Scaling of free cumulants in closed system-bath setups

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Abstract

The Eigenstate Thermalization Hypothesis (ETH) has been established as a cornerstone for understanding thermalization in quantum many-body systems. Recently, there has been growing interest in the full ETH, which extends the framework of the conventional ETH and postulates a smooth function to describe the multi-point correlations among matrix elements. Within this framework, free cumulants play a central role, and most previous studies have primarily focused on closed systems. In this paper, we extend the analysis to a system–bath setup, considering both an idealized case with a random-matrix bath and a more realistic scenario where the bath is modeled as a defect Ising chain. In both cases, we uncover a universal scaling of microcanonical free cumulants of system observables with respect to the interaction strength. Furthermore we establish a connection between this scaling behavior and the thermalization dynamics of the thermal free cumulants of corresponding observables.

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1 Introduction

Thermalization of quantum many-body systems is one of the central questions in non-equilibrium statistical mechanics. Among the various frameworks that have been developed, the Eigenstate Thermalization Hypothesis [1,2] (ETH) is one among the most promising, which receive lots of attention during the last decade.

The ETH explains the eventual thermalization of the system by postulating a universal structure for the matrix elements of physical observables in the energy eigenbasis, known as the ETH ansatz [3–5]. Despite the lack of a rigorous proof, the ETH ansatz has been verified numerically in numerous studies [6–18] and is now widely believed to hold in generic non-integrable systems for few-body observables.

However, ETH in its conventional form (the ETH ansatz) is not the end of the story. Recently, several studies have focused on correlations among the matrix elements [19–26], and a generalized form of the ETH has been proposed [27], often referred to as the full ETH. The full ETH extends the standard framework by incorporating multi-point correlations, which are described in terms of smooth functions. Furthermore, the full ETH has intriguing connections to free probability theory [28], providing a powerful mathematical perspective on many-body thermalization. In recent years, this extended framework has attracted growing attention [28–37]. However, most of the existing studies focused primarily on isolated quantum systems.

In this paper, we extend the investigation of the full ETH to open-system setups by considering a central system coupled to a quantum chaotic bath. We study the structure of smooth multi-point correlation functions in this setting, with particular emphasis on the energy scale within which such correlation functions become structureless. Using large-scale numerical simulations, we demonstrate the existence of such an energy scale and identify its scaling behavior with respect to the interaction strength and system size. Furthermore, we investigate the dynamical significance of this energy scale by analyzing the thermal free cumulants of the corresponding observables.

The remainder of the paper is organized as follows. Section 2 introduces the main framework, including the setup, the Full ETH and the definition of free cumulants. Section 3 presents the numerical results: Subsection 3.1 focuses on a random-matrix model, where we first describe the setup and then discuss microcanonical and thermal free cumulants. Subsection 3.2 follows a similar structure but considers a defect Ising chain as a bath. Finally, Section 4 provides the conclusions and outlook.

2 Framework

2.1 Model and observables

We consider an open-system setup, where the total Hamiltonian can, in full generality, be written as

$$\mathcal{H} = \mathcal{H}_S + \lambda \mathcal{H}_I + \mathcal{H}_B. \tag{1}$$

Here $\mathcal{H}_S$ refers to the Hamiltonian of a comparatively small system, $\mathcal{H}_B$ represents a bath and $\mathcal{H}_I$ describes the interaction between the two, moderated by the coupling strength parameter

λ. The eigenstates and eigenvalues of $\mathcal{H}$ are denoted by $|i\rangle$ and E_i, respectively. We consider operators of the system, generically denoted by A.

2.2 Full Eigenstate Thermalization Hypothesis

The statistical properties of the matrix elements of an operator A in the energy eigenbasis, $A_{ij} = \langle i|A|j\rangle$, can be described by the full ETH framework, which extends the conventional ETH to include multi-point correlations. Specifically, it states that the statistical averages of products of matrix elements with repeated indices factorize, while with distinct indices, $i_1 \neq i_2 \neq \ldots \neq i_n$ satisfy

$$\overline{A_{i_1 i_2} A_{i_2 i_3} \ldots A_{i_n i_1}} = \Omega_{E^+}^{1-n} F_{e^+}^{(n)} \left(\omega_{i_1 i_2}, \ldots, \omega_{i_{n-1} i_n}\right) \,, \tag{2}$$

where Ω_{E^+} is the density of states at the average energy $E^+ \equiv (E_{i_1} + \cdots + E_{i_n})/n$. Here the $F_{e^+}^{(n)}$ are *smooth functions* of the average energy density $e^+ = E^+/L$ and the eigenenergy differences $\omega_{i_1 i_2} = E_{i_1} - E_{i_2}$ [1]. These smooth functions $F_{e^+}^{(n)}(\vec{\omega})$, with $\vec{\omega} = (\omega_{i_1 i_2}, \omega_{i_2 i_3}, \ldots, \omega_{i_n, i_1})$, will be referred to as ETH function throughout the rest of the paper.

