.5), D becomes integral, resulting in an inner binding. As a result, the inequivalent solutions to the extension problem (2.62) are parameterized by

$$D_z \doteq \frac{d}{N} \begin{pmatrix} \mathbf{0}_{r-1} & \vec{0} \\ \vec{0} & z \end{pmatrix}, \qquad \text{with} \qquad z \in \mathbb{Z}_{s_d}, \qquad (3.6)$$

since the resulting D_z's are equivalent both modulo N/d and modulo d in z. To summarize, for a given d, the inequivalent $\mathrm{Sp}(2N-2, \mathbb{Z})$ representations are

$$S_{(d,z)} \doteq S_{(dd,D_z)}, \qquad d|N, \qquad z \in \mathbb{Z}_{s_d}. \qquad (3.7)$$

($S_{(d,z)}$ is a short-hand notation which we use for the rest of this subsection.)

Next, we determine which $S_{(d,z)}$'s are equivalent for different d's. A general analysis searching for integral intertwiners $M(\gamma)$ of the form (2.63) is somewhat complicated, as it depends on the specific form of the $I_{dd'}$ and $K_{dd'}$ intertwiners given in (3.2), and results in quadratic diophantine equations. But a less direct approach works. We present it in two steps:

(i) First, show that for any $S_{(d,z)}$ there is a divisor of N, d', such that $S_{(d,z)} \cong_{\mathbb{Z}} S_{(d',0)}$. This implies that each equivalence class of SK structures contains a representative $S_{(d,0)}$ for some $d|N$.

(ii) Second, determine the integer congruences between $S_{(d,0)}$ and $S_{(d',0)}$ for all d, d' pairs that each divide N.

Step (i). A solution for the equivalence $S_{(d,z)} \cong_{\mathbb{Z}} S_{(d',0)}$ is $d' = \gcd(s_d, z) = s_{\gcd(d,z)}$. To see this, take $A = d$, $B = d' = \gcd(s_d, z)$, $E = 0$, and $D = D_z$ in (2.63). Since, for $N > 2$, all the entries except for the lower right entry of $I_{d'd}$, $I_{d'd}^\vee$, $K_{d'd}$, $K_{d'd}^\vee$, and D_z are integers with gcd 1, if follows that $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ (i.e., $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d} \in \mathbb{Z}$ and $\underline{a}\underline{d} - \underline{b}\underline{c} = 1$). Then demanding that the lower right entry of each block is integral imposes the further constraints that

$$\underline{a} = \frac{d}{\gcd(s_d, z)}\alpha, \qquad \underline{b} = \frac{z}{\gcd(s_d, z)}\alpha - \frac{N/d}{\gcd(s_d, z)}\beta, \qquad \underline{\alpha}, \underline{\beta} \in \mathbb{Z}. \qquad (3.8)$$

Note that the coefficients in these expressions are all integer since $s_d|d$ and $s_d|(N/d)$. In deriving this result we have used the fact that $\gcd(N, d\gcd(s_d, z)) = d\gcd(s_d, z)$, which also follows from $s_d|(N/d)$. The determinant condition can then be written

$$1 = \underline{a}\underline{d} - \underline{b}\underline{c} = (\underline{\alpha} \; \underline{\beta})X\left(\begin{smallmatrix} \underline{c} \\ \underline{d} \end{smallmatrix}\right) \qquad \text{with} \qquad X \doteq \frac{1}{\gcd(s_d, z)} \begin{pmatrix} -z & d \\ N/d & 0 \end{pmatrix}. \qquad (3.9)$$

Since X is an integer matrix, it can be put in Smith normal form, $X = UYV$, with $U, V \in \mathrm{GL}(2, \mathbb{Z})$ such that Y is integer diagonal, $Y = \mathrm{diag}\{f_1, f_1 f_2\}$. Since it is a 2×2 matrix, its invariant factors are determined to be $f_1 = \gcd(X) = 1$ and $f_2 = f_1^2 f_2 = \det(X) = N/\gcd(s_d, z)^2$. Taking $(\underline{\alpha} \; \underline{\beta})U = (1 \; 0)$ and $V\left(\begin{smallmatrix} \underline{c} \\ \underline{d} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ then gives the desired solution.

Step (ii). Now we want to determine when $S_{(d,0)} \cong_{\mathbb{Z}} S_{(d',0)}$ for two distinct divisors of N, d and d'. The argument of step (i) applies equally well when $z = 0$, where $d' = \gcd(s_d, 0) = s_d$. Thus, for every d, $S_{(d,0)} \cong_{\mathbb{Z}} S_{(s_d,0)}$, so we need to ask about the equivalence between $S_{(s,0)}$ and $S_{(s',0)}$ when s and s' are each a square-divisor of N. We prove, by way of contradiction, that no equivalence exists between such representations. For suppose $s \neq s'$ are two distinct square-divisors of N. Then any potential intertwiner $M(\gamma)$ between $S_{(s,0)}$ and $S_{(s',0)}$ must have the form

$$M(\gamma) = \begin{pmatrix} \underline{a}I_{s's} & \underline{b}K_{s's}^{\vee} \\ \underline{c}K_{s's} & \underline{d}I_{s's} \end{pmatrix}, \qquad \gamma = \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}). \tag{3.10}$$

Since (for rank $r > 1$, i.e., $N > 2$) all but the lower right entries in the I, K, intertwiners are integers with gcd 1, we must, in fact, have $\gamma \in \mathrm{SL}(2, \mathbb{Z})$. The lower right hand entries of each block give the 2×2 matrix

$$\frac{1}{Nss'} \begin{pmatrix} \underline{a}N(s')^2 & \underline{b}(N-1)(ss')^2 \\ \underline{c}N^2(N-1) & \underline{d}Ns^2 \end{pmatrix}, \tag{3.11}$$

whose integrality requires

$$\gamma = \begin{pmatrix} \underline{\alpha}\frac{s}{\gcd(s,s')} & \underline{\beta}\frac{N}{ss'} \\ \underline{\gamma} & \underline{\delta}\frac{s'}{\gcd(s,s')} \end{pmatrix}, \qquad \underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta} \in \mathbb{Z}, \tag{3.12}$$

where we have used that $\gcd(N, ss') = ss'$ since s and s' are square-divisors of N. The determinant condition is

$$1 = \det \gamma = \underline{\alpha}\underline{\delta}\frac{ss'}{\gcd(s,s')^2} - \underline{\beta}\underline{\gamma}\frac{N}{ss'}. \tag{3.13}$$

But the right side is divisible by $ss'/\gcd(s,s')^2$ because $ss'/\gcd(s,s')$ is a square-divisor of N if s and s' are, and $ss'/\gcd(s,s')^2 > 1$ because $s \neq s'$. Thus there is no integer solution to (3.13), and so no intertwiner $M(\gamma)$ between $S_{(s,0)}$ and $S_{(s',0)}$ invertible over the integers.

We have thus shown that each $S_{(d,z)}$ SK structure is equivalent to an $S_{(s,0)}$ SK structure, where s is a square-divisor of N, and that the $S_{(s,0)}$ SK structures are all inequivalent for distinct s. The conclusion is that the $\mathcal{N}{=}4$ $\mathfrak{su}(N)$ sYM theory has distinct SK structure orbits that are in 1-to-1 correspondence with the square-divisors of N. Furthermore, by (2.49), the self-duality group of the $S_{(s,0)}$ SK structure (and every other SK structure in its orbit) is

$$\mathscr{S}_{(s,0)} \simeq \Gamma_0(N/s^2) \subset \mathrm{SL}(2, \mathbb{Z}). \tag{3.14}$$

Here we have used that $n_{ss} \doteq \gcd(K_{ss})^{-1}\gcd(K_{ss}^{\vee})^{-1} = N/s^2$.

3.1.1 The $\mathfrak{su}(2)$ special case

The case of $\mathfrak{g} = \mathfrak{su}(2)$ (or A_1) requires special discussion, as it is a kind of degenerate case. In the $\mathfrak{su}(2)$ case the dual Cartan algebra is $\mathfrak{h}^* = \mathbb{R}$ and the $\mathbb{Z}_2$ Weyl group acts by reflection

through its origin. It has just a single inequivalent integral representation, $R(1) = 1$, and $R(w) = -1$, so there is no distinction between $R_{d=1}$ and $R_{d=2}$. As a result there is just a single SK structure orbit with S-duality group which is $\mathrm{PSL}(2,\mathbb{Z})$, and conformal manifold which is the $\mathrm{SL}(2,\mathbb{Z})$ fundamental domain in the upper half-plane.

3.2 $\mathfrak{so}(12)$ example

The SK structures of the $\mathfrak{so}(2k)$ series is in some ways more intricate than that of the $\mathfrak{su}(r+1)$ series. We explain this here briefly, with the calculational details shown only in the $\mathfrak{g} = \mathfrak{so}(12)$ example.

First, the $\mathfrak{so}(4k)$ and $\mathfrak{so}(4k+2)$ series are qualitatively different, as is already apparent from table 3, reflecting the fact that the center of $\mathfrak{so}(4k+2)$ is $\mathbb{Z}_4$ and so has three subgroups and corresponding invariant lattices, while that of $\mathfrak{so}(4k)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ so has five inequivalent subgroups.

Next, and less obvious, is the pattern of Ext groups classifying the extensions of these lattices by their conjugate lattices. These are recorded in table 3, and show a different pattern for the $\mathfrak{so}(8k)$, $\mathfrak{so}(8k+4)$, and $\mathfrak{so}(4k+2)$ cases. We do not know how to predict this pattern aside from an examination of the detailed divisibility properties of the intertwiners of these lattices.

These different symplectic lattices fall into integral equivalence classes, indicated in table 1, which we determine by brute force calculation. The pattern of the resulting orbit S-duality groups is equally unobvious from this point of view. As in the $\mathfrak{su}(r+1)$ cases, they occur in the calculation as certain subgroups of $\mathrm{SL}(2,\mathbb{R})$, and can be conjugated to finite-index subgroups of $\mathrm{SL}(2,\mathbb{Z})$, shown in table 1. However, the relative way they are embedded in $\mathrm{SL}(2,\mathbb{R})$ has physical significance, since it controls the values that the matrix of low-energy couplings, $\boldsymbol{\tau}_{ij}$, takes for these SK structures. The freedom to embed the S duality groups in $\mathrm{SL}(2,\mathbb{R})$ is needed to realize the multiple S-duality orbits with isomorphic S-duality groups. For example, in the $\mathfrak{so}(8k)$ case, there are four distinct orbits each with S-duality group $\Gamma_0(2)$, but there are not four inequivalent ways of embedding a group isomorphic to $\Gamma_0(2)$ in $\mathrm{SL}(2,\mathbb{Z})$. Finally, in the $\mathfrak{so}(8k+4)$ case there is one S-duality orbit whose S-duality group is not (isomorphic to) a modular congruence subgroup. It is, instead, the index-2 subgroup of $\mathrm{SL}(2,\mathbb{Z})$ called Δ, described explicitly in appendix C.

To illustrate our construction, we work out in detail the principally polarized SK structures of type D_r with $r \equiv 2 \bmod 4$. We present the computations for $r = 6$ in order to show explicit computations and to lighten notations, but everything can be extended straightforwardly to any $r \equiv 2 \bmod 4$. This example is the richest in terms of number of SK structures and types of duality groups. We follow the steps outlined in Section 2.

Step 1: W-invariant lattices. As the algebra is simply laced, the set of lattices $\mathcal{I}$ for D_r with r even contains 5 lattices, corresponding to the 5 subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$. We use these subgroups to label the lattices, so

$$\mathcal{I} = \left\{ \mathbb{Z}_2^2, \mathbb{Z}_2^V, \mathbb{Z}_2^S, \mathbb{Z}_2^C, \mathbb{Z}_1 \right\}. \tag{3.15}$$

The lattices can be constructed as embeddings in $\mathbb{R}^r$ as

$$
\begin{array}{c|c|c}
\text{Lattice} & \text{Symbol} & \text{Construction} \\
\hline
\Gamma_{\text{root}} & \mathbb{Z}_2^2 & \mathbb{Z}^r|_{\Sigma \in 2\mathbb{Z}} \\
\Gamma_V & \mathbb{Z}_2^V & \mathbb{Z}^r \\
\Gamma_S & \mathbb{Z}_2^S & \mathbb{Z}^r|_{\Sigma \in 2\mathbb{Z}} \cup (\mathbb{Z} + \frac{1}{2})^r|_{\Sigma \in 2\mathbb{Z}} \\
\Gamma_C & \mathbb{Z}_2^C & \mathbb{Z}^r|_{\Sigma \in 2\mathbb{Z}} \cup (\mathbb{Z} + \frac{1}{2})^r|_{\Sigma \in 1 + 2\mathbb{Z}} \\
\Gamma_{\text{weight}} & \mathbb{Z}_1 & \mathbb{Z}^r \cup (\mathbb{Z} + \frac{1}{2})^r
\end{array}
\tag{3.16}
$$

which is a standard construction from Lie algebra theory. Thus this completes **Step 1** in the classification procedure. To be explicit in the computations later on, one can choose bases for the different lattices in the above table. In terms of elements of $\mathbb{Z}^6$, they are given as matrices in which basis vectors are columns:

$$
\overset{\mathbb{Z}_2^2}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}}
\overset{\mathbb{Z}_2^V}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}}
\overset{\mathbb{Z}_2^S}{\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ -1 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}}
\overset{\mathbb{Z}_2^C}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ -1 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & -1 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -1 & \frac{1}{2} \end{pmatrix}}
\overset{\mathbb{Z}_1}{\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}}
\tag{3.17}
$$

Using these bases, each simple reflection is represented as a matrix $R_A(w_i) \in \mathrm{GL}(6, \mathbb{Z})$, where $A \in \mathcal{I}$ specifies the lattice and $i = 1, \ldots, 6$ the simple reflection $w_i \in W$; the simple roots are the columns of the matrix labeled $\mathbb{Z}_2^2$ above.

Step 2: A sufficient class of symplectic integral representations. To find which representations $S_{(AA,D)}$ to consider, we have to compute, for each $A \in \mathcal{I}$, the group $\mathrm{Ext}^1_{\mathbb{Z}W}(R_A, R_A^\vee)$ of bindings modulo inner bindings (2.35), in which D is picked. For this, we follow the general procedure outlined in section 2.5. We find that the most general solution to (2.27) for $A = \mathbb{Z}_1 \in \mathcal{I}$ is

$$
D = \beta K_{\mathbb{Z}_1}^\vee + n_1 \mathfrak{D}_1 + n_2 \mathfrak{D}_2 + \cdots + n_s \mathfrak{D}_s
\tag{3.18}
$$

where $\beta \in \mathbb{R}$ and $n_1, \ldots, n_s \in \mathbb{Z}$, and

$$
K_{\mathbb{Z}_1}^\vee = \begin{pmatrix} 2 & 1 & 2 & -1 & 2 & -4 \\ 1 & 2 & 2 & 0 & 2 & -2 \\ 2 & 2 & 4 & -1 & 4 & -6 \\ -1 & 0 & -1 & 2 & -2 & 4 \\ 2 & 2 & 4 & -2 & 6 & -8 \\ -4 & -2 & -6 & 4 & -8 & 14 \end{pmatrix}, \quad
\mathfrak{D}_1 = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad
\mathfrak{D}_2 = \begin{pmatrix} -\frac{1}{4} & \frac{3}{8} & -\frac{1}{4} & -\frac{3}{8} & -\frac{1}{4} & -\frac{1}{2} \\ \frac{3}{8} & -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{8} & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{3}{8} & 0 & -\frac{3}{8} & -\frac{1}{4} & -\frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & -\frac{3}{4} & 0 \\ -\frac{1}{2} & -\frac{3}{4} & -\frac{1}{4} & -\frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}.
\tag{3.19}
$$

$K_{\mathbb{Z}_1}^\vee$ is the inverse transpose of the matrix of the Killing form in the basis specified in (3.17), and all other $\mathfrak{D}_i$ with $i > 2$ have integer coefficients, and can therefore be removed in the Ext group computation (it turns out $s = 20$ here, so there are 18 matrices $\mathfrak{D}_i$ with integer entries). One can check that for *every* $n_1, n_2, \cdots \in \mathbb{Z}$, there exists $\beta \in \mathbb{R}$ such that D in (3.18) has integer entries. Indeed, one checks that $n_1 = 1$ can be canceled by $\beta = \frac{1}{2}$ since the odd entries in (3.19) match exactly the half integers in $\mathfrak{D}_1$. Similarly, $n_2 = 1$ can be canceled by $\beta = \frac{5}{8}$. This is equivalent to saying that $\mathrm{Ext}^1_{\mathbb{Z}W}(R_A, R_A^\vee)$ is trivial for $A = \mathbb{Z}_1$.

The same computation for $A = \mathbb{Z}_2^2$ gives

$$K_{\mathbb{Z}_2^2}^\vee = \begin{pmatrix} 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 3 & \frac{3}{2} & \frac{3}{2} \\ 1 & 2 & 3 & 4 & 2 & 2 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{3}{2} & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & 1 & \frac{3}{2} \end{pmatrix}, \quad \mathfrak{D}_1 = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{D}_2 = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{4} & -\frac{3}{4} \\ -\frac{1}{2} & -1 & -\frac{1}{2} & -1 & -1 & -1 \\ -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & -1 & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & -1 & 0 & -\frac{1}{4} \end{pmatrix}. \tag{3.20}$$

In this case, for every $n_1, n_2, \dots \in \mathbb{Z}$, there exists $\beta \in \mathbb{R}$ such that D in (3.18) equals one the the four following matrices modulo 1:

$$\begin{matrix} 00 & & 11 & & 01 & & 10 \end{matrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}. \tag{3.21}$$

These generate the group[21]

$$\text{Ext}^1_{\mathbb{Z}W}(R_A, R_A^\vee) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 11, 01, 10\} \qquad \text{for} \qquad A = \mathbb{Z}_2^2. \tag{3.22}$$

We proceed similarly for each element of $\mathcal{I}$ in (3.15), and we find that $\text{Ext}^1_{\mathbb{Z}W}(R_A, R_A^\vee)$ is trivial for $A = \mathbb{Z}_2^S$ and $A = \mathbb{Z}_2^C$ while $\text{Ext}^1_{\mathbb{Z}W}(R_A, R_A^\vee) = \mathbb{Z}_2$ for $A = \mathbb{Z}_2^V$. These results are reported in table 3. We then have an explicit construction of $4 + 2 + 1 + 1 + 1 = 9$ representations $S_{\mathcal{A}}$, for $\mathcal{A} \in \mathfrak{I}$, with

$$\mathfrak{I} = (\mathbb{Z}_2 \times \mathbb{Z}_2) \sqcup \mathbb{Z}_2 \sqcup \mathbb{Z}_1 \sqcup \mathbb{Z}_1 \sqcup \mathbb{Z}_1 \tag{3.23}$$
$$= \{(\mathbb{Z}_2^2, 00), (\mathbb{Z}_2^2, 01), (\mathbb{Z}_2^2, 10), (\mathbb{Z}_2^2, 11), (\mathbb{Z}_2^V, 0), (\mathbb{Z}_2^V, 1), (\mathbb{Z}_2^S, 0), (\mathbb{Z}_2^C, 0), (\mathbb{Z}_1, 0)\}.$$

It might help at this point to recall our notation (2.10): for example, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ in the first line refers to the Ext group, while in the second line in the $(\mathbb{Z}_2^2, ij)$ entries, $\mathbb{Z}_2^2$ is the subgroup, $A = \mathbb{Z}_2^2$, of the center of $\mathfrak{so}(12)$ which we are using to label the representation, while ij labels an element of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Ext group. For instance, $S_{(\mathbb{Z}_2^2, 10)}$ is given by (2.29) with R_A the representation of the Weyl group in the basis specified by the first matrix in (3.17), $R_B^\vee = R_A^\vee$ its inverse transpose, and D the last matrix in (3.21). Hence for instance we have

$$S_{(\mathbb{Z}_2^2, 10)}(w_1) = \left(\begin{array}{ccccc|ccc} -1 & 1 & & & & -1 & -1 & -1 \\ & 1 & & & & 1 & & \\ & & 1 & & & 1 & & \\ & & & 1 & & 1 & & \\ & & & & 1 & & & \\ \hline & & & & & -1 & & \\ & & & & & 1 & 1 & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & & & & & 1 \end{array} \right). \tag{3.24}$$

---

[21]Careful not to confuse the label $A = \mathbb{Z}_2^2$ with the Ext group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The elements of the latter are denoted $\{00, 11, 01, 10\}$ and appear in tables 1 and 3.

Step 3: Integral equivalences of symplectic representations. We know from appendix B that every SK structure has a representative (S, τ) where S is one of the 9 representations $S_\mathcal{A}$ constructed above. Hence, it is enough to compute which of these 9 representations are integrally equivalent. This data is contained in the sets $\mathscr{S}_{\mathcal{A}_1 \mathcal{A}_2}$ in (2.17), which are exactly the intertwiners between $S_{\mathcal{A}_1}$ and $S_{\mathcal{A}_2}$.

