An Introduction to Loop Quantum Gravity (Lecture Notes)

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Introduction

Loop Quantum Gravity (LQG) is an approach to reconcile general relativity with quantum mechanics without imposing a fixed background geometry. In LQG, spacetime itself is treated as a physical system with quantum properties rather than a fixed stage on which physics unfolds. At extremely small (near-Planck) scales, geometric quantities such as area and volume are expected to be granular—coming in tiny, discrete quanta—while at macroscopic scales these quanta collectively reproduce the smooth geometry described by Einstein's theory.

The guiding philosophy is conservative: retain the background independence of general relativity, apply the rules of quantum theory universally, and introduce as little additional structure as possible. The aim is not a "theory of everything," but a consistent quantum description of spacetime that can be confronted with observation as the formalism matures—through arenas such as black-hole thermodynamics, early-universe cosmology, and other phenomenological windows.

These notes begin with motivation and core ideas, then build up the minimal mathematical tools needed for later sections, keeping the focus on physical meaning and on what is known, what is conjectured, and what remains open.

Acknowledgment. The bulk of these lecture notes is drawn from Carlo Rovelli's lectures in Marseille (available on YouTube: playlist link), along with the book *"An Introduction to Loop Quantum Gravity"* by Carlo Rovelli and Francesca Vidotto, and from papers I have studied over the years and talks I have attended, especially at the Perimeter Institute for Theoretical Physics, Waterloo, Canada.

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Content

1 The Imperial Basis of Loop Quantum Gravity

Loop Quantum Gravity (LQG) is a tentative theory for the quantum behaviour of gravity & of spacetime. The aim of the theory is not to have a theory of everything. We have no idea whats out there. For example, we don't know what dark matter is & we don't know what happens at high energies. This theory is only to describe the quantum behaviour of gravity, or of spacetime. For example, we see stuff falling inside black holes & what happens when they fall, but after that we have no idea & we therefore need a quantum theory of gravity.

Empirical information of QG

1. In the observation of gravitational waves in 2017 we observed that the Electromagnetic waves & the gravitational waves of the 2 Neutron stars merging arrive almost at the same time. We therefore have:

$$\frac{v_{\text{gw}}}{c} \approx 1 \ \sim \ \text{up to } 10^{-15} \tag{1.1}$$

   in which v_{gw} is the speed of gravitational waves. We already knew c & this just made it more clear.

2. No supersymmetry has been observed by the Large Hadron Collider, which was shocking for many physicist. This doesn't rule out any theory of QG but has put a constraint on it. So the degree of confidence in the theory decreases. Read below on the philosophy of science!!

   Philosophy of Science (Popper). Popper said we don't prove theories, but we falsify them. So he stated the idea that science works by proposing theories, testing them & to the extent that nothing they say contradicts reality, they might be OK. Then, when they predict something wrong, we discard them. This is a criterion separating science from non science. However, in Carlo's eyes, people have taken this too seriously. The way science works is that the more the predictions are verified, the more you believe it, like GR.

   So many theories are abandoned due to a lack of fruitfulness, which is no prediction. Every theory that cant be falsified doesn't mean its true. We have to take hints from nature we have not found things by randomly looking at things that have not been falsified. We are not smart enough to work that way.

3. There was a new theory that showed that the breaking of Lorentz invariance leads to a quantum theory of gravity. However, it was shown that the bounds on Lorentz invariance violation are $\sim 10^{-6}$. Nature shows us that Lorentz invariance doesn't break at the Planck scale. (There were theories that work on QG by breaking the Lorentz invariance.) (There could be other Lorentz invariance effects, but nature is hinting to us no.)

What do we know about the world

- **Quantum Mechanics** → Schrodinger equation doesnt fit with GR since there is no external time in GR. (It should be updated.)
- **General relativity** → Goes wrong in small actions, $\hbar \neq 0$.
- **Standard Model** → What happens at $> 15\,\text{TeV}$. We dont know maybe other particles.
- **Thermodynamics** → Is a bit different from the 3 above.

2 Space

We will now start a discussion on space and time, and what we mean by it. In QG space, time will mean something different from what it does in GR, just like the electromagnetic field of the Maxwell equation is not the same as the electromagnetic field of QED.

1) **Relational space:**

   In this definition, space is a relation between things. This notion of space has been discussed by Aristotle, Descartes, assuming space to be continuous.

   Corollaries:

   a) No empty space or vacuum, because if we take out everything, there is no space since space is a relation between things.

   b) Motion is a change of location. "Relative to other objects".

2) **Newtonian Space**

   Newton doesn't say that the previous notion of space is wrong. He states there are 2 notions of space. He makes a distinction.

   Distinction:

   a) "Common, Apparent" $\sim$ (Descartes & Aristotle)

   b) "True or Absolute"

   Newton argued that space is an entity independent of things with geometrical structure. $E^3 = (\mathbb{R}^3, \text{dist})$ with a notion of distance.

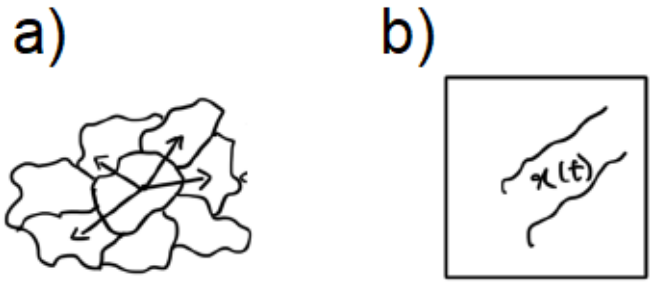

Figure 2.1: a) Aristotelian intuition of space: the position of particles with respect to one another (not connected to this one directly; no vacuum). b) Newtonian space: empty space with atoms moving in between. In this interpretation of space, to move is not to move with respect to one another, but space.

This discussion on absolute space isn't related to Galilean relativity, which Newton was wrong about. What is absolute here is the acceleration. We should also note that the geometrical structure of space allows us to distinguish a curved motion from a straight motion.

Newton comes up with the bucket experiment. You have a bucket tied from the ceiling with a rope. You twist it many times and then let it go. There will be 2 phases of motion. 1st, the water doesn't move, but then it starts moving, and its surface becomes concave due to rotation. This is because of friction. From the concave motion, we therefore know that at first water is not moving, but in the second phase it is, and we see the concave effect. By Descartes' definition, motion is defined with respect to its surroundings, and water isn't rotating in the second phase, but it's rotating in the first phase, which is wrong. Therefore, we get that motion is absolute. It is not with respect to the surroundings, but it is with respect to something else, to what? Absolute space. This is proof of the existence of absolute space. In modern terms, these are inertial forces. Newton states that the "Common" is what you see, and the "Absolute or True" you don't perceive, and you have

to do the math and use subtle observations. I can reconstruct an entity which is space "Absolute" and with this I can define acceleration.

1905, SR ⇒ Completely compatible with the Newtonian space except for the fact that we work in 4-dim Minkowski space. $E^3 = (\mathbb{R}^3) \to \mathcal{M}_4$.

3) **1915 General Relativity:** "Spacetime" is an entity and Newton was right in that regard, but it is a dynamical entity. A field very much like the Electromagnetic field ($\vec{E}(x)$, $\vec{B}(x)$). Einstein was amazed by the electromagnetic field and Maxwell's equations and realized that the space Newton defined is nothing but the field. Of course, no one, including Newton, knew about the fields.

   Now the field is a quantum gravitational field and will have quantum properties such as quantum superposition, etc . . . . So if by space you mean Newtonian space, it will disappear in QG. The continuous space will become dynamical and quantum. Very few of the same properties remain since it is not fixed, it is not continuous, it is not $E^3, \ldots$. It's a quanta of gravity interacting. But the old relation of the notion of space has not changed. There are things with respect to one another, we can say, were Newton was just adding a structure. So in QG, the old notion of where and with respect to what is still there, but the notion of continuity & . . . is no longer there. The Newtonian space has been reworked to be a field, and we know fields will have quantum properties.

In the Aristotelian sense there is always something in between. For example air. In many sense Newton was less precise & more effective. Newton in modern terms, is guessing the existence of a gravitational field. He understood that due to the inertial forces there should be a gravitational field and wrote it in the simplest way as:

$$d(\vec{x}, \vec{y}) = \sqrt{\delta_{ij}\,(x^i - y^i)(x^j - y^j)}. \tag{2.1}$$

References for further study

1. Carlo Rovelli, *Space and Time in Loop Quantum Gravity*, 2018

3 SU(2)

The conceptual discussion isn't finished, and we have yet to explain time. We'll go back to it later. For now we will discuss the math of the theory. Most of Loop Quantum Gravity is playing with $SU(2)$. $SU(2)$ is, of course, the covering group of $SO(3)$.

Definition

$SU(2)$ is a group of 2×2 matrices:

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad h^\dagger h = I, \qquad \det h = 1 \tag{3.1}$$

For example:

$$h = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \Rightarrow \quad h^\dagger = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}, \qquad h^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \quad \Rightarrow \quad h = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \tag{3.2}$$

If $\det(h) = 1 \;\Rightarrow\; |a|^2 + |b|^2 = 1$. So out of the 4 indices only 3 are independent, hence we have a 3-dimensional group.

$SU(2)$ lives in the complex space including both imaginary and real numbers. So we can write:

$$a = x + iy, \qquad b = z + ir$$

with constraint

$$x^2 + y^2 + z^2 + r^2 = 1.$$

This is a sphere in 4 dimensions, S_3, which is simply connected.

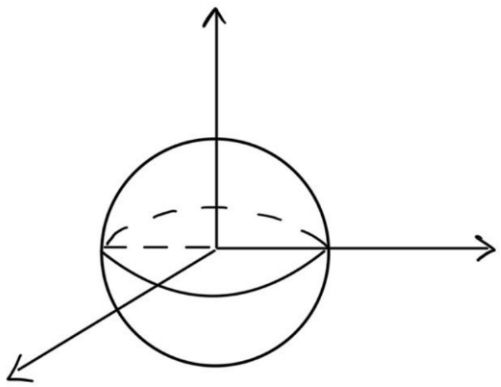

Figure 3.1: S_3 sphere in 4d, simply connected.

Topology and Haar measure

Pay attention that the sphere is in 4 dimensions S_3, not 3-dim S_2. The topology of $SU(2)$ is simply connected. From this perspective we see immediately a metric, a measure on $SU(2)$ which is natural and invariant under $SU(2)$. The Euclidean measure in 4 dimensions induces a metric on the sphere, which is also a metric on $SU(2)$ itself. According to Haar's theorem we can integrate over functions of the group:

$$\int dh\, f(h). \tag{3.3}$$

This measure is invariant under the action of $SU(2)$. Thus, we can define dh to integrate functions on the group. We can define functions φ from $SU(2)$ to complex numbers $\mathbb{C}$:

$$\varphi : SU(2) \to \mathbb{C}. \tag{3.4}$$

The function depends on h, and we can define a scalar product between two functions:

$$\int dh\, \bar{\varphi}(h)\, \phi(h) = \langle \phi | \varphi \rangle. \tag{3.5}$$

So $\varphi(h)$ lives in a Hilbert space of functions on $SU(2)$. We have a scalar product on the space of functions $SU(2)$ such as $\varphi(h) = \langle \varphi | h \rangle$. So we have a function dependent on h, $\varphi(h)$, which is a function of 2 complex numbers a & b restricted to the region $|a|^2 + |b|^2 = 1$. We can also define a scalar product between the two functions.

Hilbert space

We can consider the Hilbert space of functions on $SU(2)$. We consider the Hilbert space of square-integrable functions:

$$\mathcal{H} = L^2[SU(2)]. \tag{3.6}$$

This is the set of functions φ which are finite under the scalar product $\langle \phi | \varphi \rangle$, i.e. their integration over the group does not diverge. This is a core tool in LQG.

Relation to $SO(3)$

$SO(3)$ is a 3-dimensional group of real antisymmetric rotations. The map between $SU(2)$ and $SO(3)$ preserves the structure. It is not a $1 \leftrightarrow 1$ isomorphism, however it is a $2 \to 1$ local isomorphism:

$$SU(2) \longrightarrow SO(3), \qquad 2 \to 1 \text{ homomorphism.}$$

$$\mathrm{SO(3)} \ \ni \ R^i{}_j = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \quad \text{(real)}, \qquad \mathrm{SU(2)} \ \ni \ h^A{}_B = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, \quad A, B = 0, 1 \text{ (complex)}. \tag{3.7}$$

$$h^A{}_B \ \longrightarrow \ R^i{}_j = h^A{}_B\, h^C{}_D\, \sigma^i{}_{AC}\, \sigma_j{}^{BD}, \qquad , i, j = 1, 2, 3 \tag{3.8}$$

$$\sigma^i{}_{AB} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \qquad A, B = 0, 1. \tag{3.9}$$

Lie algebra of $SU(2)$

The matrices in $SU(2)$ can be written as:

$$h = e^{i\alpha^i \sigma_i}, \qquad \cos\alpha = 1 + i n^i \sigma_i \sin\alpha, \tag{3.10}$$

with $\vec{n} \in \mathbb{R}^3$, $|\vec{n}| = 1$. The Lie algebra of $SU(2)$ is given by commutation relations:

$$[\tau_i, \tau_j] = \epsilon_{ij}^k\, \tau_k, \qquad \tau_i = -\frac{i}{2}\sigma_i. \tag{3.11}$$

We can parametrize the rotation group with Euler angles:

$$h(\varphi, \theta, \phi) = e^{\varphi \tau_3} e^{\theta \tau_2} e^{\phi \tau_3}. \tag{3.12}$$

The invariant Haar measure is then:

$$\int dR = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{4\pi} d\phi \, \frac{\sin\theta}{16\pi^2}. \tag{3.13}$$

Key object and Hilbert space

Now let's introduce a key object, which is the main operator (like the momentum operator). The space $L^2[SU(2)]$ plays the role of the space of a wave function of a particle.

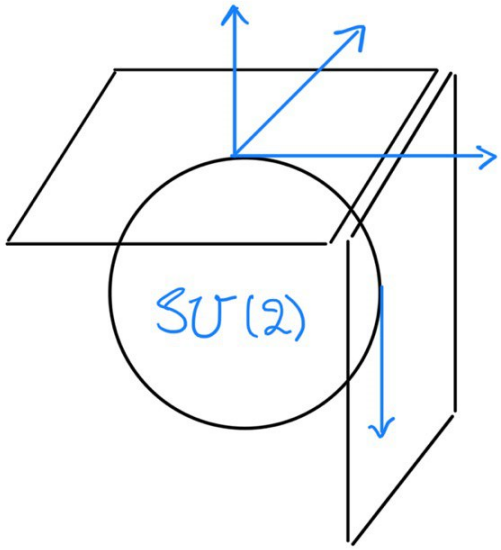

Figure 3.2: $\varphi(h)$ is a function on $SU(2)$

We then take the derivatives. The derivatives on a curved space are coordinatized funny. In Euclidean we have dx, dy, dz, which are preferred derivative operators and respect the structure of Cartesian coordinates. Here we have natural derivatives corresponding to the nature of the algebra. In the algebra we have a direction: an element of the algebra is a direction on the group. If we multiply the elements of the algebra we get a derivative, so we can define a vector field for each element of the algebra.

We now define an operator L^i on this space (the same index i as in the group element h):

$$h \;=\; e^{\,i\,\alpha^i \tau_i} \tag{3.14}$$

$$L^i\,\varphi(h) \;=\; \lim_{t\to 0} \frac{1}{t}\left[\varphi\big(h\,e^{\,t\tau_i}\big) - \varphi(h)\right] \qquad \text{(left-invariant vector field)} \tag{3.15}$$

Be careful with the order: you can't change it. If you do, it would be the right-invariant vector field.

$$\{\, L_x,\, L_y,\, L_z \,\} \qquad 3 \text{ operators (like the gradient).} \tag{3.16}$$

$$[\, L^i,\, L^j \,] \;=\; \epsilon^{ij}{}_k\, L^k \tag{3.17}$$

This is the $SU(2)$ algebra in which L^i is very much like angular momentum.

Representation theory

Representations are maps from a group G to a generalized linear group GL that act on a vector space of a given dimension D. A representation is *unitary* if you have a Hilbert structure. The irreducible representations are labeled by

$$j = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2}, \ldots$$

and this label is called the *spin*. The representation matrices are functions of a group element:

$$D^j_{mn}(h) \tag{3.18}$$

with

$$\dim V^j = 2j+1, \qquad m,n = -j, -j+1, \ldots, +j. \tag{3.19}$$

In which D is the Wigner matrix that we compute using Mathematica. It is written as

$$D^j_{mn}(\alpha, \beta, \phi). \tag{3.20}$$

$D^j_{mn}(h)$ are a very key object because of a theorem called *Peter–Weyl* theorem. It is the structure theorem of L^2 in a group. $SU(2)$ is a compact group, and you can perform a harmonic analysis on a compact group, and this is the key object. Think about the Fourier transform on a circle. You can write functions on a circle and expand it on a basis which are

$$e^{in\phi}. \tag{3.21}$$

So $e^{in\phi}$ is a basis on the function on the circle. The circle is also a compact group $U(1)$ & n labels an irreducible representation on the circle. So representation theory gives you a discrete basis with lots of good properties on the space of functions on a compact group.

Peter–Weyl theorem

$$D^j_{mn}(h) \;\text{ seen as an element of }\; L^2[SU(2)]. \tag{3.22}$$

A set of j, m, n form an orthogonal basis of $L^2[SU(2)]$:

$$\int_{SU(2)} D^j_{mn}(h)\, D^{j'}_{m'n'}(h)\, dh \;=\; \delta^{jj'}\, \delta_{mm'}\, \delta_{nn'}\, \frac{1}{2j+1} \qquad \text{(orthogonal, not orthonormal).} \tag{3.23}$$

So you can think of this object $D^j_{mn}(h)$ in the same way you think about quantum mechanics:

$$e^{\,ik\alpha} \;=\; \langle \alpha \,|\, k \rangle \tag{3.24}$$

We assume that we are on a circle that k is discrete and α is the angle; this is therefore a change in basis between the continuous basis α and the discrete basis k.

$$D^j_{mn}(h) \;=\; \langle h \,|\, j, m, n \rangle \tag{3.25}$$

Definition of Wigner D matrices.

$$D^j_{mn}(\alpha, \beta, \delta) \;=\; e^{-im\alpha}\, d^j_{mn}(\beta)\, e^{-in\delta} \tag{3.26}$$

- j is the spin quantum number ($j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$).
- m, n are the magnetic quantum numbers ($m, n = -j, -j+1, \ldots, +j$).
- α, β, δ are the Euler angles parameterizing the rotation $SO(3)$.
- $d^j_{mn}(\beta)$ is the reduced Wigner d-matrix, which contains the essential angular dependence.

$$D^{1/2}(\alpha, \beta, \delta) = \begin{pmatrix} e^{-i\alpha/2}\, \cos(\beta/2)\, e^{-i\delta/2} & -\, e^{-i\alpha/2}\, \sin(\beta/2)\, e^{\,i\delta/2} \\ e^{\,i\alpha/2}\, \sin(\beta/2)\, e^{-i\delta/2} & e^{\,i\alpha/2}\, \cos(\beta/2)\, e^{\,i\delta/2} \end{pmatrix} \tag{3.27}$$

$$D^{1/2}\Big(\frac{\pi}{2},\, \frac{\pi}{3},\, 0\Big) = \begin{pmatrix} e^{-i\pi/4}\, \cos\big(\frac{\pi}{6}\big) & -\, e^{-i\pi/4}\, \sin\big(\frac{\pi}{6}\big) \\ e^{\,i\pi/4}\, \sin\big(\frac{\pi}{6}\big) & e^{\,i\pi/4}\, \cos\big(\frac{\pi}{6}\big) \end{pmatrix} \tag{3.28}$$

For $j = 1$, the matrix is 3×3 as follows:

$$D^1_{mn}(\alpha,\beta,\delta) = \begin{pmatrix} e^{-i(\alpha+\delta)}\cos^2\frac{\beta}{2} & -\frac{1}{\sqrt{2}}\,e^{-i\alpha}\sin\beta & e^{-i(\alpha-\delta)}\sin^2\frac{\beta}{2} \\ \frac{1}{\sqrt{2}}\,e^{-i\delta}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\,e^{\,i\delta}\sin\beta \\ e^{\,i(\alpha-\delta)}\sin^2\frac{\beta}{2} & \frac{1}{\sqrt{2}}\,e^{\,i\alpha}\sin\beta & e^{\,i(\alpha+\delta)}\cos^2\frac{\beta}{2} \end{pmatrix} \tag{3.29}$$

Role of Wigner D-matrices

- **1- Spin Networks**
  - In LQG, the quantum states of the geometry are represented by spin networks, where the edges are labeled by spin j and the nodes are intertwiners.
  - Wigner D-matrices describe how these states transform under rotations.
- **2- Quantum Geometry**
  - The area and volume operators in LQG are constructed using spin representation.
  - The eigenvalues of these operators are discrete, reflecting the quantization of spacetime.
- **3- Path integrals and Spin Foams**
  - The spin-foam approach to quantum gravity involves summing over histories of spin networks.
  - Transition amplitudes are computed using Wigner D-matrices.

Wigner D-matrices, $D^j_{mn}(\alpha,\beta,\delta)$, describe how a quantum state labeled (j,m) transforms under an $SU(2)$ rotation. The second index, n, represents the final state after the transformation.

$$m \longrightarrow \text{initial state before rotation} \tag{3.30}$$

$$n \longrightarrow \text{final state after rotation} \tag{3.31}$$

The Wigner D matrix represents an element of the rotation operator in the basis of angular momentum eigenstates. If we rotate a quantum state $|j\,m\rangle$ by Euler angles (α,β,δ), the transformation is:

$$R(\alpha,\beta,\delta)\,|j,m\rangle \;=\; \sum_{n=-j}^{j} D^j_{mn}(\alpha,\beta,\delta)\,|j,n\rangle \tag{3.32}$$

References for further study

1. Robert Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*
2. J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*

4 Representations & Intertwiners

Let us try another way of thinking about the Peter–Weyl theorem and see how it makes things easier in this new language.

$$L_2[SU(2)] \;\cong\; \bigoplus_{j=0}^{\infty} \mathcal{H}^j \otimes (\mathcal{H}^j)^* \tag{4.1}$$

Here $\mathcal{H}^j$ is the spin-j representation space. The tensor product $\mathcal{H}^j \otimes (\mathcal{H}^j)^*$ is the space of "things" V^m and V^n, exactly like the matrix elements $D^j_{mn}(h)$. For each j we therefore have the space of matrices $(2j{+}1)\times(2j{+}1)$, i.e. the space of linear maps from a finite-dimensional Hilbert space to itself. So the space is a representation; the state carries representation j and we set $V^j = \mathcal{H}^j$.

$$j = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ \ldots \tag{4.2}$$

$$\rho_j : SU(2) \longrightarrow GL(V^j), \qquad \rho_j(h) \in GL(V^j) \tag{4.3}$$

$$D^j(h) \equiv \left[D^j_{mn}(h)\right]^j_{m,n=-j} \;\in\; GL(V^j) \qquad \text{(matrix components)} \tag{4.4}$$

$$C \;=\; \vec{L}\cdot\vec{L} \;=\; (L^1)^2 + (L^2)^2 + (L^3)^2 \tag{4.5}$$

Notes: C is $SU(2)$-invariant. The regular representation on $L^2[SU(2)]$ transforms under $SU(2)$ on the *right* and on the *left*. Finally, L^2 is like the Laplacian but on a curved space rather than a flat space. The manifold is $SU(2)$. $\vec{L}$ operates on $SU(2)$; it is a vector and it has indices. Its eigenspaces must transform among themselves under the $SU(2)$ representation.