For $n = 2$, the full ETH ansatz reduces to the conventional ETH ansatz [3, 4] and the ETH function $F_{e^+}^{(2)}(\omega)$ are well understood in this case. It has been found that, for generic observables in chaotic systems, there often exists an energy scale $\Delta E_{\text{eq}}^{(2)}$ within which $F_{e^+}^{(2)}(\omega)$ is a constant [38, 39], i.e.,

$$F_{e^+}^{(2)}(\omega) = \text{const.}, \ \forall |\omega| \leq \Delta E_{\text{eq}}^{(2)} \,. \tag{3}$$

A natural question is, whether a corresponding energy scale $\Delta E_{\text{eq}}^{(n)}$ exists for more generic n such that

$$F_{e^+}^{(n)}(\vec{\omega}) = \text{const.}, \quad \forall \ |\omega_{i_1 i_2}| \leq \Delta E_{\text{eq}}^{(n)}, \ldots, |\omega_{i_{n-1} i_n}| \leq \Delta E_{\text{eq}}^{(n)} \,. \tag{4}$$

Previous studies mainly focused on closed-system scenarios and evidence for the existence of $\Delta E_{\text{eq}}^{(n)}$ has been presented [40]. However, not much has been done in open system scenarios. Most questions remain open, for instance, whether such an energy scale $\Delta E_{\text{eq}}^{(n)}$ exists, and if so, how it scales with parameters such as the coupling strength and the bath sizes, as well as what impact it has on the system's dynamics. These will be the main questions explored in this paper.

2.3 ETH free cumulants

To understand the impact of the full ETH on the dynamics of the corresponding observable, let us consider the n-time thermal correlation function

$$\begin{aligned} C_n^\beta(\vec{t}) &= \frac{1}{Z} \text{Tr}[A(t_1) e^{-\frac{\beta}{n}} A(t_2) e^{-\frac{\beta}{n}} \cdots e^{-\frac{\beta}{n}} A(t_n) e^{-\frac{\beta}{n}}] \\ &= \frac{1}{Z} \left(\sum_{i_1, \ldots, i_n} e^{-\frac{\beta}{n}(E_{i_1} + \cdots + E_{i_n})} A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_n i_1} e^{i E_{i_1}(t_1 - t_n) + \cdots + i E_{i_n}(t_n - t_{n-1})} \right) \end{aligned} \tag{5}$$

where $\vec{t} = (t_1, t_2, \ldots, t_n)$ and $Z = \text{Tr}[e^{-\beta H}]$, indicating the partition function. When the sum of Eq. (5) is only taken over distinct indices, one can define the ETH free cumulants,

$$\kappa_n^{\text{ETH}}(\vec{t}) = \frac{1}{Z} \left(\sum_{\substack{i_1, \ldots, i_n \\ i_1 \neq i_2 \cdots \neq i_n}} e^{-\frac{\beta}{n}(E_{i_1} + \cdots + E_{i_n})} A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_n i_1} e^{i E_{i_1}(t_1 - t_n) + \cdots + i E_{i_n}(t_n - t_{n-1})} \right) . \tag{6}$$

[1] For $n = 1, 2$, one recovers the conventional ETH [3], where $F_e^{(1)} = \mathcal{A}(e)$ is the constant equilibrium average and $F_e^{(2)}(\omega) = |f(e, \omega)|^2$ is the dynamical correlation function.

It has been shown that, in the thermodynamic limit, $\kappa_n^{\mathrm{ETH}}(\vec{t})$ is related to the ETH function $F_{e^+}^{(n)}$ through

$$\kappa_n^{\mathrm{ETH}}(\vec{t}) = \int d\vec{\omega} F_{e_\beta}^{(n)}(\vec{\omega}) e^{i\vec{\omega}\cdot\vec{t}-\beta\vec{\omega}\cdot\vec{\ell}_n}, \tag{7}$$

where $\vec{\ell}_n = (\frac{n-1}{n}, \frac{n-2}{n}, \dots, \frac{1}{n})$ and $e_\beta = \frac{1}{ZL}\mathrm{Tr}[e^{-\beta H}H]$. Applying the Fourier transform on both sides of Eq. (7), one obtains

$$\kappa_n^{\mathrm{ETH}}(\vec{\omega}) \equiv \mathrm{FT}[k_n^{\mathrm{ETH}}(\vec{t})] = F_{e_\beta}^{(n)}(\vec{\omega}) e^{-\beta\vec{\omega}\cdot\vec{\ell}_n}. \tag{8}$$

It should be noted that, although $\kappa_n^{\mathrm{ETH}}(\vec{t})$ and $C_n^\beta(\vec{t})$ appear closely related, there is generally no one-to-one correspondence between them. For instance, $\kappa_n^{\mathrm{ETH}}(\vec{t})$ cannot be expressed in terms of $C_q^\beta(\vec{t})$.