$\underline{\text{Example 1.}}$ Let us begin with $\mathscr{S}_\mathcal{A} \doteq \mathscr{S}_{\mathcal{A}\mathcal{A}}$ for $\mathcal{A} = (\mathbb{Z}_2^2, 00)$. The results of section 2 instruct us to consider the map defined by (2.38), which reads here

$$\mathcal{M}_{(\mathbb{Z}_2^2, 00),(\mathbb{Z}_2^2, 00)}\left(\begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix}\right) = \begin{pmatrix} \underline{a}I_6 & \underline{b}K_{\mathbb{Z}_2^2}^\vee \\ \underline{c}K_{\mathbb{Z}_2^2} & \underline{d}I_6 \end{pmatrix} \tag{3.25}$$

where I_6 is the identity 6×6 matrix, $K_{\mathbb{Z}_2^2}^\vee$ is given in (3.20), and $K_{\mathbb{Z}_2^2}$ is its inverse transpose, which has only integer coefficients (it is the D_6 Cartan matrix). The set $\mathscr{S}_{(\mathbb{Z}_2^2, 00)}$ is then the set of $\mathrm{SL}(2, \mathbb{Z})$ matrices $\begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix}$ such that (3.25) has integer entries. Given (3.20), the only constraint is that $\underline{b}$ should be even. Hence

$$\mathscr{S}_{(\mathbb{Z}_2^2, 00)} = \Gamma^0(2). \tag{3.26}$$

This result is reported in table 1, where we write $\Gamma_0(2)$ and not $\Gamma^0(2)$, but this is irrelevant, as only the group up to conjugation is basis independent. Since the index of $\Gamma^0(2)$ in $\mathrm{SL}(2\mathbb{Z})$ is 3, we say that the symplectic representation $S_{(\mathbb{Z}_2^2, 00)}$ belongs to an orbit of size 3. This latter information will be useful in the next section when we compare to the field theory S-duality predictions.

$\underline{\text{Example 2.}}$ Consider now $\mathscr{S}_{(\mathbb{Z}_2^2, 11)}$. The computation is similar, except that we should now take into account non-trivial bindings in (2.38), namely $D_1 = D_2$ given by the third matrix in (3.21). A brute force evaluation of (2.38) for $\mathcal{A}_1 = \mathcal{A}_2 = (\mathbb{Z}_2^2, \mathbb{Z}_2^2; 11)$ gives for the self-intertwiner

$$\begin{pmatrix}
\underline{a}+\frac{c}{2} & -\underline{c} & \underline{c} & 0 & 0 & -\underline{c} & \underline{b}-\frac{5c}{4} & \frac{a}{2}+\underline{b}-\frac{c}{4}-\frac{d}{2} & \underline{b}-\frac{c}{4} & \frac{a}{2}+\underline{b}+\frac{3c}{4}-\frac{d}{2} & \frac{a}{4}+\frac{b}{2}+\frac{3c}{8}-\frac{d}{4} & \frac{3a}{4}+\frac{b}{2}-\frac{5c}{8}-\frac{3d}{4} \\
-\underline{c} & \underline{a}+\frac{c}{2} & 0 & \underline{c} & -\underline{c} & -\underline{c} & \frac{a}{2}+\underline{b}-\frac{c}{4}-\frac{d}{2} & 2\underline{b}-\frac{3c}{2} & 2\underline{b}-\frac{c}{2} & 2\underline{b}-\frac{c}{2} & \frac{a}{2}+\underline{b}-\frac{c}{4}-\frac{d}{2} & \frac{a}{2}+\underline{b}-\frac{5c}{4}-\frac{d}{2} \\
0 & 0 & \underline{a}+\frac{c}{2} & 0 & -\underline{c} & 0 & \underline{b}-\frac{c}{4} & 2\underline{b}-\frac{c}{2} & 3\underline{b}-\frac{3c}{4} & \frac{a}{2}+3\underline{b}+\frac{c}{4}-\frac{d}{2} & \frac{3a}{4}+\frac{3b}{2}+\frac{c}{8}-\frac{3d}{4} & \frac{a}{4}+\frac{3b}{2}+\frac{c}{8}-\frac{d}{4} \\
-\underline{c} & \underline{c} & -\underline{c} & \underline{a}+\frac{c}{2} & 0 & 0 & \frac{a}{2}+\underline{b}+\frac{3c}{4}-\frac{d}{2} & 2\underline{b}-\frac{c}{2} & \frac{a}{2}+3\underline{b}+\frac{c}{4}-\frac{d}{2} & 4\underline{b}-c & 2\underline{b}-\frac{c}{2} & 2\underline{b}-\frac{c}{2} \\
0 & 0 & -\underline{c} & \underline{c} & \underline{a}-\frac{c}{2} & 0 & \frac{a}{4}+\frac{b}{2}+\frac{3c}{8}-\frac{d}{4} & \frac{a}{2}+\underline{b}-\frac{c}{4}-\frac{d}{2} & \frac{3a}{4}+\frac{3b}{2}+\frac{c}{8}-\frac{3d}{4} & 2\underline{b}-\frac{c}{2} & \frac{a}{4}+\frac{3b}{2}-\frac{7c}{8}-\frac{d}{4} & \underline{b}-\frac{c}{4} \\
-\underline{c} & 0 & 0 & \underline{c} & 0 & \underline{a}-\frac{3c}{2} & \frac{3a}{4}+\frac{b}{2}-\frac{5c}{8}-\frac{3d}{4} & \frac{a}{2}+\underline{b}-\frac{5c}{4}-\frac{d}{2} & \frac{a}{4}+\frac{3b}{2}+\frac{c}{8}-\frac{d}{4} & 2\underline{b}-\frac{c}{2} & \underline{b}-\frac{c}{4} & \frac{3a}{4}+\frac{3b}{2}-\frac{15c}{8}-\frac{3d}{4} \\
2\underline{c} & -\underline{c} & 0 & 0 & 0 & 0 & \underline{d}-\frac{c}{2} & \underline{c} & 0 & \underline{c} & 0 & \underline{c} \\
-\underline{c} & 2\underline{c} & -\underline{c} & 0 & 0 & 0 & \underline{c} & \underline{d}-\frac{c}{2} & 0 & -\underline{c} & 0 & 0 \\
0 & -\underline{c} & 2\underline{c} & -\underline{c} & 0 & 0 & -\underline{c} & 0 & \underline{d}-\frac{c}{2} & \underline{c} & \underline{c} & 0 \\
0 & 0 & -\underline{c} & 2\underline{c} & -\underline{c} & -\underline{c} & 0 & -\underline{c} & 0 & \underline{d}-\frac{c}{2} & -\underline{c} & -\underline{c} \\
0 & 0 & 0 & -\underline{c} & 2\underline{c} & 0 & 0 & \underline{c} & \underline{c} & 0 & \frac{c}{2}+\underline{d} & 0 \\
0 & 0 & 0 & -\underline{c} & 0 & 2\underline{c} & \underline{c} & \underline{c} & 0 & 0 & 0 & \frac{3c}{2}+\underline{d}
\end{pmatrix} \tag{3.27}$$

and one can check that this has integer entries if and only if there exist integers $n_1, n_2, n_3, n_4 \in \mathbb{Z}$ such that

$$\begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} = \begin{pmatrix} 4n_1 - 2n_2 - 2n_3 - \frac{3n_4}{2} & n_2 - \frac{n_3}{2} - \frac{n_4}{4} \\ 2n_3 - n_4 & \frac{n_4}{2} \end{pmatrix}. \tag{3.28}$$

In other words,

$$\mathscr{S}_{(\mathbb{Z}_2^2, 11)} = \left\{ \begin{pmatrix} 4n_1 - 2n_2 - 2n_3 - \frac{3n_4}{2} & n_2 - \frac{n_3}{2} - \frac{n_4}{4} \\ 2n_3 - n_4 & \frac{n_4}{2} \end{pmatrix} \;\middle|\; n_j \in \mathbb{Z} \text{ and } \det = 1 \right\}. \tag{3.29}$$

What is this group? First of all, we can conjugate it to a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ as follows:

$$\Delta \doteq \begin{pmatrix} 2 & 0 \\ 1 & \frac{1}{2} \end{pmatrix} \mathscr{S}_{(\mathbb{Z}_2^2,11)} \begin{pmatrix} 2 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}^{-1} = \left\{ \begin{pmatrix} a+d & 2b+d \\ 2c+d & a \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{Z} \text{ and } \det = 1 \right\}. \tag{3.30}$$

This is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ which contains neither $\mathsf{S} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ nor $\mathsf{T} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, but does contain ST. Since $\mathrm{SL}(2,\mathbb{Z})$ is generated by ST and S, this means the group Δ has index 2 in $\mathrm{SL}(2,\mathbb{Z})$. In particular, it is not a congruence subgroup. This implies that the symplectic representation $S_{(\mathbb{Z}_2^2,11)}$ belongs to an orbit of size 2.

<u>Example 3.</u> Finally, let us quickly go over the computation of $\mathscr{S}_{\mathcal{A}_1\mathcal{A}_2}$ for $\mathcal{A}_1 \neq \mathcal{A}_2$. Proceeding in a similar way as above, we find for instance

$$\mathscr{S}_{(\mathbb{Z}_2^2,00),(\mathbb{Z}_1,0)} = \left\{ \begin{pmatrix} 2n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \;\middle|\; n_j \in \mathbb{Z} \text{ and } \det = 1 \right\}, \tag{3.31}$$

which is one of the three $\Gamma_0(2)$ cosets, while

$$\mathscr{S}_{(\mathbb{Z}_2^2,00),(\mathbb{Z}_2^C,0)} = \left\{ \begin{pmatrix} 2n_1 & 2n_2 \\ n_3 & n_4 \end{pmatrix} \;\middle|\; n_j \in \mathbb{Z} \text{ and } \det = 1 \right\} = \varnothing, \tag{3.32}$$

which is clearly empty. This means

$$S_{(\mathbb{Z}_2^2,00)} \cong_{\mathbb{Z}} S_{(\mathbb{Z}_1,0)} \qquad S_{(\mathbb{Z}_2^2,00)} \not\cong_{\mathbb{Z}} S_{(\mathbb{Z}_2^C,0)} \tag{3.33}$$

These computations are summarized in table 1.

Step 4: SK structures. The computations of the previous paragraph show that $\mathcal{I}$ is partitioned into 5 orbits, and for each orbit we can compute a duality group (up to conjugacy), whose index in $\mathrm{SL}(2,\mathbb{Z})$ give the size of the orbits. We summarized all this in the 5 lines of the D_{4k+2} entry in table 1. In order to fully characterize the SK structures, according to our definition (2.1), one needs to compute the fixed points of the representations, in which τ varies. This is given explicitly in (2.32). For instance, we compute the fixed loci

$$\mathrm{Fix}(S_{(\mathbb{Z}_2^2,00)}) = \left\{ \tau \begin{pmatrix} 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 3 & \frac{3}{2} & \frac{3}{2} \\ 1 & 2 & 3 & 4 & 2 & 2 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{3}{2} & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & 1 & \frac{3}{2} \end{pmatrix} \right\}, \qquad \mathrm{Fix}(S_{(\mathbb{Z}_1,0)}) = \left\{ \tau \begin{pmatrix} 2 & 1 & 2 & -1 & 2 & -4 \\ 1 & 2 & 2 & 0 & 2 & -2 \\ 2 & 2 & 4 & -1 & 4 & -6 \\ -1 & 0 & -1 & 2 & -2 & 4 \\ 2 & 2 & 4 & -2 & 6 & -8 \\ -4 & -2 & -6 & 4 & -8 & 14 \end{pmatrix} \right\}, \tag{3.34}$$

for $\tau \in \mathscr{H}_1$, where we denote the rank-r Siegel upper half space by $\mathscr{H}_r$. These results are, of course, consistent with the duality groups we found above. For instance, the transformation $\mathsf{S} \doteq \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ belongs to the set (3.31), which is sent via the map $\mathcal{M}_{(\mathbb{Z}_2^2,00),(\mathbb{Z}_1,0)}$ to the symplectic matrix

$$\mathcal{M}_{(\mathbb{Z}_2^2,00),(\mathbb{Z}_1,0)}(\mathsf{S}) = \begin{pmatrix} 0 & -K_{\mathbb{Z}_1,\mathbb{Z}_2^2}^{\vee} \\ K_{\mathbb{Z}_1,\mathbb{Z}_2^2} & 0 \end{pmatrix}, \tag{3.35}$$

and

$$\begin{pmatrix} 0 & -K_{\mathbb{Z}_1,\mathbb{Z}_2^2}^{\vee} \\ K_{\mathbb{Z}_1,\mathbb{Z}_2^2} & 0 \end{pmatrix} \circ \mathrm{Fix}(S_{(\mathbb{Z}_2^2,00)}) = \mathrm{Fix}(S_{(\mathbb{Z}_1,0)}), \tag{3.36}$$

as can be checked explicitly.

Step 5: Including reflection outer automorphisms of the Weyl group. From table 4 we have $\text{Out}_{\text{refl.}}(D_6) = \mathbb{Z}_2 \doteq \{0, 1\}$, in an additive notation where we denote the trivial (identity) automorphism by "0" and the non-trivial element by "1". The non-trivial element corresponds to the D_6 Coxeter diagram symmetry which interchanges the "spinor" and "conjugate spinor" nodes, leaving the rest unchanged. As such, it has the effect of interchanging the $\mathbb{Z}_2^S$ and $\mathbb{Z}_2^C$ representations as indicated in table 3.

The effect on the equivalence classes of principally polarized symplectic representations (SK orbits) is not hard to guess. For those provisional orbits found by looking for integral equivalences, and shown as sets of representations denoted $\{\dots\}$ in the 2nd column of table 1, if an orbit does not include either the $\mathbb{Z}_2^S$ or $\mathbb{Z}_2^C$ representation, then there are no further equivalences under twisted intertwiners. Most of the orbits for the D_{4k+2} cases shown in table 1 are of this type.

But if there is an orbit including a $\mathbb{Z}_2^S$ representation and no $\mathbb{Z}_2^C$ representation, then there is a twisted intertwiner identifying it with the corresponding orbit with S and C labels interchanged. (Such an orbit must exist since interchanging S and C is the expression of the automorphism of the Weyl group.) In this case those two orbits are equivalent, and so the number of orbits is reduced relative to the number found using only untwisted intertwiners. Note that in this case there are no additional self-intertwiners, so the S-duality group of the orbit stays the same relative to the group found using only untwisted intertwiners. This is the situation described in (2.59) in the general discussion of section 2.7. From table 1 we see that this situation does not occur for D_{4k+2}, but it does occur in the D_{4k} cases. (The D_4 special case will be remarked upon in section 4.)

Finally, if there is an orbit including both a $\mathbb{Z}_2^S$ representation and a $\mathbb{Z}_2^C$ representation, then there is a twisted intertwiner identification within the orbit. This does not change the number of orbits relative to the number found using only untwisted intertwiners, but does enlarge the S-duality group of the orbit relative to the group found using only untwisted intertwiners. This is the situation described in (2.60) in the general discussion of section 2.7. From table 1 we see that this situation does occur for the $\{(\mathbb{Z}_2^S, 0), (\mathbb{Z}_2^C, 0), (\mathbb{Z}_2^2, 10)\}$ orbit of D_{4k+2}.

Since $\mathbb{Z}_2^S$ and $\mathbb{Z}_2^C$ are within the same orbit, they are integrally equivalent, and since they are related by the outer automorphism, there is a twisted integral equivalence between them. Composing these two equivalences, there is therefore a twisted integral self-equivalence of $\mathbb{Z}_2^S$, i.e., there is an intertwiner between $S_{\mathbb{Z}_2^S} \circ 0$ and $S_{\mathbb{Z}_2^S} \circ 1$. This affects the S-duality group of this SK structure orbit, and we illustrate its computation here.

One can also check mechanically for the non-existence of twisted self-intertwiners for all other representations, bearing out the pattern argued above and shown in table 1, but we do not give any more details of these checks here. See the BC_2 example in the next subsection where all twisted intertwiners are described explicitly.

Here we just focus on the twisted self intertwiners for the $(\mathbb{Z}_2^S, 0)$ representation in the orbit. Since it has a trivial Ext group, we drop the irrelevant "0" and denote it as just $\mathbb{Z}_2^S$. We call the untwisted and twisted versions $S_{\mathbb{Z}_2^S} \circ 0 = S_{\mathbb{Z}_2^S}$ and $S_{\mathbb{Z}_2^S} \circ 1$. The 6 reflection generators of $\mathbb{Z}_2^S$ are easy to calculate from (2.29) in our chosen basis (3.17); we do not

list them here since they are so large. The generators of $S_{\mathbb{Z}_2^C} \circ 1$ are the same as those of $S_{\mathbb{Z}_2^S}$ but with the two reflections corresponding to the spinor and conjugate spinor roots interchanged. Using these representation matrices, or from (2.32), the fixed locus of their action on $\mathscr{H}_6$ is the 1-dimensional set

$$\text{Fix}(S_{\mathbb{Z}_2^S}) = \left\{ 2\tau K_{\mathbb{Z}_2^S}^\vee \mid \tau \in \mathscr{H}_1 \right\}, \qquad \text{with} \qquad K_{\mathbb{Z}_2^S}^\vee = \frac{1}{2} \begin{pmatrix} 3 & 0 & 1 & 2 & -4 & 3 \\ 0 & 4 & 2 & 0 & 4 & -2 \\ 1 & 2 & 3 & 2 & 0 & 1 \\ 2 & 0 & 2 & 4 & -4 & 4 \\ -4 & 4 & 0 & -4 & 12 & -8 \\ 3 & -2 & 1 & 4 & -8 & 7 \end{pmatrix}. \tag{3.37}$$

Evaluation of (2.38) gives the (untwisted) $\mathbb{Z}_2^S$ self-intertwiners

$$M_{\mathbb{Z}_2^S}^0 \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \right) = \begin{pmatrix} aI_6 & bK_{\mathbb{Z}_2^S}^\vee \\ cK_{\mathbb{Z}_2^S} & dI_6 \end{pmatrix}. \tag{3.38}$$

This is an element of $\text{Sp}(6, \mathbb{Z})$ iff $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{Z})$ and b and c are even. Furthermore, the Möbius action (A.14) of this intertwiner descends to an action on $\tau \in \mathscr{H}_1$ parameterizing $\text{Fix}(S_\mathcal{A})$ in (3.37) given by

$$\mathcal{M}_{\mathbb{Z}_2^S}^0 \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \right) \circ \tau = \frac{a\tau + b}{c\tau + d}, \tag{3.39}$$

which is the usual Möbius action. With the conditions following from the integrality of $M_{\mathbb{Z}_2^S}^0$, the transformations on $\mathscr{H}_1$ form a modular subgroup

$$\mathscr{S}_{\mathbb{Z}_2^S}^0 \doteq \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{Z}) \text{ and } b \text{ and } c \text{ even} \right\}, \tag{3.40}$$

isomorphic to $\Gamma_0(4) \subset \text{PSL}(2, \mathbb{Z})$. In appendix C we define $\Gamma_0(4)$ as the subgroup with the lower left entry divisible by 4; it is isomorphic to the subgroup with lower left and upper right entries divisible by 2 by conjugation by the $\text{SL}(2, \mathbb{R})$ matrix $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1/2 \end{smallmatrix} \right)$. Recall that $\Gamma_0(4)$ has index 6 in $\text{SL}(2, \mathbb{Z})$.

We now perform the same calculation but for twisted intertwiners $M_{\mathbb{Z}_2^S}^1$, i.e., integer determinant 1 matrices which intertwine the untwisted representation $S_{\mathbb{Z}_2^S} = S_{\mathbb{Z}_2^S} \circ 0$ with the outer automorphism twisted representation $S_{\mathbb{Z}_2^S} \circ 1$. We compute

$$M_{\mathbb{Z}_2^S}^1 \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \right) = \begin{pmatrix} a & 0 & 0 & 0 & -\frac{a}{2} & -a & b & 0 & 0 & 0 & -b & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 2b & b & 0 & 2b & -b \\ 0 & 0 & a & 0 & -\frac{a}{2} & -a & 0 & b & b & 0 & b & -b \\ 0 & 0 & 0 & a & -a & -2a & 0 & 0 & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 & 2a & 2a & -b & 2b & b & 0 & 4b & -b \\ 0 & 0 & 0 & 0 & -\frac{3a}{2} & -2a & 0 & -b & -b & -b & -b & -b \\ 2c & -c & 0 & 0 & c & 0 & d & 0 & 0 & 0 & 0 & 0 \\ -c & 2c & -c & 0 & -c & 0 & 0 & d & 0 & 0 & 0 & 0 \\ 0 & -c & 2c & -c & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & -c & 2c & 0 & -c & 0 & 0 & 0 & d & 0 & 0 \\ c & -c & 0 & 0 & c & 0 & -\frac{d}{2} & 0 & -\frac{d}{2} & -d & 2d & -\frac{3d}{2} \\ 0 & 0 & 0 & -c & 0 & 0 & -d & 0 & -d & -2d & 2d & -2d \end{pmatrix}, \tag{3.41}$$

which is in $\text{Sp}(6, \mathbb{Z})$ iff $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{Z})$ and a and d are even. It also descends to an action on $\tau \in \mathscr{H}_1$ given by the usual Möbius action (3.39). This set

$$\mathscr{S}_{\mathbb{Z}_2^S}^1 \doteq \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{Z}) \text{ and } a \text{ and } d \text{ even} \right\}, \tag{3.42}$$

does *not* form a group.