Schur lemma. An operator that commutes with the action of the group in an irreducible representation must be a multiple of the identity. Therefore, C is the identity on each of the j subspaces $\bigoplus_{j=0}^{\infty}\cdots$; it sends the space into itself multiplied by a number. Hence

$$C\, D^j_{mn}(h) = j(j+1)\, D^j_{mn}(h). \tag{4.6}$$

Let's look at $SU(2)$ representations.

$$j = 0 \;\longrightarrow\; \text{zero-spin representation.} \tag{4.7}$$

$$j = \tfrac{1}{2} \;\longrightarrow\; \text{fundamental,} \qquad \dim = 2j+1 = 2. \tag{4.8}$$

$$\mathcal{H}^{1/2} = \mathbb{C}^2, \qquad z^A = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \;\longrightarrow\; \text{spinors.} \tag{4.9}$$

So a spinor is a vector in the fundamental representation of $SU(2)$. It will turn out that $\mathbb{C}^2$ is also the fundamental representation of $SL(2,\mathbb{C})$.

$$D^{1/2}_{AB}(h) \;=\; h_{AB} \tag{4.10}$$

Since this is the unitary representation on a Hilbert space, there is a scalar product:

$$\langle z \mid w \rangle \;=\; \bar{z}^A w^B\, \delta_{AB} \;=\; \sum_{A=0}^{1} \bar{z}^A w^A. \tag{4.11}$$

Don't confuse this with the invariant product on $\mathbb{C}^2$. In quantum mechanics, when we tensor two representations, we obtain a direct sum of representations which can be split into invariant subspaces. In particular,

$$\mathcal{H}^{1/2} \otimes \mathcal{H}^{1/2} \;=\; \mathcal{H}^0 \oplus \mathcal{H}^1. \tag{4.12}$$

Since $\mathcal{H}^{1/2} \otimes \mathcal{H}^{1/2} = \mathcal{H}^0 \oplus \mathcal{H}^1$, two spin-$\frac{1}{2}$ states admit a projection onto the singlet subspace $\mathcal{H}^0$, which yields a complex number. Hence there is an $SU(2)$-invariant bilinear map

$$\mathcal{H}^{1/2} \times \mathcal{H}^{1/2} \longrightarrow \mathcal{H}^0 \cong \mathbb{C}, \qquad (z, w) \mapsto z^A w^B \, \epsilon_{AB}, \qquad \epsilon_{AB} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.13}$$

$\mathbb{C}^2$ is also a representation of $SL(2,\mathbb{C})$. But $SL(2,\mathbb{C})$ is not compact. By a general theorem, unitary representations of a non-compact group are infinite-dimensional. Therefore the $\mathbb{C}^2$ representation of $SL(2,\mathbb{C})$ is *not* unitary. So the scalar product isn't $SL(2,\mathbb{C})$ invariant, but it is $SU(2)$ invariant. If you act on the spinors z^A with $h_A{}^B$ and $h_A{}^B$ is unitary, it preserves the scalar product; but if you act with an $SL(2,\mathbb{C})$ where $h_A{}^B$ is an arbitrary 2×2 matrix with determinant 1 which is not unitary, it destroys the scalar product. But (z, w) is invariant under $SL(2,\mathbb{C})$. Let's try:

$$h_A{}^B \, h_C{}^D \, \epsilon_{BD} \;=\; \epsilon_{AC} \tag{4.14}$$

Every time $h \in SL(2,\mathbb{C})$ the determinant is 1. Don't confuse the two together. For more about spinors, study the last part of Chapter 1 of the book: *An Introduction to Loop Quantum Gravity* by Carlo Rovelli and Francesca Vidotto.

$j = 1 \Rightarrow \dim = 3 \Rightarrow$ good old Euclidean space. Given $SU(2)$ you can write an $SO(3)$ matrix that rotates $\mathbb{R}^3 \ni V^i$, V^i, $i = 1, 2, 3$. This is not the usual m, n basis. The usual m, n basis is a combination of V^1, V^2, V^3:

$$V_0 = V^3, \qquad V_{-1} = V^1 - i\, V^2, \qquad V_{+1} = V^1 + i\, V^2. \tag{4.15}$$

$$z^A w^B = \begin{cases} z^A w^B \, \epsilon_{AB} & \text{spin } 0 \\ z^A w^B \, \sigma^i{}_{AB} \;\; \mapsto \;\; V^i, & \text{vectors, spin } 1 \Rightarrow \text{mapping to 3-dim vectors} \end{cases} \tag{4.16}$$

We can write the spin–1 representation as an object with symmetric indices:

$$z^{AB} = z^{(AB)}. \tag{4.17}$$

Spin j can be written as

$$z^{A_1 \cdots A_{2j}} = z^{(A_1 \cdots A_{2j})} \qquad \text{(completely symmetric)}. \tag{4.18}$$

Spinors & some properties.

$$\vec{z} = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \in \mathbb{C}^2, \qquad \epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.19}$$

ϵ^{AB} and ϵ_{AB} can be used for raising & lowering indices ("up–right, down–left" rule):

$$z_A = \epsilon_{AB} \, z^B, \qquad z^A = \epsilon^{AB} \, z_B. \tag{4.20}$$

1. Basic contractions:

$$\epsilon_{AC} \, \epsilon^{CB} = -\delta_A^{\;B}, \qquad \epsilon_{BA} \, \epsilon^{AB} = -2, \qquad \epsilon_{AB} \, \epsilon^{AB} = 2. \tag{4.21}$$

$$\epsilon_{AC} \, \epsilon^{CB} = \epsilon_{A0} \, \epsilon^{0B} + \epsilon_{A1} \, \epsilon^{1B} = (-1)\delta_A^B + (-1)\delta_A^B \tag{4.22}$$

$$\epsilon_{BA} \, \epsilon^{AB} = \epsilon_{01} \, \epsilon^{10} + \epsilon_{10} \, \epsilon^{01} \;=\; -2 \tag{4.23}$$

$$\epsilon_{AB} \, \epsilon^{AB} = \epsilon_{01} \, \epsilon^{01} + \epsilon_{10} \, \epsilon^{10} \;=\; 2 \tag{4.24}$$

2. ϵ^{AB} is invariant under the action $SU(2)$,

$$U^A{}_C\, U^B{}_D\, \epsilon^{CD} \;=\; \epsilon^{AB} \tag{4.25}$$

3.

$$\mathrm{Det}\, U \;=\; \epsilon^{BD}\, U^0{}_B\, U^1{}_D \;=\; \tfrac{1}{2}\, \epsilon_{AC}\, \epsilon^{BD}\, U^A{}_B\, U^C{}_D \;=\; 1 \tag{4.26}$$

$$\epsilon^{BD}\, U^0{}_B\, U^1{}_D \;=\; \epsilon^{01}\, U^0{}_0\, U^1{}_1 \;+\; \epsilon^{10}\, U^0{}_1\, U^1{}_0 \;=\; U^0{}_0\, U^1{}_1 \;-\; U^0{}_1\, U^1{}_0 \;= \mathrm{Det}\, U \tag{4.27}$$

$$\frac{1}{2}\Big(\epsilon_{10}\, \epsilon^{10}\, U^1{}_1\, U^0{}_0 \;+\; \epsilon_{10}\, \epsilon^{01}\, U^1{}_0\, U^0{}_1 \;+\; \epsilon_{01}\, \epsilon^{01}\, U^0{}_0\, U^1{}_1 \;+\; \epsilon_{01}\, \epsilon^{10}\, U^0{}_1\, U^1{}_0\Big) \;=\; \mathrm{Det}\, U \tag{4.28}$$

There are 2 $SU(2)$ invariant quadratic forms:

$$\langle z \mid y\rangle \;=\; \sum_A \bar{z}^A\, y^A \;=\; \bar{z}^0 y^0 + \bar{z}^1 y^1 \tag{4.29}$$

$$(z, y) \;=\; \epsilon_{AB}\, z^A y^B \;=\; z^0 y^1 - z^1 y^0 \tag{4.30}$$

The two can be related by defining the antilinear map $J : \mathbb{C}^2 \to \mathbb{C}^2$:

$$(Jz)^A = \begin{pmatrix} \bar{z}^1 \\ -\bar{z}^0 \end{pmatrix} \quad \Rightarrow \quad \langle z \mid y\rangle = (Jz,\, y) \tag{4.31}$$

$$(Jz,\, y) = \epsilon_{AB}\, (Jz)^A y^B = \bar{z}^1 y^1 + \bar{z}^0 y^0 = \langle z \mid y\rangle \tag{4.32}$$

Most of $SU(2)$ representation theory follows directly from the invariance of ϵ_{AB}. Consider a tensor product of 2 fundamental $j = \frac{1}{2}$ representations:

$$(z \otimes y)^{AB} = z^A y^B. \tag{4.33}$$

Any 2–index spinor z^{AB} can be decomposed into its symmetric & antisymmetric parts:

$$z^{AB} = z_0\, \epsilon^{AB} + z_1^{AB}, \qquad z_0 = \frac{1}{2}\, z^A{}_A, \qquad z_1^{AB} = z^{(AB)}, \tag{4.34}$$

which are invariant under the action $SU(2)$. Scalars z_0 define the trivial representation is $j = 0$, whereas z_1^{AB} defines the adjoint representation $j = 1$. Hence, we have proved that the tensor product of two spin-$\frac{1}{2}$ representations is the sum of spin-0 and spin-1 representations:

Intertwiner

$$\mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2} \;=\; \bigoplus_{j_3 = |j_1 - j_2|}^{j_1 + j_2} \mathcal{H}^{j_3}, \qquad |\, j_1 - j_2\,| \;<\; j_3 \;<\; j_1 + j_2, \qquad j_1 + j_2 + j_3 \to \text{even (Clebsch–Gordan).} \tag{4.35}$$

$$\mathrm{Invarient}\big(\mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2} \otimes \mathcal{H}^{j_3}\big) \;\neq\; \varnothing \tag{4.36}$$

Write a tensor with explicit spinor indices

$$z^{A_1 \cdots A_{2j_1}\, B_1 \cdots B_{2j_2}\, C_1 \cdots C_{2j_3}}. \tag{4.37}$$

We want something invariant under $SL(2, \mathbb{C})$. The only invariant tensor available is ϵ. You can't use $\epsilon_{A_1 A_2}$ on the same index family since this is antisymmetric. So, for example, you contract indices as $\epsilon_{A_1 B_1}, \epsilon_{A_2 C_1}, \ldots$. Therefore you need a certain number of $\epsilon_{AB},\ \epsilon_{BC},\ \epsilon_{AC}, \ldots$ — how many?

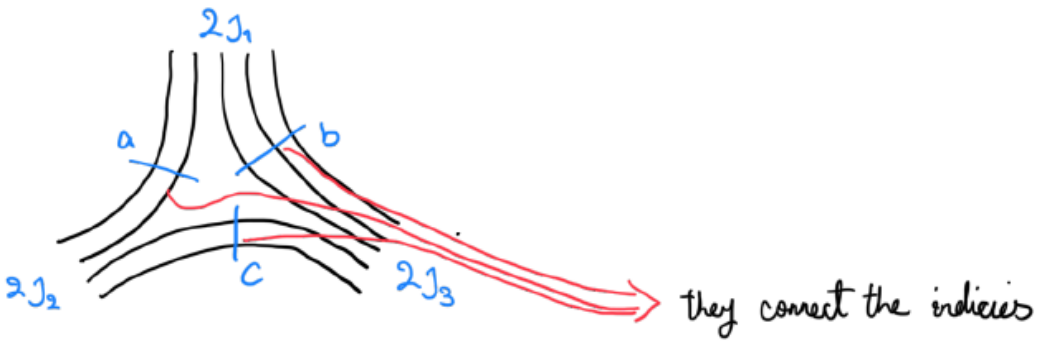

Figure 4.1: You can write something invariant only if you can close the indices with ϵ_{AB} completely.

$$\begin{cases} 2\,j_1 = a + b, \\ 2\,j_2 = a + c, \\ 2\,j_3 = b + c, \end{cases} \qquad \text{if we have this then} \quad |j_1 - j_2| < j_3 < j_1 + j_2 \text{ is satisfied.} \tag{4.38}$$

So you get this complicated object which is an invariant object:

$$z^{A_1 \cdots A_{2j_1}\, B_1 \cdots B_{2j_2}\, C_1 \cdots C_{2j_3}}\ \epsilon_{A_1 B_1}\ \cdots\ \epsilon_{B_{2j_2} C_{2j_3}}\,. \tag{4.39}$$

The mathematical interpretation of intertwiners: An intertwiner is a linear map $T : \mathcal{H}^J \to \mathcal{H}^{J'}$ such that

$$T\,D^J(g) \;=\; D^{J'}(g)\,T, \qquad \forall\, g \in G, \tag{4.40}$$

where G could be $SU(2)$.

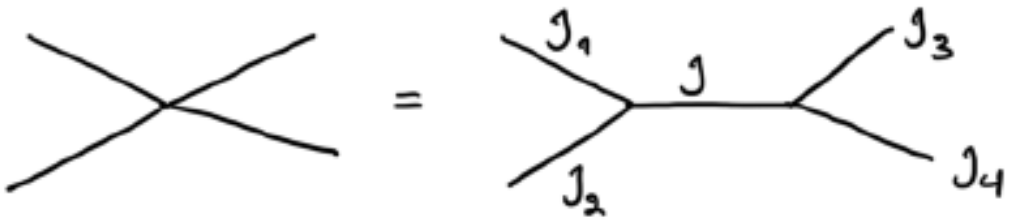

Figure 4.2: we can contract the two, but it must obey Clebsch-Gordan condition.

How we define it in LQG.

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \ \ni\ i^{\,m_1 m_2 m_3} \quad \longrightarrow \quad \text{intertwiner}, \tag{4.41}$$

$$D^{j_1}_{m_1 n_1}(h)\, D^{j_2}_{m_2 n_2}(h)\, D^{j_3}_{m_3 n_3}(h)\, i^{\,n_1 n_2 n_3} \;=\; i^{\,m_1 m_2 m_3} \qquad \text{(invariant)}. \tag{4.42}$$

The Wigner $3j$ symbol:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \;=\; i^{\,m_1 m_2 m_3}, \qquad \text{you can use this in mathematics and follow the CG condition.} \tag{4.43}$$

$$j_1 = j_2 = j_3 = 1 \;\Rightarrow\; \begin{pmatrix} 1 & 1 & 1 \\ i & j & k \end{pmatrix} = \epsilon_{ijk} \qquad \text{(the only invariant object)}. \tag{4.44}$$

Now let's look at intertwiners between 4 representations:

$$\mathrm{Inv}(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}) \ni \mathrm{V}^{m_1 m_2 m_3 m_4}. \tag{4.45}$$

Between 3 there is only one possible intertwiner, but between 4 I have a finite-dimensional linear space. I want to intertwine 4 representations. We know how to do 3, so we do the thing shown in Fig 4.2.

$$V^{m_1 m_2 m_3 m_4} = i^{m_1 m_2 m} \, i^{m m_3 m_4} \quad \Rightarrow \quad \text{we have to contract the } m \text{ indices.} \tag{4.46}$$

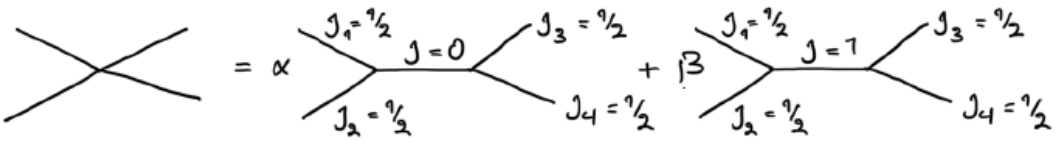

Figure 4.3: A two-dimensional space with j=1/2

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5 Loop Quantum Gravity Kinematics

We will now try to build the theory. We'll do it in two steps: the *kinematics* and the *dynamics*. In general, the structure of a quantum theory is a triple

$$(\mathcal{H},\, \mathcal{A},\, \mathcal{W}). \tag{5.1}$$

Hilbert space ($\mathcal{H}$): The space of a wave function of a particle. In QED this is the Fock space.

Algebra ($\mathcal{A}$): The position and momentum operators. In QED this is $a, a^\dagger$.

Transition amplitudes ($\mathcal{W}$): The Hamiltonian, which shows how things change in time, or the path integral (etc.). In QED you can use the Hamiltonian/action on the Fock space, or the Feynman rules.

In Loop Quantum Gravity $\mathcal{H}$, $\mathcal{A}$ represent the *kinematics* of the system and combined with $\mathcal{W}$ we have the *dynamics* of the system under study.

The first important part is the *algebra*. When Heisenberg wrote his first quantum mechanics text, it was about the algebra, not the space. He called it *matrix mechanics* (turning variables into matrices). The basic algebra was

$$[q, p] = i\hbar, \qquad i\hbar\, \dot{A} = [A, H]. \tag{5.2}$$

It is much easier, instead of working with abstract algebra, to write a concrete algebra of operators on a Hilbert space. Here, we will first define the space and then define the operators acting on that space. The algebra is what defines the variables. A physical system is specified by its variables: a particle is described by its position, velocity, momentum, energy, ... In classical mechanics, you describe a system through variables. In quantum mechanics, we also have a system of variables—called the *observables*. The term is a bit misleading, since it is more about interactions than about measurements performed by someone. What we are describing is a gravitational field, i.e., a piece of spacetime in the Newtonian sense. So we consider a region of spacetime in 4 dimensions $\mathbb{R}^4$ interacting with the rest through some variables, and the interaction is near the boundary where the variables sit. Pay attention that the variables in the algebra are separated from the Hilbert space, because $xp \neq px$; that's quantum mechanics.

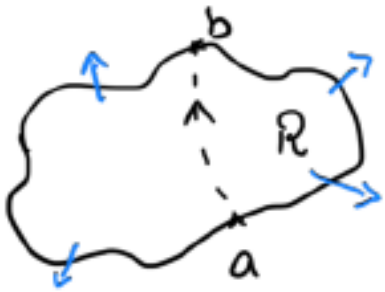

Figure 5.1: Variables sit on the boundary of a compact region. The dynamics tells us how to go from $a \to b$ and how things propagate and evolve.

How do we define the Hilbert space here? As in QED and QCD, we first define a finite set of degrees of freedom and then extend to infinity. In Fock space we define a one–particle Hilbert space, then a two–particle space, and so on, formally going to infinity. In practical calculations, you don't actually take the limit; you use Feynman graphs/paths up to a certain number of particles (a certain order).

$$\mathcal{H} \;=\; \lim_{\Gamma\to\infty} \mathcal{H}_\Gamma, \tag{5.3}$$

where $\mathcal{H}_\Gamma$ is the Hilbert space in LQG corresponding to a truncation of the number of degrees of freedom.

Γ is a graph. Graphs sit naturally inside one another, so you can always pass to larger graphs and hence to larger sets of degrees of freedom.

$$\Gamma = (\mathcal{N}, \mathcal{L}) \qquad \text{(a combinatorial graph)}.$$

- $\mathcal{N}$ is a finite set with n elements, which we call *nodes*.
- $\mathcal{L}$ is a collection of ordered pairs of elements of $\mathcal{N}$. An element $\ell \in \mathcal{L}$ is written $\ell = (n_1, n_2)$.

What matters in a graph is not how it is drawn, but how the links connect the nodes.

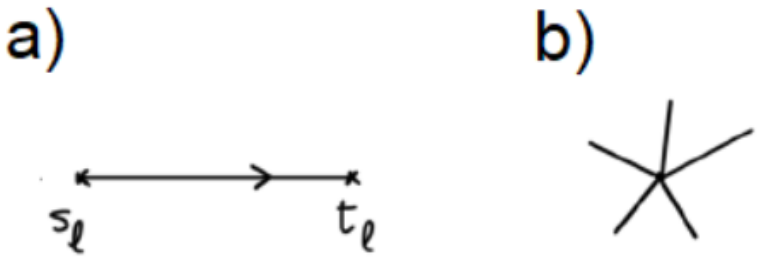

Figure 5.2: a) In this lecture, notes s_l is designated as the source link and t_l is the target link. b) The number of links is called valence of the nodes.

We now analyze curved surfaces and triangulate them.

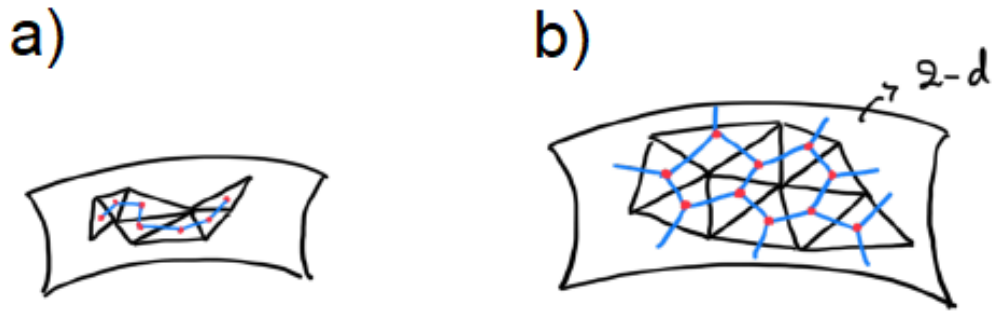

Figure 5.3: a) We put a dot in each triangle and connect them; we get a graph. It describes how the triangles are connected to one another. b) Getting the bigger picture in two dimensions with valence of the nodes.

A triangulation Δ is a set of triangles connected in a given way. The *dual triangulation* Δ^* is the graph obtained by placing a node in every triangle and connecting two nodes when the corresponding triangles are adjacent. The nodes are trivalent, i.e. each node has valence 3. We can do the same thing in 3-dimensions. You can think of a tetrahedral triangulation as shown in Figure 5.4.

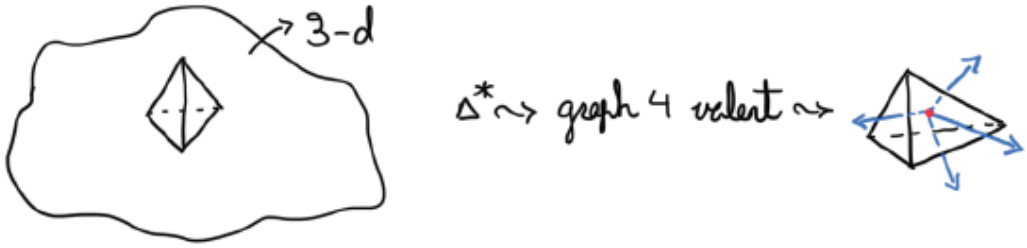

Figure 5.4: Triangulation in 3 dimensions is a tetrahedron in which in the dual space Δ^* the node has 4 valence.

A node is dual to a tetrahedron, and a link is dual to two neighboring tetrahedra separated by a triangle. This has been shown in Figure 5.5.

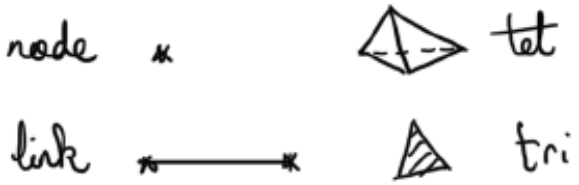

Figure 5.5: Node dual to a tetrahedron, and a link is dual to a triangle.

So, for the graph shown in Figure 5.6 we have

$$\widetilde{\mathcal{H}}_\Gamma = L^2\big[SU(2)^L\big] \ \ni \ \psi(h_\ell)\,. \tag{5.4}$$

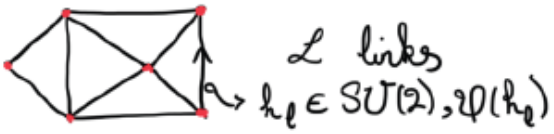

Figure 5.6: Graph in dual space.

This is the state space of a particle. In fact, it is very similar to the Hilbert space of lattice QCD, if you take the graph to be cubic.

Think of the underlying surface as curved, approximated by flat triangles as shown in Figure 5.3. Each triangle has a geometry; if we know the geometry of each triangle, we can infer the curvature of the surface. For example, take the three triangles that form a tetrahedron shown in Figure 5.7. The sum of the (red) angles around a point is not 2π, so the surface is not flat. Another way to see this is to choose an orthonormal frame on one face of the tetrahedron, parallel–transport it across the other two faces, and bring it back to the starting point; the frame returns rotated, indicating curvature. Likewise, parallel–transporting a vector on a curved surface does not return the same vector when you come back to the initial point, unlike the case of a flat surface. Also, the product of the three rotation matrices that brings us back to the same point gives the curvature of the space. It is the same with a tetrahedron: we have a rotation associated with each link, and when we go around a loop (a "tour"), if the product is not the identity (i.e. not a 2π rotation), we have curvature. We therefore describe geometry by associating a rotation to each link of the dual graph—an element of $SO(3)$ to each link of the dual graph. Thus h_ℓ are our classical variables, and a quantum state assigns an amplitude to a whole set of these variables.

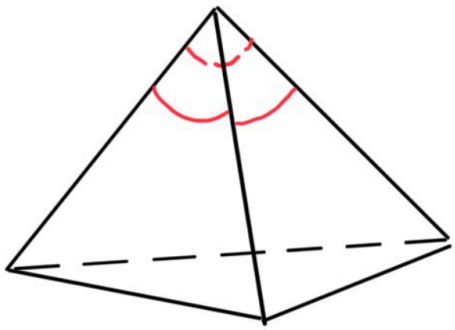

Figure 5.7: Tetrahedron with the sum of its vertex angles $\neq 2\pi$.

The Hilbert space $\mathcal{H}_\Gamma$ is a subspace of $\widetilde{\mathcal{H}}_\Gamma$. The reason is the following: if we close a reference frame on a face of a tetrahedron and rotate it, it gives us the curvature. But the reference frame itself is not important—any reference frame gives the same result. Therefore, we can write the following gauge transformation.

$$h_\ell \longmapsto \lambda_{s_\ell}\, h_\ell\, \lambda_{t_\ell}^{-1}, \tag{5.5}$$

a gauge transformation being a choice of $\lambda \in SU(2)$ for each node. We restrict to wave functions $\psi(\{h_\ell\})$ that are invariant under this action:

$$\psi(\{h_\ell\}) = \psi\big(\{\lambda_{s_\ell}\, h_\ell\, \lambda_{t_\ell}^{-1}\}\big)\,, \qquad \forall\, \lambda \in SU(2)^{N_v} \text{ (one } \lambda \text{ per node)}. \tag{5.6}$$

The set of wave functions satisfying the condition above defines $\mathcal{H}_\Gamma$. Since $SU(2)$ is compact, $\mathcal{H}_\Gamma$ is a proper subset of $\widetilde{\mathcal{H}}_\Gamma$ and there is no issue with infinities. In fact,

$$\mathcal{H}_\Gamma \subset \widetilde{\mathcal{H}}_\Gamma, \qquad \mathcal{H}_\Gamma \;\cong\; L^2\left[\frac{SU(2)^L}{SU(2)^{N_v}}\right], \tag{5.7}$$

where L is the number of links and N_v the number of nodes of the graph.