2.4 Thermal free cumulants

After introducing the ETH free cumulants, we now turn to a closely related quantity: the thermal free cumulants. The thermal free cumulants $\kappa_n^\beta(t)$ are routinely defined recursively via

$$C_n^\beta(t_1, t_2, \dots, t_n) = \sum_{\pi \in NC(n)} \kappa_\pi^\beta(t_1, t_2, \dots, t_n), \tag{9}$$

where the sum runs over all *non-crossing* partitions $NC(n)$ of the set given by $\{1, \dots, n\}$. See Ref. [28] for more details. For instance, in case of $C_1^\beta(0) = 0$

$$C_1^\beta(0) = \kappa_1^\beta(0) = 0 \tag{10}$$
$$C_2^\beta(t_1, 0) = \kappa_2^\beta(t_1, 0) \tag{11}$$
$$C_3^\beta(t_2, t_1, 0) = \kappa_3^\beta(t_2, t_1, 0) \tag{12}$$
$$C_4^\beta(t_3, t_2, t_1, 0) = \kappa_4^\beta(t_3, t_2, t_1, 0) + \kappa_2^\beta(t_3, t_2)\kappa_2^\beta(t_1, 0) + \kappa_2^\beta(t_3, 0)\kappa_2^\beta(t_2, t_1) \tag{13}$$
$$\dots = \dots \tag{14}$$

Eq. (9) can be easily inverted and the thermal free cumulants $\kappa_n^\beta(\vec{t})$ can also be expressed in terms of the correlation functions $C_n^\beta(\vec{t})$. In other words, the $\kappa_n^\beta(\vec{t})$ are unambiguously fixed by $C_q^\beta(\vec{t})$ $(q \le n)$.

The full ETH implies that thermal free cumulants $\kappa_n^\beta(\vec{t})$ are related to ETH free cumulants $\kappa_n^\beta(\vec{t})$ as [28]

$$\kappa_n^\beta(\vec{t}) = \kappa_n^{\mathrm{ETH}}(\vec{t}) + \mathcal{O}(L^{-1}) . \tag{15}$$

In the thermodynamic limit $L \to \infty$, both quantities coincide $\kappa_n^\beta(\vec{t}) = \kappa_n^{\mathrm{ETH}}(\vec{t})$. In this case, a connection between thermal free cumulants $\kappa_n^\beta(\vec{t})$ and the ETH function $F_{e^+}^{(n)}(\vec{\omega})$ can be established using Eq. (8),

$$\kappa_n^\beta(\vec{\omega}) \equiv \mathrm{FT}[k_n^\beta(\vec{t})] = F_{e_\beta}^{(n)}(\vec{\omega}) e^{-\beta\vec{\omega}\cdot\vec{\ell}_n}. \tag{16}$$

From here on, we restrict ourselves to the case of infinite temperature ($\beta = 0$), but the consideration can in principle be generalized to the finite-temperature case. At $\beta = 0$, Eq. (16) becomes

$$\kappa_n(\vec{\omega}) = F_{e_0}^{(n)}(\vec{\omega}), \tag{17}$$

where we omit the superscript $\beta = 0$ for brevity. It is clear from Eq. (17) that the non-trivial dependence of the ETH function $F^{(n)}_{e_0}(\vec{\omega})$ governs the out-of-equilibrium dynamics of the thermal free cumulants $\kappa_n(\vec{t})$. At long times t, the low-frequency structure of $F^{(n)}_{e_0}(\vec{\omega})$ becomes most relevant.

In chaotic systems, $\kappa_n(\vec{t})$ generally thermalizes after a characteristic time scale $T^{(n)}_{\text{eq}}$. Based on the preceding discussion, a connection can be drawn between the thermalization time $T^{(n)}_{\text{eq}}$ and the energy scale $\Delta E^{(n)}_{\text{eq}}$, which characterizes the energy window within which $F^{(n)}_{e_0}(\vec{\omega})$ becomes structureless:

$$\Delta E^{(n)}_{\text{eq}} \sim \frac{2\pi}{T^{(n)}_{\text{eq}}} . \tag{18}$$

This relation highlights the dynamical significance of the energy scale $\Delta E^{(n)}_{\text{eq}}$.

For $n = 2$, the thermal free cumulants $\kappa_2(t)$, which coincide with the connected autocorrelation function, have been thoroughly studied in the open system framework [41, 42], particularly in the weak-coupling regime ($\lambda \to 0$) where the Markovian approximation can be applied. Considering a system observable A that overlaps with H_S, i.e. $\text{Tr}[AH_S] \neq 0$, it can be shown that, in the Markovian regime

$$\kappa_2(t) \propto e^{-\Gamma t}, \quad \text{for } L \to \infty , \tag{19}$$

where $\Gamma \propto \lambda^2$. Correspondingly, using Eq. (16), the ETH function $F^{(2)}_{e_0}(\omega)$ takes the form