But the union of these two sets of identifications on the conformal manifold of the $\mathbb{Z}_2^S$ SK structure orbit,

$$\mathscr{S}_{\mathbb{Z}_2^S} \doteq \mathscr{S}_{\mathbb{Z}_2^S}^0 \cup \mathscr{S}_{\mathbb{Z}_2^S}^1 = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}(2,\mathbb{Z}) \text{ and } a \text{ and } d \text{ even, or } b \text{ and } c \text{ even} \right\}, \qquad (3.43)$$

does form a group, as is easy to check. Since it is twice the size of $\mathscr{S}_{\mathbb{Z}_2^S}^0 \cong \Gamma_0(4)$ which has index 6 in $\mathrm{SL}(2,\mathbb{Z})$, it follows that $\mathscr{S}_{\mathbb{Z}_2^S}$ must have index 3, and so it must be $\cong \Gamma_0(2)$ since that is the only index 3 modular subgroup. More explicitly, conjugating the group (3.43) in $\mathrm{SL}(2,\mathbb{Z})$ by $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right)$ takes it to $\Gamma_0(2)$. This shows that, upon including outer automorphism twisted intertwiners, the S-duality group of the $\{(\mathbb{Z}_2^S,0),(\mathbb{Z}_2^C,0),(\mathbb{Z}_2^2,10)\}$ orbit of D_6 is $\Gamma_0(2)$, as claimed in table 1.

3.3 $\mathfrak{so}(5) = \mathfrak{sp}(4)$ example

In this subsection, as in the previous one, we carry out the explicit computations for $W = BC_2$. There are two lattices to consider, which we call the root lattice and the weight lattice, defined by the following matrices (as above, the basis vectors are the columns of the matrices):

Lattice	Symbol	Construction	Basis
Γ_{root}	$\mathbb{Z}_2$	$\mathbb{Z}^2\vert_{\Sigma\in 2\mathbb{Z}}$	$\left(\begin{smallmatrix} 1 & 0 \\ -1 & 2 \end{smallmatrix} \right)$
Γ_{weight}	$\mathbb{Z}_2^\vee$	$\mathbb{Z}^2$	$\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix} \right)$

(3.44)

Hence $\mathcal{I} = \{\mathbb{Z}_2, \mathbb{Z}_2^\vee\}$. The Ext group computations follows exactly the same lines as in the example in section 3.2. We find $\mathrm{Ext}^1_{\mathbb{Z}W}(R_A, R_A^\vee) = \mathbb{Z}_2$ for both $A = \mathbb{Z}_2, \mathbb{Z}_2^\vee$, so

$$\mathfrak{I} = \mathbb{Z}_2 \sqcup \mathbb{Z}_2 = \{(\mathbb{Z}_2,0),(\mathbb{Z}_2,1),(\mathbb{Z}_2^\vee,0),(\mathbb{Z}_2^\vee,1)\}. \qquad (3.45)$$

The explicit matrix form of the generating reflections in all these cases is

$\mathcal{A} \in \mathfrak{I}$	$S_{\mathcal{A}}(w_1)$	$S_{\mathcal{A}}(w_2)$
$(\mathbb{Z}_2,0)$	$\left(\begin{smallmatrix} -1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{smallmatrix} \right)$	$\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{smallmatrix} \right)$
$(\mathbb{Z}_2,1)$	$\left(\begin{smallmatrix} -1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{smallmatrix} \right)$	$\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{smallmatrix} \right)$
$(\mathbb{Z}_2^\vee,0)$	$\left(\begin{smallmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{smallmatrix} \right)$	$\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{smallmatrix} \right)$
$(\mathbb{Z}_2^\vee,1)$	$\left(\begin{smallmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{smallmatrix} \right)$	$\left(\begin{smallmatrix} 1 & 0 & 0 & -1 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{smallmatrix} \right)$

(3.46)

The bindings can be read from the top right 2×2 matrices.

The sets of $\boldsymbol{\tau} \in \mathscr{H}_2$ fixed by these representations is easily computed. For instance, we find

$$\mathrm{Fix}(S_{(\mathbb{Z}_2,1)}) = \left\{ \tau \left(\begin{smallmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{smallmatrix} \right) - \left(\begin{smallmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{smallmatrix} \right), \, \tau \in \mathscr{H}_1 \right\}. \qquad (3.47)$$

From here we can compute the duality groups. The intertwiners $\mathcal{M}_{\mathcal{AB}}$ defined in (2.63) and (2.38) are given in the untwisted section of table 5. The duality groups (2.44)

are therefore defined by requiring that these matrices have integer entries. For instance, the Möbius action (A.14) of $\mathcal{M}^0_{\mathcal{A}\mathcal{A}}$ descends to a Möbius action on $\tau \in \mathscr{H}_1$ parameterizing $\text{Fix}(S_{\mathcal{A}})$ given by

$$\mathcal{M}^0_{\mathcal{A}\mathcal{A}}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \circ \tau = \frac{a\tau + b}{c\tau + d}, \tag{3.48}$$

where, from table 5, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \text{SL}(2,\mathbb{Z})$ and b is even in the $\mathcal{A} = (\mathbb{Z}_2, 1)$ case and c is even in the $\mathcal{A} = (\mathbb{Z}_2^\vee, 1)$ case. Hence the provisional duality groups are isomorphic to $\Gamma_0(2)$ for $\mathcal{A} = (\mathbb{Z}_2, 1)$, $(\mathbb{Z}_2^\vee, 1)$, while the provisional duality group for $\mathcal{A} = (\mathbb{Z}_2, 0)$, $(\mathbb{Z}_2^\vee, 0)$ is $\text{PSL}(2,\mathbb{Z})$. (These modular groups are reviewed in appendix C.) There also exist integral intertwiners between $\mathcal{A} = (\mathbb{Z}_2, 1)$ and $\mathcal{B} = (\mathbb{Z}_2^\vee, 1)$, shown in the last untwisted row of table 5. There are no solutions for integral untwisted intertwiners between any other pair of symplectic representations.

The untwisted rows of table 5 thus partition the set of representations $\mathcal{A}$ into provisional equivalence classes, or orbits:

$$
\begin{array}{ccc}
\text{PSL}(2,\mathbb{Z}) \;\; \big(\;\; (\mathbb{Z}_2, 0) & (\mathbb{Z}_2^\vee, 0) \;\; \big) \;\; \text{PSL}(2,\mathbb{Z}) \\[2mm]
\Gamma_0(2) \;\; \big(\;\; (\mathbb{Z}_2, 1) \text{——} (\mathbb{Z}_2^\vee, 1) \;\; \big) \;\; \Gamma_0(2)
\end{array}
\tag{3.49}
$$

The gray rectangles each indicates a provisional SK orbit. We write the provisional duality groups on the lines connecting a representation to itself. There are thus three orbits: two with provisional duality group $\text{PSL}(2,\mathbb{Z})$ and one with provisional duality group $\Gamma_0(2)$. For the line connecting distinct representations within the $\Gamma_0(2)$ orbit, the set of intertwiners is not a group, but is explicitly described in table 5.

Outer automorphisms. The orbits and duality groups in the last paragraph were only provisional because the discussion has so far omitted outer automorphisms. For BC_2 the reflection outer automorphism group is $\mathbb{Z}_2$ whose elements we denote in an additive notation as $\phi \in \{0, 1\} = \mathbb{Z}_2$. The $\phi = 0$ superscript in table 5 refers to the fact that those intertwiners involve no outer automorphism twist, meaning that they are twisted by the identity element $\phi \doteq 0$ of the outer automorphism group. If we look for intertwiners between untwisted representations $\mathcal{A} \circ (\phi = 0)$ and their twisted versions, $\mathcal{A} \circ (\phi = 1)$, we find the additional intertwiners listed in the twisted part of table 5.

The actions of these intertwiners and the resulting self-duality groups can be repre-

untwisted intertwiners $\mathcal{A} \to \mathcal{B}$	$\mathcal{M}_{\mathcal{AB}}^{\phi=0}$	for $ad-bc=1$ and
$\text{PSL}(2,\mathbb{Z}) \circlearrowright (\mathbb{Z}_2,0)$	$\begin{pmatrix} a & 0 & 2b & b \\ 0 & a & b & b \\ c & -c & d & 0 \\ -c & 2c & 0 & d \end{pmatrix}$	$a,b,c,d \in \mathbb{Z}$
$\Gamma_0(2) \circlearrowright (\mathbb{Z}_2,1)$	$\begin{pmatrix} a & 0 & b & \frac{b}{2} \\ c & a-2c & \frac{b}{2} & \frac{a}{2}+\frac{b}{2}-c-\frac{d}{2} \\ 2c & -2c & d & -c \\ -2c & 4c & 0 & 2c+d \end{pmatrix}$	$a,c,d \in \mathbb{Z},\ b \in 2\mathbb{Z}$
$(\mathbb{Z}_2^\vee,0) \circlearrowleft \text{PSL}(2,\mathbb{Z})$	$\begin{pmatrix} a & 0 & b & b \\ 0 & a & b & 2b \\ 2c & -c & d & 0 \\ -c & c & 0 & d \end{pmatrix}$	$a,b,c,d \in \mathbb{Z}$
$(\mathbb{Z}_2^\vee,1) \circlearrowleft \Gamma_0(2)$	$\begin{pmatrix} a-c & \frac{c}{2} & \frac{a}{2}+b-\frac{c}{2}-\frac{d}{2} & b \\ 0 & a & b & 2b \\ 2c & -c & c+d & 0 \\ -c & c & -\frac{c}{2} & d \end{pmatrix}$	$a,b,d \in \mathbb{Z},\ c \in 2\mathbb{Z}$
$(\mathbb{Z}_2,1) \longrightarrow (\mathbb{Z}_2^\vee,1)$	$\begin{pmatrix} a-c & c & b-\frac{d}{2} & \frac{b}{2}+\frac{c}{2} \\ 0 & 2a & b & a+b \\ 2c & -2c & d & -c \\ -c & 2c & 0 & c+\frac{d}{2} \end{pmatrix}$	$a,b,c \in \mathbb{Z},\ d \in 2\mathbb{Z}$
twisted intertwiners $\mathcal{A} \circ 0 \to \mathcal{B} \circ 1$	$\mathcal{M}_{\mathcal{AB}}^{\phi=1}$	for $ad-bc=1$ and
$(\mathbb{Z}_2,0) \longrightarrow (\mathbb{Z}_2^\vee,0)$	$\begin{pmatrix} 0 & a & b & 2b \\ a & 0 & b & b \\ -c & c & 0 & d \\ 2c & -c & d & 0 \end{pmatrix}$	$a,b,c,d \in \mathbb{Z}$
$\circlearrowright (\mathbb{Z}_2,1)$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 2a & b & a+b \\ a-2c & 2c & b-d & \frac{b}{2}+c \\ -2c & 4c & 0 & 2c+d \\ 4c & -4c & 2d & -2c \end{pmatrix}$	$a,b,d \in \sqrt{2}\mathbb{Z},\ c \in \frac{\mathbb{Z}}{\sqrt{2}}$
$(\mathbb{Z}_2^\vee,1) \circlearrowleft$	$\frac{1}{\sqrt{2}}\begin{pmatrix} c & a-c & b+\frac{c}{2} & 2b-d \\ 2a & 0 & a+2b & 2b \\ -2c & 2c & -c & 2d \\ 2c & -c & c+d & 0 \end{pmatrix}$	$a,c,d \in \sqrt{2}\mathbb{Z},\ b \in \frac{\mathbb{Z}}{\sqrt{2}}$
$(\mathbb{Z}_2,1) \longrightarrow (\mathbb{Z}_2^\vee,1)$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & a & b & 2b \\ a-2c & c & \frac{a}{2}+b-c-d & b \\ -2c & 2c & -c & 2d \\ 4c & -2c & 2c+2d & 0 \end{pmatrix}$	$a,b,c \in \sqrt{2}\mathbb{Z},\ d \in \frac{\mathbb{Z}}{\sqrt{2}}$

Table 5. The $\text{Sp}(4,\mathbb{Z})$ matrices $\mathcal{M}_{\mathcal{AB}}^{\phi \in \{0,1\}}$ for untwisted and twisted intertwiners among the four symplectic representations $\mathcal{A}, \mathcal{B} \in \{(\mathbb{Z}_2,0), (\mathbb{Z}_2,1), (\mathbb{Z}_2^\vee,0), (\mathbb{Z}_2^\vee,1)\}$ of Weyl(BC_2). In column 1, the $\circlearrowright$ ($\circlearrowleft$) arrows indicate self-intertwiners between symplectic representations $\mathcal{A} = \mathcal{B}$.

sented schematically as

The gray rectangles are to aid the eye in seeing which representations are equivalent (related by untwisted or twisted intertwiners) and so belong to the same orbit. The left columns show both the representations $\mathcal{A} \circ 0$ and their outer automorphism twisted counterparts, $\mathcal{A} \circ 1$. But, since a representation $\mathcal{A} = \mathcal{A} \circ 0$ and its automorphism twisted version, $\mathcal{A} \circ 1$, describe isomorphic SK structures $(\mathcal{A} \circ 0) \cong (\mathcal{A} \circ 1)$, they must be identified.[22] This isomorphism induces additional equivalences among the provisional 3 orbits shown in (3.49). Diagrammatically, this is represented by identifying the two rows of gray rectangle to one row and is indicated by the red arrows. The resulting orbits of representations and their (self-)equivalences are shown in the right column. We see that of the provisional 3 orbits shown in (3.49), the two with provisional self-duality group $\mathrm{PSL}(2, \mathbb{Z})$ are identified by a twisted intertwiner and become a single orbit with unchanged self-duality group $\mathrm{PSL}(2, \mathbb{Z})$. This is the situation (2.59) described in the general discussion of section 2.7. The third provisional orbit in (3.49), with provisional self-duality group $\Gamma_0(2)$, remains an orbit of the same two representations, but now has the enlarged self-duality group H_4 by virtue of additional twisted self-intertwiners. This is the situation described earlier in (2.60).

It may be useful to describe explicitly how H_4 arises from the $\mathcal{M}_{\mathcal{A}\mathcal{A}}^{\phi}$ intertwiners shown in table 5 with $\mathcal{A} = (\mathbb{Z}_2, 1)$. We have already remarked that the $\mathcal{M}_{\mathcal{A}\mathcal{A}}^{\phi=0}$ action on $\mathrm{Fix}(\mathcal{A})$ (3.47) descends to the action (3.48) on $\tau \in \mathscr{H}_1$ by $\mathrm{PSL}(2, \mathbb{Z})$ matrices. Noting that $\mathrm{Fix}(\mathcal{A} \circ 0) = \mathrm{Fix}(\mathcal{A} \circ 1)$ (almost by definition), a similar calculation shows that the $\mathcal{M}_{\mathcal{A}\mathcal{A}}^{\phi=1}$ action on $\mathrm{Fix}(\mathcal{A})$ also descends to the action (3.48) on $\tau \in \mathscr{H}_1$ but by matrices in $\mathrm{SL}(2, \mathbb{R})$. (We chose the parameterization of the intertwiners in table 5 just so that this would be the case.) Thus, we have computed

$$\mathscr{S}_{\mathcal{A}}^0 \doteq (\mathcal{M}_{\mathcal{A}\mathcal{A}}^{\phi=0})^{-1}(\mathrm{Sp}(4, \mathbb{Z})) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}(2, \mathbb{R}) \;\middle|\; a, c, d \in \mathbb{Z}, b \in 2\mathbb{Z} \right\} \cong \Gamma^0(2), \qquad (3.50)$$

$$\mathscr{S}_{\mathcal{A}}^1 \doteq (\mathcal{M}_{\mathcal{A}\mathcal{A}}^{\phi=1})^{-1}(\mathrm{Sp}(4, \mathbb{Z})) = \left\{ \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix} \right) \in \mathrm{SL}(2, \mathbb{R}) \;\middle|\; a', b', d' \in \sqrt{2}\mathbb{Z}, c' \in \tfrac{1}{\sqrt{2}}\mathbb{Z} \right\}.$$

As indicated in (3.50), $\mathscr{S}_{\mathcal{A}}^0$ forms the congruence subgroup $\Gamma^0(2) \subset \mathrm{SL}(2, \mathbb{Z})$, but $\mathscr{S}_{\mathcal{A}}^1$ by itself is not a group. Nonetheless, their union $\mathscr{S}_{\mathcal{A}}^0 \cup \mathscr{S}_{\mathcal{A}}^1$ is a group generated by $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} 0 & \sqrt{2} \\ -1/\sqrt{2} & 0 \end{smallmatrix} \right)$, which is isomorphic to the Hecke group H_4, reviewed in appendix C.

4 SK structure as a low-energy test of S-duality of N=4 sYM

A set of sharp S-duality conjectures for absolute $\mathcal{N}=4$ sYM theories were proposed and worked out in [4, 5, 8]. These specify which of the possible global forms of the gauge group — decorated with some further discrete labels (the "discrete theta angles" of [4]) — occur as weak coupling limits of a connected conformal manifold. These connected pieces of the conformal manifold are the called *S-duality orbits of global structures* of the sYM theory. Furthermore, the group of self identifications of the conformal manifold of each global structure orbit are also predicted to be certain Hecke subgroups. A summary of these predictions is given in the first three columns of table 6.

A comparison of this table with our results for SK structures given in table 1 shows broad agreement between the S-duality groups of SK structure orbits and of global structure

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[22]This isomorphism is argued for in appendix A.3.