As an example, let

$$\mathcal{S} = \{\{1,2,3\}\} \quad \text{(identity ordering)}, \qquad \mathcal{G} = S_3 = \big\{\{1,2,3\},\{1,3,2\},\{2,1,3\},\{2,3,1\},\{3,1,2\},\{3,2,1\}\big\}.$$

$$G' = \{\text{identity},\ (1\,2)\}.$$

$$\mathcal{G}/G' = \big\{\{1,2\},\{3\}\big\} \qquad \text{(quotient group)}.$$

We can exchange 1 and 2 with one another here, but not 3. Therefore this is a smaller group.

Now, let's start with the algebra:

$$\mathcal{A}_\Gamma : \text{ it has two parts, creation and annihilation, } x \text{ and } p.$$

$\mathcal{A}_\Gamma$

$$f(h_\ell) : \text{ any function of the group is an operator in } \mathcal{A}_\Gamma.$$

The wavefunctions $\psi(h_\ell)$ are functions of group elements, so we can multiply by any function of a group element; this is the diagonal operator, exactly like $\hat{x}\,\psi(x) = x\,\psi(x)$ in ordinary quantum mechanics. Here the analogue is

$$\hat{h}_\ell\,\psi(h_\ell) = h_\ell\,\psi(h_\ell).$$

Since h_ℓ is a group element (not a number), this means "matrix element by matrix element":

$$\big(\hat{h}_\ell\big)^A{}_B\,\psi(h_\ell) = \big(h_\ell\big)^A{}_B\,\psi(h_\ell).$$

For the momentum–type operator we have the analogue of $\hat{p}\,\psi(x) = -i\hbar\,\frac{d}{dx}\psi(x)$, namely the left-invariant vector fields $\vec{L}_\ell$. We have one for every group element (i.e., for every link), so the operators are defined link by link.

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6 The Geometrical Operators

In $\mathcal{A}_\Gamma$ we have

$$f(h_\ell), \qquad \vec{L}_\ell.$$

The operator $\vec{L}_\ell$ will now be our main topic. It is the basic operator of the theory. They are the derivatives on the groups and turn variables $h_\ell \in SU(2)$ in three directions of the algebra. It should be noted that both $f(h_\ell)$and$\vec{L}_\ell$ are self–adjoint with respect to the scalar product of the Hilbert space.

$$\begin{aligned}\left[L^i_\ell,\, L^j_{\ell'}\right] = \epsilon^i{}_{jk}\, L^k_\ell\, \delta_{\ell,\ell'} \;\Rightarrow\; &\text{if they are on the same link they commute;}\\ &\text{however, on different links they don't.}\\ &\text{They also don't commute with} f(h_\ell).\end{aligned} \tag{6.1}$$

$$\left(\vec{L}_{\ell_1} + \vec{L}_{\ell_2} + \vec{L}_{\ell_3} + \vec{L}_{\ell_4}\right)\psi = 0 \;\Rightarrow\; \left[\sum_{\ell\in n} \vec{L}_\ell\right]\psi = 0. \tag{6.2}$$

Furthermore, we define the closure equation or the constraint equation seen above. The sum of the links coming out of a node is zero. Similar to Gauss' law. This can be seen in Figure 6.1.

Figure 6.1: The 4 links coming out of a node in the dual space of a tetrahedron.

Action of $\vec{L}_\ell$ and node operators: So basically the operator $\vec{L}_\ell$ acts on the 4 h_ℓ link originating from a node, rotating it a bit in the same direction. It is like multiplying all the 4 by the same group element, which is exactly what gauge invariance does,

$$\psi(h_\ell) = \psi\left(\lambda_{s_\ell}\, h_\ell\, \lambda^{-1}_{t_\ell}\right).$$

Hence $\sum_\ell \vec{L}_\ell\, \psi$ is a generator of gauge transformations at the node n. These operators are defined on $\widetilde{\mathcal{H}}_\Gamma$, not on $\mathcal{H}_\Gamma$. Let us see what we can do on $\mathcal{H}_\Gamma$.

$$A^2_\ell = \vec{L}_\ell \cdot \vec{L}_\ell, \qquad G_{\ell\ell'} = \vec{L}_\ell \cdot \vec{L}_{\ell'}, \qquad V^2_n = L^i_{\ell_1} L^j_{\ell_2} L^k_{\ell_3}\, \epsilon_{ijk}.$$

V_n actually depends only on the node n. If, instead of$\vec{L}_{\ell 1}\vec{L}_{\ell 2}\vec{L}_{\ell 3}$, we use $\vec{L}_{\ell 1}\vec{L}_{\ell 2}\vec{L}_{\ell 4}$, we get the same object (up to a sign) because of the closure/constraint

$$\sum_{\ell\in n} \vec{L}_\ell\, \psi = 0. \tag{6.3}$$

Indeed,

$$\begin{aligned} V^2_n &= L^i_{\ell_1} L^j_{\ell_2} L^k_{\ell_4}\, \epsilon_{ijk} = -\, L^i_{\ell_1} L^j_{\ell_2} \left(L^k_{\ell_1} + L^k_{\ell_2} + L^k_{\ell_3}\right)\epsilon_{ijk} \\ &= -\, L^i_{\ell_1} L^j_{\ell_2} L^k_{\ell_3}\, \epsilon_{ijk}, \end{aligned} \tag{6.4}$$

so $\vec{L}_{\ell 3}$ and $\vec{L}_{\ell 4}$ give the same thing up to a sign. Therefore V^2 is uniquely determined by the node; the sign only depends on whether the parity (orientation) of the triple is positive or negative.

Theorem: The set $\{A_\ell,\, V_n\}$ forms a maximally commuting set of operators in the Dirac space, for all links ℓ and nodes n. This is analogous to the set $\{E, \mathbf{J}^2, J_z\}$ in the hydrogen atom. In other words, A_ℓ

and V_n can be diagonalized simultaneously: there exists a basis in which they are both diagonal, and there is no degeneracy.

Figure 6.2: Geometrical Picture.

Back to the geometrical picture: Assume we have a metric space with a three–dimensional curved Riemannian metric $q_{ab}(x)$. We approximate this metric by a triangulation, taking each triangle to be flat. Curvature is when we go around on the red segment shown in Figure 6.3. It is concentrated on going around loops and is supported on the corresponding (dual) segment. Therefore, if we know the size of each triangle we know the geometry: from the sizes we can compute the angles, and conversely.

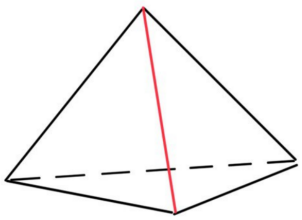

Figure 6.3: Going around the tetrahedron.

If I have the geometry, I can compute the area of each triangle and the volume of the tetrahedron shown in Figure 6.4.

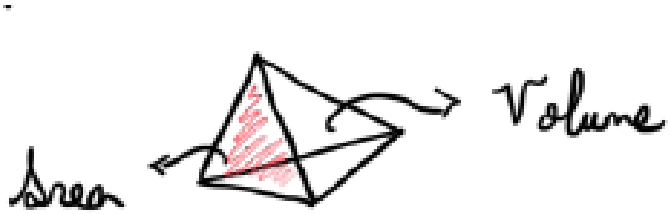

Figure 6.4: Four surface area and one volume for a total of 5 variables.

It turns out that A_ℓ is the operator associated with the area, and V_n is the operator associated with the volume. It is much better to work in GR with tetrads rather than the metric. We will do the same since the metric is not good for fermions. Therefore we write:

$$q_{ab} \longrightarrow e^i_a(x), \qquad a = 1,2,3, \quad i = 1,2,3,$$

$$q_{ab} = e^i_a \, e^j_b \, \delta_{ij}. \tag{6.5}$$

The inverse of $e_a^{\ i}$ is $e^a{}_i$, and it satisfies

$$q_{ab}\, e^a{}_i\, e^b{}_j = \delta_{ij}, \qquad e_a^{\ i}\, e^a{}_j = \delta_{ij}, \tag{6.6}$$

so they are orthogonal with unit length. Thus, instead of writing the geometry as a metric, we write at each point three orthogonal vectors (a tetrad). If you know the tetrad at each point, you know the metric. So, you have a tetrad on each face of the tetrahedron. The amount of changes that occur by going from one surface to another measures the curvature.

$$e^{\,i}{}_a \;\Rightarrow\; e^{\,i}{}_a\, dx^a = e^{\,i} \qquad \text{(a one–form)}. \tag{6.7}$$

$$\int_{\triangle} e^{\,i} \wedge e^{\,j}\, \varepsilon_{ij}{}^k = E^k, \qquad \text{i.e. a 2–form can be integrated over the triangle to give } E^k. \tag{6.8}$$

Assume we are in flat Cartesian coordinates. Then

$$e^{\,i}{}_a = \delta^i{}_a \;\Rightarrow\; \int_{\triangle} e^1 \wedge e^2\, \varepsilon_{12}{}^3 = E^3, \tag{6.9}$$

which is the component along the third direction, normal to the face. The proper covariant way of writing this is

$$\int_{\triangle} e^{\,j}{}_a \wedge e^{\,k}{}_b\, \varepsilon^{abc}\, \varepsilon_{jk}{}^{\,i} \;=\; E^i. \tag{6.10}$$

Hence, we have four such flux vectors for a tetrahedron, satisfying the closure relation

$$\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 = 0. \tag{6.11}$$

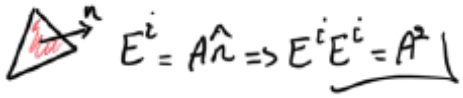

Figure 6.5: Relationship between E^i and the area operator A.

So take the tetrahedron, put it in a bucket of water, and move up and down. You will see that nothing changes. The pressure on the four faces is constant. Hence, the force is normal to the surface–area direction, and since the pressures and surface areas are equal, the forces sum to zero. It's the same as above.

What about the volume?

$$V \;=\; \alpha\, \varepsilon_{ijk}\, E_1^i E_2^j E_3^k, \qquad \alpha \text{ is a constant we have to determine.} \tag{6.12}$$

Take a regular configuration with unit lengths as shown in Fig 6.6.

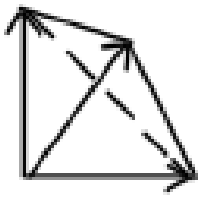

Figure 6.6: The three coordinate lengths are one. So the angle between E^is is $\pi/2$ and the cosines are one.

The area is

$$1 \times 1 \times \frac{1}{2} \;\Rightarrow\; S = \frac{1}{2} \;\Rightarrow\; S^2 = \frac{1}{4},$$

and the volume is

$$1 \times 1 \times 1 \times \frac{1}{3} \;\Rightarrow\; V = \frac{1}{3} \;\Rightarrow\; V^2 = \frac{1}{9}.$$

Therefore,

$$V^2 \;=\; \alpha\, \varepsilon_{ijk}\, E_1^i E_2^j E_3^k \;=\; \alpha\, A^3 \;=\; \frac{9}{8}\, A^3. \tag{6.13}$$

Thus we can describe the geometry of the tetrahedron with the four vectors $\vec{E}$. This mirrors the operators $\vec{L}_\ell$ and the associated V_n^2 and A_ℓ^2: $\vec{L}_\ell$ is attached to links, which are dual to triangles. These are exactly the quantization of E^i. This can actually result from a canonical quantization: If you write the action with a connection, the conjugate momentum of the action turns out to be E^i. So we write

$$E^i_\triangle \;=\; \alpha\, G\, \hbar\, \vec{L}_\ell, \tag{6.14}$$

where α is a numerical constant (to be figured out later).

For $c = 1$ we have $G\hbar\,\alpha \sim \mathrm{m}^2$, hence $E^i \sim \mathrm{m}^2$ (area), which fits dimensionally.

$$\int e^J \wedge e^K\, \epsilon_{JK}{}^i \;=\; E^i_\triangle \;=\; \alpha\, G\, \hbar\, \vec{L}_\ell. \tag{6.15}$$

$$E^a{}_i \;=\; \det(e)\, e^a{}_i \;=\; \epsilon^{abc}\, \epsilon_{ijk}\, e^j{}_b\, e^k{}_c, \tag{6.16}$$

i.e., the densitized triad (the inverse, as it first appeared).

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7 Quanta of Space, Spin Networks & Discreteness

What is the main discovery of QM? *Discreteness*. It is the core: the discrete quanta of the photon, black–body radiation (BBR), atomic spectra, angular momentum in QFT, and the discreteness of the field. In the phase space of classical theory, you cannot localize a system better than a Planck–sized cell. So if you have a region of space, the variables are discrete. If it were continuous, it would allow you to localize space better than the Planck scale. Thus QM is really the discovery of discreteness in nature.

At first, the creators of wave mechanics, such as Schrödinger, hoped to find a way to interpret the discreteness in terms of a continuous wave function, but they were misled. They simply moved it from the continuous evolution of the wave function to the measurement. If you have a particle, the wave function spreads continuously, but when it interacts with the screen it becomes a point. So the initial hope of resolving the discreteness is dead. It not only can't be avoided, but it is what nature is telling us. There is a reason why atoms are stable: because the orbit is discrete and it can't spiral down to zero.

How do you compute discreteness in QM? *Noncommutative algebra.*

$$A \in \Delta, \qquad A\,|\psi\rangle = a\,|\psi\rangle. \tag{7.1}$$

$$\widetilde{\mathcal{H}}_\Gamma = L^2\big[SU(2)^L\big], \qquad \mathcal{H}_\Gamma = L^2\left[\frac{SU(2)^L}{SU(2)^N}\right]_\Gamma. \tag{7.2}$$

So as you remember we had:

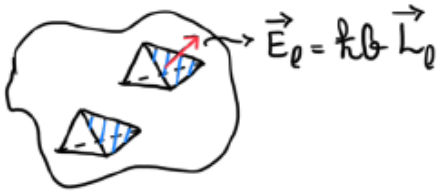

Figure 7.1: The $\vec{L}_\ell$ operator in real space.

Where $\vec{L}_\ell$ is the link between the nodes in the dual space of the graph as can be seen in the following Figure:

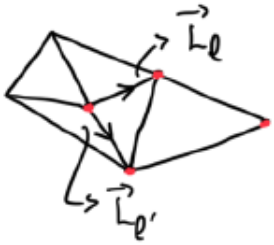

Figure 7.2: The $\vec{L}_\ell$ operator in dual space in which each node has 4 links and the sum of the 4 links is equal to zero.

$$A_\ell^2 \;=\; \vec{L}_\ell \cdot \vec{L}_\ell \qquad \longrightarrow \qquad \text{area.} \tag{7.3}$$

$$V_\ell^2 \;=\; L^i_{\ell_1}\, L^j_{\ell_2}\, L^k_{\ell_3}\, \epsilon_{ijk} \qquad \longrightarrow \qquad \text{volume.} \tag{7.4}$$

$$G_{\ell\ell'} \;=\; \vec{L}_\ell \cdot \vec{L}_{\ell'} \qquad \longrightarrow \qquad \text{scalar product of two links.} \tag{7.5}$$

So, if you have two faces of a tetrahedron you will have the two perpendicular vectors coming out of the two faces, and when you take their scalar product you get the area of each surface and the cosine of the dihedral angle between the two faces (the angle between the two triangles). So if you have the volume, the areas, and the angle, you have the full geometry of the tetrahedron; and if you have the

geometry of all of the tetrahedra, you can reconstruct the space. If you approximate the geometry with big tetrahedra you obtain a rough geometry; if you approximate it with small tetrahedra you obtain a smooth geometry.

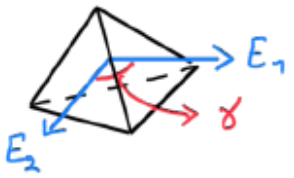

Figure 7.3: The two perpendicular vectors E_1 and E_2 coming out of each face and the diheadral angle γ between them.

For each node, we have three vectors. We actually have four, but due to $\sum_{\ell\in n} \vec{L}_\ell\, \psi(h_\ell) = 0$ we have 3. Every vector has three components, making a total of nine components. We choose an arbitrary basis which is invariant under $SO(3)$ which is a 3–dimensional rotation group. Therefore, from the 9 components (of three vectors) we subtract the 3 common rotations and obtain $9 - 3 = 6$. These 6 are precisely the number of degrees of freedom of the geometry of a tetrahedron: you can specify it either by the 6 edge lengths, or by three vectors with three components each, modulo the 3 common rotations.

Now we will compute the eigenvalues for the area and the volume.

Peter–Weyl decomposition (reminder). Let $D^j_{mn}(h)$ be the Wigner D–matrix elements. Then

$$D^j_{mn}(h) = \langle h \mid j, m, n\rangle, \qquad L^2\big[SU(2)\big] = \bigoplus_j \mathcal{H}_j \otimes \mathcal{H}_j^*, \tag{7.6}$$

so that the D–matrices form an orthonormal basis of $L^2[SU(2)]$. For a graph with L links we have

$$L_2[SU(2)] \;\to\; D^j_{mn} \;=\; \langle\, h \mid j, m, n\rangle, \qquad \mathcal{H}_j \;=\; |j, m\rangle \tag{7.7}$$

$$\begin{aligned} L_2\big[SU(2)^L\big] \;\to D^{j_1}_{m_1 n_1}(h_{\ell_1}) \cdots D^{j_L}_{m_L n_L}(h_{\ell_L}) \;&=\; \langle\, h_\ell \mid j_\ell, m_\ell, n_\ell\rangle = \\ \bigotimes_{\ell=1}^{L}\Big(\bigoplus_{j_\ell} \mathcal{H}_{j_\ell} \otimes \mathcal{H}^*_{j_\ell}\Big) \;&=\; \bigoplus_{\{j_\ell\}} \bigotimes_{\ell=1}^{L} (\mathcal{H}_{j_\ell} \otimes \mathcal{H}^*_{j_\ell}), \end{aligned} \tag{7.8}$$

with basis vectors

$$D^{j_1}_{m_1 n_1}(h_{\ell_1}) \cdots D^{j_L}_{m_L n_L}(h_{\ell_L}) \;=\; \langle \{h_\ell\} \mid \{j_\ell, m_\ell, n_\ell\}\rangle. \tag{7.9}$$

Thus, on each link ℓ we have a basis labeled by j_ℓ and by the magnetic indices m_ℓ, n_ℓ.

Consider a link ℓ. We have the matrix element

$$D^j_{mn}(h_\ell) \;\in\; \bigotimes_\ell \left(\bigoplus_{j_\ell} (\mathcal{H}_{j_\ell} \otimes \mathcal{H}^*_{j_\ell}) \right). \tag{7.10}$$

Here, the first factor $\mathcal{H}_{j_\ell}$ is the space where the index m lives, and the second factor $\mathcal{H}^*_{j_\ell}$ is the space where the index n lives. When you act with gauge transformations, the transformations at the two nodes act on two different indices, m and n.

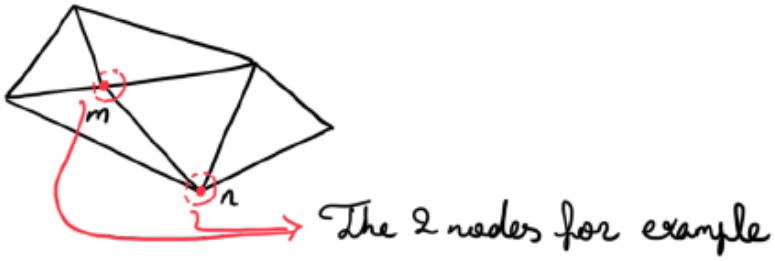

Figure 7.4: The two nodes m and n in the dual space.

When you act on the left and right on $D^j_{mn}(h)$, you are acting on two different indices. Thus the gauge transformation at the source node acts on m, and the one at the target node acts on n. Therefore it is a good idea to regroup all the Hilbert spaces that live on the links by node.

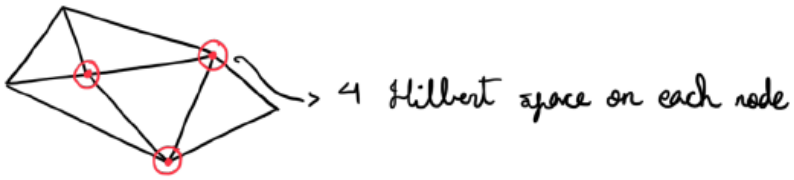

Figure 7.5: We have a 4 Hilbert space on each node.

We have:

$$\langle\, h_\ell \,|\, j_\ell,\, m_{s_\ell},\, n_{t_\ell} \,\rangle \;=\; \langle\, h_\ell \,|\, j_\ell,\, (m_1,\dots,m_4),\, (n_1,\dots,n_4) \,\rangle. \tag{7.11}$$

To say it better:

$$\bigoplus_{j_\ell}\left(\bigotimes_{\ell}(\mathcal{H}_{j_\ell}\otimes\mathcal{H}_{j_\ell})\right) \;=\; \bigoplus_{j_\ell}\left(\bigotimes_{\ell}(\mathcal{H}_{\ell_1}\otimes\mathcal{H}_{\ell_2}\otimes\mathcal{H}_{\ell_3}\otimes\mathcal{H}_{\ell_4})\right), \tag{7.12}$$

where the four links meeting at a node are grouped together.

We are juggling because we want a basis in $\mathcal{H}_\Gamma$, not in $\widetilde{\mathcal{H}}_\Gamma$. We found a basis on $\widetilde{\mathcal{H}}_\Gamma$; now we want a basis of gauge invariant states. So let us construct them—states invariant under $SU(2)$ transformations at the nodes.

$$\widetilde{\mathcal{H}}_\Gamma \;=\; \bigoplus_{j_\ell}\left(\bigotimes_{n}(\mathcal{H}_{j_{\ell_1}}\otimes\cdots\otimes\mathcal{H}_{j_{\ell_4}})\right) \;\Longrightarrow\; \mathcal{H}_\Gamma \;=\; \bigoplus_{j_\ell}\left(\bigotimes_{n}\mathrm{Inv}\big[\mathcal{H}_{j_{\ell_1}}\otimes\cdots\otimes\mathcal{H}_{j_{\ell_4}}\big]\right), \tag{7.13}$$

where Inv[·] denotes the (node) intertwiner space.

$$|\, j_\ell,\, (m_1,\dots,m_4)\,\rangle \in \widetilde{\mathcal{H}}_\Gamma, \qquad \text{(for each node we have four indices)}.$$

We want to write gauge–invariant states, which are only a subset of $\widetilde{\mathcal{H}}_\Gamma$; hence we will take suitable linear combinations of these basis states.

$$|\psi\rangle \;=\; \sum_{\{m_1,\dots,m_4\}_n}\left(\prod_{n=1}^{4} i_{(n)}^{m_1\cdots m_4}\right)|\, j_\ell,\, (m_1,\dots,m_4)\rangle \;\equiv\; |\, j_\ell,\, k_n\rangle, \tag{7.14}$$

where $i_{(n)}^{m_1\cdots m_4}$ is a coefficient (an intertwiner) for each node n. This state is gauge invariant only if it is

invariant under local rotations of the setup. If $(m'_1, \ldots, m'_4)$ is invariant, then $i^{m'_1 \cdots m'_4}$ are invariant. In particular, the intertwiners must satisfy the $SU(2)$-invariance condition

$$D^{j_1}(h)^{n_1}{}_{m_1}\, D^{j_2}(h)^{n_2}{}_{m_2}\, D^{j_3}(h)^{n_3}{}_{m_3}\, D^{j_4}(h)^{n_4}{}_{m_4}\, i_{(n)}^{m_1 m_2 m_3 m_4} \;=\; i_{(n)}^{n_1 n_2 n_3 n_4}, \qquad \forall\, h \in SU(2). \quad (7.15)$$

If (7.15) holds, the state (7.14) is gauge invariant. To obtain a basis of gauge–invariant states we pick, at each node, an independent set of solutions to (7.15). For a four–valent node this can be built by recoupling two three–valent intertwiners:

$$i^{m_1 m_2 m_3 m_4} \;=\; i^{m_1 m_2 m}\, g_{m m'}\, i^{m' m_3 m_4}, \quad (7.16)$$

where $g_{mm'}$ is the invariant metric in the intermediate representation. For example, when all four spins are $j_1 = j_2 = j_3 = j_4 = \frac{1}{2}$, the space of four–valent intertwiners is two–dimensional and can be labelled by the intermediate spin $k = 0, 1$.

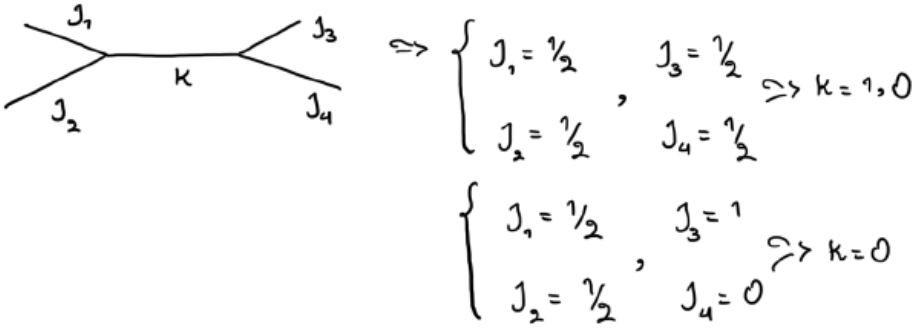

Figure 7.6: Four spin intertwiner example.