$$F^{(2)}_{e_0}(\omega) \propto \frac{1}{\omega^2 + \Gamma^2} . \tag{20}$$

The function $F^{(2)}_{e_0}(\omega)$ can be regarded as constant at $\omega \ll \Gamma \propto \lambda^2$, thus it is expected that $\Delta E^{(2)}_{\text{eq}} \propto \Gamma \propto \lambda^2$. The standard open-system framework cannot be straightforwardly generalized to the study of higher cumulants, and the general properties of $\kappa_n(\vec{t})$ and $F^{(n)}_{e_0}(\vec{\omega})$, as well as $\Delta E^{(n)}_{\text{eq}}$ remain unknown. The question we aim to address is whether the scaling $\Delta E^{(2)}_{\text{eq}} \propto \lambda^2$, established for $n = 2$ in the weak-coupling regime, also carries over to higher-order cumulants with $n > 2$.

2.5 Microcanonical free cumulants

Studying the energy scale $\Delta E^{(n)}_{\text{eq}}$ by direct calculation of the ETH function $F^{(n)}_{e_+}(\vec{\omega})$ is very challenging, particularly for system sizes beyond the limits of exact diagonalization (ED). To this end, we employ an approach used in Ref. [40] and consider the microcanonically truncated operator

$$A_{\Delta E} = P_{\Delta_E} A\, P_{\Delta E}, \quad \text{where } P_{\Delta E} = \sum_{|E_i - E_0| < \Delta E/2} |E_i\rangle\langle E_i| . \tag{21}$$

Here E_0 indicates the center of the energy window which is chosen as $E_0 = Le_0$, corresponding to the infinite temperature $\beta = 0$. The microcanonical free cumulants are then given as a combination of the moments

$$\mathcal{M}_n(\Delta E) = \frac{\text{tr}(A_{\Delta E})^n}{d_{\Delta E}} \tag{22}$$

and read

$$\Delta_n(\Delta E) = \mathcal{M}_n - \sum_{j=1}^{n-1} \Delta_j \sum_{a_1 + a_2 + \cdots + a_j = n-j} \mathcal{M}_{a_1} \ldots \mathcal{M}_{a_j}, \tag{23}$$

where $d_{\Delta E} = \mathrm{Tr}[P_{\Delta E}]$. In the eigenbasis Δ_n can be expressed as

$$\Delta_n(\Delta E) = \frac{1}{d_{\Delta E}} \sum_{\substack{i_1,\ldots,i_n, i_1 \neq i_2 \cdots \neq i_n \\ |E_{i_k} - E_0| < \frac{\Delta E}{2} \ \forall k}} A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_n i_1}. \tag{24}$$

If the number of eigenstates within the energy windows is sufficiently large, one can replace $A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_n i_1}$ by its average $\overline{A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_n i_1}}$. Inserting Eq. (2), Eq. (24) becomes

$$\Delta_n(\Delta E) = \frac{1}{d_{\Delta E}} \sum_{\substack{i_1,\ldots,i_n, i_1 \neq i_2 \cdots \neq i_n \\ |E_{i_k} - E_0| < \frac{\Delta E}{2} \ \forall k}} F_{e^+}^{(n)}(\omega_{i_1 i_2}, \ldots, \omega_{i_n i_1}). \tag{25}$$

In the case of $\Delta E \leq \Delta E_{\mathrm{eq}}^{(n)}$, the condition $|\omega_{i_k i_{k+1}}| \leq \Delta E_{\mathrm{eq}}^{(n)}$ holds for all k (with $i_{n+1} = i_1$). Making use of Eq. (4), one obtains

$$\Delta_n(\Delta E) = \frac{1}{d_{\Delta E}} \sum_{\substack{i_1,\ldots,i_n, i_1 \neq i_2 \cdots \neq i_n \\ |E_{i_k} - E_0| < \frac{\Delta E}{2} \ \forall k}} 1 \simeq (d_{\Delta E})^{n-1}. \tag{26}$$

For sufficiently small windows ΔE, within which the density of states Ω_E can be regarded as constant, one has $d_{\Delta E} \propto \Delta E$, and Eq. (26) becomes

$$\Delta_n(\Delta E) \propto \Delta E^{n-1}. \tag{27}$$

Eq. (27) will serve as our primary indicator for identifying the energy scale $\Delta E_{\mathrm{eq}}^{(n)}$.

3 Numerical investigation

Below we expand on the models and observables considered in this work. Further details on the numerical methods may be found in App. A.

3.1 A random-matrix bath

As a first model we consider a setup where a central spin is coupled to a random-matrix bath, also known as the spin-Gaussian orthogonal random matrices (GORM) model [43]. The Hamiltonian reads

$$\mathcal{H} = \omega_S \sigma_x^S + \lambda \sigma_z^S \otimes \mathcal{H}_I + \mathcal{H}_B, \tag{28}$$

where $\mathcal{H}_I$ and $\mathcal{H}_B$ are random matrices from the Gaussian orthogonal ensemble (GOE). Their entries are drawn from a Gaussian distribution with zero mean and variance $\sigma_0^2 = \frac{1}{4d}$ with d the dimension of the full system. To facilitate discussions we typically address the *size* L of the system, with $d = 2^L$, rather than its dimension to treat it in the same language as other spin models. We fix the parameter ω_S to $\omega_S = 0.05$, while varying the interaction strength λ.