$\mathfrak{g}$	S-duality orbits of global structures	S-duality group	Index in $H_{q(\ell)}$
A_1	$\{\mathrm{SU}(2), \mathrm{SO}(3)_\pm\}$	$\Gamma_0(2)$	3
$A_r\ (r\geq 2)$ $N \doteq r+1$	one S-duality orbit for each $s^2 \mid N$	$\Gamma_0(N/s^2)$	$\frac{N}{s^2} \cdot \prod\limits_{p \mid \frac{N}{s^2}} \left(1 + \frac{1}{p}\right)$
$B_2 = C_2$	$\{\mathrm{SO}(5)_-\}$	H_4	1
	$\{\mathrm{Sp}(4), \mathrm{SO}(5)_+\}$	$\Gamma_0(2)$	2
B_{2k}/C_{2k} $(k\geq 2)$ $N \doteq 4k$	$\{(\mathrm{Sp}(N)/\mathbb{Z}_2)_+, \mathrm{Spin}(N+1)\}$	$\Gamma_0(2)$	2
	$\{(\mathrm{Sp}(N)/\mathbb{Z}_2)_-, \mathrm{SO}(N+1)_-\}$	$\Gamma_0(2)$	2
	$\{\mathrm{Sp}(N), \mathrm{SO}(N+1)_+\}$	$\Gamma_0(2)$	2
B_{2k+1}/C_{2k+1} $N \doteq 4k+2$	$\{(\mathrm{Sp}(N)/\mathbb{Z}_2)_\pm, \mathrm{Spin}(N+1), \mathrm{SO}(N+1)_-\}$	$\Gamma_0(4)$	4
	$\{\mathrm{Sp}(N), \mathrm{SO}(N+1)_+\}$	$\Gamma_0(2)$	2
D_{4k} $N \doteq 8k$	$\{\mathrm{Spin}(N), (\mathrm{SO}(N)/\mathbb{Z}_2)_{\pm+ \atop \mp\pm}\}$	$\Gamma_0(2)$	3
	$\{\mathrm{SO}(N)_-, (\mathrm{SO}(N)/\mathbb{Z}_2)_{\pm- \atop \mp\pm}\}$	$\Gamma_0(2)$	3
	$\{\mathrm{Ss}(N)_-, (\mathrm{SO}(N)/\mathbb{Z}_2)_{\pm+ \atop \pm\mp}\}$	$\Gamma_0(2)$	3
	$\{\mathrm{Sc}(N)_-, (\mathrm{SO}(N)/\mathbb{Z}_2)_{\pm- \atop \mp\pm}\}$	$\Gamma_0(2)$	3
	$\{\mathrm{SO}(N)_+\}$	$\mathrm{PSL}(2,\mathbb{Z})$	1
	$\{\mathrm{Ss}(N)_+\}$	$\mathrm{PSL}(2,\mathbb{Z})$	1
	$\{\mathrm{Sc}(N)_+\}$	$\mathrm{PSL}(2,\mathbb{Z})$	1
D_{4k+2} $N \doteq 8k+4$	$\{\mathrm{Spin}(N), (\mathrm{SO}(N)/\mathbb{Z}_2)_{\pm+ \atop \mp\pm}\}$	$\Gamma_0(2)$	3
	$\{\mathrm{SO}(N)_-, (\mathrm{SO}(N)/\mathbb{Z}_2)_{\pm- \atop \mp\pm}\}$	$\Gamma_0(2)$	3
	$\{\mathrm{Ss}(N)_\pm, \mathrm{Sc}(N)_\pm, (\mathrm{SO}(N)/\mathbb{Z}_2)_{\pm\pm \atop \mp\mp}\}$	$\Gamma_0(4)$	6
	$\{(\mathrm{SO}(N)/\mathbb{Z}_2)_{\pm- \atop \mp\mp}\}$	Δ	2
	$\{\mathrm{SO}(N)_+\}$	$\mathrm{PSL}(2,\mathbb{Z})$	1
D_{2k+1} $N \doteq 4k+2$	$\{\mathrm{Spin}(N), \mathrm{SO}(N)_-, (\mathrm{Spin}(N)/\mathbb{Z}_2)_{1,2,3,4}\}$	$\Gamma_0(4)$	6
	$\{\mathrm{SO}(N)_+\}$	$\mathrm{PSL}(2,\mathbb{Z})$	1
E_6	$\{E_6, (E_6)_{1,2,3}\}$	$\Gamma_0(3)$	4
E_7	$\{E_7, (E_7)_\pm\}$	$\Gamma_0(2)$	3
E_8	$\{E_8\}$	$\mathrm{PSL}(2,\mathbb{Z})$	1
F_4	$\{F_4\}$	H_4	1
G_2	$\{G_2\}$	H_6	1

Table 6. The first 3 columns summarize the S-duality orbit and group predictions from field theory [4, 8]. The red entries highlight the places where these predictions disagree with our classification of SK structures given in table 1. The $\pm$ signs in the subscripts for the $\mathrm{SO}(N)/\mathbb{Z}_2$ global structures are correlated; thus, for example, $(\mathrm{SO}(N)/\mathbb{Z}_2)_{\pm+ \atop \mp+}$ denotes two global structures, not four. The last column gives the index of the S-duality group as a subgroup of the Hecke group H_q where $q = 3, 4, 6$ is determined (C.1) by the lacing number, $\ell = 1, 2, 3$, respectively, of the gauge algebra $\mathfrak{g}$. It counts the number of global structures in each orbit. Note that $H_3 = \mathrm{PSL}(2,\mathbb{Z})$.

orbits, and a looser qualitative agreement between our listing of SK structure representations and global structures of gauge groups. Indicated in red in both tables are the entries where either S-duality groups do not agree, or where the counting of the number of orbits

do not agree.

In this section we discuss these discrepancies and argue that

1. The disagreement of the S-duality groups for certain orbits in the BC_r series (including A_1) is an "IR accident": the SK structure is a low energy observable which, in these cases, simply fails to distinguish between inequivalent field theories. Indeed, it is a striking and unexpected fact that in so many cases the SK structure successfully distinguishes $\mathcal{N}{=}4$ sYM theories, whereas there are many other examples of $\mathcal{N}{=}2$ theories where the scale invariant CB geometry fails to distinguish them.

2. The counting of orbits disagreement in the D_{4k} cases, as well as the disagreement of the S-duality group for an orbit of the D_{4k+2} cases, are due to an error in the field theory literature: the SK structure computation gives the correct answer.

3. Finally, the pattern observed that, in general, the number of field theory global structures in a given orbit is greater than the number of SK structure representations in the corresponding orbit, is to be expected, and does not indicate a physical contradiction. Instead, only a correspondence between equivalence classes (orbits) of SK structures and S-duality orbits of global structures is expected.

Before addressing these three points, we pause to emphasize the logic of comparing low energy effective actions on the moduli space of vacua (SK structures) to S-duality conjectures in the field theory. Field theory S-duality conjectures the exact equivalence between CFTs described by various weak coupling limits of their exactly marginal coupling. These conjectures are supported, of course, by an overwhelming amount of evidence by now, but have not been proven in field theory. A way of testing these conjectures is to compute all inequivalent moduli space geometries consistent with unbroken $\mathcal{N}{=}4$ supersymmetry without making assumptions about S-duality properties or about the spectrum of massive BPS states out on the moduli space. But mismatches between moduli space geometries and field theories can occur for two reasons:

▸ moduli space geometries fail to distinguish inequivalent field theories, or

▸ moduli space geometries exist which do not arise from any (known) field theory.

We limit the second possibility by putting a few additional constraints on the geometries. These are:

(i) Do not allow complex singularities on a CB slice of the $\mathcal{N}{=}4$ moduli space.

(ii) Assume the existence of at least one free field theory point with a simple gauge group on the conformal manifold of the structure.

(iii) Restrict to principally polarized SK structures.

These then restrict to the Weyl group orbifolds studied in this paper, as explained in appendix A.

Constraint (i) is imposed since it is known that all $\mathcal{N}=4$ sYM theories have such CBs. Though there also exist CBs of $\mathcal{N}=4$ sYM theories with complex singularities [13, 14], they all arise from gauging certain discrete 0-form symmetries of theories without such singularities. This constraint eliminates these geometries as well as other possible "exotic" singular geometries compatible with $\mathcal{N}=4$ supersymmetry, but possessing no known field theory origin.

Constraint (ii) is imposed since all $\mathcal{N}=4$ sYM theories have an exactly marginal coupling with, by definition, a weak coupling limit. From the discussion in appendix A, the Kähler structure of an SK geometry which satisfies the constraint (i) and which has a weak coupling point on its conformal manifold where it is described by a sYM lagrangian with simple gauge group $\mathfrak{g}$, will be that of a Weyl group orbifold. By adopting this constraint we are eliminating geometries that could correspond to possible "exotic" $\mathcal{N}=4$ SCFTs which have no weak coupling limits [26].

The third and final restriction, (iii), derives from wanting to compare to absolute $\mathcal{N}=4$ theories. Absolute $4d$ gauge theories, such as $\mathcal{N}=4$ sYM theories, are defined by a choice of maximal mutually local probe lines [4, 6], meaning that the Dirac pairing between them is principal.[23] This choice of probe lines in the sYM SCFT corresponds to a choice of probe lines that renders the low-energy $\mathfrak{u}(1)^r$ gauge theory at a generic point on the CB to be absolute. Such abelian probe lines are specified by a charge lattice that is maximal and mutually local with respect to the Dirac pairing and contains the physical BPS charge lattice of the sYM theory [9, 27]. A lattice of probe lines with its principal Dirac pairing is referred to as a *line lattices* in [9], and it's identified with the homology lattice and symplectic pairing, respectively, of the principally polarized abelian variety fiber of a (necessarily) principal SK structure. It is for this reason that we focus on SK structures with principal polarization.

In section 5 we discuss some possible consequences of lifting constraint (iii) as well as of allowing non-simple $\mathfrak{g}$ in constraint (ii).

4.1 The BC_r disagreement

In the $A_1 = BC_1$ case, though there is a single SK structure orbit and a single S-duality orbit of $\mathfrak{g} = \mathfrak{su}(2)$ global structures, the S-duality groups of the two orbits do not match. The SK orbit duality group is $\mathrm{PSL}(2,\mathbb{Z})$ while the field theory S-duality group is $\Gamma_0(2)$, an index 3 subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Correspondingly, the field theory conformal manifold is a 3-fold cover of that of the SK orbit. This is not a contradiction: the CB geometry, after all, is just some observable of the field theory, and there is no reason it need uniquely characterize the field theory, so may fail to distinguish inequivalent field theories.

Conversely, the SK structure result cannot be interpreted as a mistake in the field theory S-duality conjectures. For there are three weak coupling global structures, $\mathrm{SU}(2)$ and $\mathrm{SO}(3)_\pm$ in the notation of [4], and two of them, namely $\mathrm{SO}(3)_\pm$, are continuously connected at weak coupling in $\mathcal{N}=4$ sYM. Thus there can only be either: a single S-duality orbit containing all three weak coupling limits (which is the usual S-duality conjecture);

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[23]This global data is often referred to as the global structure or global variant of the gauge theory.

or two S-duality orbits with one containing the $SU(2)$ limit and the other containing the $SO(3)_\pm$ limits. The SK structure result cannot distinguish between these two since it identifies the SK geometries of all three weak coupling limits.

The S-duality group discrepancies for the $BC_{r>1}$ theories are similar. In these cases there is an SK structure which fails to distinguish between two global structures, $Sp(N)$ and $SO(N+1)_+$, instead of three as in the A_1 case.[24] The SK orbit duality group is once again a 3-fold cover of the field theory S-duality group. The fact that this 3-fold cover fails to distinguish only 2 weak coupling limits indicates that it actually makes a 2-fold self-identification of one of the weak coupling limits. In this case it is the $SO(N+1)_+$ limit, where the SK structure cannot distinguish between gauge couplings τ and $\tau + 1$ in the normalization where the theta angle periodicity is $\tau \sim \tau + 2$. This purely weak coupling discrepancy makes it clear that the SK orbit result is simply an "IR accident" and cannot be taken as an indication of a mistake in the field theory S-duality conjecture.

In more detail, S-duality transformations of $\mathcal{N}$=4 sYM theories form subgroups of the level-q Hecke groups H_q for $q = 3, 4, 6$ corresponding to the lacing number for the gauge Lie algebra $\ell = 1, 2, 3$, respectively [28]. As a reminder, the lacing number is $\ell = 1$ for simply-laced $\mathfrak{g}$'s, $\ell = 2$ for BC_r and F_4, and $\ell = 3$ for G_2. Also, $H_3 = PSL(2, \mathbb{Z})$, but H_4 and H_6 are inequivalent discrete subgroups of $PSL(2, \mathbb{R})$; they are defined in appendix C. The maximal possible field theory S-duality group, H_q, would be one that fixes the gauge algebra $\mathfrak{g}$. But precisely for the $BC_{r>2}$ cases, GNO duality [29] relates two different Lie algebras, $\mathfrak{sp}(2N)$ and $\mathfrak{so}(2N+1)$. So in these cases the maximal S-duality group is only the index-2 subgroup of H_4 [7], which happens to be isomorphic to $\Gamma_0(2)$, which is an index-3 subgroup of $H_3 = PSL(2, \mathbb{Z})$ as reviewed in appendix C. So the fact that the H_4 group is not a subgroup of the SK orbit duality group, $PSL(2, \mathbb{Z})$, is not a contradiction since the predicted $\Gamma_0(2)$ S-duality group is a subgroup.

These failures of the SK structure result to distinguish between inequivalent field theories raises the question of what other observables, beyond the SK structure of the moduli space, could faithfully distinguish them. Since this failure is detected even at weak coupling, such additional observables are easy to provide. One, described in [4, 6, 9], is the lattice of EM charges of (finite energy) BPS states in the field theory as a sublattice of the probe line charge lattice. This clearly distinguishes the weak coupling limits since it is used as the definition of the gauge theory global structure. There are hints, coming from studies of rank-1 $\mathcal{N}$=2 geometries [30–32], that the BPS charge sublattice of the probe line lattice can be detected from some subtle arithmetic properties of the SK structures we have presented. We will comment on these in the context of non-principally polarized SK structures in section 5.

A more speculative proposal for a moduli space observable which could distinguish field theory global structures are the SK structures of $\mathcal{N}$=2 supersymmetry-preserving mass deformed moduli spaces. The evidence that this might be a rich enough observable comes mostly from the study of rank-1 CB geometries [10, 19] where the 3-fold degeneracy of the (principally polarized) scale invariant A_1 SK structure is lifted, as are many similar

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[24]In the BC_2 case $\mathfrak{sp}(4) = \mathfrak{so}(5)$ so global structure $Sp(4) = Spin(5)$ which is still distinct from $SO(5)_+$.

degeneracies in other rank-1 $\mathcal{N}$=2 geometries.

Finally, there is a disagreement of our SK structure results with a claim in [33], who, in their study of rank-2 $\mathcal{N}$=4 Seiberg-Witten curves, claim that the two SW curve orbits for BC_2 have duality groups which are both $\Gamma_0(2)$. This is an index-2 subgroup of the H_4 duality group of the $\{(\mathbb{Z}_2, 1), (\mathbb{Z}_2^\vee, 1)\}$ SK orbit reported in table 1, and a 3-fold cover of the $\mathrm{PSL}(2, \mathbb{Z})$ duality group of the $\{(\mathbb{Z}_2, 0), (\mathbb{Z}_2^\vee, 0)\}$ SK orbit reported there. (Comparing to the field theory predictions of table 6, they disagree with the $\{\mathrm{SO}(5)_-\}$ orbit, but agree with the $\{\mathrm{Sp}(4), \mathrm{SO}(5)_+\}$ orbit: the opposite of what we have computed here.) Both claims of [33] are mistaken; in the case of the curve for the first orbit it is due to missing the isometry of the CB resulting from the BC_2 reflection outer automorphism, and in the case of the curve for the second orbit it is due to missing a set of identifications hidden by the particular algebraic form of the Seiberg-Witten curve that they used.

4.2 The D_{2k} disagreement

The automorphism-twisted equivalences of SK structures are crucial in obtaining the Hecke group S-duality groups that appear in the BC_2, G_2 and F_4 non-simply laced cases, and this was illustrated in detail in the BC_2 case in section 3.3. But in the D_{2k} Weyl groups also have a non-trivial reflection outer automorphism, and including their associated equivalences leads to a modification of the counting of global structures relative to that given in the field theory literature, and also to a modification of the S-duality group of one orbit of the D_{4k+2} theories relative to the field theory prediction. This modification of the S-duality group in the case of D_6 was also illustrated in detail in section 3.2.

While one could interpret this mismatch between our geometric classification and the field theory S-duality conjectures as another instance of the geometry failing to distinguish between inequivalent field theories, in this case it turns out the geometrical classification is correct, and a mistake — a double counting of certain field theories — was made in the field theory literature. The reason is very simple. They Weyl group reflection outer automorphism in the D_{2k} cases, unlike in the non-simply-laced cases, lifts to a gauge algebra outer automorphism. In particular, it is the automorphism which interchanges the spinor and conjugate spinor representations of $\mathfrak{so}(4k)$. This symmetry means that at the level of the sYM gauge theory lagrangian there is no way of distinguishing, even in principle, between an $\mathcal{N}$=4 sYM theory in which we put probe lines charged in the spinor representation or in the conjugate spinor representation: they differ just by the arbitrary choice of a name. In the $\mathfrak{so}(8k)$ theories this just means that the pairs of orbits in table 6 with $\mathrm{Ss}(N)_\pm$ and $\mathrm{Sc}(N)_\pm$ global structures (shown in red) are indistinguishable, so two of the S-duality orbits should be removed from the list. In the $\mathfrak{so}(8k+4)$ theories the $\mathrm{Ss}(N)_\pm$ and $\mathrm{Sc}(N)_\pm$ global structures appear in the same orbit, so their equivalence implies a 2-fold identification of its conformal manifold, and so a 2-fold enlargement of its S-duality group from $\Gamma_0(4)$ (shown in red in table 6) to $\Gamma_0(2)$.

One might object that once we have a theory with, say, semi-simple gauge algebra $\mathfrak{so}(8k) \times \mathfrak{so}(8k)$ then though the overall distinction between spinor and conjugate spinor representations is a matter of convention, the *relative* distinction (i.e., whether they are the same or opposite) between the two spinor representations in each factor is observable. This

is incorrect because for $\mathcal{N}$=4 sYM theories the two simple gauge factors of the theory are decoupled, and so there is no observable that can distinguish between a (spinor × spinor) and a (spinor × conjugate spinor) theory. If, on the other hand, there exist semi-simple versions of $\mathcal{N}$=4 theories whose SK geometry does not factorize because of some twisting of their abelian variety fibers, then in these theories the relative spinor versus conjugate spinor distinction may be observable. The possibility of such twisted geometries is discussed further in section 5.

D_4 is a special case where the reflection outer automorphism group is enlarged to S_3. However, these automorphisms, permuting the S, C, and V representations as well as some of the bindings of the $\mathbb{Z}_2^2$ or root lattice representations, always act among the untwisted orbits, so only identify them without enlarging their self-duality groups, as shown in table 1. This symmetry makes the D_4 global structure orbits in table 6 with $\mathrm{SO}(8)_\pm$, $\mathrm{Ss}(8)_\pm$, and $\mathrm{Sc}(8)_\pm$ indistinguishable.

4.3 SK structures versus global structures of the $\mathfrak{g}$ sYM gauge theory

A result of the discussion so far in this section is that S-duality orbits of sYM global structures are in 1-to-1 correspondence to the orbits of CB SK structures under EM duality. One may wonder whether there is a canonical way of matching subsets of SK structures to the individual global structures that make up their S-duality orbits. Comparing our SK structure representations $\mathcal{A} = (R_A, D)$ shown in the second column of table 1 to the global forms of the sYM gauge group shown in the second column of table 6, we can see that, heuristically, different choices of the representation R_A can be identified with the global form of the gauge group, while the binding matrix Ext class D plays the role of the "discrete theta angles". But these identifications are not unique, and in many cases there are more global structures than SK representations.

On the one hand, this mismatch is not physically significant, and reflects the fact that gauge invariance is merely a redundancy in the description of the field theory. On the field theory side different gauge group global structures are related by S-duality equivalences, while on the SK geometry side, different constructions of (R_A, D) symplectic representations are related by physically unobservable basis changes (integral equivalences). Another way of saying this is that a division of an S-duality orbit into global structures corresponds to a tiling of their conformal manifolds by copies of H_q fundamental domains in $\mathscr{H}_1$. There are infinitely many arbitrary choices that can be made in such a tiling. As a simple example, even in a weak coupling limit, if w copies of H_q fundamental domains have a common weak coupling cusp on the conformal manifold corresponding to an enlarged $\theta \sim \theta + 2\pi w$ theta angle periodicity there, there is a continuous family of tilings by w intervals in θ of width 2π as $\theta \in [\theta_0 + 2\pi n, \theta_0 + 2\pi(n+1)]$ parameterized by θ_0.

On the other hand, at a weak coupling limit, the global form of the gauge group is a physical property of the gauge theory that can be observed on the CB as a relation between the charge lattice of probe lines and its charge sublattice of BPS states [4, 6, 9]. This does not uniquely specify the associated discrete theta angles, but does pick out the global form of the gauge group. At a weak coupling point on the conformal manifold one relates the global form of the gauge group to the integral representation of the Weyl group,

since they are both related to choices of subgroups of the center of the simply connected gauge group, as described in section 2.3. This explains the qualitative matching between global structures and symplectic Weyl representations within the orbits listed in tables 1 and 6.

Furthermore, the counting of the number of global structures in each orbit is also an invariant observable. Since the conformal manifolds described above are the fundamental domains of finite-index subgroups of the Hecke groups H_q in the τ upper half-space, they are (well-studied) *modular curves*. These are Riemann surfaces with marked points of three types: $\mathbb{Z}_2$ orbifold points, $\mathbb{Z}_3$ orbifold points, and cusps. Physically the orbifold points (a.k.a., elliptic points) correspond to couplings where the 0-form symmetry group is enhanced by the orbifold group, while the cusps correspond to weak coupling limits. Each cusp also has a positive integer, w, the *width*, associated to it. Physically, the width at a cusp corresponds to the θ angle periodicity $\theta \sim \theta + 2\pi w$ at that weak coupling point in the standard normalization. The geometry of the conformal manifold (modular curve) is described by the number and types of the orbifold points, the number and widths of the cusps, and the genus of the curve. These are computed for the modular curves of many modular congruence subgroups; see e.g., [34]. The sum of all the widths is the index of the modular subgroup in H_q which counts the number of global structures in that S-duality orbit. This number is recorded for the field results of [4] in the 4th column of table 6. (As discussed above, this has one incorrect entry for the D_{2k+1} case, shown in red, which should be 3 instead of 6.)

Nevertheless, although there is no *canonical* way of relating symplectic representations of the Weyl group to global structures, we can still make such a correspondence "by hand". We illustrate this for $\mathfrak{su}(N)$; similar identifications can be made for the other simply-laced algebras, but it does not work in as simple a way for the non-simply-laced algebras, where there is no longer a simple correspondence between representations and central subgroups, as mentioned in section 2.3. This illustration will also serve to clarify the arbitrary choices that are made in making such a correspondence.