Now let's solve a problem. The area operator on a link is

$$A_\ell^2 \;=\; \vec{L}_\ell \cdot \vec{L}_\ell, \quad (7.17)$$

which is diagonal under the Peter–Weyl representation.

It has a value in each representation, and the value is $j(j+1)$.

$$A_\ell^2\, |j_\ell, \kappa_n\rangle = J_\ell(J_\ell + 1)\, |j_\ell, \kappa_n\rangle = \alpha^2\, (\hbar G)^2\, J_\ell(J_\ell + 1)\, |j_\ell, \kappa_n\rangle. \quad (7.18)$$

$$A_\triangle\, |j_\ell, \kappa_n\rangle = \alpha\, (\hbar G)\, \sqrt{J_\ell(J_\ell + 1)}\, |j_\ell, \kappa_n\rangle, \qquad \hbar G \sim \ell_{\rm P}^2 \approx 10^{-66}\, {\rm cm}^2. \quad (7.19)$$

The area operator is completely blind to intertwiners (κ_n). It only sees the spins associated with the links (j_ℓ) and returns the eigenvalue shown above.

Thus we observe that the area has a discrete spectrum: you cannot have an arbitrarily small area. Quantum mechanics tells us that, when constructing space, you cannot go arbitrarily small with the tetrahedra you have.

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8 Volume Operator & Quantum Space

$$V^2 \;=\; \epsilon_{ijk}\, L^i_{\ell_1}\, L^j_{\ell_2}\, L^k_{\ell_3}$$

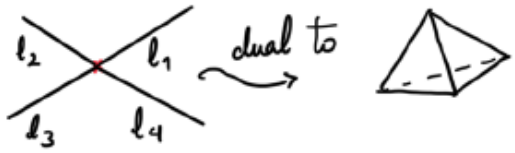

Figure 8.1: Four–valent node, dual to a tetrahedron.

$\vec{L}_\ell$ is the rotation generator on link ℓ. Its action on Wigner D-matrices is (acting on the right index)

$$\vec{L}\, D^j_{mn}(h) \;=\; D^j_{mn'}(h)\, \vec{L}^j{}_{n'\,n}, \quad \text{acting on this index.} \tag{8.1}$$

$$|\psi\rangle \equiv |j_\ell, \kappa_n\rangle \;=\; \sum_{m_1,\ldots,m_4} i^{m_1\cdots m_4}_{n_1} \ldots i^{m_1\cdots m_4}_{n_4}\, |\, j_\ell, (m_1,\ldots,m_4)\rangle, \tag{8.2}$$

and the operators $\vec{L}_{\ell_1}, \vec{L}_{\ell_2}, \vec{L}_{\ell_3}$ act on the three indices living on the same node. In words: $\vec{L}$ acts on these indices.

The volume acts only on the node indices $i^{m_1,m_2,m_3,m_4}_{n_i}(m_1,\ldots,m_4)$; hence the volume is a linear operator on the finite–dimensional Hilbert space

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4},$$

which contains as a linear subspace the *intertwiner space*. Since it is gauge–invariant, it transforms the intertwiner to itself. So this can be done explicitly and we write it as:

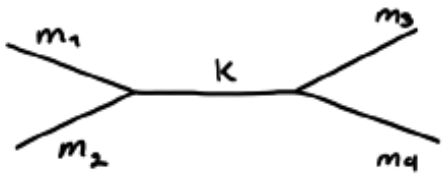

Figure 8.2: You have a matrix acting on the four branches. So you have to take a linear combination of similar intertwiners.

The linear combination of similar intertwiners can be seen in the following Figure

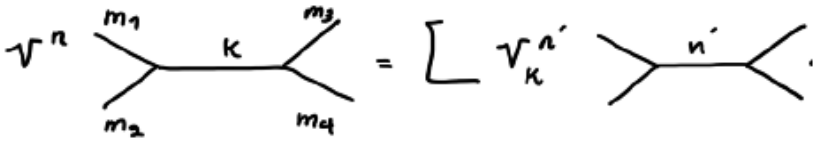

Figure 8.3: Similar intertwiners. The calculations are tedious. Look at the book of *Carlo Rovelli* and *Francesca Vidotto* on the representation on SU(2)

To diagonalize then means to diagonalize a matrix, as in standard practice. The space becomes large for big j's. There are software packages to compute the eigenvalues of the volume. Instead of the basis

$|j_\ell, \kappa\rangle$ we can use $|j_\ell, V_n\rangle$, where we write

$$|V\rangle \;=\; \sum_k c_k^V \,|k\rangle. \tag{8.3}$$

Nothing particularly interesting comes out of this, but the point is that these are finite–dimensional spaces, so the eigenvalues are discrete.

So we have: $|j_\ell,\, V_n\rangle$ — discrete quantum numbers.

The area and the volume have discrete eigenvalues. If you diagonalize a matrix in a finite–dimensional space, the eigenvalues are discrete. This means that we cannot have a tetrahedron with an arbitrarily small area or volume. There is also something more interesting: the number of degrees of freedom of a tetrahedron is six. What we write for each tetrahedron is one *volume* quantum number and four *area* quantum numbers:

$|j_\ell,\, V_n\rangle$, where V_n is the volume quantum number and the j_ℓ are the area quantum numbers.

So what is missing? A typical example from QM would be:

$$(L_x,\, L_y,\, L_z) \;\Rightarrow\; \text{you can only diagonalize two, e.g. } (L_z,\, L^2),\; \text{not all three.}$$

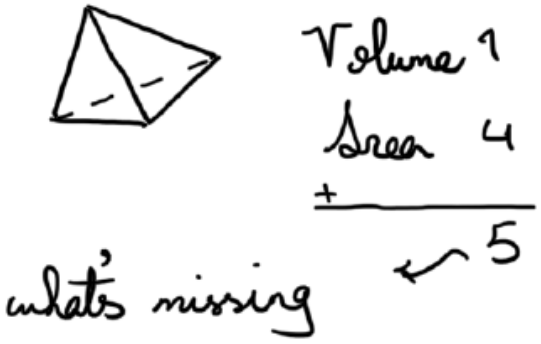

Figure 8.4: 1 Volume and 4 surface areas satisfy the 5 out of 6 degrees of freedom of a tetrahedron.

For a macroscopic object you can think of the particle as having a vector. But for a small object such as an electron, we can't treat it in the same manner. This is because it does not have a fixed angular momentum. In one direction, the angular momentum is sharp, and in the other directions, it is spread, so it would be more complicated. The same holds here in the sense that you have 6 variables describing a tetrahedron, but only 5 variables describing the system.

Of course, there are other variables that can describe the system. For example, you can fix the angle by

$$G_{\ell\ell'} \;=\; \vec{L}_\ell \cdot \vec{L}_{\ell'}. \tag{8.4}$$

But in this case, some of the other variables in $|j_\ell, V_n\rangle$ will not be sharp anymore. So you can fix four areas and one volume, or you can fix four areas and one dihedral angle. By doing the latter—and fixing one dihedral angle on the base—we can also know the other angles on the base, but we will not know the angle between the three triangles on top of the base with one another. That means that all the tetrahedra that have been drawn are not exact tetrahedra. They are fuzzy ones—just like the electron, which is a fuzzy rolling stone.

Quantum Aspects of the Geometry

1. **Discreteness:** $|j_\ell, V_n\rangle \;\Rightarrow$ the eigenvalues of the area and the volume are discrete.

2. **Fuzziness:** The geometry of the elementary operators does not commute with one another.

3. **Superposition of geometries:** The possible states of the geometry are linear combinations of the operators ⇒ a particle can go through two holes of the double–slit experiment at the same time, since position is a quantum variable that can be spread. The same holds for geometry. We therefore have a linear combination of the geometry in which it can be spread.

All these are explained and described by the Hilbert space and algebra operators. The spin–network state $|j_\ell, V_n\rangle$ can be described by the following graph.

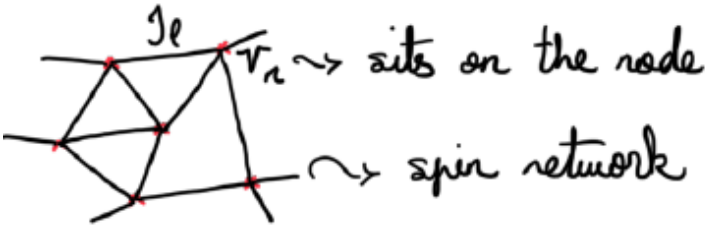

Figure 8.5: The spin network and the V_n operator that sits on the nodes.

The basis of volume eigenstates is a natural basis, since it diagonalizes some good macroscopic observables, but it is not easy to work with. It is hard to compute in that basis; the easier basis to work with, is the following

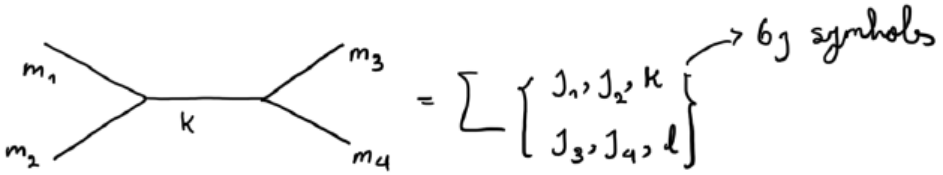

Figure 8.6: The basic intertwiner basis to work with. The linear coefficients come from standard $SU(2)$ representation theory. Six numbers can be constructed out of three–dimensional intertwiners, which allow you to perform a change of basis, as we observed.

This basis is easy for calculations, but it's not unique, in the sense that if you have a 4-valent intertwiner you can write it in different ways, such as

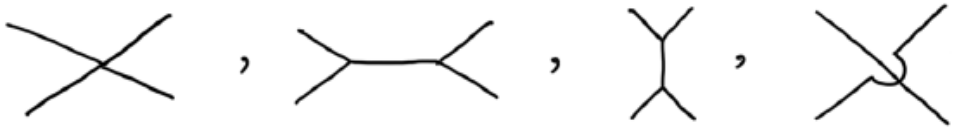

Figure 8.7: Splitting the basis in different ways.

The geometrical interpretation of Fig. 8.8 is the *dihedral angle* on the tetrahedron. The dihedral angle is the scalar product of the two normal vectors on the surface, $(\vec{L}\cdot\vec{L}')$ devided by the area. So what we have is that the object labeled by k is an eigenstate acting on the indices m_1, m_2. Hence k is actually the quantum number associated with the dihedral angle. Thus, one can write states where we specify the four faces and one dihedral angle. To change basis from one to the other we apply the procedure described above; in this way we obtain another dihedral angle from the same construction.

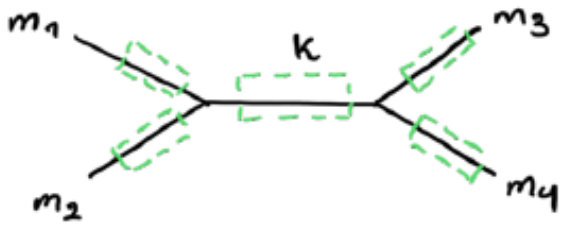

Figure 8.8: Geometrical interpretation.

How do we write a state that is not too spread over either of the dihedral angles? The most classical thing we have done in the past is to write wave packets in which we have x and p minimally spread, although they cannot both be sharp. For example, we can write a wave packet for a harmonic oscillator with x and p minimally spread. Can we do the same here? Yes: there is already a theory of *coherent states* in $SU(2)$, developed by mathematicians. Thus the desired state will be a linear combination of the states $|j_\ell, V_n\rangle$ that describe the classical geometry as well as possible. One takes a classical geometry and discretizes it (the geometry is a tetrahedron). Then we have the six quantities we want, and we write states that are as minimally spread as possible, and the quantities have all the expectation values you want. This is described in detail in the book.

Intuitively, we can think of all the nodes as not being sharp—minimal elements of geometry. Thus spacetime, in the Newtonian sense, is not continuous but quantum-like; on top of it, matter lives. The adjacencies are what the links encode. The quantum numbers describe how large they are in the geometric sense. This has been shown in Fig. 8.9.

Figure 8.9: The nodes are not sharp but minimally spread in spacetime.

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9 Penrose Spin Network, Clebsch–Gordan, ...

1. **Penrose spin network:** "evaluation of a spin network" and explaining what it is.
2. What is the metric ϕ^j_{nm} in the Hilbert space?
3. Relation between the Clebsch–Gordan coefficients $\langle JM \mid j_1 m_1; j_2 m_2\rangle$ and intertwiners. The C–G coefficients allow two particles with spins j_1, j_2 and magnetic numbers m_1, m_2, and decompose the composite system into a representation J with total angular momentum M.
4. What is the numerical coefficient α in $\vec{E}_\ell = \alpha G\hbar \vec{L}_\ell$?
5. What is the exact interpretation of the diagonal operator $f(h_\ell)$?

1 - So what are Penrose spin networks, and why did he come up with them? First, Penrose considered trivalent graphs, as shown

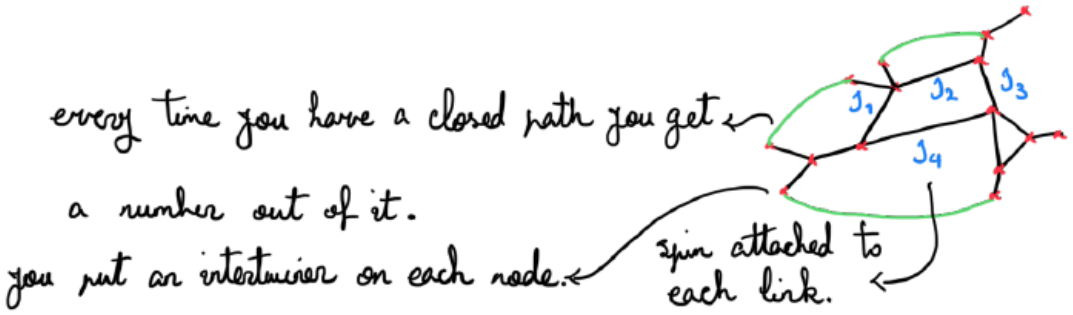

Figure 9.1: Penrose trivalent graph.

To get a number, we contract all the intertwiners—this is the *evaluation* of the spin network, $i^{m_1 m_2 m_3} \ldots i^{m'_1 m'_2 m'_k}$. Furthermore, if you have some open legs instead of closed ones, you obtain, for example, two vectors in two different representations. These behave like vectors, and Penrose proved a number of theorems showing that, with this structure, one can construct a three–dimensional geometry.

So why did Penrose do this? You can obtain three–dimensional geometry more precisely by taking larger and larger j—that is, a bigger and bigger spin network. At first, one hoped to formulate QM as a continuous theory, but discreteness is unavoidable because of measurement. Thus, the idea that space is discrete at small scales has led to much research in quantum gravity. Penrose, therefore, tried to write a discrete version of a three–dimensional space that still has all the symmetries of a 3-D space. From a classical point of view, this is impossible because once you discretize, you break continuous symmetries. But quantum mechanics allows you to do that, in the sense that it's the representation theory of a rotation group: the group is continuous, but L_z, L_y, L_x have eigenvalues. So it is discrete in all directions, but in a funny way in which, when you rotate, the eigenvalues remain discrete yet you continuously change the probability of having one or the other. So Roger tried to recover three–dimensional Euclidean space from something discrete in the QM sense. We know how to do it for a single vector (angular momentum), but can we do it for the whole space? So he created a set of things called *tensor networks*.

LQG was developed in a completely different manner, starting from GR. Performing the canonical analysis of GR starting with Dirac, the loop was embedded in the 3–dimensional manifold. Then it was realized that the states inside the manifold must intersect, and what was important was the *graph* and not the loop. Then it was proven that the graph could be compressed, irrespective of where it was in the manifold. That gave the idea to forget the manifold and just give the graph. In the graph there were the operators acting (volume and area). It was realized then that the graph and eigenstates were behaving exactly like the Penrose networks.

So how is it related to the spin–network states? We have $|\Gamma, j_\ell, i_n\rangle$,

$$\begin{cases} \Gamma & \text{graph,} \\ j_\ell & \text{spin on the link, like a spin network,} \\ i_n & \text{since it can be more than trivalent, we should specify which} \\ & \text{intertwiner sits on the nodes.} \end{cases}$$

These are elements of the Hilbert space written as follows:

$$\psi_\Gamma(\{h_\ell\}) \;=\; \langle \{h_\ell\} \mid \Gamma, j_\ell, i_n \rangle \;=\; \underbrace{\bigotimes_\ell D^{j_\ell}_{m_\ell n_\ell}(h_\ell)_\Gamma}_{\text{the state of } h_\ell} \quad \underbrace{\bigotimes_n i_n}_{\substack{\text{contracted by} \\ \text{the product} \\ \text{of intertwiners}}} \quad . \tag{9.1}$$

So what is the Penrose evaluation of the spin network? It is contracting the intertwiners with the identity instead of $D^{j_\ell}_{m_\ell n_\ell}(h_\ell)$. The value of the state on the identity is the Penrose evaluation of a spin network. Therefore, the evaluation of the spin network is $\mathrm{Ev}(\Gamma, \mathbf{n}) = \psi_\Gamma(\mathbf{1})$. In Penrose's theory, you only have graphs with labels and a single number as the outcome, but here in LQG we have graphs with labels, each giving a function on $SU(2)$. A 4–valent intertwiner can be thought of as two 3–valent intertwiners. Be careful: they are not completely the same.

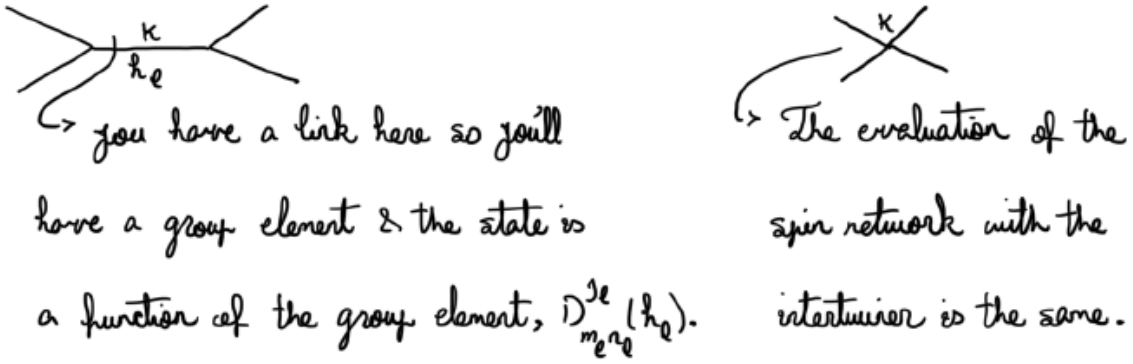

Figure 9.2: 4-valent intertwiner.

These two, as a state, are not the same thing. This is because a state with a 4-valent intertwiner does not depend on a group element in between. If you have one point, there is nothing to go from one to the other, and the state does not depend on the group element. If you have two points, there is some degree of freedom in how to go from one point to the other.

2 - What is $\Phi^j_{mn} = \delta_{m,-n}\,(-1)^{j-m}$?

We have:

$\mathcal{H}_j$ (representation space), $\quad v^m$ (components of a vector), $\quad |j,m\rangle$ (basis of the j representation).

A generic vector can be written as

$$|V\rangle = \sum_m v^m\,|j,m\rangle. \tag{9.2}$$

There are two quadratic operations one can define. Namely, there are two ways to map $\mathcal{H}_j \otimes \mathcal{H}_j$ into the complex numbers; these should not be confused with one another.

First, $\mathcal{H}_j$ is a Hilbert space, and the matrices that transform $|j,m\rangle$ into one another are unitary with respect to a scalar product. Hence there is a map from a pair of states to a scalar product, with $\{|j,m\rangle\}$ orthonormal:

$$\boxed{\langle j,m \mid j,m'\rangle = \delta_{mm'}\ , \qquad \langle v \mid w\rangle = \bar v^{\,m} w^n\,\delta_{mn} = \sum_m \bar v^{\,m} w^m} \qquad \text{(one quadratic structure).}$$

Think about angular momentum,

$$\mathcal{H}_i \otimes \mathcal{H}_j \;=\; \bigoplus_{k=|i-j|}^{i+j} \mathcal{H}_k, \qquad \mathcal{H}_0 \cong \mathbb{C}$$

(this is a complex number, i.e. the trivial representation). Given two vectors, instead of the usual scalar product we can decompose a *bilinear* form into components, with one component invariant under $SU(2)$. This is not an inner product (it is linear in both, not antilinear):

$$(v, w) \;=\; v^m w^n \, g_{mn}, \tag{9.3}$$

where g_{mn} is a 2-index matrix that makes the form invariant under rotations.

If you get confused, think about irreducible representations of $SU(2)$ as symmetric tensors of fundamentals. If you think of a symmetrized tensor, you can write it as,

$$\mathcal{H}^j \;\cong\; \underbrace{z^2 \vee \cdots \vee z^2}_{2j} \qquad \text{(spinor with } 2j \text{ indices)}, \tag{9.4}$$

there is a corresponding version of g in the following form,

$$g_{a_1 \ldots a_{2j},\, b_1 \ldots b_{2j}}, \tag{9.5}$$

which is (v, w) in that basis.

$$g_{a_1 \ldots a_{2j}\, b_1 \ldots b_{2j}} \;=\; \epsilon_{a_1 b_1}\, \epsilon_{a_2 b_2} \cdots \epsilon_{a_{2j} b_{2j}} \qquad \Longrightarrow \qquad \epsilon \text{ is the only invariant tensor under } SU(2).$$

These $\epsilon_{a_1 b_1}, \ldots, \epsilon_{a_{2j} b_{2j}}$ translate into

$$\Phi^j_{mn} \;=\; \delta_{m,-n}\, (-1)^{\,j-m}.$$

Landau presents another viewpoint. If you have a state with $L_z = m$ and you want to combine it with another state in the fundamental, note that in the fundamental L_z has only eigenvalue zero. So the only possibility of putting a particle with angular momentum j together with something that gives angular momentum j is to combine it with something with the opposite angular momentum. The 3–valent intertwiner can be written as

$$i^{\,m_1 m_2 m_3} \;=\; \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \qquad \text{(Wigner } 3j \text{ symbol)}.$$

For four legs one may write

$$i^{\,m_1 m_2 m_3 m_4}_{\;\;k} \;=\; \begin{pmatrix} j_1 & j_2 & k \\ m_1 & m_2 & m \end{pmatrix} g_{mn} \begin{pmatrix} k & j_3 & j_4 \\ n & m_3 & m_4 \end{pmatrix},$$

where k is the intermediate spin. The range of k is fixed by the consistent Clebsch–Gordan conditions together with m_1, m_2, m_3, m_4.

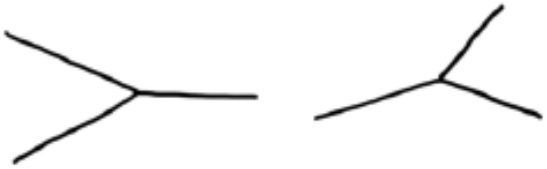

Figure 9.3: You have to dualize in an invariant manner (not a scalar product). So we have a g in between.

3 - What is the relation between $i^{m_1 m_2 m_3}$ and the Clebsch-Gordon coefficient?
They are exactly the same up to a g

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \;=\; g_{m_3 m} \left\langle j_3\, m \,\middle|\, j_1\, m_1,\, j_2\, m_2 \right\rangle. \tag{9.6}$$

Look for books on atomic and nuclear physics where $SU(2)$ representation theory is used.

Figure 9.4: Comparison between the Clebsch-Gordon coefficient and the intertwiner.

4 - $\vec{E}_\ell = \alpha\, \hbar G\, \vec{L}_\ell$, what is α?

$$\alpha \;=\; 8\pi\gamma \qquad \text{(Barbero–Immirzi parameter)}.$$

By doing the canonical analysis we get the value (more on this later).

5 - What is $f(h_\ell)$?

h_ℓ is a rotation when you go from the frame of one tetrahedron to the frame of another tetrahedron (this is an incomplete interpretation). More on this ahead. It is the holonomy of a three–dimensional $SU(2)$ connection, but it is not uniquely determined by the triad.

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10 Time

The confusion about time is because of mixing different notions. So the first distinction about the notion of time is **When**?

1. **Aristotle** Each time we answer this question, we refer to things happening at the same time. We locate events one with respect to the other. This is the way Aristotle defined time: "Time is the order of change with respect to before and after." So it is like space; it is a way of numbering change, and if nothing happens, there is no time. There is always a process going on, and what we call time is this process, according to Aristotle. So time is numbering processes and events.

2. **Newton** Newton states that there is something more, which is called the absolute time. It is a variable that passes regardless of any change. If everything, including your thought processes, freezes, Newtonian time still marches ahead. Newton states that it is not (1) or (2) alone, but that both notions have to come together, although Newtonian time is not directly observable. Newtonian time is a mathematical abstraction derived from the equations, such that when we parametrize everything with respect to time t, systems follow the Newtonian equation $F = ma$.