The out-of-equilibrium dynamics of the central spin have been well studied, e.g. in Refs. [43–46], which can also be straightforwardly generalized to the study of the second thermal free cumulant of the system observables. For instance, for $A = \sigma_x^S$, which coincides with $\mathcal{H}_S$ up to a prefactor and will be the observable of interest in this paper, one has in the weak-coupling limit that $\kappa_2(t) \propto e^{-\Gamma_R t}$ with $\Gamma_R \propto \lambda^2$. This leads to the prediction $\Delta E_{\mathrm{eq}}^{(2)} \propto \lambda^2$, see also the preceding discussion in Subsection 2.4. In App. B we also investigate σ_z^S. To verify the

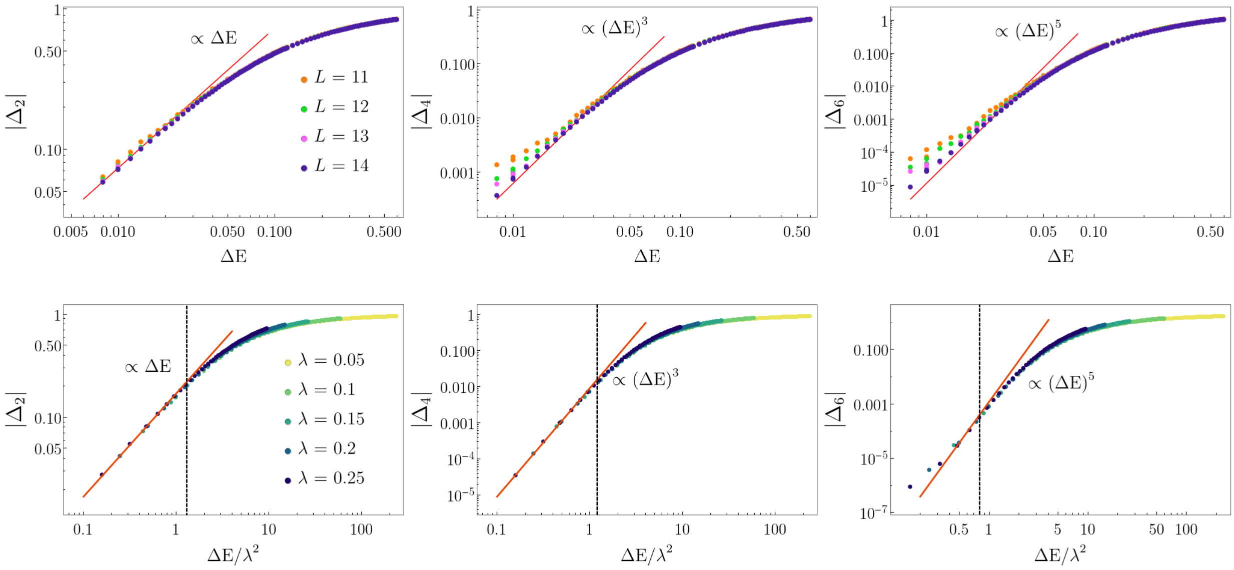

Figure 1: **Top:** Scaling of the microcanonical free cumulants Δ_n (with n even) for the random-matrix model with interaction strength $\lambda = 0.15$ for different system sizes. **Bottom:** Scaling of the free cumulants Δ_n in the random-matrix model with size $L = 14$ for different interaction strengths $\lambda = 0.05, \dots, 0.25$. Here the dashed black line indicates ΔE_U as a guide to the eye. Note that L is related to the Hilbert space dimension of the bath as $d = 2^L$.

prediction for A, and to examine whether the same scaling persists for $n > 2$, we employ the criterion introduced in Eq. (27) and analyze the microcanonical free cumulants of A.

To this end, we first consider the scaling of the microcanonical free cumulants Δ_n with respect to the microcanonical energy window ΔE for different system-bath interaction strengths λ, see Fig. 1. We find that energy scales $\Delta E_{\text{eq}}^{(n)}$ exist, within which the microcanonical cumulants to exhibit a power-law scaling with respect to ΔE, as in Eq. (27). From the top panel, $\Delta E_{\text{eq}}^{(n)}$ appears to be almost independent of the system size. For small energy windows, deviations from Eq. (27) can be observed, which are probably due to the finite size effect, since the validity of Eq. (27) requires $d_{\Delta E} \propto 2^L \Delta E \gg 1$. However, it is evident that, the agreement with Eq. (27) seems to extend to smaller ΔE, for larger system sizes. It suggests that Δ_n will follow the power-law scaling indicated by Eq. (27) to arbitrary small energy windows for $L \to \infty$.