For $\mathfrak{su}(N)$ the lattices are labeled by divisors $d|N$, with $d = N$ the root lattice and $d = 1$ the weight lattice, corresponding to the central groups $H = \mathbb{Z}_d$; see table 3. Let's associate symplectic representations to gauge group global forms by the rule

$$S_{(d,z)} \leftrightarrow (\mathrm{SU}(N)/\mathbb{Z}_{N/d})_z, \qquad d|N, \qquad z \in \mathbb{Z}_d, \qquad (4.1)$$

where the z subscript on the right is the "discrete theta angle" introduced in [4]. Note that we are labeling the representations by divisors d and by binding index z mod d, even though we saw in section 3.1 that most of these representations are, in fact, integrally equivalent.

Why did we make the "wrong" correspondence, $S_{(d,0)} \leftrightarrow \mathbb{Z}_{N/d}$, which inverts d? For no good (i.e., gauge invariant) reason: we could have as easily made instead a $S_{(d,0)} \leftrightarrow \mathbb{Z}_d$ correspondence, or even more complicated ones. Another way of thinking about this is that the correspondence (4.1) is the "natural" one for the GNO dual "magnetic" gauge algebra instead of the "electric" gauge algebra. All this is just to emphasize that the following correspondence to global forms is not in any way canonical.
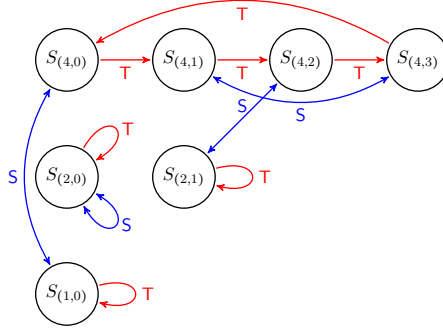
Figure 1. The S-duality orbits of $\mathfrak{g} = \mathfrak{su}(4)$ $\mathcal{N}$=4 sYM global structures. These are depicted by intertwiners S and T in (4.2) acting on the symplectic representations $S_{(d,z)}$ associated to the global forms by (4.1).

With the choice (4.1), the S and T generators of the $\mathrm{PSL}(2,\mathbb{Z})$ group act on the (d,z) representation labels as [8]

$$\mathsf{T} : (d,z) \mapsto (d',z') = \left(d, z + \frac{N}{d}\right), \qquad\qquad z' \in \mathbb{Z}_{d'}, \qquad (4.2)$$

$$\mathsf{S} : (d,z) \mapsto (d',z') = \left(\frac{N}{\gcd(d,z)}, \frac{N(d-z)}{d\gcd(d,z)}\right), \qquad z' \in \mathbb{Z}_{d'},$$

where in the (d',z')'s on the right sides, z' is defined modulo d'. Indeed, it is possible to show that these actions on (d,z) obey $\mathsf{S}^2 = (\mathsf{ST})^3 = 1$. Thus S and T generate $\mathrm{PSL}(2,\mathbb{Z})$.

We claim these S and T maps correspond to integral equivalences of the $S_{(d,z)}$ symplectic representations. It is complicated to write the explicit forms of the $M(\gamma)$ in (2.63) intertwining $S_{(d,z)}$ with $S_{(d',z')}$ for the T and S actions defined in (4.2). However, it is easy to show that such equivalences must exist, using the result of section 3.1 that $S_{(d,z)} \cong S_{(\gcd(s_d,z),0)}$. Indeed it is a short calculation to show that $\gcd(s_{d'},z') = \gcd(s_d,z)$ for the (d',z') on the right side of (4.2), and thus that they are equivalent representations.

For a concrete example, for $\mathfrak{su}(4)$ this set of representations are acted on by S and T as in figure 1. This gives the two S-duality orbits and the action of $\mathrm{PSL}(2,\mathbb{Z})$ on the global structures within each orbit, and reproduces the S-duality orbit diagram familiar from [4].

5 Non-principally polarized SK geometries and twisted products

In addition to the principally polarized geometries we have discussed so far, there are also geometries whose Dirac pairing J has invariant factors other than 1. These have the interpretation as corresponding to *relative* QFTs [35], i.e., field theories which are the boundary of a non-invertible topological field theory. The low energy physics on the CB reflects the absolute versus relative distinction as follows. The Dirac pairing on the lattice, $\Lambda_{\mathrm{charge}}$, of charges of finite energy states may be non-principal, and may be refined to a principally polarized "line lattice", $\Lambda_{\mathrm{line}} \supset \Lambda_{\mathrm{charge}}$, by including a maximal set of genuine probe line operators [6]. So the CB geometry of the relative theory is that with the $\Lambda_{\mathrm{charge}}$ pairing, J, so has an SK geometry in which the Weyl group is represented in $\mathrm{Sp}_{J^\vee}(2r,\mathbb{Z})$.

By contrast, the SK geometry of the CB of an absolute theory is one in which the Weyl group is represented in $\mathrm{Sp}(2r, \mathbb{Z})$, as discussed in the previous three sections, where we saw that the different S-duality orbits of the global structures of [4] correspond to the inequivalent $\mathrm{Sp}(2r, \mathbb{Z})$ representations of the Weyl group.

The Dirac pairing on $\Lambda_{\mathrm{charge}}$ is an observable property of a QFT with a Coulomb vacuum, so not all of the SK geometries we can construct using Weyl group orbifolds are physically realized as the geometries of absolute or relative QFTs. Using semiclassical techniques, the Dirac pairing on $\Lambda_{\mathrm{charge}}$ of $\mathcal{N}=4$ sYM theories can be determined, as reviewed in [9]. For instance, the charge lattice Dirac pairing for the $\mathfrak{su}(N)$ theory has invariant factors

$$\text{invariant factors of } J_{\mathrm{charge}} = (1, \ldots, 1, N), \tag{5.1}$$

where there are $N-2$ "1" entries.

On the other hand we can form symplectic representations $S_{(A,B;D)}$ (2.29) with pairing given by (2.4) with (2.30).

Here we chose to normalize the pairing so that the gcd of its invariant factors is 1, i.e., so that its smallest invariant factor is 1. But since the SK structure does not depend on the overall normalization of the pairing, we can just as well work with $\mathbf{j} = I_{AB}^{\vee}$, with the understanding that we may have to multiply by an overall factor to clear denominators.

For $\mathfrak{su}(N)$, the possible inequivalent irreducible representations of the Weyl group are labeled by the divisors d of N, i.e., $A, B \in \{d \text{ such that } d|N\}$. The invariant factors of the invariant Dirac pairing on $S_{(A,B;D)} = S_{(dd',D)}$ are the invariant factors of $I_{dd'}$ which are

$$\text{invariant factors of } I_{dd'} = (1, \ldots, 1, d/d') \tag{5.2}$$

as computed from $I_{dd'}$ given in (3.2). This shows there is a unique integral symplectic representation of the Weyl group with the physical charge lattice polarization, namely that with $(d, d') = (1, N)$. Indeed, there is a unique $\mathrm{Sp}_{J^{\vee}}(2r, \mathbb{Z})$ representation of the Weyl group with these values of (d, d') — i.e., the Ext group is trivial, so we can set the binding matrix to $D = 0$. This corresponds to the expectation that there is a unique "maximally relative" $\mathfrak{su}(N)$ $\mathcal{N}=4$ sYM theory. It therefore has a single connected conformal manifold with S-duality group $\mathrm{PSL}(2, \mathbb{Z})$ (as opposed to a subgroup).

SK geometries with other values of d'/d therefore do not correspond to the maximally relative field theory. Clearly those with $d'/d = 1$ are principally polarized, and correspond to the absolute theories. If $1 < d'/d < N$ is integral, then these geometries correspond to "intermediate relative" field theories, which are those for which the charge lattice has been refined to a line lattice, but without choosing a *maximal* set of probe line charges. By contrast, values of $d'/d \notin \mathbb{Z}$ cannot be found in this way. Since the overall normalization of J is not determined by the SK structure, we can clear denominators to make J integral with invariant factors to $(1, d, \ldots, d, dd')$ (assuming d and d' coprime for simplicity). In general, such Dirac pairings do not occur for any intermediate line lattices. On the other hand, non-integral d'/d could occur as the polarizations of lattices of non-mutually-local probe lines, i.e., of lines which are non-genuine line operators, which means lines which

are boundaries of topological surface operators. These make sense as low energy CB $\mathfrak{u}(1)^r$ gauge theories, but it is difficult to see how they can be defined as sensible kinds of local QFTs relative to a higher-dimensional symmetry TFT.

Similar statements apply to the other simple Lie algebras.

Even if one is only interested in absolute field theories, non-principally polarized SK geometries are still relevant. For there may be principally polarized geometries for *non-simple* — i.e., product — gauge algebras which are built from non-principally polarized factor geometries. (Indeed, examples of absolute product field theories which are formed from the product of relative field theory factors are known in the case of 6d (2,0) SCFTs [36].)

Consider two orbifold CB geometries

$$\mathrm{CB}_i \simeq_{\mathbb{C}} \mathbb{C}^{r_i}/(W_i)_{\mathbb{C}}, \qquad\qquad i = 1, 2, \qquad\qquad (5.3)$$

where W_i are two Weyl groups. To define these geometries we need to also specify integral symplectic representations

$$S_i : (W_i)_{\mathbb{C}} \to \mathrm{Sp}_{J_i^\vee}(2r_i, \mathbb{Z}) \qquad\qquad (5.4)$$

which leave invariant lattices $\Lambda_i \simeq \mathbb{Z}^{2r_i}$ with invariant symplectic pairings

$$J_i : \Lambda_i \times \Lambda_i \to \mathbb{Z}. \qquad\qquad (5.5)$$

We can obviously form the direct product geometry, $\mathrm{CB}_\times \doteq \mathrm{CB}_1 \times \mathrm{CB}_2$, where $(W_1)_{\mathbb{C}} \times (W_2)_{\mathbb{C}}$ acts via the reducible representation $S_\times = (S_1, 1) \oplus (1, S_2)$. Then the induced invariant symplectic pairing is $J_\times = J_1 \oplus J_2$, whose invariant factors are, more or less, the union of those of J_1 and J_2. In particular, if either of the J_i are not principally polarized, then neither is $J_\times$.

We raise the question: are there principally-polarized "twisted" product CBs,

$$\mathrm{CB}_D \doteq \mathrm{CB}_1 \times_D \mathrm{CB}_2 \qquad\qquad (5.6)$$

which, when viewed as *Kähler* geometries are product geometries, $\mathrm{CB}_D \simeq_{\mathbb{C}} \mathrm{CB}_\times$? In other words, the "twisting" involves only the integral symplectic representation specifying the SK structure. That means that the integral symplectic representation of the orbifold group,

$$S_D : (W_1)_{\mathbb{C}} \times (W_2)_{\mathbb{C}} \to \mathrm{Sp}(2(r_1{+}r_2), \mathbb{Z}), \qquad\qquad (5.7)$$

should be principally polarized (as shown) and should be rationally equivalent to the product representation but not integrally equivalent,

$$S_D \simeq_{\mathbb{Q}} S_\times, \qquad\qquad \text{but} \qquad\qquad S_D \not\simeq_{\mathbb{Z}} S_\times. \qquad\qquad (5.8)$$

That is, as should be familiar now and is explained in appendix B, over the integers S_D should be reducible but indecomposable,

$$S_D(g_1, g_2) = \begin{pmatrix} S_1(g_1) & L(g_1, g_2) \\ 0 & S_2(g_2) \end{pmatrix}, \qquad \text{with} \qquad L(g_1, g_2) = S_1(g_1)D - DS_2(g_2) \qquad (5.9)$$

with D some fixed rational matrix independent of $(g_1, g_2) \in W_1 \times W_2$.

Now, a short calculation shows that a symplectic pairing invariant under (5.9) must have the form

$$J_D = \begin{pmatrix} \kappa_1 J_1 + \kappa_2 D J_2 D^t & -\kappa_2 D J_2 \\ -\kappa_2 J_2 D^t & \kappa_2 J_2 \end{pmatrix}, \qquad \kappa_i \in \mathbb{Q}. \qquad (5.10)$$

Assuming there exists a non-vanishing D such that S_D is an integral representation, then there are some minimal value(s) of $\kappa_{1,2}$ such that J_D is integral and has smallest invariant factor 1. The question of the existence of twisted principally polarized representations S_D is thus equivalent to the existence of a non-vanishing extension D and rational numbers κ_i such that J_D in (5.10) is principal, i.e., all its invariant factors are 1.

It follows from (5.10) that

$$\det J_D = \kappa_1^{2r_1} \kappa_2^{2r_2} \det J_1 \det J_2, \qquad (5.11)$$

independent of D. We are interested in the case where J_D is principally polarized, so $\det J_D = 1$, while either or both J_i are not, so $\det J_1 \det J_2 > 1$ and integral. This, (5.11), rationality of the κ_i, and the integrality of J_D put strong constraints on the possible invariant factors of J_i. In general, this is an algebraically complicated question. We leave the exploration of such "twisted" product $\mathcal{N}{=}4$ sYM theories to a later work. But we show now that non-trivial solutions can exist, by demonstrating a simple, familiar, though somewhat degenerate, example.

5.1 $\mathfrak{u}(N)$ $\mathcal{N}{=}4$ sYM

We now construct the twisted product (5.9) of the $\mathfrak{u}(1)$ and $\mathfrak{su}(N)$ $\mathcal{N}{=}4$ sYM geometries.

Take $r_1 = 1$ and $r_2 = N - 1$ with

$$J_1 = \epsilon \otimes \mathrm{diag}\{N\} = N\epsilon, \qquad J_2 = \epsilon \otimes \mathrm{diag}\{1, \ldots, 1, N\} \doteq \epsilon \otimes \mathbf{j}, \qquad (5.12)$$

where ϵ is the 2×2 antisymmetric unit matrix.[25] Here we have chosen the non-principal pairing, J_2, of the $\mathfrak{su}(N)$ theory to be that of the "maximally relative" $\mathfrak{su}(N)$ sYM CB geometry described earlier — i.e., that with symplectic representation $S_2 \doteq S_{(d,d';D)} = S_{(1,N;0)}$. The normalization of the pairing J_1 is not constrained by physics since the $\mathfrak{u}(1)$ theory is free. We have chosen it to be proportional to N so that a solution to (5.11) with square rational but non-integer κ_1 exists. We will see below that this allows for potentially non-trivial twistings.

Indeed, (5.11) implies $1 = \det J_D = \kappa_1^2 \kappa_2^{2N-2} N^4$ so

$$\kappa_1 = N^{-2} \kappa_2^{1-N}. \qquad (5.13)$$

Integrality of the lower right block of J_D implies κ_2 is integral. Taking $\kappa_2 > 1$ dramatically increases the denominator of κ_1 making arranging the integrality of the upper left block of

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[25]Note that if you reverse the roles of r_1 and r_2, no solution exists.

J_D increasingly difficult, especially if we are looking for a solution which is uniform in N. This leads us to guess that

$$\kappa_2 = 1, \qquad\qquad \text{and so} \qquad\qquad \kappa_1 = N^{-2}. \qquad (5.14)$$

Now analyze the integrality of the upper right block of J_D. Write the binding matrix in terms of $r_1 \times r_2 = 1 \times (N-1)$ row vectors as

$$D \doteq \begin{pmatrix} v & w \\ x & y \end{pmatrix}, \qquad (5.15)$$

so that $DJ_2 = \begin{pmatrix} w\mathbf{j} & -v\mathbf{j} \\ y\mathbf{j} & -x\mathbf{j} \end{pmatrix}$. Then, since the first $N-2$ diagonal entries of $\mathbf{j}$ are 1, the first $N-2$ entries of the row vectors v, w, x, and y are all integers. And since integer entries of D can be set to zero by a basis change, we can set the first $N-2$ entries of these rows to 0. Since the last entry in $\mathbf{j}$ is N, the last entry of the rows can have denominator N. So set the binding to the $2 \times (N-1)$ matrix

$$D \doteq \frac{1}{N} \begin{pmatrix} \vec{0} & \alpha & \vec{0} & \beta \\ \vec{0} & \gamma & \vec{0} & \delta \end{pmatrix}, \qquad (5.16)$$

where the $\vec{0}$'s are $(N-2)$-component zero row vectors, and $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_N$.

Now compute the 2×2 upper left block in J_D to find that it is $\epsilon(1 + \alpha\delta - \beta\gamma)/N$. So for it to be integral we must have

$$\alpha\delta - \beta\gamma = -1 \ (\text{mod } N). \qquad (5.17)$$

Putting (5.14) – (5.17) together gives an integral and principal solution to (5.10).

We now have to check whether there are values of $\alpha, \beta, \gamma, \delta$ satisfying (5.17) which is a non-trivial binding for the appropriate $\text{Weyl}(\mathfrak{u}(1))$ and $\text{Weyl}(\mathfrak{su}(N))$ integral representations. $\text{Weyl}(\mathfrak{u}(1)) = 1$ is trivial, so its only irreducible representation is $\rho_1(1) = 1$, and its rank-2 symplectic representation is just $S_1 = \rho_1 \oplus \rho_1$, so

$$S_1(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (5.18)$$

On the other hand, the invariant Dirac pairing, J_2, of the "maximally relative" symplectic integral representation, S_2, of $\text{Weyl}(\mathfrak{su}(N))$, is given in (5.12), and

$$S_2 = S_{(1,N;0)} = \begin{pmatrix} R_1^\vee & 0 \\ 0 & R_N \end{pmatrix}. \qquad (5.19)$$

Then from (5.9), (5.18), and (5.19), the upper right $2 \times 2(N-1)$ block of S_D is

$$D(1 - S_2) = \frac{1}{N} \begin{pmatrix} \alpha(1 - R_1^\vee)_{N-1,j} & 0 \\ 0 & \delta(1 - R_N)_{N-1,j} \end{pmatrix}, \qquad (5.20)$$

where only the last rows of the $1 - R_1^\vee$ and $1 - R_N$ matrices appear because of the form (5.16) of the D binding matrix.

Using the explicit form of the R_d representation matrices computed in the bases used in section 3.1, we find

$$
\begin{aligned}
(1 - R_1^{-t}(w_k))_{N-1,j} &= -\delta_{j,N-2}\delta_{k,N-1} \\
(1 - R_N(w_k))_{N-1,j} &= +\delta_{j,1}\delta_{k,1}.
\end{aligned}
\tag{5.21}
$$

These are not divisible by N, so we must have $\alpha = \delta = 0 \pmod{N}$ for S_D to be integral. This implies from (5.17) that we must have $\beta\gamma = 1 \pmod{N}$ if S_D is to be principally polarized. Since β and γ do not appear in (5.20), we can consistently choose them to be $\beta = \gamma = 1$.

Thus we have constructed a "twisted" $\mathfrak{u}(N)$ SK structure suitable for describing an absolute $\mathfrak{u}(N)$ sYM theory. Moreover, it is likely that it is the unique solution which is uniform in N. The existence of this solution is expected on physical grounds because the rank-$2(N-1)$ $\mathfrak{su}(N)$ charge lattice is embedded in the rank-$2N$ charge lattice of the $\mathfrak{u}(N)$ theory in such a way that the induced Dirac pairing on the $\mathfrak{su}(N)$ sublattice is the non-principal one given by J_2 [9].

Acknowledgments

A SK structure of an N=4 sYM CB

We recall the definition of an SK structure that defines the CB geometry of a generic $\mathcal{N}=2$ SCFT relative to the Kähler structure on the moduli space of vacua. We then adapt it to the CB geometry of an $\mathcal{N}=4$ sYM theory which results in additional constraints on the SK structure due to the simplicity of its moduli space of vacua. In particular, they are complex orbifolds with isotrivial SK structure and come in one-parameter families with a

weak-coupling limit. For a given $\mathcal{N}$=4 sYM theory with simple gauge algebra $\mathfrak{g}$, we claim that such SK structures are determined by

- the Dirac pairing, J, on the EM charge lattice,

- an integral representation, S, of the orbifold group which preserves the Dirac pairing, and

- a positive definite matrix, $\boldsymbol{\tau}(\tau)$, that is the low-energy EM coupling matrix and is fixed by the action of this representation of the orbifold group for all values of the complex parameter $\tau \in \mathscr{H}_1$.

In particular, in this appendix we show that the SK structures of $\mathcal{N}$=4 sYM theories are in 1-to-1 correspondence with the orbits of triples $(J, S, \boldsymbol{\tau})$ formed from equivalences that are defined by SK structure isomorphism. In the body of the paper, we classify all distinct orbits of triples for simple $\mathfrak{g}$ with principal J.

A.1 Review of SK geometry

The Coulomb branch (CB), $\mathcal{C}$, is the subset of the moduli space of vacua of 4d $\mathcal{N}$=2 supersymmetric theories which have $\mathfrak{u}(1)^r$ gauge fields coupled to massive charged fields.[26] The vevs of the complex scalars in the r vector multiplets are coordinates on $\mathcal{C}$, and their kinetic terms endow $\mathcal{C}$ with a Kähler structure in the usual way. The relation of these scalar fields to the gauge fields via supersymmetry endows $\mathcal{C}$ with an SK structure. In particular, the gauge fields couple to massive electrically and magnetically charged states, encoded by the symmetric complex $r \times r$ matrix $\boldsymbol{\tau}$ of coefficients of the gauge field kinetic terms. This is in a basis of gauge fields with respect to which the charges of the massive states span a rank-$2r$ lattice, $\Lambda = \Lambda_e \oplus \Lambda_m$, of integer electric and magnetic charges. This quantization of the charges is due to the Dirac quantization condition, which states that Λ carries a physically observable Dirac pairing, $\mathbf{j} : \Lambda_e \times \Lambda_m \to \mathbb{Z}$, which is non-degenerate.