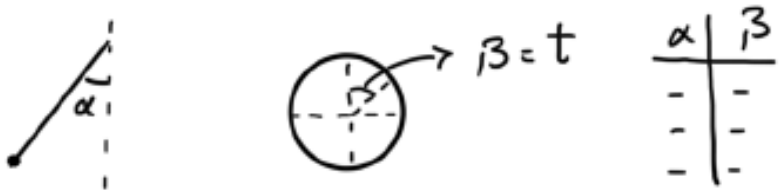

Figure 10.1: Evolution of α and β with respect to one another.

What is the evolution of $\alpha(t)$? What we actually mean is the evolution of α with respect to β. We don't see time; we see the change in the angle, $\alpha(\beta)$. A clock is the closest to Newton's ideal time. We have $\alpha(t)$ and $\beta(t)$, and we write them with respect to one another as $\alpha(\beta)$. Newtonian time is approximated by clocks.

3. **1905 (SR)** It merges Newtonian time and space, bringing them together in Minkowski spacetime.

4. **1915 (GR)** Minkowski spacetime is a gravitational field, $g_{\mu\nu}$. A clock is a geometrical object along its worldline, reading a feature of the gravitational field

$$T = \int_\gamma \sqrt{g_{\mu\nu}\, dx^\mu\, dx^\nu}\,. \tag{10.1}$$

So Newton's intuition was right: the clock measures an entity of the world—namely $g_{\mu\nu}$, the gravitational field—not something external. The question now is: if T is an entity of the world, why does it behave so differently? Why can we go back and forward in space but not in time? Aspects of time not captured by the above are *irreversibility* and the fact that it *flows*.

5. **Irreversibility** This is not in mechanics, but in thermodynamics. All phenomena with no heat exchange are reversible. Heat has to do with microscopic variables, so it is related to statistical mechanics. We do understand the second law of thermodynamics—except the reason why entropy was lower in the past. If you have a statistical system and look at a macrostate realized by a huge number of microstates, then, if you evolve with the classical equations of motion, almost all of them grow in entropy both toward the future and toward the past. Going forward into the future sounds right, but what is annoying is: why doesn't the same thing happen in reality when we go backward? If you take a compressed box of gas, it expands. If you take a picture midway and, based on the picture, infer where the gas is going, it is expanding. But, based on the same picture, the backward equations of motion tell us it will also be expanding. So that state is very

interesting because it comes from a low–entropy state. Why was it low in entropy? Because I was low in entropy; why was I low in entropy? Because ... and it goes on like this. The more you go backward, the more you are in an improbable situation.

Questions such as "why do we remember the past but not the future lies in the entropy. It's not, of course, obvious. One can show that, in an entropy gradient, you can have traces of the past but not the future. All the phenomenology around us is directed by this, including QM. We state that the quantum state changes at measurement: we measure $+z$ at some time and say that the state "remembers" it until a future measurement. This is perspectival (relative to the observer). QM doesn't distinguish between the past and the future. If we have a sequence of Stern–Gerlach measurements and repeat them many times and get the results, we cannot know the direction of time. QM is blind to the direction of time. You cannot know the future; you cannot either predict the past. You have a probabilistic relation for the past as well. The reason you can make better predictions about the past is that there are traces, due to the fact that entropy was lower in the past. Since we have memories, we have traces, and therefore we perceive the past as fixed—while this is only a matter of perception.

6. **Flows** The way that the flow can be understood is by understanding how the brain works. This has nothing to do with QG; it has to do with the brain, thermodynamics, and statistical mechanics.

   In GR you have processes and events, and you number them by the proper time that runs along a worldline (there is no common time in GR, no simultaneity, no direction of time). In classical theory an initial value is given, and we see how the variables evolve with respect to one another, not with respect to time. In quantum theory, by knowing the value of some quantities we can predict the values of other quantities, and how they are related to one another, which may be parameterized by the proper time.

Now lets formulate what we have said until now

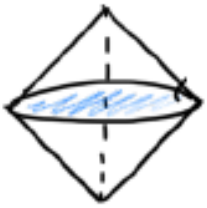

Figure 10.2: Light cone.

If we know the state in the blue region of Fig 10.2, we can uniquely predict everything inside the light cone, future and past.

Another way of looking at this

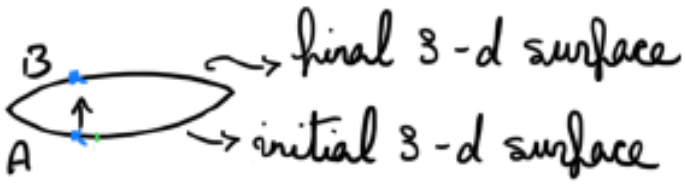

Figure 10.3: Transitioning from A to B.

Looking at Fig. 10.3, classically given the data in A we can predict the data in B. In QM, given the data in A we have a probabilistic prediction of the data in B. This is how we will work: we have quantum states on the initial and final boundary surfaces, and an amplitude that gives the probability of the transition,

$$\langle w \,|\, (\,|i\rangle \otimes |f\rangle\,)\rangle. \tag{10.2}$$

The spin network sits on the boundary surface which points A and B lie, and we have to find and compute the amplitude related to these spin networks.

The space of initial and final states is called the *boundary Hilbert space*, and the dynamics assigns an amplitude to any boundary. The dynamics is a bra on the given Hilbert space.

$$\langle w \underbrace{|\,(\,|i\rangle \otimes |f\rangle\,)\,}_{\mathcal{H}_boundary} \rangle.$$

Where is time? We do not need time. Time is encoded in the variables written on the initial and final boundary surfaces, which include the gravitational field, and any clock is a function of the gravitational field.

The state tells you where the particle is in Minkowski spacetime and how much time T it takes for it to go from "down" to "up". So let's do it in GR. We have to specify the fields all around space on the boundary. If we have the gravitational field, the distance is given by the value of the metric along a path γ. Time is what was written before along the green path on the left,

$$T = \int_\gamma \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}\,.$$

So we do not need to give a time or a position; we just need to give a gravitational field and the position of the particle with respect to the gravitational field. So we are going to give a spin network and specify where the particles are on the spin network. The spin network *is* the spacetime. The bra $\langle W|$ can be thought of as the path integral over all the spacetimes that are possible inside. Keep in mind that the words "initial" and "final" are a bit misleading.

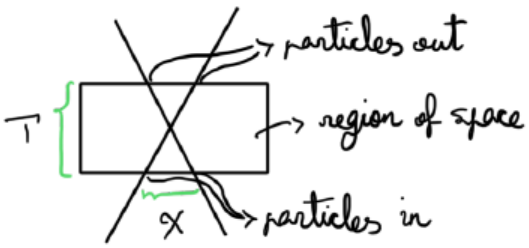

Figure 10.4: The spacetime structure in GR.

Hamilton function. It is a function of variables defined as

$$q(t) \;\longmapsto\; S[q] \;=\; \int dt\; \mathcal{L}\big(q(t), \dot{q}(t)\big), \qquad \mathcal{L} = \mathcal{L}(q, \dot{q}).$$

Varying the action and setting the variation to zero gives the equations of motion.

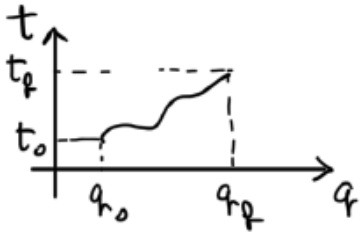

Figure 10.5: Path Integral.

The solutions can be determined by the boundary conditions q_0, $\dot{q}_0$ or q_f, $\dot{q}_f$.

$$S(q_0, t_0;\, q_f, t_f) \;=\; \int_{t_0}^{t_f} \mathcal{L}\big(q(t), \dot{q}(t)\big)\, dt, \qquad \text{with } q(t_0) = q_0,\; q(t_f) = q_f. \tag{10.3}$$

It is a *function of the variables*, not of the path.

Given the four numbers $(q_0, t_0; q_f, t_f)$ we can solve the equations of motion. This is called the *Hamilton function*. "Mr. Lagrange's function defines the mechanical problem; Mr. Hamilton's function solves the problem." — Hamilton (in his paper).

We have the following Hamilton–Jacobi relations:

$$\frac{\partial S(q_i, q_f; t_i, t_f)}{\partial q_{i,f}} \;=\; p_{i,f}\big(q_i, t_i;\, q_f, t_f\big) \;\to\; q_f \;=\; q_f\big(q_i;\, p_{i,f};\, t_i, t_f\big) \quad \text{(invert to use initial data)}. \tag{10.4}$$

$$\frac{\partial S(q_i, q_f; t_i, t_f)}{\partial t_{i,f}} \;=\; -\,E_{i,f}. \tag{10.5}$$

In the Hamilton function, the two variables q and t are treated on the same footing. The Hamilton function does not require us to specify which variable is "the time variable." In GR, time is a clock: it is an integral of $g_{\mu\nu}$, hence it is hidden in the components of $g_{\mu\nu}$. We do not have it separated from the other variables. What Einstein's equations give us is how the various components of $g_{\mu\nu}$ evolve together. In the Lagrangian formulation, q and t have two completely different statuses: t is a parameter with respect to which you evolve, and q is another parameter. So it is an evolution of q and t.

How can we find the Hamilton function without going through the Lagrangian? Another property of the Hamilton function is that it is a solution of the Hamilton–Jacobi equation:

$$\frac{dS}{dt} \;=\; H\left(q, \frac{\partial S}{\partial q}\right), \tag{10.6}$$

which can also be written as the constraint

$$C\left(q, t, \frac{\partial S}{\partial q}, \frac{\partial S}{\partial t}\right) = 0 \;\implies\; C(q, t, S_q, S_t) = S_t - H(q, S_q)\,. \tag{10.7}$$

The transition amplitude in QM in the semiclassical limit is

$$W \;\propto\; e^{\,iS/\hbar}. \tag{10.8}$$

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11 The Structure of Timeless Mechanics

Newtonian and special relativistic physics always have a time variable $t \in \mathbb{R}$. It is, more or less, what a clock measures. We also have a set of variables in the configuration space:

$$q_i \in \mathcal{C} \qquad \text{(configuration space)}.$$

So the game in dynamics is: how do they vary in time?

$$q_i(t), \qquad (q_i, p_i) \in \Gamma \qquad \text{(phase space)}.$$

Dynamics can be given in many ways; one is via the Hamiltonian:

$$\text{Hamiltonian} \ \to \mathcal{H}(q_i, p_i).$$

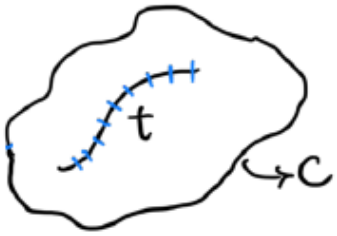

Figure 11.1: The configuration space $\mathcal{C}$ contains the generalized coordinates q_i; their trajectories $q_i(t)$ describe the canonical motion, with t denoting the clock time.

In quantum theory, $q_i(t)$ become operators $(\hat{q}_i, \hat{p}_i)$ forming a non-commutative algebra, and they have eigenstates. Let $|q_i\rangle$ denote an eigenstate. The quantum dynamics is captured by the transition amplitude

$$W\big(q_i^{\text{in}}, t_{\text{in}}; q_i^{\text{f}}, t_{\text{f}}\big) = \big\langle q_i^{\text{f}} \big| e^{-\frac{i}{\hbar}(t_{\text{f}} - t_{\text{in}})\,\mathcal{H}} \big| q_i^{\text{in}} \big\rangle. \tag{11.1}$$

The transition amplitude can be written as a path integral over histories. In the limit in which one takes $\hbar$ to be small, the functional integral is dominated by classical trajectories, and the dominant trajectory is the classical one. Therefore,

$$\big\langle q_i^{\text{f}} \big| e^{-\frac{i}{\hbar}(t_{\text{f}} - t_{\text{in}})\,\mathcal{H}} \big| q_i^{\text{in}} \big\rangle \ \sim \ \exp\left[\frac{i}{\hbar}\, S\big(q_i^{\text{in}}, t_{\text{in}}; q_i^{\text{f}}, t_{\text{f}}\big)\right], \tag{11.2}$$

where S is the classical action of the physical trajectory. This concludes the summary of classical and quantum mechanics when time is present.

Now, going forward: if you have two clocks showing the same time and then you throw one up and catch it again, they will not be showing the same time. Which one is showing the right time? Which one is the real time? The one going along the geodesic is the one that was thrown up and has come down. This is the one running slower.

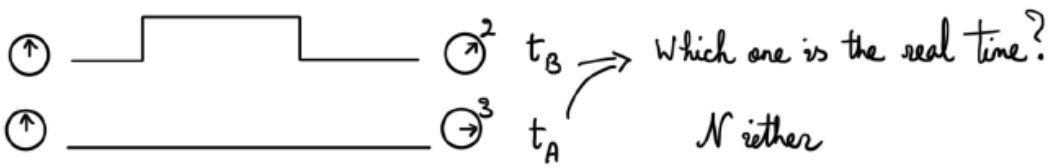

Figure 11.2: Determining the real time. Two initially synchronized clocks follow different worldlines—one is launched upward and later retrieved, the other stays on the ground—and, upon reunion, they register different times (they have accumulated different proper times).

There is not a single time in GR. As in the classical case, we do not have a preferred time $t \in \mathbb{R}$. The Aristotelian time—time understood as locating events with respect to other events—is still there (we are talking about physics, mainly events), but these events and processes do not have a single variable t to label them in general. Therefore, we need a QG formulation of QM in which we do not start from a preferred Newtonian time, because in GR there is no preferred time. So the stated structure doesn't work and needs to be changed. We won't change it a lot; we'll make it a bit more general so that we can treat t_A and t_B on equal grounds (or, more generally, treat a set of variables on equal grounds). To do this, we take the time variable and put it on the same footing as q_i (the other variables).

We start from an *extended* configuration space,

$$q_a \in \mathcal{C}_{\text{ext}} \quad \longleftrightarrow \quad (q_i, t).$$

We will not distinguish between q_i and t. Thus we have

$$q_i \in \mathcal{C}, \qquad i = 1, \dots, N \quad \text{(number of degrees of freedom)},$$

$$q_a \in \mathcal{C}_{\text{ext}}, \qquad a = 1, \dots, N+1.$$

The *extended phase space* is

$$(q_a, p^a) \in \Gamma_{\text{ext}}. \tag{11.3}$$

If there is a Newtonian time, we would have a point of extended phase space of the form

$$\Gamma_{\text{ext}} \ni (q_i, t, p_i, \underbrace{p^t}_{-E}). \tag{11.4}$$

Now, the dynamics in this new language is not given by the Hamiltonian as in the previous case, but rather by an extended function on the phase space.

$$C(q_a, p^a) \approx 0 \tag{11.5}$$

is the "relativistic Hamiltonian" (the Hamiltonian constraint).

How do we do the dynamics now? We start from the Hamilton–Jacobi relation

$$C\left(q_a, \frac{\partial S(q_a)}{\partial q_a}\right) = 0, \tag{11.6}$$

i.e., the relativistic (covariant) Hamilton–Jacobi equation. If we know how to solve this equation, and if we have a *family* of its solutions, then we can reconstruct the solution of the equations of motion.

Define the Hamilton (principal) function

$$S\big(q_a^{\text{in}},\, q_a^{\text{f}}\big)\,, \tag{11.7}$$

namely a member of that family which solves the Hamilton–Jacobi equation in both sets of variables—evaluated at the initial variables q_a^{in} and the final variables q_a^{f}.

How does this determine the evolution? From the Hamilton–Jacobi function we obtain

$$p^a_{\text{in}} = \frac{\partial S\big(q_a^{\text{in}},\, q_a^{\text{f}}\big)}{\partial q_a^{\text{in}}} = p^a\big(q_a^{\text{in}},\, q_a^{\text{f}}\big)\,. \tag{11.8}$$

Thus we get a relation between the initial position and momentum and the final positions. Therefore, for each set of initial position and momentum—i.e., for each point in Γ_{ext}—we obtain a relation fixing all the final variables.

Since S satisfies the Hamilton–Jacobi equation, all variables q_a and p_a that enter this relation obey the constraint $C(q_a, p_a) = 0$. Hence the motions under consideration are determined by points in phase

space that form a subset of the extended phase space $\Gamma_{\rm ext}$, specified by the equation

$$C(q_a, p_a) = 0.$$

Another way to view this is the following: the phase space $\Gamma_{\rm ext}$ is symplectic (it carries Poisson brackets), and the pairs (q_a, p^a) are conjugate. The constraint $C(q_a, p_a) = 0$ defines a surface in phase space, called the *constraint surface*. In a symplectic space, when a constraint surface is present the symplectic form has null directions, and the integral curves of these null directions are the motions. From these, we obtain motions in the configuration space.

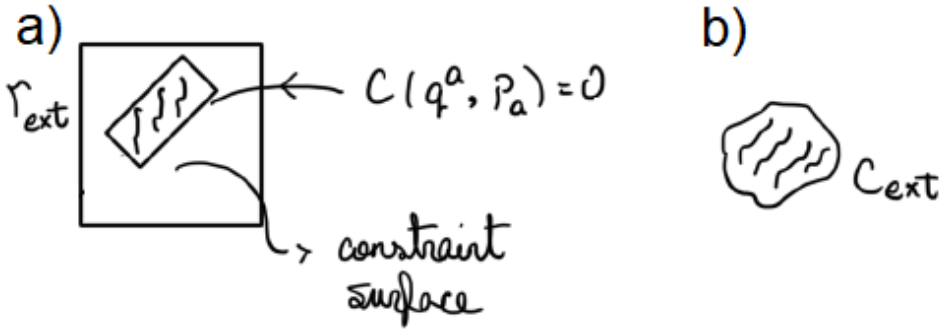

Figure 11.3: a) The symplectic space inside the constranit surface. b) The motions in the configuration space.

These motions are very similar to what we had in the standard configuration space, except that there we used a given time to label points. In $\mathcal{C}_{\rm ext}$ there is no preferred parameter along which to evolve.

Figure 11.4: $N+1$ dimensions in $\mathcal{C}_{\rm ext}$ vs N dimensions in C

Newtonian case:

Let $q_a = (q_i, t)$. The constraint reads

$$C(q_a, p^a) \;=\; p^t + \mathcal{H}(q_i, p_i) = 0, \tag{11.9}$$

in which the Hamilton–Jacobi equation is

$$\frac{\partial S}{\partial t} + \mathcal{H}\left(q^i,\, \frac{\partial S}{\partial q_i}\right) = 0, \tag{11.10}$$

and whose solution is the Hamilton principal function

$$S\big(q_i^{\rm in}, t_{\rm in};\; q_i^{\rm f}, t_{\rm f}\big)\,. \tag{11.11}$$

If we further impose

$$\frac{\partial S}{\partial q_i^{\rm in}} = p_i^{\rm in}\big(q_i^{\rm in}, t_{\rm in};\; q_i^{\rm f}, t_{\rm f}\big) = \text{const}, \tag{11.12}$$

then we can extract $q_i^{\rm f}(t)$. For each point $(q_i^{\rm in}, p_i^{\rm in})$ in the usual phase space, this procedure yields a trajectory.

Thus we recover the standard story when the relativistic Hamiltonian takes the form $p^t + \mathcal{H} = 0$. The framework, however, is more general: we never need to declare in advance which variable is the independent one; variables are determined relationally—one as a function of another. This is the key language required in a GR context, where there is no preferred time reference.

Example 1: Free particle with Newtonian time.
Let $(q_i, t) \in \mathcal{C}_{\text{ext}}$ and

$$\mathcal{H}_{\text{rel}} \equiv C(q_a, p^a) \;=\; p^t + \frac{1}{2m}\, p^2 = 0. \tag{11.13}$$

The Hamilton–Jacobi equation reads

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 = 0, \tag{11.14}$$

whose solution is

$$S = \frac{m}{2}\,\frac{(q_{\text{f}} - q_{\text{in}})^2}{t_{\text{f}} - t_{\text{in}}}\,. \tag{11.15}$$

Hence the equations of motion give

$$\frac{\partial S_{\text{in}}}{\partial q} = p_{\text{in}} \qquad \Rightarrow \qquad q_{\text{f}} = \frac{p_{\text{in}}}{m}\,(t_{\text{f}} - t_{\text{in}}) + q_{\text{in}}. \tag{11.16}$$

Thus, given $(q_{\text{in}}, p_{\text{in}})$, we obtain a relation between q and t for all times. This is not an evolution *with respect to a preferred* t; it is simply a relation between the two.

Example 2: without a preferred Newtonian time.
Consider a free *relativistic* particle, with $\mathcal{C}_{\text{ext}}$ the Minkowski spacetime and Hamiltonian constraint

$$\mathcal{H}_{\text{rel}} \equiv C = p^2 - m^2 = 0. \tag{11.17}$$

One can again relate the variables q and "t" and solve as in the previous case. In Minkowski space the motions are straight worldlines. The point is that in one Lorentz frame x^0 is the time coordinate, while in another frame $\tilde{x}^0 = \alpha\, x^0 + \beta\, x^i$ (a Lorentz transformation). Which one is "time"? We do not need to decide: the formalism is relational and the dynamics is well defined.

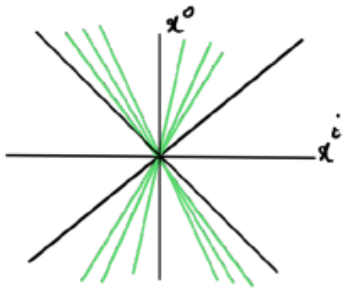

Figure 11.5: Relativistic motion in Minkowski space.

Example 3: without Newtonian time at all.
If there is a Newtonian time, the motion goes on forever: time runs from $-\infty$ to $+\infty$; it is global. What we have here is not like this. It can be formulated with the previous method, but it does not work the same way. The extended configuration space has two variables:

$$(a, b) \in \mathcal{C}_{\text{ext}}, \tag{11.18}$$

which you might be tempted to call "time" and "position," but we will not call it time here, because there is no preferred time.

The constraint is

$$C(a, b, p_a, p_b) = p_a^2 + p_b^2 + a^2 + b^2 - E = \text{const.} \tag{11.19}$$

Solving the equations gives a closed orbit (a circle/ellipse) in the (a, b)–plane. This can be seen in Figure 11.6.

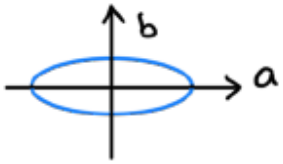

Figure 11.6: Elliptical motion derived from solving Eq. 11.9.

One could say that b "evolves" with respect to a and try to take a as a time parameter, but the trajectory returns to its initial point. Hence we cannot cast this as a basic evolutionary system with a globally monotonic time variable as we wrote in the beginning.

The formalism sketched above is not only valid for Newtonian mechanics; it also works in GR.

Some more points.

Figure 11.7: Motion in the configuration space with coordinates q_1 and q_2 and time coordinate *tau*.

The motion in Figure 11.7 gives a relation among the different variables $q_1, q_2, \ldots$ in $\mathcal{C}_{\text{ext}}$. However, nothing prevents us from choosing an arbitrary parametrization and calling it τ. In the literature, this is called *parameter time*. We can choose it as we wish, and therefore it need not have any geometrical or physical meaning.

We are used to this in SR. In SR, for a moving particle, the motion is often given by the position as a function of the coordinate time x^0,

$$\vec{x}\big(x^0\big),$$

namely, the position of the particle with respect to some reference frame. But this is not always a convenient way to describe the particle's motion. A good way is to write all four variables as a function of an arbitrary parameter τ:

$$x^\mu(\tau), \qquad \mu = 0, 1, 2, 3, \qquad \dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}. \tag{11.20}$$

The action is

$$S = m \int d\tau \, \sqrt{\dot{x}^\mu \dot{x}_\mu}. \tag{11.21}$$

The action is invariant under reparametrizations $\tau \to \tau'$; if we change the parameter τ, the action does not change. Therefore, the action between the initial and final points is only a function of x^μ_{in} and x^μ_{f}; it depends only on these endpoints and not on any chosen parametrization of the path between them. This can be seen in Figure 11.8.

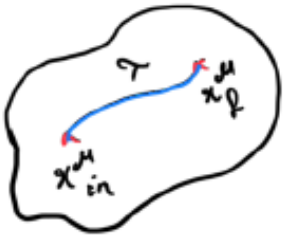

Figure 11.8: The action between the initial and final points depends only on the endpoints and not on any chosen parametrization of the path between them.

Writing the Hamiltonian for this action gives zero:

$$\mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L}, \qquad p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = m\,\frac{\dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}}, \qquad p^2 = m^2. \tag{11.22}$$

Indeed,

$$\mathcal{H} = m\,\frac{\dot{x}^\mu \dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} - m\sqrt{\dot{x}^\nu \dot{x}_\nu} = 0. \tag{11.23}$$

How, then, does one formulate the Hamiltonian theory? We have

$$\mathcal{H} = 0, \qquad \Gamma_{\text{ext}} = \{(x^\mu, p_\mu)\} \to \textit{phase space}, \qquad C \equiv p^2 - m^2 = 0 \to \textit{constrain}. \tag{11.24}$$

We arrive at the general–covariance structure in which the "relativistic Hamiltonian" is the constraint $p^2 - m^2 = 0$ on an extended phase space. This extended phase space carries all the information, and we do not need to specify which variable is "time."