From the lower panel, a scaling law $\Delta_n(\Delta E) = f\left(\frac{\Delta E}{\lambda^2}\right)$ with respect to the system–bath interaction strength λ seems to hold for all cumulants considered. The observation reveals a scaling of the energy scale ΔE_{eq} with system–bath coupling strength $\Delta E_{\text{eq}}^{(n)} \propto \lambda^2$, which highlights the first main result of our paper.

In addition, we investigate specific thermal free cumulants $\kappa_n(t) \equiv \kappa(t, 0, \dots, t, 0)$, often considered in study of the long-time freeness [28, 32, 47–49]. From Eq. (9) we infer for $n = 2, 4, 6$

$$\kappa_2(t) = \langle A(t)A \rangle, \tag{29}$$

$$\kappa_4(t) = \langle A(t)A\,A(t)A \rangle - \kappa_2(t)^2, \tag{30}$$

$$\kappa_6(t) = \langle A(t)A\,A(t)A\,A(t)A \rangle - 6\kappa_2(t)\kappa_4(t) - 4\kappa_3(t)^2 - 5\kappa_2(t)^3. \tag{31}$$

In Fig. 2 we depict the autocorrelation function for $A = \sigma_x^S$, as well as the higher cumulants $\kappa_4(t)$ and $\kappa_6(t)$ in the random-matrix model with size $L = 14$. An approximate data collapse of all thermal free cumulants $\kappa_n(t)$ for $n = 2, 4, 6$, with respect to the rescaled time $\lambda^2 t$, can be observed, most clearly after an initial transient time. This behavior indicates that $T_{\rm eq}^{(n)} \propto \lambda^{-2}$, in agreement with Eq. (18), $T_{\rm eq}^{(n)} \sim 2\pi/\Delta E_{\rm eq}^{(n)}$, together with the scaling $\Delta E_{\rm eq}^{(n)} \sim \lambda^2$. This constitutes the second main result of our paper. Considering $T_{\rm eq}^{(n)}$ of the thermal free cumulants $\kappa_n(t)$, from Fig. 2 we infer the equilibration times scales to be largely independent of n, i.e. $T_{\rm eq}^{(2)} \sim T_{\rm eq}^{(4)} \sim T_{\rm eq}^{(6)}$.

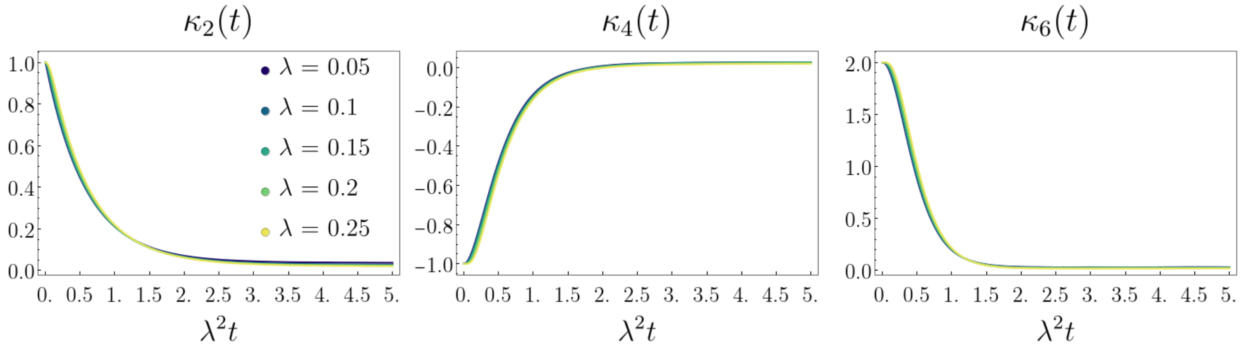

Figure 2: Free cumulants $\kappa_n(t)$ for $n = 2, 4, 6$ in the random-matrix model with $L = 14$ and different interaction strengths λ.

3.2 A chaotic Ising bath

Further we investigate a quantum spin chain of similar structure. Its Hamiltonian is given by

$$\mathcal{H} = \sigma_x^S + \lambda \sigma_z^S \otimes \mathcal{H}_I + \mathcal{H}_B, \tag{32}$$

$$\mathcal{H}_I = \frac{1}{\sqrt{L-1}} \sum_{n=1}^{L-1} (-1)^n \sigma_z^n, \tag{33}$$

$$\mathcal{H}_B = J \sum_l \sigma_z^l \sigma_z^{l+1} + h_x \sigma_x^l + h_2 \sigma_z^2 + h_5 \sigma_z^5. \tag{34}$$

Due to the precise form of the coupling, we restrict ourselves to odd sizes of the full system. The bath is given by a transverse Ising chain with two defects imposed at sites 2 and 5 to break symmetries and render the Hamiltonian chaotic. The parameters are chosen as $(J, h_x, h_2, h_5) = (1.0, 1.0, 1.11, 1.61)$ and periodic boundary conditions are imposed. As with the random-matrix model above, we consider the observable $A = \sigma_x^S$. Overall, similar results are observed in Figs. 3 and 4 when compared to the results for the random–matrix bath shown in Figs. 1 and 2, as detailed below.