These physical properties of the low energy effective action on the CB can be encoded in an SK geometry of $\mathcal{C}$, which is described in a coordinate-invariant way as the r-complex-dimensional base of an *algebraic integrable system* $(\mathcal{A}, \pi, J, \Omega)$ [37–40]. We review how, with an appropriate choice of bases, the low energy effective action on the CB is derived from the integrable system data.

The complex phase space, $\mathcal{A}$, of the integrable system is a connected $2r$-dimensional symplectic manifold with holomorphic symplectic form Ω and a proper holomorphic lagrangian fibration $\pi : \mathcal{A} \to \mathcal{C}$. Its fibers, $A_u = \pi^{-1}(u)$ for $u \in \mathcal{C}$, are abelian varieties, and the fibration being lagrangian means that $\Omega|_{A_u} = 0$. The rank-$2r$ 1-homology lattice of the fiber, $\Lambda = H_1(A_u)$, is the EM charge lattice of the low-energy theory on the CB. Since Λ is discrete, it is locally constant, so forms a linear system over $\mathcal{C}$ which captures the monodromies $S_{\mathbb{Z}}(\gamma)$ it experiences after traversing a linking 1-cycle $\gamma \in \pi_1(\mathcal{C})$ in $\mathcal{C}$. Specifically,

---

[26]Here we mean by $\mathcal{C}$ the *smooth* locus of the CB, i.e., excluding for the moment subspaces where some charged states become massless on the CB. We incorporate these massless particle subspaces — which correspond to non-analyticities of the Kähler geometry — after eqn. (A.9) below.

if we fix a basis $(\widehat{\lambda}_a, a = 1, \ldots, 2r)$ of the charge lattice at a point $u \in \mathcal{C}$, and then drag it around a closed path $\gamma \in \pi_1(\mathcal{C})$, it is allowed to come back to itself up to the linear action of the monodromy matrix $S_{\mathbb{Z}}(\gamma)$, $S_{\mathbb{Z}}(\gamma) \cdot \widehat{\lambda}_a \doteq S_{\mathbb{Z}}(\gamma)_a{}^b \widehat{\lambda}_b$, that must produce a basis change of Λ, so $S_{\mathbb{Z}}(\gamma) \in \mathrm{GL}(2r, \mathbb{Z})$. The set of these monodromies define a *monodromy map* $S_{\mathbb{Z}} : \pi_1(\mathcal{C}) \to \mathrm{GL}(2r, \mathbb{Z})$ that produces an integral representation of $\pi_1(\mathcal{C})$. We put the $\mathbb{Z}$ subscript on the monodromy $S_{\mathbb{Z}}$ to emphasize that it defines an integral representation of $\pi_1(\mathcal{C})$, and to contrast it with other representations we will introduce shortly.

Furthermore, the fibers come with a choice of a positive polarization, $J \in H^{1,1}(A_u) \cap H^2(A_u, \mathbb{Z})$, which can be viewed as a nondegenerate integral skew-symmetric pairing on the charge lattice, $J : \Lambda \times \Lambda \to \mathbb{Z}$. As it is discrete, it is also locally constant on $\mathcal{C}$ like the charge lattice Λ. But because J is identified with the Dirac pairing on the charge lattice in the low-energy theory on the CB, whose value is a physical observable, it must extend to a constant, and therefore global, section over $\mathcal{C}$. So, unlike the charge lattice, this implies it does not experience monodromies, which constrains $S_{\mathbb{Z}}(\gamma)$ to be valued in the subgroup $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$.

More concretely, define the symplectic form associated to J by the $2r \times 2r$ matrix with entries $J_{ab} \doteq J(\widehat{\lambda}_a, \widehat{\lambda}_b)$ relative to a charge lattice basis $(\widehat{\lambda}_a, a = 1, \ldots, 2r)$, which we'll often denote just by J when the choice of basis is understood. Since the polarization J is globally defined, this matrix form of J must be preserved by the monodromy $S_{\mathbb{Z}}(\gamma)$ associated to a linking 1-cycle γ: $J \mapsto J' = S_{\mathbb{Z}}(\gamma) \cdot J \cdot S_{\mathbb{Z}}(\gamma)^t = J$. This restricts the image of the monodromy map to $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$, the set of automorphisms of the symplectic form $J^\vee$,[27]

$$ S_{\mathbb{Z}} : \pi_1(\mathcal{C}) \to \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z}), \qquad \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z}) \doteq \{ M \in \mathrm{GL}(2r, \mathbb{Z}) \mid M^t J^\vee M = J^\vee \}. \tag{A.1} $$

The image of the monodromy map, $S_{\mathbb{Z}}(\pi_1(\mathcal{C})) \subset \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$, is the *monodromy group* of $\mathcal{C}$, which we also denote by $S_{\mathbb{Z}}$ or S when the context is understood. That the image of the monodromy map is in the symplectic group of the dual symplectic form $J^\vee$ to J, where $J^\vee \doteq J^{-t}$, is an artifact of our definition of the symplectic group of a symplectic form, see footnote 27. In particular, if M is a basis change of the charge lattice Λ that preserves the matrix form of J, then $M \in \mathrm{Sp}_{J^\vee}$ as opposed to Sp_J relative to our definition of the symplectic group J. Note that since the special coordinates are valued in the (complexification of the) dual lattice of the homology (charge) lattice Λ, under a monodromy $S_{\mathbb{Z}}(\gamma)$ they transform linearly under $S_{\mathbb{Z}}(\gamma) \in \mathrm{Sp}_{J^\vee}$ as opposed to an element of Sp_J. Note that if J is principal, there is no distinction between J and $J^\vee$, so $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z}) = \mathrm{Sp}_J(2r, \mathbb{Z}) = \mathrm{Sp}(2r, \mathbb{Z})$. But for non-principal polarizations $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ and $\mathrm{Sp}_J(2r, \mathbb{Z})$, though isomorphic as abstract groups, are not integrally equivalent as subgroups of $\mathrm{GL}(2r, \mathbb{Z})$ unless a condition on their invariant factors (that is specified below) is satisfied.[28]

Because symplectic groups relative to non-principal symplectic forms may be less familiar, we pause here to review them and the way they appear in an SK geometry. Write an element of the charge lattice as $\Lambda \ni \lambda = \ell^a \widehat{\lambda}_a$ for some integers ℓ^a — the "charge vector".

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[27]Note that this definition of $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ is convention and defines for us what we call the *symplectic group* $\mathrm{Sp}_X(2r, \mathbb{Z})$ of a generic symplectic form X.

[28]The two matrix groups are related by the inverse transpose of their elements, i.e. $M \in \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ iff $M^{-t} \in \mathrm{Sp}_J(2r, \mathbb{Z})$, which establishes the isomorphism between them as abstract groups.

Then the Dirac pairing is

$$J(\ell_1, \ell_2) \doteq (\ell_1)^a J_{ab} (\ell_2)^b \qquad\qquad\qquad (A.2)$$

thought of as a skew form on the charge vectors. This is the traditional definition of the *matrix* of the Dirac pairing. The value of the Dirac pairing between two charges is a physical observable, so the normalization of J is physical. However, the individual matrix elements J_{ab} depend on the arbitrary choice of lattice basis.

We can choose a *symplectic basis* $(\widehat{m}_i, \widehat{e}^i)$, $i = 1, \ldots, r$, of Λ such that the sublattices $\Lambda_m \doteq \langle \widehat{m}_i \rangle$ and $\Lambda_e \doteq \langle \widehat{e}^i \rangle$ are lagrangian with respect to J, i.e., $J(\widehat{m}_i, \widehat{m}_j) = J(\widehat{e}^i, \widehat{e}^j) = 0$. J in this basis is the non-degenerate integral $2r \times 2r$ matrix (2.4), i.e., $J = \left(\begin{smallmatrix} 0 & \mathbf{j} \\ -\mathbf{j}^t & 0 \end{smallmatrix} \right)$, with $(\mathbf{j})_j{}^i \doteq J(\widehat{m}_j, \widehat{e}^i)$. This symplectic basis can be specialized to *invariant factor form* (a.k.a., Smith normal form),

$$\mathbf{j} = \mathrm{diag}(d_1, \ldots, d_r), \qquad d_i \in \mathbb{Z}_{>0}, \qquad d_i | d_{i+1}. \qquad (A.3)$$

The d_i are the *invariant factors* of J, and uniquely characterize it. In particular, there exist change of bases $\mathbf{u}, \mathbf{v} \in \mathrm{GL}(r, \mathbb{Z})$ respectively of the Λ_e and Λ_m sublattices, such that $\mathbf{u}\mathbf{j}\mathbf{v} = \mathbf{d}$ with $\mathbf{d}$ of the form (A.3). Then $\left(\begin{smallmatrix} 0 & \mathbf{u}^t \\ -\mathbf{v} & 0 \end{smallmatrix} \right)^t J \left(\begin{smallmatrix} 0 & \mathbf{u}^t \\ -\mathbf{v} & 0 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 0 & \mathbf{d} \\ -\mathbf{d} & 0 \end{smallmatrix} \right)$.

We can define the EM duality monodromies to be either linear transformations acting on the charge vector components as $\ell^a \mapsto N^a{}_b \ell^b$, or as linear transformations acting on the components of the special coordinate vector as $\sigma_a \mapsto M_a{}^b \sigma_b$. Since our main focus is on the special coordinates, we define EM duality transformations to be the linear transformations acting as basis changes on the special coordinate components which also define basis transformations acting on the basis elements of the charge lattice. The special coordinates are a vector of holomorphic functions on the CB which appear in the central charge of the low energy $\mathcal{N}{=}2$ super Poincaré algebra on the CB as $Z_\lambda(u) = \sigma_a \ell^a$; we will give a more detailed definition of the special coordinates that connects them to the algebraic integrable system, shortly. But since the central charge is a physical observable, this implies that the special coordinates are coordinates on the dual vector space to the (complexification of) the charge lattice. Thus, components of vectors in the special coordinate vector space inherit a symplectic pairing using $J^\vee$,

$$J^\vee(\sigma_1, \sigma_2) \doteq (\sigma_1)_a (J^\vee)^{ab} (\sigma_2)_b, \qquad\qquad J^\vee \doteq J^{-t}. \qquad (A.4)$$

Since the monodromies $S_{\mathbb{Z}}(\gamma)$ preserve the dual symplectic pairing and act as basis changes on the special coordinates components, special coordinate monodromies are in $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ while the charge monodromies are in $\mathrm{Sp}_J(2r, \mathbb{Z})$. Note, also, that (A.1) incorporates the choice of defining $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ to be the set of $M \in \mathrm{GL}(2r, \mathbb{Z})$ such that $M^t J^\vee M = J^\vee$ rather than $M J^\vee M^t = J^\vee$. The second choice is, in our convention, equivalent to the definition of the group $\mathrm{Sp}_J(2r, \mathbb{Z})$, as is easily seen by taking the inverse transpose of its defining relation and using the fact that the inverse of any element of a group is also in the group.

Since the definitions of $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ (A.1), or the similar definition of $\mathrm{Sp}_J(2r, \mathbb{Z}) \doteq \{ N \in \mathrm{GL}(2r, \mathbb{Z}) \mid N^t J N = J \}$, are linear in $J^\vee$ and J, the normalizations of J or $J^\vee$ are irrelevant. So one can always normalize a rational J by dividing by the rational gcd of its elements,

so that $J/\gcd(J_{ab})$ is an integral symplectic form. In this case its leading invariant factor in (A.3) is $d_1 = 1$. The invariant factors, $d_i^\vee$, of a normalized $J^\vee$ are related to those of J, (A.3), by $d_i^\vee = d_r/d_{r-i+1}$. So, in general, Sp_J and $\mathrm{Sp}_{J^\vee}$ are not isomorphic over the integers for non-principal J.[29] Relatedly, for non-principal $J^\vee$, in general $M^t \notin \mathrm{Sp}_{J^\vee}$ when $M \in \mathrm{Sp}_J$.

We now return to describing the connection of the algebraic integrable system data to the low energy effective action on the CB. In a symplectic basis there is always a choice of basis $(\mathrm{d}z_j)$ of $H^{1,0}(A_u)$ such that its period matrix takes the form

$$\int_{\widehat{m}_i} \mathrm{d}z_j = (\boldsymbol{\tau}_u)_{ij}, \qquad\qquad \int_{\widehat{e}^i} \mathrm{d}z_j = (\mathbf{j})_j^{\ i}, \qquad\qquad \text{(A.5)}$$

where $\boldsymbol{\tau}_u$ is symmetric and has positive definite imaginary part, so is in the degree-r Siegel half space, $\boldsymbol{\tau}_u \in \mathscr{H}_r$. This is the content of the Riemann conditions following from A_u being an abelian variety. Physically, $\boldsymbol{\tau}_u$ is the holomorphically varying matrix of low energy $\mathfrak{u}(1)^r$ gauge couplings. The condition that the A_u fibers are lagrangian implies that

$$(\boldsymbol{\tau}_u)_{ij} = \frac{\partial a_i^D}{\partial a^j}, \qquad \text{with} \qquad \mathrm{d}_u a_i^D \doteq \int_{\widehat{m}_i} \Omega, \qquad (\mathbf{j}^t)^i_{\ j} \mathrm{d}_u a^j \doteq \int_{\widehat{e}^i} \Omega, \qquad \text{(A.6)}$$

where d_u is the exterior derivative on $\mathcal{C}$ and the integrals are fiber-wise integrals of the symplectic form Ω, which are well-defined because the fibers are lagrangian. These fiber periods of Ω can be written locally as total derivatives because Ω is closed. In particular, $\{\mathrm{d}_u a^j\}$ forms a basis of $(1,0)$ forms at each point on $\mathcal{C}$, and in this basis

$$\Omega = \mathrm{d}_u a^j \wedge dz_j. \qquad\qquad \text{(A.7)}$$

For SCFTs, the CB has a complex scale symmetry which fixes only the superconformal vacuum and under which the special coordinates scale homogeneously with weight 1. Thus the superconformal vacuum is the origin of the special coordinates,

$$a^i = a_i^D = 0, \qquad i = 1, \ldots, r, \qquad \text{at the superconformal vacuum.} \qquad \text{(A.8)}$$

They can be determined everywhere on $\mathcal{C}$ by integrating along paths from this origin their differentials given by the fiber periods of Ω. These so-defined (dual) special coordinates on the CB, a^j (a_i^D), are the vevs of the complex scalar superpartners of the low energy $\mathfrak{u}(1)^r$ gauge fields. The $2r$-component vector of dual special coordinates and special coordinates, (a_i^D, a^j), is what we referred to as the vector of special coordinates σ_a in (A.4). The metric on the CB — describing the kinetic terms of the scalars — is given by

$$g = (\det \mathbf{j})\,(\mathrm{Im}\boldsymbol{\tau}_u)_{ij}\,\mathrm{d}_u a^i \overline{\mathrm{d}_u a}^j. \qquad\qquad \text{(A.9)}$$

In this way we have translated the data of an algebraic integrable system into that of an SK geometry in special coordinates.

Although the above definition treated $\mathcal{A}$ and $\mathcal{C}$ as smooth complex manifolds, in fact, the CB, as a metric space, has complex codimension-1 finite-distance non-analyticities

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[29]J and $J^\vee$ are isomorphic over the integers if they have the same invariant factors, which happens when $d_{i+1}d_{r-i} = d_i d_{r-i+1}$ for all $1 \le i \le r-1$.

along a subvariety $\mathcal{D} \subset \mathcal{C}$.[30] The smooth part of $\mathcal{C}$, which we now denote $\mathcal{C}^* \doteq \mathcal{C} \smallsetminus \mathcal{D}$, is what is described by the algebraic integrable system. In particular, the monodromy map (A.1) is a map from the fundamental group of the non-simply connected $\mathcal{C}^*$.

The image of $\gamma \in \pi_1(\mathcal{C}^*)$ under the monodromy map $S_{\mathbb{Z}}$ is an element of $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$. Write it in terms of $r \times r$ block matrices (which we denote by bold letters) as

$$S_{\mathbb{Z}}(\gamma) \doteq \begin{pmatrix} \mathbf{m} \ \mathbf{n} \\ \mathbf{p} \ \mathbf{q} \end{pmatrix} \in \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z}). \tag{A.10}$$

The definition of $\mathrm{Sp}_{J^\vee}$ in (A.1) implies

$$(\mathbf{j}^{-1}\mathbf{m})^t \mathbf{p} = \mathbf{p}^t(\mathbf{j}^{-1}\mathbf{m}), \qquad (\mathbf{j}^{-1}\mathbf{n})^t \mathbf{q} = \mathbf{q}^t(\mathbf{j}^{-1}\mathbf{n}), \qquad \mathbf{q}^t(\mathbf{j}^{-1}\mathbf{m}) - (\mathbf{j}^{-1}\mathbf{n})^t \mathbf{p} = \mathbf{j}^{-1}. \tag{A.11}$$

$S_{\mathbb{Z}}(\gamma)$ acts linearly on the basis of the homology lattice of the fiber (by definition), as well as on the special coordinates (by (A.6), using the invariance of Ω) as

$$S(\gamma): \qquad \begin{pmatrix} \widehat{m} \\ \widehat{e} \end{pmatrix} \mapsto S_{\mathbb{Z}}(\gamma) \begin{pmatrix} \widehat{m} \\ \widehat{e} \end{pmatrix}, \qquad \begin{pmatrix} \mathrm{d}_u a^D \\ \mathbf{j} \, \mathrm{d}_u a \end{pmatrix} \mapsto S_{\mathbb{Z}}(\gamma) \begin{pmatrix} \mathrm{d}_u a^D \\ \mathbf{j}^t \, \mathrm{d}_u a \end{pmatrix}, \tag{A.12}$$

in a notation where $\widehat{m}$, $\widehat{e}$, a^D, and a are treated as r-component column vectors.

Define also the following $\mathrm{GL}(r, \mathbb{C})$ matrices

$$S_{\mathbb{C}}(\gamma) \doteq \mathbf{j}^{-t}\big(\mathbf{p}(\gamma)\boldsymbol{\tau} + \mathbf{q}(\gamma)\mathbf{j}^t\big), \qquad S_{\mathbb{C}}^\vee(\gamma) \doteq \mathbf{j}\big(\mathbf{p}(\gamma)\boldsymbol{\tau} + \mathbf{q}(\gamma)\mathbf{j}^t\big)^{-t}. \tag{A.13}$$

Then the monodromies act on the dz basis of $(1,0)$-forms on A_u, on the special coordinates on $\mathcal{C}$, and on $\boldsymbol{\tau}$ as

$$S(\gamma): \qquad \begin{aligned} dz &\mapsto S_{\mathbb{C}}^\vee(\gamma) dz, \\ \mathrm{d}_u a &\mapsto S_{\mathbb{C}}(\gamma) \mathrm{d}_u a, \\ \boldsymbol{\tau} &\mapsto S(\gamma) \circ \boldsymbol{\tau} \doteq (\mathbf{m}\boldsymbol{\tau} + \mathbf{n}\mathbf{j}^t)(\mathbf{p}\boldsymbol{\tau} + \mathbf{q}\mathbf{j}^t)^{-1}\mathbf{j}^t. \end{aligned} \tag{A.14}$$

The last gives a group action on $\mathscr{H}_r$, while the first two are only group actions if combined with the action on $\boldsymbol{\tau}$. These actions follow from the definitions (A.5) and (A.6), and from the $S_{\mathbb{Z}}$ actions (A.12). All these actions of the monodromy group — integral, complex, and Möbius — will play an important role shortly.

There is an additional condition on the special coordinates that stems from the physical origin of the divisor $\mathcal{D}$ as the locus where the $\mathfrak{u}(1)^r$ low energy effective action on $\mathcal{C}$ breaks down due to charged states becoming massless. The central charge, $Z_\lambda(u)$, of the $\mathcal{N}=2$ Poincaré supersymmetry algebra measures the BPS lower mass bound on states of charge $\lambda \in \Lambda$ at $u \in \mathcal{C}$. Symmetry considerations in a SCFT (where there are no dimensionful parameters and a broken complex scale symmetry) imply that $Z_\lambda(u) = m^t a^D(u) + e^t a(u)$ in a notation where m and e are the r-component column vectors of magnetic and electric

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[30]All singularities occur in the closure of the codimension-1 singularities for physical reasons described around (A.15) below; see [41] for a detailed description of the stratification of physical SK geometries by singular submanifolds.

charges relative to the basis $(\widehat{m}_i, \widehat{e}^i)$ of Λ. Thus $\lambda = m^i \widehat{m}_i + e^j \widehat{e}_j$, and $m^i, e_j \in \mathbb{Z}$.[31] Then the physical requirement is that there must be at least one non-zero charged state becoming massless along each component of $\mathcal{D}$,

$$\forall u \in \mathcal{D}, \qquad \exists \lambda \in \Lambda, \quad \lambda \neq 0 \qquad \text{such that} \qquad Z_\lambda(u) = 0. \qquad (A.15)$$

This is an additional requirement that needs to be imposed, in principle, on the algebraic integrable system data $(\mathcal{A}, \pi, J, \Omega)$ for an SK geometry for it to be physical. We will see below that this condition is automatically satisfied by the SK structures of $\mathcal{N}{=}4$ sYM theories.