The parameter τ is arbitrary and unimportant. The Hamilton function $S[x^\mu_{\text{in}}, x^\mu_{\text{f}}]$ depends only on the initial and final endpoints; the intermediate parametrization should not be confused with time. In general relativity the dynamical variable is the metric $g_{\mu\nu}(x)$. Writing it as $g_{\mu\nu}(\vec{x}, t)$ merely chooses a coordinate time t, which—like τ—is an arbitrary parametrization of spacetime. It is not one of the physical variables of the theory: no measurement corresponds to it, and it has nothing to do with a physical clock.

The clock reading is given by an integral along a worldline γ; there is a (clock) proper time associated with every worldline:

$$T_\gamma = \int_\gamma \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}\,. \tag{11.25}$$

Thus, this is an observable defined on phase space (or, equivalently, on the configuration space). Writing the metric as $g_{\mu\nu}(x, t)$ is like using the parameter τ: it corresponds to no direct measurement and has nothing to do with the clock. The metric components $g_{\mu\nu}$ are the physical variables that can be observed in the corresponding (extended) phase space—not the coordinates x and t.

Einstein's major struggle in developing GR was to understand the meaning of the coordinates. In GR, x and t have completely different meaning than in SR. In SR there are physical clocks and rods; in GR these are merely coordinate artifacts. One does not need t in GR, and the entire Hamilton–Jacobi derivation in the next section will proceed without t. This does not mean that time is frozen.

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12 Hamilton–Jacobi in General Relativity

Hamilton–Jacobi (GR) [Penrose 1969]

Dirac was the first to realize that Hamiltonian mechanics should be generalized because of general relativity. He has a paper on generalized Hamiltonian mechanics that builds on what was discussed in the last lecture.

Extended configuration space: It can be taken to be the space of 3-dimensional Riemannian metrics

$$\mathcal{C}_{\text{ext}} = \big\{\, q_{ab}(x^i) \;\big|\; a,b = 1,2,3 \,\big\}. \tag{12.1}$$

Here $q_{ab}(x^i)$ is a field, and its geometrical interpretation is that it is a 3-dimensional metric.

Extended phase space:

$$\Gamma_{\text{ext}} = \big\{\, \big(g_{ab}(x^i),\, p^{ab}(x^i)\big) \,\big\} \in \Gamma_{\text{ext}}, \tag{12.2}$$

i.e., it can be coordinatized by the metric and its conjugate momentum.

H–J equation: The Hamilton–Jacobi functional $S[q_{ab}]$ for GR satisfies

$$G^2\Big(q^{ac}q^{bd} - \tfrac{1}{2}q^{ab}q^{cd}\Big)\frac{\delta S[q]}{\delta q^{ab}(x)}\,\frac{\delta S[q]}{\delta q^{cd}(x)} \;-\; Det[g(\vec{x})]\,R[q_{ab}](\vec{x}) = 0, \tag{12.3}$$

where $R[q]$ is the 3-dimensional Ricci scalar of the metric q_{ab}.

Moreover, S is diffeomorphism invariant:

$$S[q_{ab}] = S[\tilde{q}_{ab}] \quad \text{if} \quad \tilde{q}_{ab}(\tilde{x}) = \frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^b}\,q_{cd}\big(x(\tilde{x})\big), \tag{12.4}$$

i.e., if q_{ab} and $\tilde{q}_{ab}$ are related by a diffeomorphism.

Thus (12.4) simply states that S, the Hamilton–Jacobi functional, is not really a function of the field $q_{ab}(x^i)$ as written in any particular coordinates; rather, it is a function of the equivalence class of q_{ab} under diffeomorphisms. In other words, it depends on the geometry, not on the coordinate description. Notice that obtaining (12.3) is equivalent to obtaining Einstein's equations (here written without matter, though matter can be included).

Also note that there is no explicit time t here, and we do not need it. This is not an easy equation to solve. If we knew a general solution of (12.3), it would be equivalent to knowing the solution of Einstein's equations, which we do not.

Hamilton function. The Hamilton function of GR is a special class of solutions of the equation. It is the functional

$$S\big[q^{\text{in}}_{ab}(x^i),\, q^{\text{f}}_{ab}(x^i)\big]\,, \tag{12.5}$$

such that it is a solution of the Hamilton–Jacobi equation in both sets of variables (initial and final).

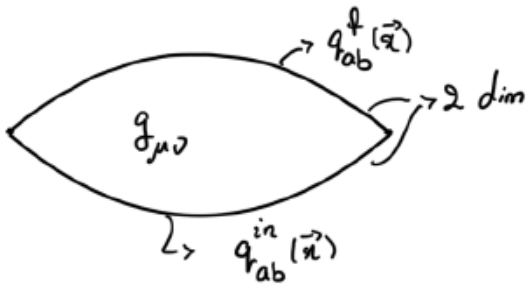

Figure 12.1: Consider two 3-dimensional spatial hypersurfaces joined along a common 2-dimensional boundary. We are interested in how the spacetime geometry evolves from one hypersurface to the other.

We therefore use the H–J function to solve the dynamics in a closed set of variables. In particular, one can compute $q^{\rm f}_{ab}(\vec{x})$ as a function of the initial position and momentum.

From the boundary variation one has

$$\frac{\delta S}{\delta q^{\rm in}_{ab}(x)} = p^{ab}_{\rm in}(x) = p^{ab}_{\rm in}\big[q^{\rm in},\, q^{\rm f}\big]\,(x), \tag{12.6}$$

so, given the initial momentum and the initial metric, you obtain a condition on the final metric. (Note that there is not a unique final metric since many can solve the problem.)

Moreover,

$$S\big[q^{\rm in}_{ab}(x^i),\, q^{\rm f}_{ab}(x^i)\big] = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\,R[g], \tag{12.7}$$

computed on the 4-dimensional metric $g_{\mu\nu}$ that solves Einstein's equations and is bounded by the given initial and final 3-dimensional metrics.

Notice that everything is local in spacetime: we do not need to go to infinity. GR is a local theory, and everything is local in space and time. If we push this reasoning further, the momentum has a geometrical interpretation related to the time derivative of the metric, and it is related to the extrinsic curvature. The extrinsic curvature describes how the hypersurface is curved in the 4-dimensional geometry.

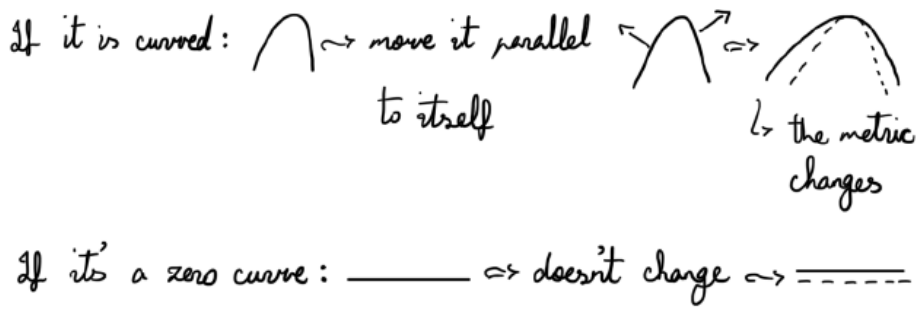

Figure 12.2: A schematic explanation on the curvature of space.

It is clear that $\dot{q}$, the time derivative of the spatial metric, is related to the extrinsic curvature; there are differential-geometric formulas showing this.

In the quantum theory the coordinates—namely the 3-metric $q_{ab}(x^i)$—and its conjugate momentum $p^{ab}(x^i)$ are promoted to a noncommutative algebra, i.e., to operators. Our goal is to turn the previously ill-defined classical expressions into well-defined quantum ones:

$$\begin{cases} \hat{q}_{ab}(x) \\ \hat{p}^{ab}(x) \end{cases} \qquad \text{noncommutative operators.}$$

For simplicity, let us take $\hat{q}_{ab}(x)$ to form a maximal commuting set. We want to compute the amplitude for evolving from an initial 3-geometry to a final 3-geometry:

$$\big\langle\, q^{\rm f}_{ab} \,\big|\, q^{\rm in}_{ab} \,\big\rangle = \int_{q^{\rm in}}^{q^{\rm f}} \mathcal{D}g\ \exp\left[\frac{i}{\hbar}\int d^4x\,\sqrt{g}\,R[g]\right] \ \sim\ \exp\left[\frac{i}{\hbar}\,S\big(q^{\rm in}_{ab}, q^{\rm f}_{ab}\big)\right]. \tag{12.8}$$

We expect this to satisfy the quantum version of the Hamilton–Jacobi equation, (12.3).

Finite-dimensional analogy: For a system with constraints $C(q^a, p_a) = 0$,

$$C\left(q^a,\, \frac{\partial S}{\partial q^a}\right) = 0, \qquad S\big(q^{\rm in}_a, q^{\rm f}_a\big)\,, \tag{12.9}$$

and the transition amplitude is

$$W\big(q_a^{\rm in}, q_a^{\rm f}\big) = \int \mathcal{D}q(\tau)\ \exp\!\left[\frac{i}{\hbar}\, S\big[q(\tau)\big]\right], \quad \text{with } q(\tau) \text{ fixed on the boundary.} \tag{12.10}$$

Quantum constraints. The amplitude satisfies the operator constraints in each boundary argument:

$$C\!\left(q^a, -i\hbar\,\frac{\partial}{\partial q^a}\right) W\big(q_a^{\rm in}, q_a^{\rm f}\big) = 0, \qquad \text{(in both sets of variables)}, \tag{12.11}$$

which, in GR, is the Wheeler–DeWitt equation for the transition amplitude.

In the special case where the constraint takes the Newtonian form $C = p_t - \mathcal{H}(p_i, q^i)$, the quantum constraint (12.11) becomes

$$\Big[-\,i\hbar\,\frac{d}{dt} - \mathcal{H}\Big(q^i, -i\hbar\,\frac{d}{dq^i}\Big)\Big]\, W\big(q_i^{\rm in}, q_i^{\rm f}\big) = 0, \tag{12.12}$$

which can be written as the Schrödinger equation.

Thus the Schrödinger equation is a special case of the Wheeler–DeWitt equation, which is the equation satisfied by the transition amplitude in the general covariant formalism when the relativistic Hamiltonian $C(q^a, p_a)$ reduces to the nonrelativistic Newtonian form $C = p_t - \mathcal{H}(q^i, p_i)$. Accordingly, ordinary quantum mechanics is valid when there is a Newtonian time; its covariant generalization is the Wheeler–DeWitt equation, where, instead of Newtonian time, one uses the generalized coordinates q^a.

In quantum gravity the transition amplitude $\langle q_{ab}^{\rm f} \,|\, q_{ab}^{\rm in}\rangle$ is

$$\big\langle q_{ab}^{\rm f} \,\big|\, q_{ab}^{\rm in}\big\rangle = \int \mathcal{D}g\ \exp\!\left[\frac{i}{\hbar}\int d^4x\,\sqrt{-g}\,R[g]\right] \ \simeq\ \exp\!\left[\frac{i}{\hbar}\, S\big(q_{ab}^{\rm in}, q_{ab}^{\rm f}\big)\right] =:\ W\big(q_a^{\rm in}, q_a^{\rm f}\big)\,, \tag{12.13}$$

and the Hamiltonian constraint of the main of the Hamilton–Jacobi equation, (12.3), acts as

$$C\!\left(q^a, -i\hbar\,\frac{d}{dq_a}\right) W\big(q_a^{\rm in}, q_a^{\rm f}\big) = 0. \tag{12.14}$$

However, in the quantum Hamilton–Jacobi equation, (12.3), we no longer have $\delta S/\delta q_a$. Instead we have functional derivatives acting as $\partial/\partial q_a$. This is precisely the equation Wheeler and DeWitt wrote in 1963. Notice that the "funny" role of time lies in the classical theory.

Thus,

$$\big\langle q_{ab}^{\rm f} \,\big|\, q_{ab}^{\rm in}\big\rangle = \int \mathcal{D}g\ \exp\!\left[\frac{i}{\hbar}\int d^4x\,\sqrt{-g}\,R[g]\right] \ \sim\ \exp\!\left[\frac{i}{\hbar}\, S\big(q_{ab}^{\rm in}, q_{ab}^{\rm f}\big)\right] =: W\big(q_{ab}^{\rm f}, q_{ab}^{\rm in}\big)\,. \tag{12.15}$$

Introduce the DeWitt supermetric G_{abcd} on the space of 3-metrics (where for example $G_{abcd} = G^2(q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd})$). Then the Wheeler–DeWitt equation reads

$$\left[G_{abcd}\,\frac{\delta^2}{\delta q_{ab}\,\delta q_{cd}} - \sqrt{q}\,R[q]\right] W\big(q_{ab}^{\rm f}, q_{ab}^{\rm in}\big) = 0, \qquad \text{(Wheeler–DeWitt, 1963/64)}. \tag{12.16}$$

Question: The operator equation involves "time," but where is the time variable in $W\big(q_{ab}^{\rm f}, q_{ab}^{\rm in}\big)$? The answer is that the states evolve: $q_{ab}^{\rm f}$ differs from $q_{ab}^{\rm in}$, and what we call "time variation" is encoded in this evolution of boundary data.

In Copenhagen quantum mechanics we have the following:

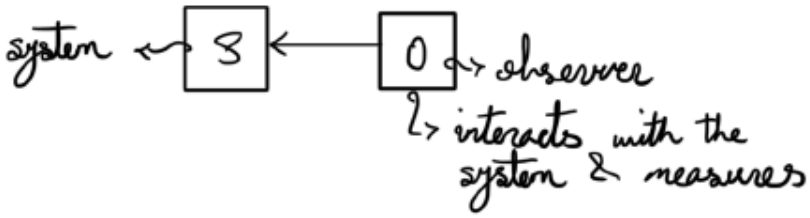

Figure 12.3: The Copenhagen approach. Pay attention that nature doesn't know about the interaction.

So quantum mechanics describes interactions between systems in which the variables q_a of a system take definite values when they interact with something else (the "observer"). It provides probability distributions for such values. In essence, QM does two things:

1. It gives the possible values a variable can take—i.e., the spectra of the $\hat{q}$'s and $\hat{p}$'s.
2. It gives the probabilities of these values as functions of previously measured values (conditional probabilities).

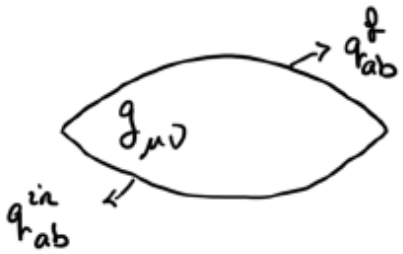

Figure 12.4: In quantum gravity, the relevant values are specified on two boundaries. Given data fixed at q^{in}_{ab}, one obtains the probability distribution for the value of q^{f}_{ab}.

Quantum mechanics concerns relations between interacting systems, whereas general relativity concerns relations between spacetime regions that are contiguous to one another. Hence, the boundary data q^{f}_{ab} and q^{in}_{ab} are important—though arbitrary. In quantum gravity, the emphasis is on spacetime regions interacting with one another, while in quantum mechanics, it is systems that interact.

In QG a "system" can be identified with a spacetime region, and "interaction" corresponds to contiguity of regions:

$$\text{system} \longleftrightarrow \text{spacetime region}, \qquad \text{interaction} \longleftrightarrow \text{contiguity}.$$

One of the great lessons of twentieth-century physics is locality: to interact, entities must be adjacent—nearby, contiguous. Thus, QG can be formulated on finite regions of spacetime, and the way such regions interact is analogous to how systems interact in QM. Physics is local in this sense; we do not need "infinity." In this way the standard languages of QM and GR speak to each other well.

Our task is to determine the transition amplitude $W\big(q^{\text{f}}_{ab}, q^{\text{in}}_{ab}\big)$. It must reproduce general relativity in the classical limit, namely

$$W\big(q^{\text{f}}_{ab}, q^{\text{in}}_{ab}\big) \xrightarrow[\hbar\to 0]{} \exp\left[\frac{i}{\hbar}\, S\big(q^{\text{in}}_{ab}, q^{\text{f}}_{ab}\big)\right]. \tag{12.17}$$

Can we define quantities that are bona fide transition amplitudes and behave as in (12.17)?

Why is this ill-defined?

1. The formal path-integral measure $\mathcal{D}g$ is ill-defined and must be made precise.

2. The functional derivatives $\delta/\delta q_{ab}(x)$ and $\delta/\delta q_{cd}(x)$ in the Wheeler–DeWitt equation are distributionally ill-defined.

3. We must specify the operator realization of $\hat{q}_{ab}(x)$ and $\hat{p}^{ab}(x)$. These operators define the quantum geometry of a 3-dimensional region (as treated in the kinematical part). The bra–ket $\langle q^{\rm f}_{ab} \,|\, q^{\rm in}_{ab}\rangle$ should not be read as "geometries" but as labels of the eigenstates that diagonalize a maximal commuting set of operators. Thus

$$\langle q^{\rm f}_{ab} \,|\, q^{\rm in}_{ab}\rangle \;\rightsquigarrow\; \text{(quantum numbers of the } \hat{q} \text{ operators)}.$$

Accordingly, what we need is not a transition from one classical 3-geometry to another, but from an eigenstate on the initial boundary to an eigenstate on the final boundary. Hence,

$$W\big(q^{\rm in}_{ab}, q^{\rm f}_{ab}\big) \;\longrightarrow\; W\Big[\{\Gamma^{out}, j^{\rm out}_\ell, V^{\rm out}_\ell\};\, \{\Gamma^{in}, j^{\rm in}_\ell, V^{\rm in}_\ell\}\Big], \tag{12.18}$$

i.e., a function of spin–network data $\{\Gamma, j_\ell, V_\ell\}$ on the two boundaries. In the semiclassical limit,

$$W\Big[\{\Gamma^{out}, j^{\rm out}_\ell, V^{\rm out}_\ell\};\, \{\Gamma^{in}, j^{\rm in}_\ell, V^{\rm in}_\ell\}\Big] \;\xrightarrow[\hbar\to 0]{}\; \exp\left[\frac{i}{\hbar}\, S\big(q_{ab}[\text{spin network}]\big)\right]. \tag{12.19}$$

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13 $\vec{k} = \gamma\,\vec{\ell}$

We discuss classical GR and build the main ingredient needed to structure the transition amplitudes.

Classical GR $\Rightarrow$ $\vec{k} = \gamma\,\vec{\ell}$ — we will discuss this relation.

1)

Einstein was the first to give a formulation of GR based on the field $g_{\mu\nu}(x)$, which is the Riemannian metric field. The main point about GR is the identification of the gravitational field with the Riemann metric. The proper way of looking at GR is not that the gravitational field is the metric of spacetime, but rather that the metric of spacetime *is* the gravitational field. If you reduce the gravitational field to "the metric of spacetime," you may become confused about what its quantum version is; it is clearer to proceed the other way around. If we instead adopt relational spacetime notions and regard the spacetime metric as something emergent, then we can say that the metric is nothing more than the gravitational field viewed as a quantum field—and we know how to deal with quantum fields.

From the metric connection $\Gamma^{\mu}{}_{\nu\rho}(x)$ and the Riemann tensor we obtain Einstein's equations

$$R_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}R + \lambda\,g_{\mu\nu} = 8\pi G\,T_{\mu\nu}. \tag{13.1}$$

Assumptions for now.

- We assume no matter: $T_{\mu\nu} = 0$, for simplicity. (In the book you can see how matter couples to gravity.)
- The cosmological-constant term: λ primarily affects very large distances. We will neglect it for now and bring it back later.

2)

Hilbert wrote Einstein's equations in the form of an action. The Einstein–Hilbert action is

$$S[g] = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R[g] \;=\; \frac{1}{2}\int d^4x\,\sqrt{-g}\,R[g] \quad \text{(in units } 8\pi G = 1\text{)}, \tag{13.2}$$

where R is the Ricci scalar (see Appendix).

3)

Palatini realized one can do a trick: take $g_{\mu\nu}(x)$ and $\Gamma^{\mu}{}_{\nu\rho}(x)$ as *independent* in the Einstein–Hilbert action. This works in finite dimensions. We write

$$S[g,\Gamma] = \frac{1}{2}\int d^4x\,\sqrt{-g}\,g^{\mu\nu}\,R_{\mu\nu}(\Gamma) = \frac{1}{2}\int d^4x\,\sqrt{-g}\,g^{\mu\nu}\,R^{\rho}{}_{\mu\rho\nu}(\Gamma). \tag{13.3}$$

It can be shown that varying with respect to Γ yields the Levi–Civita connection (i.e., Γ as a function of g), and varying with respect to g gives Einstein's equations.

The problem with this formulation is that it is impossible to couple fermions directly. To couple fermions, something must be added; to this end we work with tetrads (vielbeins) and a spin connection, following Weyl and Cartan.

4)

Introducing tetrads (Cartan & Weyl). Let η_{IJ} be the Minkowski metric, $\eta_{IJ} = \mathrm{diag}(+1,-1,-1,-1)$, with internal indices $I, J = 0, 1, 2, 3$. The spacetime metric is written as

$$g_{\mu\nu}(x) = e^{I}{}_{\mu}(x)\,e^{J}{}_{\nu}(x)\,\eta_{IJ}. \tag{13.4}$$

With tetrads, the action becomes

$$S[e] \;=\; \frac{1}{2}\int d^4x\ \det(e)\ R[e], \tag{13.5}$$

where $R[e]$ is the Ricci scalar built from the Levi–Civita connection of the metric (13.4) (in units $8\pi G = 1$). However, this introduces a redundancy: under a local Lorentz transformation on the internal indices (over the I and J indices), the metric $g_{\mu\nu}$ remains unchanged, so the tetrad description carries extra (gauge) fields in the action.

$$e^I{}_\mu(x) \;\longrightarrow\; \Lambda^I{}_J(x)\, e^J{}_\mu(x), \qquad \Lambda(x) \in SO(3,1), \tag{13.6}$$

In this sense, $g_{\mu\nu}(x)$ alone is not the most convenient description of the gravitational field when fermions are present, since it does not couple directly to spinors. The tetrad $e^I{}_\mu(x)$ provides the proper framework for including fermions.

5)

We can apply the Palatini trick again and treat the tetrad $e^I{}_\mu$ and the spin connection $\omega_\mu{}^{IJ} = -\omega_\mu{}^{JI}$ as *independent* fields. The first–order (Einstein–Cartan/Palatini) tetrad formulation is:

$$e^I{}_\mu(x) \qquad \text{tetrad,} \tag{13.7}$$

$$\omega_\mu{}^{IJ}(x) \qquad \text{SO(3,1) connection.} \tag{13.8}$$

The action reads

$$S[e,\omega] = \frac{1}{2}\int d^4x\ \det(e)\, e^\mu{}_I\, e^\nu{}_J\, F_{\mu\nu}{}^{IJ}[\omega], \tag{13.9}$$

with the (Yang–Mills–like) curvature

$$F^{IJ}{}_{\mu\nu}(\omega) = \partial_\mu\omega_\nu{}^{IJ} - \partial_\nu\omega_\mu{}^{IJ} + \omega_\mu{}^I{}_K\,\omega_\nu{}^{KJ} - \omega_\nu{}^I{}_K\,\omega_\mu{}^{KJ} \quad \big(\text{ equivalently } F^{IJ} = d\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}\big). \tag{13.10}$$

This is still exactly GR and enjoys local $SO(3,1)$ gauge invariance.

Using the identity

$$\det(e)\, e^\mu{}_I\, e^\nu{}_J = \frac{1}{2}\,\epsilon_{IJKL}\,\epsilon^{\mu\nu\rho\sigma}\, e^K{}_\rho\, e^L{}_\sigma, \tag{13.11}$$

we can rewrite the action (the $\det(e)$ in the denominator cancels the one in the numerator) as

$$S[e,\omega] = \frac{1}{2}\int d^4x\ \epsilon_{IJKL}\,\epsilon^{\mu\nu\rho\sigma}\, e^K{}_\rho\, e^L{}_\sigma\, F_{\mu\nu}{}^{IJ}(\omega). \tag{13.12}$$

In differential-form notation this is the compact Einstein–Cartan form

$$S[e,\omega] \;=\; \frac{1}{2}\int \epsilon_{IJKL}\, e^I \wedge e^J \wedge F^{KL}(\omega). \tag{13.13}$$

We also summarize the form notation:

$$e^I = e^I{}_\mu\, dx^\mu, \qquad \omega^{IJ} = \omega_\mu{}^{IJ}\, dx^\mu, \qquad F^{IJ} = F_{\mu\nu}{}^{IJ}\, dx^\mu \wedge dx^\nu = d\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}. \tag{13.14}$$

We can also write (13.13) using the internal Hodge dual. Define

$$F^{*IJ} \;:=\; \tfrac{1}{2}\,\epsilon^{IJ}{}_{KL}\, F^{KL}, \tag{13.15}$$

then the Einstein–Cartan action becomes

$$S[e,\omega] \;=\; \tfrac{1}{2}\int e \wedge e \wedge F^*. \tag{13.16}$$

6)

Ashtekar–Barbero–Immirzi/Holst form. (Ashtekar, Rovelli, Holst.) Ashtekar observed that a useful canonical transformation corresponds to adding to the action a term built from the non–dual curvature. This yields the Holst action

$$S_\gamma[e,\omega] \;=\; \tfrac{1}{2}\int \Big(e\wedge e\wedge F^* \;+\; \tfrac{1}{\gamma}\, e\wedge e\wedge F\Big), \tag{13.17}$$

where γ is the Barbero–Immirzi parameter. This is the action on which LQG is based.