The energy scale $\Delta E_{\rm eq}^{(n)}$, within which Eq. (27) holds, can be identified here for $n = 2, 4, 6$, and its value appears to be nearly independent of the system size (upper panel of Fig. 3). We again observe a scaling law $\Delta_n(\Delta E) = f(\frac{\Delta E}{\lambda^2})$, which indicates the scaling $\Delta E_{\rm eq}^{(n)} \propto \lambda^2$ (lower panel of Fig. 3). As for the dynamics of the thermal free cumulants $\kappa_n(t)$ (Fig. 4), an approximate data collapse is observed as a function of $\lambda^2 t$, although it is less pronounced than in the random–matrix bath model. This implies $T_{\rm eq}^{(n)} \propto \lambda^{-2}$, consistent with the scaling of energy scale $\Delta E_{\rm eq}^{(n)} \propto \lambda^2$. Further, we again observe the equilibration times scales $T_{\rm eq}^{(n)}$ to depend only weakly on n.

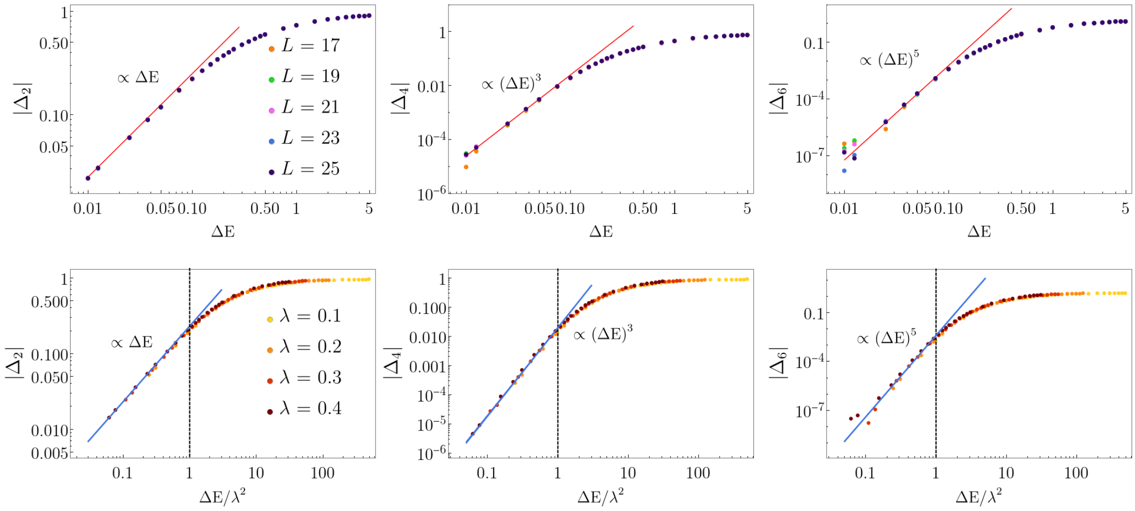

Figure 3: **Top:** System-size scaling of the microcanonical free cumulants Δ_n in the chaotic Ising bath model with interaction strength $\lambda = 0.3$. **Bottom:** Scaling the microcanonical free cumulants Δ_n in the chaotic Ising-bath model with system-size $L = 23$ and different system-bath couplings λ. Here the dashed black line indicates ΔE_U as a guidance to the eye.

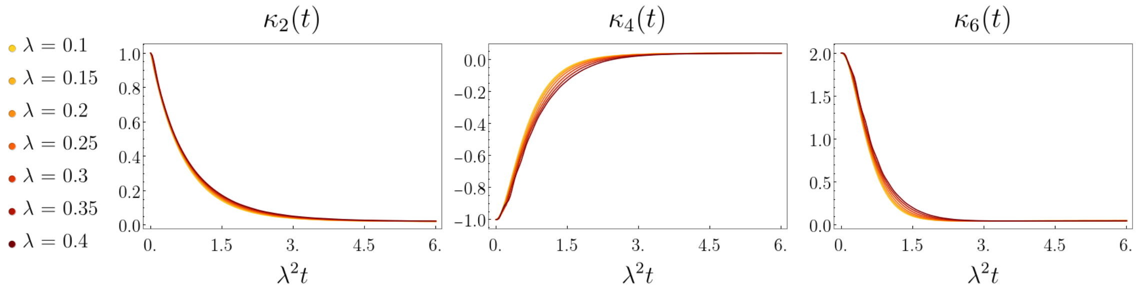

Figure 4: Thermal free cumulants $\kappa_n(t)$, where $n = 2, 4, 6$, in the model with an Ising bath and total size $L = 23$ and different coupling strengths λ.