Two monodromy maps, S and S', which differ just by a choice of charge lattice basis correspond to equivalent SK structures. Since a change of lattice basis is given by an element $Z \in \mathrm{GL}(2r, \mathbb{Z})$, we introduce the notion of *integral equivalence*, denoted $\cong_\mathbb{Z}$, of representations by

$$S \cong_\mathbb{Z} S' \qquad \text{iff} \qquad \left\{ \exists\ Z \in \mathrm{GL}(2r, \mathbb{Z}) \ \big|\ ZS(\gamma) = S'(\gamma)Z \quad \forall\ \gamma \in \pi_1(\mathcal{C}^*) \right\}. \qquad (A.16)$$

Note that if S preserves the symplectic form J, then S' preserves the integrally equivalent symplectic form $J' = ZJZ^t$. Analogously, if $S \cong_\mathbb{Z} S'$ using $Z \in \mathrm{GL}(2r, \mathbb{Z})$ and $\boldsymbol{\tau} \in \mathrm{Fix}(S)$, then $Z \circ \boldsymbol{\tau} \in \mathrm{Fix}(S')$.

$\boldsymbol{\tau}$ is only defined up to $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ transformations that correspond to the changes of symplectic basis preserving a given block-skew symplectic form (2.4). Points of the Siegel upper half-space are identified by the $J^\vee$-symplectic action (A.14) because they correspond to equivalent SK structures. Therefore, the points in the quotient space $\mathscr{H}_r/\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ parameterize the inequivalent values for the low-energy couplings $\boldsymbol{\tau}$. This space is an $r(r+1)/2$-dimensional connected complex space with orbifold and cusp-like singularities. Then $[\boldsymbol{\tau}(u)]$, thought of as the $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ equivalence class of a holomorphic function on $\mathcal{C}^*$, defines the holomorphic map

$$[\boldsymbol{\tau}] : \mathcal{C}^* \to \mathscr{H}_r/\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z}). \qquad (A.17)$$

A.2 N=4 sYM SK structures

$\mathcal{N}{=}4$ sYM theories with gauge algebra $\mathfrak{g}$ have an exactly marginal complex gauge coupling, τ, and have CBs which are orbifolds by the Weyl group of $\mathfrak{g}$, $W(\mathfrak{g})$, and are isotrivial. The SK structure depends on τ, which should not be confused with $\boldsymbol{\tau} \equiv (\boldsymbol{\tau})_{ij}$ which is the $r \times r$ matrix of low energy $\mathfrak{u}(1)^r$ couplings.

If the $\mathcal{N}{=}4$ sYM theory has gauge algebra of rank $r = \mathrm{rk}(\mathfrak{g})$ with Weyl group acting as the real reflection group $W(\mathfrak{g}) \subset \mathrm{GL}(r, \mathbb{R})$, then its "CB stratum" is an orbifold,

$$\mathcal{C}_\mathfrak{g} = \mathbb{C}^r / W_\mathbb{C}, \qquad (A.18)$$

with $W_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} W(\mathfrak{g}) \subset \mathrm{GL}(r, \mathbb{C})$ a finite group acting linearly, holomorphically, and faithfully on $\mathbb{C}^r$. Actually, the total $\mathcal{N}{=}4$ moduli space $\mathcal{M}_\mathfrak{g}$ defined in (1.1) has additional

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[31]More invariantly, the $2r$-component complex vector field (a^D, a) is naturally a global section of the vector bundle whose fiber is the complexification of the dual charge lattice, $\mathbb{C} \otimes_\mathbb{Z} \Lambda^*$, and the central charge is its dual pairing with Λ.

structure that $\mathcal{C}_\mathfrak{g}$ is an $\mathcal{N}{=}2$ component of. By decomposing $\mathcal{M}_\mathfrak{g}$ into Higgs, mixed, and Coulomb branches (CBs) with respect to an $\mathcal{N}{=}2$ subalgebra of the $\mathcal{N}{=}4$ algebra, one finds that the CB stratum is the complex orbifold (A.18), and carries an SK structure. The triple-SK structure of $\mathcal{M}_\mathfrak{g}$ is uniquely reconstructed from the SK structure of $\mathcal{C}_\mathfrak{g}$ using the $SU(3) \subset SO(6)_R$ symmetry action; see [2].

Isotrivial SK geometries [21] are particularly simple ones in which the map (A.17) is constant. All $\mathcal{N} \geq 3$ SCFT moduli spaces are isotrivial by virtue of an $\mathcal{N}{=}2$ selection rule [42] which says that hypermultiplet effective actions do not depend on vector multiplet scalar vevs; by rotating the choice of $\mathcal{N}{=}2$ subalgebra in $\mathcal{N} \geq 3$ theories, this implies their isotriviality. There are many $\mathcal{N}{=}2$ SCFT moduli spaces which are isotrivial as well [21, 43].

Constancy of $\boldsymbol{\tau}$ implies it is fixed by the monodromy group action (A.14), so

$$\boldsymbol{\tau} \in \mathrm{Fix}(S) \subset \mathscr{H}_r. \tag{A.19}$$

$\mathrm{Fix}(S)$ is determined by the set of quadratic matrix equations

$$\boldsymbol{\tau} \in \mathrm{Fix}(S) \qquad \text{iff} \qquad \mathbf{m}\boldsymbol{\tau} + \mathbf{n}\mathbf{j}^t = \boldsymbol{\tau}\mathbf{j}^{-t}(\mathbf{p}\boldsymbol{\tau} + \mathbf{q}\mathbf{j}^t) \qquad \text{for all} \qquad \left(\begin{smallmatrix} \mathbf{m} & \mathbf{n} \\ \mathbf{p} & \mathbf{q} \end{smallmatrix}\right) \in S, \tag{A.20}$$

and is connected [44]. Note that it is enough to solve (A.20) for a generating set of S corresponding to a set of simple reflections that define a generating set of reflections for $W(\mathfrak{g})$. This fixed point set is thus determined by the monodromy group. If the dimension of the fixed point set is greater than zero, then the geometry is not isolated, and the associated SCFT has a conformal manifold (with singularities) of exactly marginal deformations. We therefore expect fixed point sets of dimension 1 for $\mathcal{N}{=}4$ sYM theories.

Furthermore, isotriviality implies the SK metric (A.9) is flat on $\mathcal{C}^*$ and the special coordinates are flat coordinates. Also in $\mathcal{N}{=}4$ sYM theories, the orbifold group acts linearly on the Coulomb branch (special) coordinates as a finite reflection group. This follows from the existence of a weak coupling limit where the action of the Weyl group gauge identifications on CB vevs are calculable from the classical Higgs mechanism. The monodromy map actions $S_\mathbb{C}$ and $S_\mathbb{C}^\vee$ are now representations (group homomorphisms) of S and $S^\vee$, respectively, by virtue of (A.20). They are faithful, dual, r-complex-dimensional representations of the monodromy group. Invariance of the symplectic form Ω (A.7) requires that the orbifold group acts in the dual (a.k.a., contragredient) representation $W_\mathbb{C}^\vee$ on the abelian variety fiber. Also, because the CB geometry is isotrivial, the abelian variety fiber is constant, so $A_u \doteq A_\tau$ for some fixed $\boldsymbol{\tau} \in \mathrm{Fix}(S)$. Thus the orbifold description of the CB extends to an orbifold description of the algebraic integrable system,

$$\mathcal{A} = (A_{\boldsymbol{\tau}} \times \mathbb{C}^r)/(W_\mathbb{C}^\vee \oplus W_\mathbb{C}). \tag{A.21}$$

We now argue that the Weyl group $W_\mathbb{C}$ action must coincide with the monodromy group $S_\mathbb{C}$ action on the special coordinates. This argument has two ingredients. The first is a general result on the connection between the orbifold group $W_\mathbb{C}$ and $\pi_1(\mathcal{C}^*)$; though elementary, we give a detailed argument for this below, since we do not know where to find it in the literature. The second is the observation that the monodromy map is trivial in

the covering space of the orbifold (A.21), since it is a direct product of the covering space of $\mathcal{C}$ with a fixed abelian variety.

The singular locus, $\mathcal{D} \subset \mathcal{C}$, is given by the fixed points of non-identity elements of $W_{\mathbb{C}}$. Weyl groups are reflection groups, so are generated by reflections, which are elements, $r_A \in W_{\mathbb{C}}$, which fix a codimension-1 hyperplane $\widetilde{\mathcal{D}}_A \subset \mathbb{C}^r$. $\widetilde{\mathcal{D}}_A$ are the preimages under the orbifold quotient map of the singular points of $\mathcal{C}$. Denote their union by $\widetilde{\mathcal{D}} \doteq \cup_A \widetilde{\mathcal{D}}_A$. The singular locus of the CB is thus $\mathcal{D} = \widetilde{\mathcal{D}}/W_{\mathbb{C}}$.

Define $\widetilde{\mathcal{C}}^* \doteq \mathbb{C}^r \setminus \widetilde{\mathcal{D}}$, an r-dimensional complex vector space minus some finite number of distinct $(r-1)$-dimensional subspaces. Then the smooth points of the CB are $\mathcal{C}^* = \widetilde{\mathcal{C}}^*/W_{\mathbb{C}}$. Since we have removed the fixed points, $W_{\mathbb{C}}$ acts freely on $\widetilde{\mathcal{C}}^*$, and so $\varpi : \widetilde{\mathcal{C}}^* \to \mathcal{C}^*$ is a connected cover. For a point $p \in \mathcal{C}^*$, pick one preimage $\widetilde{p} \in \varpi^{-1}(p)$. Then $\varpi^{-1}(p)$ is the discrete set $\{W_{\mathbb{C}} \cdot \widetilde{p}\} \subset \widetilde{\mathcal{C}}^*$. By forming the fundamental groups of $\widetilde{\mathcal{C}}^*$ and $\mathcal{C}^*$ with base points $\widetilde{p}$ and p, respectively, an elementary argument shows that there is an exact sequence of groups

$$1 \to \pi_1(\widetilde{\mathcal{C}}^*) \xrightarrow{\iota_\varpi} \pi_1(\mathcal{C}^*) \xrightarrow{\phi} \pi_1(\mathcal{C}^*)/\pi_1(\widetilde{\mathcal{C}}^*) \cong W_{\mathbb{C}} \to 1. \tag{A.22}$$

To show this we need that: (1) the map ι_ϖ induced by the covering map ϖ is an injective group morphism; and (2) the resulting quotient is isomorphic to the orbifold group $W_{\mathbb{C}}$.

1. Since $\widetilde{\mathcal{C}}^*$ is a cover of $\mathcal{C}^*$, ϖ maps homotopic paths to homotopic paths, and maps concatenations of paths to concatenations, so ι_ϖ is a group morphism. The identity in $\pi_1(\mathcal{C}^*)$ is homotopic to the trivial path based at p, and so lifts by ϖ^{-1} to the trivial path based at $\widetilde{p}$. Thus $\iota_\varpi^{-1}(1) = 1$ in $\pi_1(\widetilde{\mathcal{C}}^*)$, and ι_ϖ is injective.

2. Any $\gamma \in \pi_1(\mathcal{C}^*)$ lifts to a unique homotopy class of paths $\widetilde{\gamma}$ in $\widetilde{\mathcal{C}}^*$ with endpoints $\widetilde{\gamma}(0) = \widetilde{p}$ and $\widetilde{\gamma}(1) = g_\gamma \cdot \widetilde{p}$ for some $g_\gamma \in W_{\mathbb{C}}$. Any representative of coset element $[\gamma] \in \pi_1(\mathcal{C}^*)/\pi_1(\widetilde{\mathcal{C}}^*)$ can be written $\gamma \nu$ for some $\nu \in \mathrm{im}(\iota_\varpi)$. Its pre-image in $\pi_1(\widetilde{\mathcal{C}}^*)$ is $\iota_\varpi^{-1}(\gamma\nu) = \iota_\varpi^{-1}(\gamma) \iota_\varpi^{-1}(\nu) = \widetilde{\gamma}\widetilde{\nu}$ which is a path starting at $\widetilde{p}$ and ending at $g_\gamma \cdot \widetilde{p}$. Furthermore, any path $\widetilde{\delta}$ in $\widetilde{\mathcal{C}}^*$ starting at $\widetilde{p}$ and ending at $g_\gamma \cdot \widetilde{p}$ can be written as $\widetilde{\delta} = \widetilde{\gamma}\widetilde{\nu}$ with $\widetilde{\nu} \doteq \widetilde{\gamma}^{-1}\widetilde{\delta} \in \pi_1(\widetilde{\mathcal{C}}^*)$ since it starts and ends at $\widetilde{p}$. Thus the cosets $[\gamma] \in \pi_1(\mathcal{C}^*)/\pi_1(\widetilde{\mathcal{C}}^*)$ are in 1-to-1 correspondence with elements of $g_\gamma \in W_{\mathbb{C}}$.

Because of the direct product structure of the orbifold covering space in (A.21), any closed path $\widetilde{\nu} \subset \mathbb{C}^r$ in the $\mathcal{C}$ orbifold covering space is accompanied by a trivial monodromy on Λ (the 1-homology of $A_{\boldsymbol{\tau}}$), and therefore also for the image of this path in the $\mathcal{C}$ orbifold (2.2). Thus

$$\mathrm{im}(S \circ \iota_\varpi) = 1, \tag{A.23}$$

the trivial subgroup of $\mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$. It follows that the monodromy map S determines and is determined by an integral symplectic representation of the orbifold group,

$$W_{\mathbb{Z}} : W_{\mathbb{C}} \to \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z}). \tag{A.24}$$

$W_{\mathbb{Z}}$ determines S by $S \doteq W_{\mathbb{Z}} \circ \phi$, and S determines $W_{\mathbb{Z}}$ by $W_{\mathbb{Z}}(g) = S \circ \phi^{-1}(g)$, where $\gamma = \phi^{-1}(g)$ is any representative of the quotient equivalence class. This makes sense since if

γ and γ' are in the same class, $\phi(\gamma) = \phi(\gamma') = g$, then $S(\gamma') = S(\gamma'\gamma^{-1}\gamma) = S(\gamma'\gamma^{-1})S(\gamma) = S(\gamma)$ where we used that $\gamma'\gamma^{-1} \in \ker\phi = \mathrm{im}\iota_\varpi$ together with (A.23).

This tells us not only that $S \cong W_{\mathbb{C}}$ as abstract groups, but also, since S has the linear action $S_{\mathbb{C}}$ on the same $\mathbb{C}^r$ space of special coordinates, that $S_{\mathbb{C}} \cong_{\mathbb{C}} W_{\mathbb{C}}$. (Representations are equivalent over the complexes, $\rho \cong_{\mathbb{C}} \rho'$, if there exists a $C \in \mathrm{GL}(r,\mathbb{C})$ such that for all $g \in G$, $\rho(g) = C\,\rho'(g)\,C^{-1}$.) Although the $S_{\mathbb{C}}$ representation (A.13) depends on the value of $\boldsymbol{\tau}$, it is easy to see that the condition $S_{\mathbb{C}} \cong_{\mathbb{C}} W_{\mathbb{C}}$ is satisfied for any $\boldsymbol{\tau} \in \mathrm{Fix}(S)$ as long as

$$S \cong_{\mathbb{R}} W_{\mathbb{R}} \oplus W_{\mathbb{R}}, \tag{A.25}$$

by Matschke's theorem (representations of a finite group S are completely decomposable over $\mathbb{R}$) and since $S_{\mathbb{C}} \cong_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{Z}} S$ and $W_{\mathbb{C}} \cong_{\mathbb{R}} W_{\mathbb{R}} \oplus W_{\mathbb{R}}$. Thus, the integral symplectic representation of the monodromy group is equivalent over the reals to two copies of the real reflection representation of the orbifold group.

Conversely, since the linear coordinates of the $\mathbb{C}^r$ space upon which the Weyl groups acts are the special coordinates, given a choice of $\boldsymbol{\tau} \in \mathrm{Fix}(S)$, the monodromy action then determines the special coordinates, and so the whole SK geometry of the CB.

A.3 Equivalence of N=4 sYM SK structures

Two integral symplectic representations, S and S', correspond to equivalent SK structure orbits if they are integrally equivalent, $S \cong_{\mathbb{Z}} S'$. In particular, a map implementing integral equivalence is just a change of choice of basis of the charge lattice fibers, and therefore has no effect on any low energy physical observables on the CB.

But there can be other sources of equivalence between SK structures. In particular, two CB geometries may be equivalent under a map which has an action on the points of the CB in addition to a change of basis of the homology lattice of the fibers. To be considered equivalent, all physical observables in the low energy effective action on the CB, or, equivalently, all the coordinate- and basis-independent ingredients of the CB SK structures should be isomorphic. In coordinate-free language, such an equivalence of SK structures is a polarization- and fiber-preserving holosymplectomorphism of the algebraic integrable system.

Relative to a choice of charge lattice basis (and a choice of base point on the CB), this means that an equivalence of CB geometries is a bijection of the CB to itself, $f : \mathcal{C} \to \mathcal{C}$, which maps the charge lattice, symplectic pairing, monodromy map, and special coordinates as

$$
\begin{aligned}
&f : \Lambda_u \to \Lambda'_u \doteq Z\Lambda_{fu}, && u \in \mathcal{C} \\
&f : J \to J' \doteq Z^{-1}JZ^{-t}, && \\
&f : S(\gamma) \to S'(\gamma) \doteq ZS(f\gamma)Z^{-1}, && \gamma \in \pi_1(\mathcal{C}^*), \\
&f : \sigma(u) \to \sigma'(u) \doteq Z^\vee \sigma(fu), && u \in \mathcal{C},
\end{aligned}
\tag{A.26}
$$

for some $Z \in \mathrm{GL}(2r,\mathbb{Z})$. The subscript on Λ_u labels the point in $\mathcal{C}$ over which the charge lattice is a fiber. Since the charge lattice experiences nontrivial monodromies, (A.26)

should be understood to apply on simply connected subsets of $\mathcal{C}^*$ containing the base point; similarly for the special coordinate maps. But Z implements an "ordinary" $\mathbb{Z}$-equivalence (A.16) of SK structures, so, up to an ordinary $\mathbb{Z}$-equivalence, we are free to take $Z = \mathbf{1}$. This then clearly leaves all physical observables — like the EM duality conjugacy classes of the EM monodromies, the matrix $\boldsymbol{\tau}$ of low energy EM couplings, the CB metric, and the $\mathcal{N}{=}2$ central charge function — invariant. We refer, somewhat sloppily, to such equivalences which act on the points of the CB as *SK structure isometries.*

In the case of $\mathcal{N}{=}4$ sYM CBs, their SK structure isometries are easy to describe explicitly. We have seen that their CBs are orbifolds by Weyl groups and their monodromy maps are essentially integral symplectic representations of the Weyl group W (lifted by the quotient map in (A.22)). Any automorphism, $\phi : W \to W$, of W as a reflection group (that is, any group automorphism which maps reflections to reflections) induces a map $f : \mathcal{C} \to \mathcal{C}$ which is an SK structure isometry. In particular, ϕ determines a $W_{\mathbb{C}}$-equivariant map $\widetilde{f} \in \mathrm{GL}(r, \mathbb{C})$ of the vector space covering the CB orbifold, $\mathcal{C} = \mathbb{C}^r/W_{\mathbb{C}}$, which descends to the isometry $f : \mathcal{C} \to \mathcal{C}$ upon quotienting by $W_{\mathbb{C}}$.

This follows because, as we review in section 2.6, all reflection automorphisms of W are either inner automorphisms or are Coxeter diagram automorphisms. Inner automorphisms, $\phi(W) = vWv^{-1}$ for some $v \in W$, act as $\widetilde{f} = R(v)$ on $\mathbb{C}^r$, where $R : W_{\mathbb{C}} \to \mathrm{GL}(r, \mathbb{C})$ is a complexified reflection representation of W. (All such representations are equivalent over $\mathbb{C}$; different embeddings in $\mathrm{GL}(r, \mathbb{C})$ differ by linear coordinate changes.) By taking R to be the complexification of one of the integral representations R_A, it is clear that these inner automorphism SK structure isometries are equivalent, up to a change of coordinates, to "ordinary" $\mathbb{Z}$-equivalences (A.16).