Remark. The additional term does not change the classical theory: the difference between (13.16) and (13.17) is a boundary term, so varying either action yields the same classical equations of motion.

We can understand the additional Holst term by simple inspection (it does not change the classical theory). Let us see what the piece $\frac{1}{\gamma}\, e\wedge e\wedge F$ becomes in Riemannian geometry. In index notation,

$$\frac{1}{\gamma}\, e\wedge e\wedge F \;\;\longleftrightarrow\;\; \frac{1}{\gamma}\,\epsilon^{\mu\nu\rho\sigma}\, e_\mu{}^I e_\nu{}^J F_{\rho\sigma\, IJ},$$

which vanishes by symmetry of the lower indices in the purely Levi–Civita (torsionless) case.
Thus, classically, the action $S[e,\omega]$ still reproduces GR and nothing has changed. **Caution:** although adding the Holst term does not alter the classical equations, it *does* modify the quantum theory's structure; in particular, one cannot naively take $\gamma\to\infty$ in the quantum theory without running into difficulties.

Writing the action (13.17) a bit differently,

$$S[e,\omega] = \frac{1}{2}\int \Big[\,(e\wedge e)^* \;+\; \frac{1}{\gamma}\,(e\wedge e)\,\Big]\wedge F, \tag{13.18}$$

we can cast it into a BF-type form by defining a Lie-algebra–valued 2-form B^{IJ}:

$$S[e,\omega] = \frac{1}{2}\int B\wedge F, \qquad B^{IJ} \text{ is a 2-form (antisymmetric in } I,J) \text{ valued in } \mathfrak{sl}(2,\mathbb{C}), \tag{13.19}$$

where F is the curvature of an $\mathrm{SL}(2,\mathbb{C})$ connection ω. Explicitly,

$$S_{BF} = \frac{1}{2}\int B\wedge F, \qquad B \;=\; (e\wedge e)^* \;+\; \frac{1}{\gamma}\,(e\wedge e). \tag{13.20}$$

We will examine this B-field expression in more detail next.

We want to study the theory in a *compact* region of spacetime.

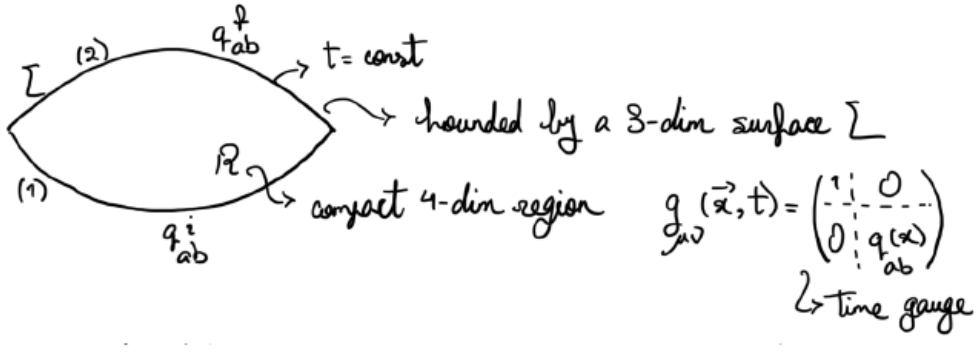

Figure 13.1: The compact region under study. We're interested in the *transition* amplitude from (1) to (2).

In the Einstein formulation the field is $g_{\mu\nu}(\vec{x},t)$. On the boundary surfaces we can always choose coordinates with $t=$ const. If we use tetrads, the boundary data are tetrads rather than the three–metric $q_{ab}(x)$.

Appendix

Start from the Einstein–Hilbert action

$$S \;=\; \frac{1}{2}\int d^4x\,\sqrt{|g|}\,R, \qquad g \;=\; \det(g_{\mu\nu}). \tag{13.21}$$

Write the metric in terms of tetrads (coframes) $e^a{}_\mu$ with $\eta_{ab} = \mathrm{diag}(+1,-1,-1,-1)$:

$$g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu, \qquad \sqrt{|g|} = \det(e^a{}_\mu) \equiv e. \tag{13.22}$$

The curvature 2-form and the scalar curvature are

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \qquad R = e^\mu{}_a\, e^\nu{}_b\, R^{ab}{}_{\mu\nu}. \tag{13.23}$$

Hence

$$S \;=\; \frac{1}{2}\int d^4x\; e\; e^\mu{}_a\, e^\nu{}_b\, R^{ab}{}_{\mu\nu}, \qquad e^a := e^a{}_\mu\, dx^\mu. \tag{13.24}$$

Using the identity for the volume form,

$$e\, d^4x \;=\; \frac{1}{4!}\,\epsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge e^d, \tag{13.25}$$

$$S \;=\; \frac{1}{2}\int \frac{1}{4!}\,\epsilon_{abcd}\; e^a \wedge e^b \wedge e^c \wedge e^d\; e^\mu{}_a\, e^\nu{}_b\; R^{ab}{}_{\mu\nu}. \tag{13.26}$$

By definition of the curvature 2–form,

$$R^{ab} \;=\; \frac{1}{2}\, R^{ab}{}_{\mu\nu}\; dx^\mu \wedge dx^\nu \qquad \Longrightarrow \qquad R^{ab}{}_{\mu\nu} \;=\; 2\,\frac{\partial R^{ab}}{\partial(dx^\mu \wedge dx^\nu)}. \tag{13.27}$$

Substituting (13.27) into (13.26) gives

$$S \;=\; \frac{1}{2}\int \frac{1}{4!}\,\epsilon_{abcd}\; e^a \wedge e^b \wedge e^c \wedge e^d\; e^\mu{}_a\, e^\nu{}_b\, \Big(2\,\frac{\partial R^{ab}}{\partial(dx^\mu \wedge dx^\nu)}\Big). \tag{13.28}$$

Recombining the differentials yields the compact Einstein–Cartan form

$$S \;=\; \frac{1}{2}\int \epsilon_{abcd}\; e^a \wedge e^b \wedge R^{cd}. \tag{13.29}$$

Using the definition of the curvature in terms of the spin connection,

$$R^{cd} \;=\; d\omega^{cd} + \omega^c{}_e \wedge \omega^{ed}, \tag{13.30}$$

we can also write

$$S \;=\; \frac{1}{2}\int \epsilon_{abcd}\; e^a \wedge e^b \wedge \big(d\omega^{cd} + \omega^c{}_e \wedge \omega^{ed}\big). \tag{13.31}$$

In components, the curvature tensor is

$$R^{ab}{}_{\mu\nu} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^a{}_c\, \omega_\nu{}^{cb} - \omega_\nu{}^a{}_c\, \omega_\mu{}^{cb}. \tag{13.32}$$

Metric compatibility (vanishing tetrad covariant derivative) reads

$$\nabla_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b\, e^b{}_\nu - \Gamma^\lambda{}_{\mu\nu}\, e^a{}_\lambda = 0, \tag{13.33}$$

and we recall the 1–form notation for the spin connection,

$$\omega^{ab} \;=\; \omega_\mu{}^{ab}\, dx^\mu. \tag{13.34}$$

and,

$$\Gamma^{\lambda}_{\mu\nu} \;=\; \frac{1}{2}\, g^{\lambda\rho}\,(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{13.35}$$

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14 $\vec{k} = \gamma\,\vec{\ell}$ (Continue of the previous section)

Continuing from the last session. Starting from the Holst form,

$$S[e,\omega] = \frac{1}{2}\int \Big[\,(e^K \wedge e^L)^* + \frac{1}{\gamma}\,(e^K \wedge e^L)\Big] \wedge F_{KL}, \tag{14.1}$$

we can package it as a BF theory:

$$S[e,\omega] = \frac{1}{2}\int B^{KL} \wedge F_{KL}, \qquad B^{KL} := (e^K \wedge e^L)^* + \frac{1}{\gamma}\,(e^K \wedge e^L). \tag{14.2}$$

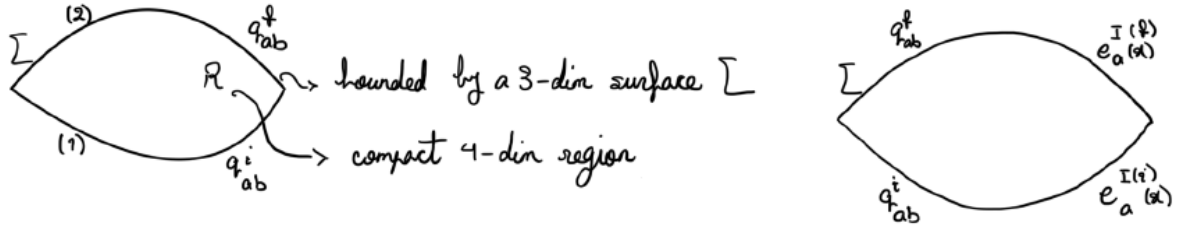

Figure 14.1: The compact region under study in the 3 metric and the tetrad form.

When we use tetrads, we have tetrads instead of the 3-metric $q_{ab}(x)$.

$$g_{\mu\nu}(\vec{x},t) = \begin{pmatrix} 1 & 0 \\ 0 & q_{ab}(x) \end{pmatrix}, \qquad \text{(time gauge, symmetric components)}. \tag{14.3}$$

In the tetrad formulation, we write

$$e^I{}_\mu(\vec{x},t) = \begin{pmatrix} 1 & 0 \\ 0 & e^I{}_a(x) \end{pmatrix}, \qquad a = 1,2,3 \ \text{(spacetime index)}, \quad I = 1,2,3 \ \text{(internal index)}. \tag{14.4}$$

Here, $e^I{}_a(x)$ represents the *triad*. Unlike the spatial metric components $q_{ab}(x)$, the tetrads are not symmetric—each carries distinct geometric and algebraic properties.

The *second* time gauge is stronger. Not only does it set the 00 component to 1 and the $0i$ and $i0$ components to 0, it also orients the Lorentz frame appropriately. When you sit on a 3-dimensional surface, the surface itself defines a frame for you, so at the boundary the symmetry $\mathrm{SO}(3,1)$ is broken down to $\mathrm{SO}(3)$. That is, a preferred subgroup of $\mathrm{SO}(3,1)$ is selected—the subgroup that is preserved on the boundary. The boundary thus breaks the full Lorentz invariance $\mathrm{SO}(3,1)$ to rotational invariance $\mathrm{SO}(3)$ because local rotations tangential to the surface remain possible.

Equivalently, at the level of Lie algebras the breaking

$$\mathrm{SO}(3,1) \longrightarrow \mathrm{SO}(3) \qquad \text{corresponds to} \qquad \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{su}(2),$$

which is characteristic of working *on the boundary*.

Since B is a 2-form, we restrict it to the boundary:

$$B \longrightarrow B\big|_\Sigma, \qquad \text{with a corresponding split of indices (Lorentz vs. spatial)}. \tag{14.5}$$

For example, one may write schematically $B^{IJ}_{\mu\nu}\big|_\Sigma = (\cdots)^{IJ}{}_{ab}\,(\cdots)^{ab}_{\mu\nu}$, to indicate the decomposition into internal Lorentz indices I,J and spatial indices a,b.

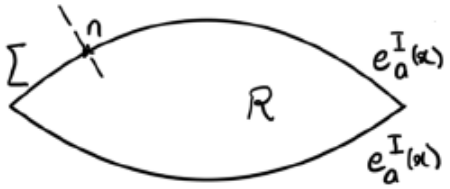

Figure 14.2: The compact region under study in which at each point we have a 1-form n.

At each point of the boundary surface Σ we have a (co)normal 1-form n_μ defined from the embedding $x^\mu(\sigma^1,\sigma^2,\sigma^3)\colon \Sigma \to \mathcal{M}$ by

$$n_\mu = \epsilon_{\mu\nu\rho\sigma}\,\frac{\partial x^\nu}{\partial\sigma^1}\,\frac{\partial x^\rho}{\partial\sigma^2}\,\frac{\partial x^\sigma}{\partial\sigma^3}, \qquad (\sigma^1,\sigma^2,\sigma^3) \text{ are coordinates on the surface } \Sigma. \tag{14.6}$$

If the surface is at constant time, $t = \text{const}$, and the chosen coordinates are $\sigma^i = x^i$ $(i = 1,2,3)$, then

$$n_\mu = (1,0,0,0). \tag{14.7}$$

Since we work with tetrads, we can convert the spacetime co-normal into an internal (Minkowski) vector by

$$n_I \;=\; e^\mu{}_I\, n_\mu. \tag{14.8}$$

Thus n_I is a bona fide object in the internal Minkowski space at each point of Σ.

Restrict the Lie–algebra–valued 2-form B^{IJ} to the boundary, and project it along the internal normal:

$$k^I \;:=\; \big(B^{IJ}\big|_\Sigma\big)\, n_J, \qquad \ell^I \;:=\; \big(B^{*IJ}\big|_\Sigma\big)\, n_J \;=\; \tfrac{1}{2}\,\epsilon^{IJ}{}_{KL}\,\big(B^{KL}\big|_\Sigma\big)\, n_J. \tag{14.9}$$

In the time gauge $n_I = (1,0,0,0)$ these become

$$k^I = B^{I0}\big|_\Sigma, \qquad \ell^I = \tfrac{1}{2}\,\epsilon^{I0}{}_{kl}\,B^{kl}\big|_\Sigma, \qquad k,l \in \{1,2,3\}. \tag{14.10}$$

Equivalently, the antisymmetric matrix $B^{IJ}\big|_\Sigma$ can be organized so that its "electric" components are k^I and its "magnetic" components are ℓ^I:

$$B^{IJ}\big|_\Sigma \;\sim\; \begin{pmatrix} 0 & k^1 & k^2 & k^3 \\ -k^1 & 0 & \ell^3 & -\ell^2 \\ -k^2 & -\ell^3 & 0 & \ell^1 \\ -k^3 & \ell^2 & -\ell^1 & 0 \end{pmatrix}.$$

If we think of B^{IJ} as the analogue of $F^{\mu\nu}$, then $\vec{K}$ plays the role of the electric field and $\vec{L}$ that of the magnetic field. The split of B^{IJ} into "electric" and "magnetic" parts is frame–dependent: $F^{\mu\nu}$ splits into electric and magnetic components only after choosing a Lorentz frame, and a different frame mixes the two. Thus breaking Lorentz $SO(3,1)$ down to $SO(3)$ allows us to separate $F^{\mu\nu}$ into electric and magnetic parts. Here, on the boundary, the gauge group $SL(2,\mathbb{C})$ is broken to $SU(2)$, which breaks the B-field into K and L. The components K^I are associated with the boost generators, while L^I correspond to the rotation generators. Note that both are spatial vectors (their time component is zero).

From the boundary definitions we have

$$K^I \;=\; B^{IJ}\big|_\Sigma\, n_J, \qquad L^I \;=\; (B^*)^{IJ}\big|_\Sigma\, n_J, \tag{14.11}$$

and using $B = (e\wedge e)^* + \frac{1}{\gamma}(e\wedge e)$,

$$K^I \;=\; \Big[\,\big(e\wedge e\big)^{*\,IJ}\Big|_\Sigma \;+\; \frac{1}{\gamma}\,\big(e\wedge e\big)^{IJ}\Big|_\Sigma\Big]\, n_J. \tag{14.12}$$

On the boundary we impose the time gauge. Writing the triad as $e^I{}_a(x) = (e^i{}_a,\, e^0{}_a)$ with $e^0{}_a = 0$, and using $n_I = (1,0,0,0)$,

$$\frac{1}{\gamma}\,(e\wedge e)^{IJ}\Big|_\Sigma n_J \;=\; \frac{1}{\gamma}\,(e\wedge e)^{I0}\Big|_\Sigma \;=\; 0, \tag{14.13}$$

since $(e\wedge e)^{I0} \propto e^I \wedge e^0$ and $e^0\big|_\Sigma = 0$. Hence, in time gauge the Holst contribution drops out of K^I on the boundary.

$\big[(e\wedge e)^{*\,IJ}\big]_\Sigma n_J$ contains an $\epsilon^I{}_{KL0}$ and survives: the zero component in n_J contracts with $(e\wedge e)^{*\,IJ}$, and the $\epsilon^I{}_{KL0}$ further contracts with this zero, saving the day.

$$K^I = \big[(e\wedge e)^{*\,IJ}\big]_\Sigma\, n_J = \begin{cases} K^0 = 0, \\ K^i = \epsilon^i{}_{jk}\, e^j \wedge e^k. \end{cases} \tag{14.14}$$

For L^I it is the opposite:

$$L^I = \big[(e\wedge e)^{IJ}\big]_\Sigma\, n_J = \begin{cases} L^0 = 0, \\ L^i = \dfrac{1}{\gamma}\,\epsilon^i{}_{jk}\, e^j \wedge e^k. \end{cases} \tag{14.15}$$

We therefore have

$$\vec{K} = \gamma\,\vec{L} \qquad \text{(rotation} \leftrightarrow \text{boost)}. \tag{14.16}$$

Thus GR can be viewed as an $\mathrm{SL}(2,\mathbb{C})$ BF theory: on the boundary the generators of $\mathrm{SL}(2,\mathbb{C})$ become proportional to one another. Consequently, $\vec{K}$ and $\vec{L}$ are 2-forms that can be integrated on different surfaces (e.g., over a triangle).

$$\int_\Delta \vec{L} \;\equiv\; \int_\Delta L^i \;=\; \frac{1}{2\gamma}\int_\Delta \epsilon^i{}_{jk}\, e^j \wedge e^k. \tag{14.17}$$

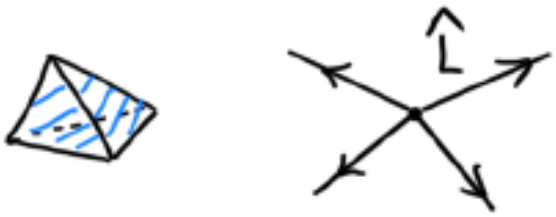

Figure 14.3: The tetrahedron in the real space and its equivalent with the designated $\widehat{L}$ operator in the dual space.

This is (almost) the same operator $\vec{L}$ discussed in kinematics; we now define it by $\widehat{\vec{L}}$.

$$\widehat{L}^{\,i} \;=\; \frac{1}{8\pi\hbar G\,\gamma}\left(\tfrac{1}{2}\,\epsilon^i{}_{jk}\, e^j \wedge e^k\right). \tag{14.18}$$

$$\mathcal{A}^2 \;=\; \vec{L}\cdot\vec{L} \;=\; \big(8\pi\hbar G\,\gamma\big)^2\,\widehat{\vec{L}}\cdot\widehat{\vec{L}}. \tag{14.19}$$

$$\mathcal{A} \;=\; 8\pi\hbar G\,\gamma\,\sqrt{j\,(j+1)}. \tag{14.20}$$

$$|K|^2 = \gamma^2\,|L|^2 = \gamma^2\,\frac{\mathcal{A}^2}{\big(8\pi\hbar G\,\gamma\big)^2} \quad\Rightarrow\quad |K| = \frac{|\mathcal{A}|}{8\pi\hbar G}. \tag{14.21}$$

(Here $|K|$ denotes the absolute value of the boost generator.)

Black–hole hovering example. Frodden–Ghosh–Perez (FGP) showed that if you are at a distance d from a black hole, to avoid falling in you must maintain the proper acceleration

$$a = \frac{1}{d} \qquad (c = 1), \tag{14.22}$$

i.e., the acceleration required to hover at fixed d. This has been shown in the following Figure.

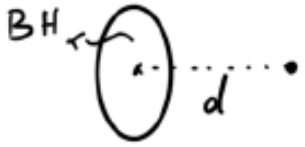

Figure 14.4: The distance d for not falling inside a black hole.

Now, to consider a generator that advances *proper time* (rather than the dimensionless boost angle), we scale the boost generator $\vec{K}$ by the proper acceleration a. Using $|K| = \mathcal{A}/(8\pi G)$, we have the Hamiltonian

$$\mathcal{H} \;=\; a\,|K| \;=\; \frac{a\,\mathcal{A}}{8\pi G}. \tag{14.23}$$

Frodden–Ghosh–Perez further showed that when an object with energy dE falls into a black hole, the horizon area changes by dS according to

$$dE \;=\; \frac{a}{8\pi G}\,dS. \tag{14.24}$$

This can be viewed as a rewriting of the relation $\vec{K} = \gamma\,\vec{L}$ in a different guise.

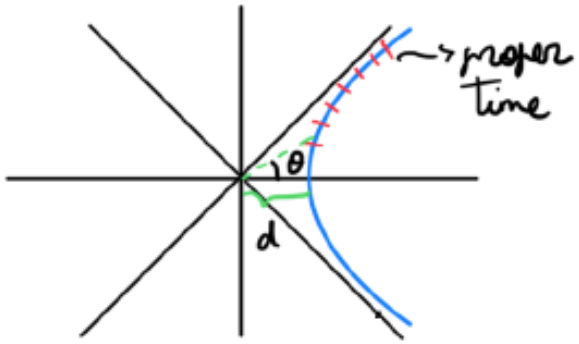

Figure 14.5: Locally, you are in Minkowski spacetime, accelerating with $a = \frac{1}{d}$. The generator of this trajectory is a boost, $\vec{K}$. The vector $\vec{K}$ generates translations in the boost parameter θ.

If we compute the ADM energy of a black hole as defined at infinity and redshift it down to a distance d, as shown in the Figure, the change in energy is

$$dE \;=\; \frac{a}{8\pi G}\,dS. \tag{14.25}$$

We also have

$$E \;=\; T\,S, \tag{14.26}$$

for the relationship between the energy and the entropy. In the accelerated frame, the Unruh temperature is

$$T \;=\; \frac{\hbar\,a}{2\pi k_B}, \tag{14.27}$$

hence

$$a \;=\; \frac{2\pi k_B}{\hbar}\,T. \tag{14.28}$$

Substituting (14.28) into (14.25) gives

$$dE \;=\; \frac{k_B\,T}{4\,\hbar\,G}\,dS. \tag{14.29}$$

Therefore, using $\vec{K} = \gamma\,\vec{L}$ and the relations above, one recovers the Hawking–Bekenstein entropy for a black hole,

$$S_{\mathrm{BH}} \;=\; \frac{k_B\,A}{4\,\hbar\,G}\,, \tag{14.30}$$

where A is the horizon area.

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15 Unitary Representation of $SL(2,\mathbb{C})$ and the γ-map

$$S[B,\omega] = \frac{1}{2}\int B \wedge F, \tag{15.1}$$

where

ω is an $SL(2,\mathbb{C})$ connection, B is an $SL(2,\mathbb{C})$ two-form, F is the curvature of ω.

The field B has boost and rotation parts, analogous to the electric and magnetic fields.

Restriction to the boundary. On the boundary we have

$$\vec{K} = \gamma\,\vec{L}. \tag{15.2}$$

The boundary breaks $SL(2,\mathbb{C})$ down to $SU(2)$ (equivalently, $SO(3,1)$ down to $SO(3)$), with

$$SU(2) \subset SL(2,\mathbb{C}).$$

To pass to the quantum theory we introduce Hilbert spaces; unsurprisingly, these Hilbert spaces carry representations of the relevant groups. We now develop the mathematics of $SL(2,\mathbb{C})$ and its representations.

The $SL(2,\mathbb{C})$ representations that physicists use (those related to the Lorentz group) are not finite-dimensional unitary representations—$SL(2,\mathbb{C})$ is non-compact, so its unitary representations are infinite-dimensional. Following the classic treatment of Rühl (1970), we begin by defining a group element

$$g \in SL(2,\mathbb{C}), \qquad g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \qquad \det g = 1. \tag{15.3}$$

Here a, b, c, d are complex numbers (four complex = eight real parameters), and the condition $\det g = 1$ imposes one complex (i.e. two real) constraint. Hence

$$8 \text{ real parameters } - \text{ 2 real constraints } = \text{ 6 real degrees of freedom,}$$

so $SL(2,\mathbb{C})$ is a 6-dimensional Lie group (corresponding to 3 boosts and 3 rotations).

An $SU(2)$ group element can be written as

$$g = \exp\bigl(i\,\alpha^i \sigma_i\bigr), \qquad \alpha^i \in \mathbb{R}, \tag{15.4}$$

with σ_i the Pauli matrices. Complexifying the three parameters α^i leads to $SL(2,\mathbb{C})$. Taking the parameters to be purely imaginary picks out the boost sector, while real parameters correspond to rotations.

Let us derive the commutation relations between the antisymmetric Lorentz generators J^{ij}:

$$\bigl[J^{ij},\, J^{k\ell}\bigr] \;= ? \tag{15.5}$$

The J^{ij} generate the Lorentz group $SO(3,1)$ and are antisymmetric,

$$J^{ij} = -\,J^{ji}. \tag{15.6}$$

They generate infinitesimal Lorentz transformations acting on spacetime vectors, spinors, and tensors. In $d = 4$ we have

$$\dim SO(3,1) = \frac{4\,(4-1)}{2} = 6, \tag{15.7}$$

so there are six independent generators: three rotations J^{ij} $(i,j = 1,2,3)$ and three boosts J^{0i}.