4 Conclusion and Discussion

In this work, we extended the discussion of the full Eigenstate Thermalization Hypothesis (ETH) framework to open quantum systems, focusing on a central system coupled to a quantum chaotic bath. Through numerical analysis of free cumulants, we identified a clear and universal scaling with respect to the interaction strength, valid for observables that have a nonvanishing overlap with the central system Hamiltonian. Moreover, we established a connection between this scaling behavior and the thermalisation time scales of thermal free cumulants of the corresponding observables, and further support it with numerical evidence. These findings highlight the versatility of the full ETH in describing thermalization dynamics in realistic system–bath setups and point toward promising directions for future studies of open many-body systems.

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A Details on the numerics

In the random-matrix model (28) the computation the microcanonical and the time-dependent cumulants Δ_n and $\kappa_n(t)$ were carried out using exact diagonalisation (ED). For the microcanonical free cumulants, $N = 10$ realizations were considered.

For the Ising model (32) the system sizes considered are beyond the reach of ED. Therefore both Δ_n and $\kappa_n(t)$ were obtained via techniques exploiting quantum typicality. Concretely, for the time-dependent quantities we use the concept of dynamical quantum typicality and time-evolve the respective quantities via Chebyshev-integrators. For the microcanonical free cumulants we employed the scheme suggested in Ref. [40] that expands the microcanonical projector (21) in terms of Chebyshev polynomials and then computes the moments (22) on the basis of typicality. For more details on the method we refer to [40]. For the expansion of the projectors we used $N_{\text{trunc}} = 10a\frac{2\pi}{\Delta E}$, with ΔE the width of the energy window and $a = (E_{\text{max}} - E_{\text{min}})/2$ and $E_{\text{min}}, E_{\text{max}}$ the edges of the spectrum. For every system size L study, we consider $N_{\text{typ}} \geq 2^{26-L}$ different realisations.

B Another observable in the random-matrix model

Here we illustrate the investigation of another observable in the random-matrix model (28), given by $A = \sigma_S^z$. In Fig. 5 we depict the autocorrelation function $\langle A(t)A \rangle$, i.e. the second cumulant $\kappa_2(t)$ in this model. As becomes apparent in Fig. 5 $\kappa_2(t)$ does not show a coupling strength-dependent scaling with respect to time as for different λ the overall behavior of the functions changes. The picture is similar for the microcanonical free cumulants Δ_n, see Fig. 6 for the first three even cumulants. While for every interaction strength λ the microcanonical free cumulants individually obey a power law as Eq. (27), they do not allow for a rescaling like in the case for $A = \sigma_S^x$ expanded on in the main text.

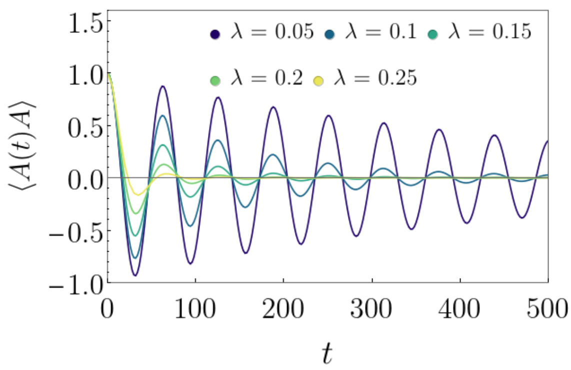

Figure 5: Autocorrelation function of the observable $A = \sigma_S^z$ in the random-matrix model (28) with system size $L = 14$ and different interaction strengths λ.

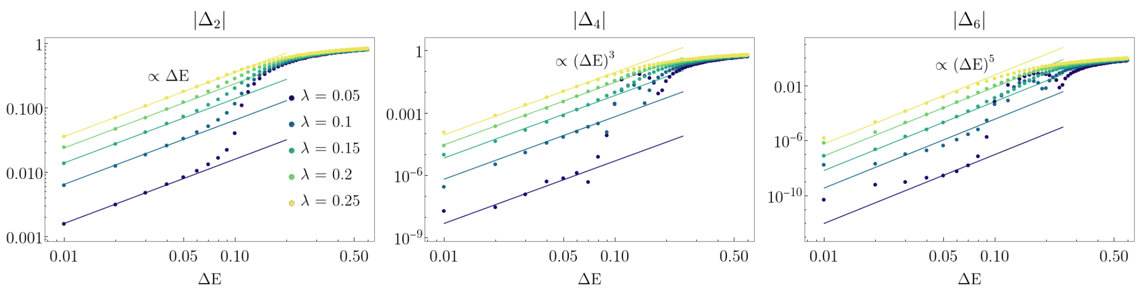

Figure 6: The first three even microcanonical free cumulants Δ_n for $A = \sigma_S^z$ in the random-matrix model. The solid lines serve as a guide to the eye for the scaling with respect to the width of the microcanonical energy window ΔE, as described in Eq. (27).