Coxeter diagram automorphisms are defined to be certain permutations of a basis of simple roots of W.[32] Such a permutation of r linearly independent lines in $\mathbb{R}^r$ can always be implemented by an orthogonal transformation, $\widetilde{f}(\phi) \in \mathrm{O}(r)$, which is interpreted as an element of $\mathrm{GL}(r, \mathbb{C})$ when acting on $\mathbb{R}^r \otimes \mathbb{C} = \mathbb{C}^r$. This definition of $\widetilde{f}(\phi)$ is not unique: there are $|W|$ choices corresponding to composing any given choice of $\widetilde{f}(\phi)$ with all combinations of simple reflections. This corresponds to composing the Coxeter diagram automorphism with the inner automorphisms. Note, however, that Coxeter diagram automorphisms need not be inner automorphisms. Even though both give (complexified) orthogonal actions on $\mathbb{C}^r$, the inner ones are characterized by these transformations being in $W_{\mathbb{C}}$, while it is possible that $\widetilde{f}(\phi) \notin W_{\mathbb{C}}$ for the Coxeter diagram automorphisms. We list the cases in which this occurs in table 4.

In summary, if $\phi \in \mathrm{Aut}_{\mathrm{refl.}}(W)$ is any reflection automorphism of W, then there is an associated SK structure isometry, relating two physically indistinguishable CB geometries, but which relates two *a priori* distinct SK structures,

$$(J, S, \boldsymbol{\tau}) \cong_\phi (J, S \circ \phi, \boldsymbol{\tau}). \tag{A.27}$$

If $(J, S \circ \phi, \boldsymbol{\tau}) \cong_{\mathbb{Z}} (J', S', \boldsymbol{\tau}')$, by combining it with the reflection automorphism isometries

---

[32]For reflection groups, the roots are the directions in $\mathbb{R}^r$ perpendicular to the reflection hyperplanes; they are not vectors (with specific norms) as for Lie algebra roots.

we get a potentially larger set of equivalences

$$(J, S, \boldsymbol{\tau}) \cong_{\mathbb{Z},\phi} (J', S', \boldsymbol{\tau}'). \tag{A.28}$$

We call this whole set of possible equivalences of $\mathcal{N}{=}4$ sYM CB geometries *automorphism-twisted $\mathbb{Z}$-equivalences*, and they are defined more generally and formally in appendix B. If $\phi \in \mathrm{Inn}(W)$ is an inner automorphism, then the SK structure isometry $\mathcal{C} \cong_\phi \mathcal{C}'$ can be traded by a change of coordinates for a $\mathbb{Z}$-equivalence between two associated symplectic representations, so $\mathrm{Inn}(W)$ do not give additional equivalences. But if $\phi \in \mathrm{Out}_{\mathrm{refl.}}(W) \doteq \mathrm{Aut}_{\mathrm{refl.}}(W)/\mathrm{Inn}(W)$, then the associated SK structure isometry gives a new equivalence of geometries whose associated symplectic representations may or may not be integrally equivalent. (An explicit description of all these equivalences in the $W = BC_2$ case is given in section 3.3.)

Thus, we have shown in this appendix that:

> The possible SK structures of an $\mathcal{N}{=}4$ sYM CB with Dirac pairing J are in 1-to-1 correspondence with automorphism-twisted $\mathbb{Z}$-equivalence classes of pairs $(S, \boldsymbol{\tau})$ with S an integral $J^\vee$-symplectic representation of the orbifold group satisfying (A.25), and $\boldsymbol{\tau} \in \mathrm{Fix}(S)$.

Finally, we mentioned earlier that the central charge condition (A.15) is an additional condition that an algebraic integrable system must satisfy for it to describe a physical SK structure. This condition requires that the monodromy $S(\gamma)$ for loops linking $\mathcal{D}$ fixes integral linear combinations of special coordinates that must vanish there. But this is automatic for $\mathcal{N}{=}4$ sYM orbifold geometries, because the integral symplectic representation $W_\mathbb{Z}$ acts on the $2r$ component vector $(a^D\ a)$ of (dual) special coordinates, and $\mathcal{D}$ is the fixed-point locus of this action. Thus $\mathcal{D}$ is the union of the eigenvalue 1 eigenspaces of $W_\mathbb{Z}$, and since $W_\mathbb{Z}$ is integral, these eigenspaces are spanned by integral combinations of the (dual) special coordinates.

B Integer and rational representations

B.1 Basic definitions

We first introduce the usual notions of finite group representations over rings, and equivalence. In what follows, let W be a *finite* group and R be a commutative ring (think of $\mathbb{Z}$ or $\mathbb{Q}$).

Definition (R-representation). An *r-dimensional R-representation* of W is a group morphism $\rho: W \to \mathrm{GL}(r, R)$.

Definition (Reducible and decomposable representations). An R-representation ρ is *reducible*, respectively *decomposable*, if there exists a matrix $B \in \mathrm{GL}(r, R)$ such that for all $w \in W$, $B\rho(w)B^{-1}$ is block triangular, respectively block diagonal, with at least two non-trivial blocks.

Definition (R'-equivalence). Let $R' \subseteq R$ be a subring of R. Two R-representations ρ, ρ' of W are called R'-*equivalent*, denoted $\rho \sim_{R'} \rho'$, if there is a matrix $B \in \mathrm{GL}(r, R')$ such that for all $w \in W$, $B\rho'(w) = \rho(w)B$.

We also twist the above definition of equivalence using automorphisms of the group W.

Definition (Automorphism-twisted R'-equivalence). Let $\Phi \subseteq \mathrm{Aut}(W)$ be a subgroup of the group of automorphisms of W. Two R-representations ρ, ρ' of W are called (R', Φ)-*equivalent*, denoted $\rho \sim_{(R', \Phi)} \rho'$, if there is a matrix $B \in \mathrm{GL}(r, R')$, and an automorphism $\phi \in \Phi$, such that for all $w \in W$, $B\rho'(w) = \rho(\phi(w))B$.

B.2 Indecomposable reducible representations over $\mathbb{Z}$ and $\mathbb{Q}$

We repeatedly use the two following results in the bulk of the paper.

Theorem [23, Th. 73.5]. Every r-dimensional $\mathbb{Q}$-representation of W is $\mathbb{Q}$-equivalent to an r-dimensional $\mathbb{Z}$-representation of W.

Theorem [23, Th. 73.12]. Let ρ be a reducible finite-dimensional $\mathbb{Q}$-representation of W of the form

$$\rho = \begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix} \tag{B.1}$$

with $\rho_{1,2}$ some $\mathbb{Q}$-representations of W. Then there exist $\mathbb{Z}$-representations $\widetilde{\rho}_{1,2}$ of W such that

$$\widetilde{\rho}_i \sim_{\mathbb{Q}} \rho_i \qquad \text{and} \qquad \rho \sim_{\mathbb{Z}} \begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix}. \tag{B.2}$$

These follow from Burnside's argument, which we reproduce here since it is key to the arguments of this paper.

Lemma 1 (Burnside [45]). *Let G be a finite subgroup of $\mathrm{GL}(n, \mathbb{Q})$. Then there exists a matrix of change of basis $A \in \mathrm{GL}(n, \mathbb{Q})$ such that $A^{-1} \cdot G \cdot A \subset \mathrm{GL}(n, \mathbb{Z})$.*

Proof. Let $g_k \in \mathrm{GL}(n, \mathbb{Q})$ be generators of G, indexed by $k \in K$, where K is a finite set. Let d be the greatest common divisor of all the entries of all the matrices g_k. The matrices dg_k have only integer entries. Define the lattice

$$I = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \forall k \in K, \quad g_k \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{Z}^n \right\}. \tag{B.3}$$

Define, for $i = 1, \ldots, n$,

$$a_{i,i} = \min \left\{ x_i \in \mathbb{R}_{>0} \;\middle|\; \exists (x_1, \ldots, x_{i-1}) \in \mathbb{R}^{i-1}, \, (x_1, \ldots, x_i, 0, \ldots, 0) \in I \right\}. \tag{B.4}$$

From the construction of $a_{i,i}$, there exists $(a_{i,1}, \ldots, a_{i,i-1}) \in \mathbb{R}^{i-1}$ such that

$$\mathbf{a}_i := (a_{i,1}, \ldots, a_{i,i}, 0, \ldots, 0) \in I. \tag{B.5}$$

Define then the matrix

$$
A = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ 0 & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n,n} \end{pmatrix} .
\tag{B.6}
$$

The matrix A has the following crucial property:

$$
X \in \mathbb{Z}^n \quad \Leftrightarrow \quad A \cdot X \in I .
\tag{B.7}
$$

The direct implication immediately follows from the lattice structure of I. Consider then

$$
x = A \cdot X \in I .
\tag{B.8}
$$

We want to show that $X \in \mathbb{Z}^n$, and we use induction on the components X_i of X, starting from the last entry X_n. The n-th entry of equation (B.8) gives $x_n = a_{n,n} X_n$, and by definition of $a_{n,n}$, we have

$$
\frac{x_n}{a_{n,n}} = X_n \in \mathbb{Z} .
\tag{B.9}
$$

Assume now that $X_n, \ldots, X_{r+1} \in \mathbb{Z}$. Then

$$
x - X_{r+1} \mathbf{a}_{r+1} - \cdots - X_n \mathbf{a}_n \in I
\tag{B.10}
$$

since the $\mathbf{a}_i \in I$ using (B.5). Moreover only the first r entries of that vector are non vanishing, so by definition of $a_{r,r}$ we have

$$
\frac{x_r - X_{r+1} a_{r+1,r} - \cdots - X_n a_{n,r}}{a_{r,r}} \in \mathbb{Z} .
\tag{B.11}
$$

This concludes the proof of (B.7), and as a direct consequence, $A^{-1} \cdot g_k \cdot A$ has integer coefficients, which proves the lemma.

Lemma 2. *Using the notations of the previous lemma, if every $g \in G$ has a block diagonal form with blocks of sizes $n_1, \ldots, n_r$ such that $n_1 + \cdots + n_r = n$, then so does the matrix A.*

Proof. Assume the block decomposition is given by $n = n_1 + n_2$. The set I then splits into $I = I_1 \oplus I_2$. We can then apply the proof of the previous lemma independently on I_1 and I_2, constructing matrices A_1 and A_2. The final matrix A has A_1 and A_2 as its two blocks.

B.3 Symplectic representations

Let J be a non-degenerate antisymmetric $2r \times 2r$ matrix with integer coefficients. The skew normal form theorem asserts that there exists an integer matrix U with determinant 1 such that

$$
J = U^t \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} U
\tag{B.12}
$$

with $\delta = \mathrm{diag}(\delta_1, \ldots, \delta_r)$ and the δ_i are positive integers such that $\delta_i | \delta_{i+1}$. These integers are uniquely determined by the matrix J. We say that J is principally polarized if $\delta_1 = \cdots = \delta_r = 1$. The symplectic group $\mathrm{Sp}_J(2r, \mathbb{Z})$ is the group of $2r \times 2r$ matrices M such that $M^t J M = J$, and we call J its *symplectic form*.

Theorem. Let W be a Weyl group of rank r. Let $\mathcal{R} : W \to \mathrm{Sp}_J(2r, \mathbb{Z})$ be a group morphism such that $\mathcal{R} \simeq_{\mathbb{Q}} R_1 \oplus R_2$ with $R_i : W \to \mathrm{GL}(r, \mathbb{Z})$ (for $i = 1, 2$) two representations which are $\mathbb{Q}$-equivalent to the standard reflection representation of W. Then there exists a matrix $P \in \mathrm{GL}(2r, \mathbb{Z})$ such that

$$P\mathcal{R}(w)P^{-1} = \begin{pmatrix} S(w) & L(w) \\ 0 & \delta S^{-t}(w)\delta^{-1} \end{pmatrix} \tag{B.13}$$

for some representation $S : W \to \mathrm{GL}(r, \mathbb{Z})$ and some matrix $L(w) \in \mathrm{GL}(r, \mathbb{Z})$, and (B.13) is a symplectic representation for the symplectic form $P^{-t}JP^{-1} = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}$.

Proof. [23] defines a *binding function* for the pair (R_1, R_2) as a function $L : W \to \mathrm{Mat}(r, \mathbb{Z})$ such that the mapping

$$\mathcal{R}' : W \to \mathrm{GL}(2r, \mathbb{Z}), \qquad \mathcal{R}'(w) = \begin{pmatrix} R_1(w) & L(w) \\ 0 & R_2(w) \end{pmatrix}, \tag{B.14}$$

is a group morphism. This is equivalent to saying that for each $w, w' \in W$, we have $L(ww') = R_1(w)L(w') + L(w)R_2(w')$. The binding function L is *inner* if there exists a matrix $D \in \mathrm{GL}(r, \mathbb{Z})$ such that for all $w \in W$, $L(w) = R_1(w)D - DR_2(w)$. Theorem 73.22 in [23] asserts that $|W|L$ is an inner binding for any binding L.

This guarantees that there exists a matrix $D \in \mathrm{GL}(r, \mathbb{Z})$ such that, if we define

$$L(w) := \frac{1}{|W|} \left[R_1(w)D - DR_2(w) \right], \tag{B.15}$$

then $L(w) \in \mathrm{GL}(r, \mathbb{Z})$ for all $w \in W$, and $\mathcal{R}$ is $\mathbb{Z}$-equivalent to the group representation (B.14). Let $P \in \mathrm{GL}(2r, \mathbb{Z})$ such that for all $w \in W$, $\mathcal{R}(w) = P^{-1}\mathcal{R}'(w)P$. Define

$$J' = P^{-t}JP^{-1} = \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \tag{B.16}$$

with $A^t = -A$ and $C^t = -C$. Then $\mathcal{R}'(w) \in \mathrm{Sp}_{J'}(2r, \mathbb{Z})$ for all $w \in W$. This implies

$$\begin{aligned}
A &= R_1^t(w)AR_1(w), \\
B &= R_1^t(w)AL(w) + R_1^t(w)BR_2(w), \\
C &= L^t(w)AL(w) - R_2^t(w)B^tL(w) + L^t(w)BR_2(w) + R_2^t(w)CR_2(w).
\end{aligned} \tag{B.17}$$

We now use the fact that the R_i are reflection representations of the real Weyl group W, so the matrices $R_i(w)$ are orthogonal. Intertwiners of orthogonal representations are symmetric so $A = 0$, and the last equation in (B.17) can be rewritten as

$$\left[C + \frac{B^tD - D^tB}{|W|} \right] R_2^{-1}(w) = R_2^t(w) \left[C + \frac{B^tD - D^tB}{|W|} \right]. \tag{B.18}$$

This means the matrix in square brackets is symmetric, so $C = \frac{D^tB - B^tD}{|W|}$, and we have found that

$$J' = \begin{pmatrix} 0 & B \\ -B^t & \frac{D^tB - B^tD}{|W|} \end{pmatrix}. \tag{B.19}$$

Now let us put B in Smith normal form, i.e., we write $B = U^t \delta V$ with U, V integer matrices with determinant 1, and δ integer and diagonal with each diagonal entry dividing the next one. Then define

$$J'' = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}^{-t} J' \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \delta \\ -\delta & \frac{(D'')^t \delta - \delta D''}{|W|} \end{pmatrix} \tag{B.20}$$

$$\mathcal{R}''(w) = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \mathcal{R}'(w) \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}^{-1} = \begin{pmatrix} R_1''(w) & L''(w) \\ 0 & R_2''(w) \end{pmatrix} \tag{B.21}$$

with $R_1'' = U R_1 U^{-1}$, $R_2'' = V R_2 V^{-1}$, $D'' = U D V^{-1}$ and $L'' := \frac{1}{|W|} [R_1'' D'' - D'' R_2''] = U L V^{-1}$. In particular, the middle equation in (B.17) becomes

$$\delta = (R_1'')^t(w) \, \delta \, R_2''(w) \,. \tag{B.22}$$

We perform one more transformation. Write the lower right entry of J'' as

$$\frac{(D'')^t \delta - \delta D''}{|W|} = \Delta - \Delta^t \,, \tag{B.23}$$

with Δ some matrix with integer coefficients. Then we transform J'' and $\mathcal{R}''(w)$ to

$$J''' = \begin{pmatrix} 1 & \delta^{-1}\Delta^t \\ 0 & 1 \end{pmatrix}^{-t} J'' \begin{pmatrix} 1 & \delta^{-1}\Delta^t \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}, \tag{B.24}$$

$$\mathcal{R}'''(w) = \begin{pmatrix} 1 & \delta^{-1}\Delta^t \\ 0 & 1 \end{pmatrix} \mathcal{R}''(w) \begin{pmatrix} 1 & \delta^{-1}\Delta^t \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R_1'''(w) & L'''(w) \\ 0 & R_2'''(w) \end{pmatrix}.$$

with $R_1''' = R_1''$, $R_2''' = R_2''$ and $D''' = D'' - |W|\delta^{-1}\Delta^t$. This finishes the proof, using (B.22).

Corollary. Assume J is principally polarized. Any representation $\mathcal{R} : W \to \mathrm{Sp}_{J^\vee}(2r, \mathbb{Z})$ of the rank r Weyl group W which is $\mathbb{Q}$-equivalent to two copies of the standard reflection representation is $\mathbb{Z}$-equivalent to a representation of the form

$$\begin{pmatrix} S(w) & L(w) \\ 0 & S^{-t}(w) \end{pmatrix}, \qquad L(w) = \frac{1}{|W|}(S(w)D - DS^{-t}(w)) \in \mathrm{GL}(r, \mathbb{Z}) \tag{B.25}$$

for some representation $S : W \to \mathrm{GL}(r, \mathbb{Z})$ and some matrix $D \in \mathrm{GL}(r, \mathbb{Z})$.

C Hecke groups and subgroups

The level-q Hecke groups for integer $q > 2$ are

$$H_q \doteq \left\langle \mathsf{S}_\ell, \mathsf{T} \,|\, \mathsf{S}_\ell^2 = (\mathsf{S}_\ell\mathsf{T})^q = 1 \right\rangle, \qquad \text{where} \qquad \ell \doteq 2[\cos(2\pi/q) + 1], \tag{C.1}$$

which act on $\tau \in \mathscr{H}_1$ as $\mathrm{PSL}(2, \mathbb{R})$ Möbius transformations[33]

$$\mathsf{S}_\ell : \tau \mapsto -1/(\ell\tau), \qquad\qquad \mathsf{T} : \tau \mapsto \tau + 1. \tag{C.2}$$

---

[33]This presentation is equivalent to the ordinary definition of the Hecke groups [46], with generators $\mathsf{S}, \mathsf{T}_\lambda$ acting as $\mathsf{S} : \tau' \to -1/\tau'$, $\mathsf{T}_\lambda : \tau' \to \tau' + \lambda$, where $\lambda \doteq \sqrt{\ell}$, provided we rescale $\tau' = \lambda\tau$.

A double cover in $\mathrm{SL}(2,\mathbb{R})$ is

$$\mathsf{S}_\ell = \begin{pmatrix} 0 & -1/\sqrt{\ell} \\ \sqrt{\ell} & 0 \end{pmatrix}, \qquad\qquad \mathsf{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad\qquad \text{(C.3)}$$

though in this case $\mathsf{S}_\ell^2 = (\mathsf{S}_\ell\mathsf{T})^q = -1$. We often use the presentation as a $\mathrm{SL}(2,\mathbb{R})$ subgroup rather than as a $\mathrm{PSL}(2,\mathbb{R})$ one in what follows.

When $q = 3, 4, 6$ then ℓ are the integers $\ell = 1, 2, 3$, respectively. Note that $H_3 = \mathrm{PSL}(2,\mathbb{Z})$, while H_4 and H_6 are other inequivalent discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$. These groups and some of their finite index subgroups appear as S duality groups. In particular, subgroups of H_q appear as the S duality groups of theories whose gauge Lie algebra has lacing number $\ell(q)$ [4, 7, 28].

The proper subgroups which appear are the subgroups of $\mathrm{SL}(2,\mathbb{Z})$

$$\Delta \doteq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, a + b + d = a + c + d = 0 \ (\mathrm{mod}\ 2) \right\},$$
$$\Gamma_0(n) \doteq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, c = 0 \ (\mathrm{mod}\ \mathrm{n}) \right\}. \qquad\qquad \text{(C.4)}$$

Also, considered as a projective subgroup, $\Gamma_0(\ell) \subset H_q$ is the subgroup generated by $\langle \mathsf{S}_\ell\mathsf{T}\mathsf{S}_\ell, \mathsf{T} \rangle$. Their indices as subgroups of H_q are computed for general n in, for example, [47]. The indices for most of the groups appearing in table 1 are

group	Δ	$\Gamma_0(2)$	$\Gamma_0(3)$	$\Gamma_0(4)$
index in $\mathrm{PSL}(2,\mathbb{Z}) \simeq H_3$	2	3	4	6
index in H_4	–	2	–	4
index in H_6	–	–	2	–

$$\text{(C.5)}$$

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