An infinitesimal Lorentz transformation acts on a vector x^i as

$$x'^i = x^i + \omega^i{}_j\, x^j, \qquad \omega^{ij} = -\,\omega^{ji}. \tag{15.8}$$

This is generated by the antisymmetric generators J^{ij} through

$$\Lambda = \mathbb{I} + \frac{1}{2}\,\omega_{ij}\, J^{ij}. \tag{15.9}$$

The action of a generator on a vector is

$$\big(J^{ij}V\big)^m = \eta^{jm}V^i - \eta^{im}V^j, \tag{15.10}$$

with η^{ij} the Minkowski metric. Using (15.10), we compute

$$\big[J^{ij}, J^{k\ell}\big]\, V^m = J^{ij}\big(J^{k\ell}V^m\big) - J^{k\ell}\big(J^{ij}V^m\big)\,, \tag{15.11}$$

where

$$J^{ij}\big(J^{k\ell}V^m\big) = J^{ij}\big(\eta^{\ell m}V^k - \eta^{km}V^\ell\big)\,. \tag{15.12}$$

Continuing the derivation, expanding the action of the generators gives

$$\begin{aligned}
[J^{ij}, J^{k\ell}]V^m &= J^{ij}\big(J^{k\ell}V^m\big) - J^{k\ell}\big(J^{ij}V^m\big) \\
&= \eta^{\ell m}\big(\eta^{jk}V^i - \eta^{ik}V^j\big) - \eta^{km}\big(\eta^{j\ell}V^i - \eta^{i\ell}V^j\big) \\
&\quad - \eta^{jm}\big(\eta^{\ell k}V^i - \eta^{ik}V^\ell\big) + \eta^{im}\big(\eta^{\ell k}V^j - \eta^{jk}V^\ell\big) \\
&= -\,\eta^{ik}\big(\eta^{\ell m}V^j - \eta^{jm}V^\ell\big) + \eta^{i\ell}\big(\eta^{km}V^j - \eta^{jm}V^k\big) \\
&\quad -\,\eta^{jk}\big(\eta^{\ell m}V^i - \eta^{im}V^\ell\big) + \eta^{j\ell}\big(\eta^{km}V^i - \eta^{im}V^k\big)\,.
\end{aligned} \tag{15.13}$$

Hence, the commutator takes the covariant form

$$\boxed{\big[J^{IJ}, J^{KL}\big] = -\,\eta^{IK}J^{JL} + \eta^{IL}J^{JK} - \eta^{JK}J^{IL} + \eta^{JL}J^{IK}.} \tag{15.14}$$

Here η^{IJ} is the Minkowski metric. This covariant expression does not depend on the choice of an $SU(2)$ subgroup, though it is not necessarily the most practical way to write it. If we pick an $SU(2)$ subgroup by breaking $SL(2,\mathbb{C})$ down to $SU(2)$, this amounts to selecting a preferred time direction or Lorentz frame. Let

$$t_I = (1,0,0,0), \tag{15.15}$$

so that we can decompose the generators into their *magnetic* (rotation) and *electric* (boost) parts:

$$L^I = \frac{1}{2}\,\epsilon^{IJKL}\, J_{JK}\, t_L = \frac{1}{2}\,\epsilon^I{}_{JKL}\, J^{KL}\, t^L = \epsilon^I{}_{JK}\, L^K, \tag{15.16}$$

$$K^I = J^{0I} = \frac{1}{2}\,\epsilon^I{}_{JKL}\, J^{KL}\, t^L = (0, \vec{K}). \tag{15.17}$$

The commutation relations can then be obtained using the Lorentz algebra:

$$[L^i, L^j] = \frac{1}{4}\,\epsilon^{ipq}\,\epsilon^{jrs}\,[J^{pq}, J^{rs}], \tag{15.18}$$

$$= \epsilon^{ijk}\, L^k, \tag{15.19}$$

which shows that $\vec{L}$ generates an $SU(2)$ subgroup that sits inside the Lorentz group.

Next, we compute the mixed commutator:

$$\begin{aligned}[L^i, K^j] &= \frac{1}{2}\,\epsilon^{ipq}\,[J^{pq}, J^{0j}] \\ &= -\,\eta^{p0}\,J^{qj} + \eta^{q0}\,J^{pj} + \eta^{pj}\,J^{q0} - \eta^{qj}\,J^{p0} \\ &= \epsilon^{ijk}\,K^k. \end{aligned} \tag{15.20}$$

Finally, for the commutator of two boosts:

$$\begin{aligned}[K^i, K^j] &= [J^{0i}, J^{0j}] \\ &= -\,\eta^{00}\,J^{ij} + \eta^{0j}\,J^{0i} - \eta^{i0}\,J^{0j} + \eta^{ij}\,J^{00} \\ &= -(-1)\,J^{ij} + 0 + 0 + 0 \\ &= -\,J^{ij} = \epsilon^{ijk}\,L^k. \end{aligned} \tag{15.21}$$

Thus,

$$[L^i, L^j] = \epsilon^{ijk}\,L^k, \tag{15.22}$$

$$[L^i, K^j] = \epsilon^{ijk}\,K^k, \tag{15.23}$$

$$[K^i, K^j] = -\,\epsilon^{ijk}\,L^k. \tag{15.24}$$

It is not by chance that the L^k's here are named the same as the L^k's in general relativity: they represent the generators of spatial rotations. The commutator of two boosts is a rotation. If you boost, for example, along the x-direction and then along the y-direction—and then perform the two in reverse order—you obtain the same result up to an additional rotation.

Now we have two Casimirs. For $SU(2)$, we had only one Casimir, given by the square of $\vec{L}$. For $SL(2,\mathbb{C})$, we seek invariants under Lorentz (or $SL(2,\mathbb{C})$) transformations. The two Casimir operators are:

$$J^{IJ}J_{IJ}, \qquad J^{IJ}J^*_{IJ} = \epsilon_{IJKL}\,J^{IJ}J^{KL}. \tag{15.25}$$

Writing these in terms of $\vec{K}$ and $\vec{L}$, we have:

$$J^{IJ}J_{IJ} \;\longrightarrow\; C_1 = \vec{K}^{\,2} - \vec{L}^{\,2}, \tag{15.26}$$

$$J^{IJ}J^*_{IJ} \;\longrightarrow\; C_2 = \vec{K}\cdot\vec{L}. \tag{15.27}$$

(Check all calculations carefully — there might be missing factors of $\frac{1}{2}$ or sign conventions.)

Now let us consider the representations of $SL(2,\mathbb{C})$. Among the finite-dimensional representations, the simplest one is the *fundamental*. A group element $g \in SL(2,\mathbb{C})$ acts on a complex vector space $\mathbb{C}^2$, whose elements are called *spinors*.

The group element acts on a spinor $\mathbf{z} \in \mathbb{C}^2$ by

$$g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \qquad \mathbf{z}' = g\,\mathbf{z} \;=\; \begin{pmatrix} z'^0 \\ z'^1 \end{pmatrix} = \begin{pmatrix} a\,z^0 + c\,z^1 \\ b\,z^0 + d\,z^1 \end{pmatrix}. \tag{15.28}$$

Define the Hermitian scalar product between two spinors $\mathbf{z}$ and $\mathbf{w}$:

$$\langle \mathbf{z}, \mathbf{w} \rangle = \bar{z}^A\,w^B\,\delta_{AB}. \tag{15.29}$$

This product is *not* invariant under $SL(2,\mathbb{C})$ transformations, although it is invariant under $SU(2)$. There is no other scalar invariant product under $SL(2,\mathbb{C})$. A general theorem states that no non-compact group admits finite-dimensional unitary representations.

If we act with an $SU(2)$ matrix at the given starred positions, the matrices are unitary, and bringing them next to one another gives $UU^\dagger = \mathbb{1}$. This is true for $SU(2)$ but not for $SL(2,\mathbb{C})$.

However, there exists another quadratic antisymmetric bilinear form defined as

$$(\mathbf{z}, \mathbf{w}) = z^A w^B \, \epsilon_{AB}, \tag{15.30}$$

which *is* invariant under both $SU(2)$ and $SL(2,\mathbb{C})$ transformations. The determinant of the transformation satisfies

$$\det g = 1, \tag{15.31}$$

so although the form (15.30) is invariant, the representation itself is not unitary.

The Hermitian scalar product on spinor space is

$$\langle \mathbf{z}, \mathbf{w} \rangle = \bar{z}^A w^B \delta_{AB}. \tag{15.32}$$

This structure is *not* invariant under $SL(2,\mathbb{C})$ transformations and therefore breaks $SL(2,\mathbb{C})$ symmetry down to $SU(2)$. In fact, defining such a scalar product on the spinor space corresponds to selecting a Lorentz frame.

There exist many other finite-dimensional representations of $SL(2,\mathbb{C})$, but they are not suitable for our purposes—we require infinite-dimensional unitary representations. For $SU(2)$, the representation is labeled by

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \tag{15.33}$$

and within the j-representation we have a basis labeled by m:

$$|j, m\rangle \,, \qquad m = -j, -j+1, \dots, j. \tag{15.34}$$

The matrix elements of the representation are written as

$$D^j_{mn}(g), \qquad g \in SU(2). \tag{15.35}$$

For $SL(2,\mathbb{C})$, there are two Casimirs, C_1 and C_2 (see Eqs. (15.26)–(15.27)), so the corresponding unitary representations are infinite-dimensional and are characterized by two quantum numbers.

The unitary irreducible representations of $SL(2,\mathbb{C})$ are labeled by two real parameters:

$$(p, k), \qquad p \in \mathbb{R}, \quad k \in \tfrac{1}{2}\mathbb{Z}. \tag{15.36}$$

The representation space carries states denoted by

$$|p, k; j, m\rangle \,, \tag{15.37}$$

which form the *canonical basis*. The matrix elements of the representation are written as

$$D^{(p,k)}_{jm,j'm'}(g), \qquad g \in SL(2,\mathbb{C}). \tag{15.38}$$

The way we construct the basis for the (p,k) representation is based on a theorem which states that the representation labeled by (p,k) is defined on a Hilbert space $\mathcal{H}^{(p,k)}$ that is infinite-dimensional, and also carries a representation of $SU(2)$. Therefore, it can be decomposed into irreducible $SU(2)$ representations:

$$\mathcal{H}^{(p,k)} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j, \tag{15.39}$$

where each $\mathcal{H}_j$ corresponds to an $SU(2)$ irreducible subspace with basis vectors

$$|p, k; j, m\rangle \,. \tag{15.40}$$

The basis elements are therefore labeled by j, and within each j-subspace we can diagonalize L_z.

For $SU(2)$ we have

$$C = L^2, \qquad L^2\,|j,m\rangle = j(j+1)\,|j,m\rangle\,. \tag{15.41}$$

For $SL(2,\mathbb{C})$, the Casimir operators act on the canonical basis as

$$C_1 = \vec{K}^{\,2} - \vec{L}^{\,2}, \qquad C_1\;|p,k;j,m\rangle = \left(p^2 - k^2 - 1\right)\;|p,k;j,m\rangle\,, \tag{15.42}$$

$$C_2 = \vec{K}\cdot\vec{L}, \qquad C_2\;|p,k;j,m\rangle = -p\,k\;|p,k;j,m\rangle\,. \tag{15.43}$$

We are not proving these results here—they can be found in standard mathematical physics references on the representation theory of non-compact groups. Now, consider the Hilbert space $\mathcal{H}^{(p,k)}$ and impose the following constraint on it. We choose a subset of $\mathcal{H}^{(p,k)}$ where the representations satisfy:

$$p = \gamma\,k, \qquad \gamma \in \mathbb{R}, \tag{15.44}$$

that is, we relate the parameters p and k by a real constant γ (the Barbero–Immirzi parameter). We also select the lowest spin representation in the decomposition (15.39), namely

$$k = j, \qquad \mathcal{H}^{(p,k)} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j. \tag{15.45}$$

Thus, we focus on the subset of states

$$|\gamma j; j, m\rangle\,, \tag{15.46}$$

for which

$$\langle\vec{K}\rangle = \gamma\,\langle\vec{L}\rangle. \tag{15.47}$$

For such states, this condition holds identically. The parameter γ must be real because in the $SL(2,\mathbb{C})$ representation the quantity p is real, while k is a half-integer.

For $SU(2)$, the representation space is

$$\mathcal{H}_j \;=\; \mathrm{span}\{|j,m\rangle\}, \tag{15.48}$$

while for $SL(2,\mathbb{C})$ the representation space is

$$\mathcal{H}^{(p,k)} \;=\; \mathrm{span}\{|p,k;j,m\rangle\}. \tag{15.49}$$

The γ-map establishes a correspondence between these two representations:

$$\gamma_{_\gamma} \;:\; \mathcal{H}_j \;\longrightarrow\; \mathcal{H}^{(p,k)}, \qquad \gamma_{_\gamma}\,|j,m\rangle = |\gamma j, j; j, m\rangle\,. \tag{15.50}$$

This map $\gamma_{_\gamma}$ embeds the $SU(2)$ representation into the $SL(2,\mathbb{C})$ representation under the constraint

$$k = j, \qquad p = \gamma\,j. \tag{15.51}$$

The version of the theory with *imaginary* γ is still an open question in the literature.

The γ-map will play a key role in writing the transition amplitude in the next section.

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16 Loop Quantum Gravity Dynamics

We now have the following tools:

$$|p, k; j, m\rangle \quad \longrightarrow \quad \text{representation of } SL(2,\mathbb{C}), \tag{16.1}$$

together with the γ-map

$$\gamma_\gamma : |j, m\rangle \mapsto |\gamma j, j; j, m\rangle\,, \tag{16.2}$$

which defines a map that embeds representations of $SU(2)$ into those of $SL(2,\mathbb{C})$. In addition, we have the constraint

$$\vec{K} = \gamma\, \vec{L}, \tag{16.3}$$

which is the same relation that connects boosts to rotations in the Lorentz algebra. This is precisely how the boost-to-rotation correspondence (the γ-map) is implemented.

BF Theory and the Action

Now let us turn to the BF theory, whose action reads:

$$S_{\text{BF}}[B, \omega] = \int B \wedge F. \tag{16.4}$$

Here B is a $\mathfrak{g}$-valued 2-form (for some Lie group G), and ω is a generic connection 1-form taking values in the same Lie algebra $\mathfrak{g}$. We want to quantize this theory, i.e. to construct the quantum theory associated with the action (16.4). This case is particularly simple because BF theory is a *topological* field theory: its equations of motion imply

$$F = 0, \qquad DB = 0, \tag{16.5}$$

that is, the curvature F is flat and the 2-form B is covariantly constant. Consequently, the theory contains *no local degrees of freedom*. Everything is rigid and global—there are no local excitations. Next, we will see how such a theory can be defined and quantized through the formalism of quantum path integrals.

We define the quantum amplitude as a path integral over the fields B and ω:

$$Z = \int \mathcal{D}B\, \mathcal{D}\omega\; e^{\frac{i}{\hbar}\int B\wedge F}. \tag{16.6}$$

We want to define it in the same way that Feynman originally defined the path integral in his Ph.D. thesis. In one-dimensional quantum mechanics, Feynman discretized time and constructed the transition amplitude step by step, then reinterpreted it as a sequence of small integrals—each representing a transition—whose composition in the continuum limit defines the path integral.

Thus, the path integral can be understood as the limit of multiple integrals over intermediate configurations:

$$Z = \int \mathcal{D}B\, \mathcal{D}\omega\; e^{\frac{i}{\hbar}\int B\wedge F} \;\approx\; \text{(discretize and write as multiple finite integrals)}. \tag{16.7}$$

We therefore aim to follow the same logic here—discretize the fields B and ω, write the integral as a product of local contributions, and then take the continuum limit to recover the path integral form. Now let us discretize the fields appearing in Z.

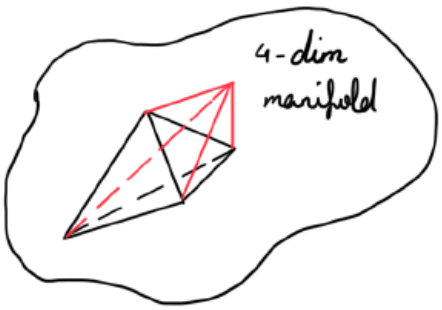

Figure 16.1: The continuous fields are defined on the following 4-dimensional space. We discretize the space with a 4-simplex which is a tetrahedron with an extra point.

We then introduce a *triangulation* of the four-dimensional manifold and define the fields on it. Why do we use a triangulation? In quantum chromodynamics (QCD), when we wish to perform a similar discretization—approximating the integrals by sums—we typically use a *cubic lattice*. Since in QCD there exists a fixed background metric, the cubic lattice is natural and convenient because it respects the Euclidean metric structure and is geometrically simple. However, in our present case we do *not* have a background metric. Therefore, we must employ the most covariant and general type of discretization available—namely, a triangulation. This allows the construction to remain independent of any background geometry. As was done in the three-dimensional case, it is often more convenient to work with the *dual complex* associated with this triangulation.

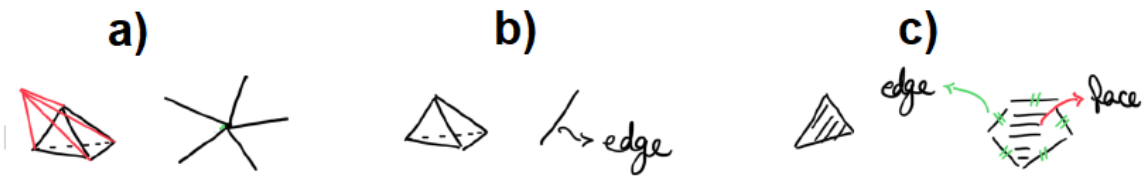

Figure 16.2: Given a 4D triangulation, it is convenient to pass to its dual space. a) A **4–simplex** σ^4 is represented by a **vertex** v in the dual space. b) The **five tetrahedra** τ_a^3 $(a = 1, \ldots, 5)$ bounding σ^4 are represented by the **edges** e_a incident at v. Thus, each tetrahedron corresponds to one dual edge. c) A **triangle** Δ^2 shared by several tetrahedra gives a **face** f of the dual complex. One obtains f by circling around Δ^2 through the sequence of adjacent tetrahedra; in the dual picture this closes a loop made of the corresponding dual edges.

Intuitively: starting from a triangle Δ^2, follow the tetrahedra that meet at Δ^2; the associated dual edges form a closed loop, which is the dual face f.

In the dual space we have **vertices**, **edges**, and **faces**. Surfaces are connected along edges, and edges meet at points. This structure is called a **2–complex**. Abstractly, a 2–complex can be viewed as a graph together with a definition of faces, where the faces are closed curves built along the links of the graph.

$$Z = \int \mathcal{D}B\, \mathcal{D}\omega\, e^{\frac{i}{\hbar}\int B\wedge F} \tag{16.8}$$

The field B is a 2–form. To discretize a 2–form, it is natural to integrate it over a triangle. Therefore, to each triangle—or equivalently, to each **face** of the dual 2–complex—we can associate a discrete variable obtained by integrating B:

$$B \longrightarrow B_{\text{faces}} = \int_{\triangle} B \quad \in \mathfrak{g}, \tag{16.9}$$

where $\mathfrak{g}$ denotes the Lie algebra of some group G.

Similarly, the connection ω is a 1–form and can be integrated along **edges** (which correspond to tetrahedra):

$$\omega \longrightarrow \omega_{\text{edges}} \sim \int_{\text{edge}} \omega \quad \Rightarrow \quad g = e^{\int_{\text{edge}} \omega} \in G. \tag{16.10}$$

When we integrate the connection, we do not simply add the 1–form values; because the integration is taken over the group G, we must instead take the **exponential of the integral**. This gives the corresponding group element $g \in G$.

We are simply performing a **discretization of spacetime**, replacing the continuous fields with discrete ones—exactly as in lattice QCD—except here the procedure is carried out in a more convenient manner, and its significance will become clear later.

The BF path integral is written as:

$$Z = \int \mathcal{D}B\, \mathcal{D}\omega\, e^{\frac{i}{\hbar}\int B\wedge F}, \tag{16.11}$$

where F is the curvature of the connection ω.

We now wish to define and discretize the curvature. When the theory is discretized on a 2–complex, the integral over the connection becomes a product over group elements g_e associated with each edge e, and the curvature becomes a holonomy around each face f. The discrete version of the partition function then reads:

$$Z = \int \prod_f \mathrm{d}B_f \int \prod_e \mathrm{d}g_e \, \exp\left[\tfrac{i}{\hbar}\sum_f \left(B_f \prod_{e\subset f} g_e\right)\right], \tag{16.12}$$

where the product $\prod_{e\subset f} g_e$ runs over all edges e forming the boundary of each face f.

Figure 16.3: The curvature is defined on the **edges** of the dual 2–complex. Geometrically, curvature represents the **anomaly of a connection** when parallel transporting something around a small closed loop.

To discretize the idea described in the caption of Fig. 16.3, we take the **group element** corresponding to the connection along the edges forming the boundary of a face and multiply them together to return to the same point. If there is no curvature, this holonomy gives the **identity element** of the group. We recall the analogous relation from Fourier analysis:

$$\int \mathrm{d}p\, e^{ipx} \;\propto\; \delta(x), \tag{16.13}$$

which connects integration over momentum space to the delta function enforcing flatness. In the BF theory, a similar mechanism ensures that the curvature is constrained to vanish on-shell.

The discrete BF partition function can be written as:

$$Z = \int_G \prod_e \mathrm{d}g_e \prod_f \delta(g_{e_1} g_{e_2} \cdots g_{e_n}), \tag{16.14}$$

where the product in the argument of the delta function runs over all edges $e_1, e_2, \ldots, e_n$ forming the boundary of each face f. This expression depends on the underlying manifold (or equivalently, its triangulation) and on how it is connected. It can be interpreted as a **discretization of the integral**

over all possible flat connections, since the delta functions enforce that the holonomy around each face is trivial, corresponding to vanishing curvature.

We can now rewrite the expression in a slightly different form, doing some cosmetics as shown in the following Figure.

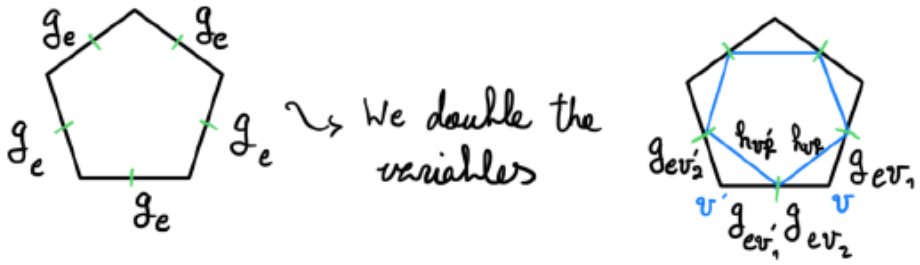

Figure 16.4: A **triangle** Δ^2 shared by several tetrahedra gives a **face** f of the dual complex. We further divide each edge into two parts doubling the number of variables.

We therefore have:

$$\begin{aligned} Z &= \int_G \prod_e \mathrm{d}g_e \prod_f \delta(g_{e_1} \cdots g_{e_n}) \\ &= \int \prod_{e,v} \mathrm{d}g_{ev} \prod_f \delta\left(\prod_{e \subset f} g_{ev}\right) \\ &= \int \prod_{e,v} \mathrm{d}g_{ev} \int \prod_{v,f} \mathrm{d}h_{vf} \prod_f \delta(h_{v_1 f} \cdots h_{v_n f}) \prod_{v,f} \delta(h_{vf} g_{ev} g_{e'v}) \,, \end{aligned} \tag{16.15}$$

where h_{vf} are group elements associated with the pairs of vertices v and faces f.

Now, since we have the product over all vertices and faces, we can collect all the vertex terms and write:

$$Z = \int \prod_{v,f} \mathrm{d}h_{vf} \prod_f \delta(h_{v_1 f} \cdots h_{v_n f}) \prod_v A_v, \tag{16.16}$$

where each vertex amplitude A_v is defined as:

$$A_v = \int \prod_{e \ni v} \mathrm{d}g_{ve} \prod_{f \ni v} \delta(h_{vf} g_{ve} g_{v'e}) \,, \tag{16.17}$$

which represents an integral over all the group elements g's attached to the edges connected to the vertex v, as well as over all faces adjacent to v.

If the group G is compact, this expression is well-defined and can be interpreted as the **path integral of the BF theory** written in a fully discrete form.

We further have the following formulation, which works for any compact group:

$$\delta(g) = \sum_j (2j+1) \operatorname{Tr}_j(g), \tag{16.18}$$

where the trace is taken in the representation j. This relation also holds for $SL(2,\mathbb{C})$ representations, appropriately generalized. Using this expansion, the vertex amplitude $A_v(h)$ can be expressed as:

$$S_v(h) = \int \prod_{e \ni v} \mathrm{d}g_{ve} \prod_{f \ni v} \sum_j \operatorname{Tr}_j(h_{vf} g_{ve} g_{v'e}) \,, \tag{16.19}$$

where the group elements g_{ve} are associated with the edges meeting at vertex v, and h_{vf} with the faces

adjacent to v.

Now we wish to interpret this construction in a group that is $SL(2,\mathbb{C})$, while on the boundary we have $SU(2)$, imposed by the condition $\vec{K} = \gamma \vec{L}$. Each vertex corresponds to a **4-simplex**, and the vertex amplitude A_v represents the amplitude associated with that simplex. It depends on the boundary group elements h_{vf} which sit on its boundary. The boundary of a 4-simplex is a **3-dimensional surface** which itself is triangulated. For this triangulated boundary, we have the set of boundary data that defines the full spin foam amplitude.

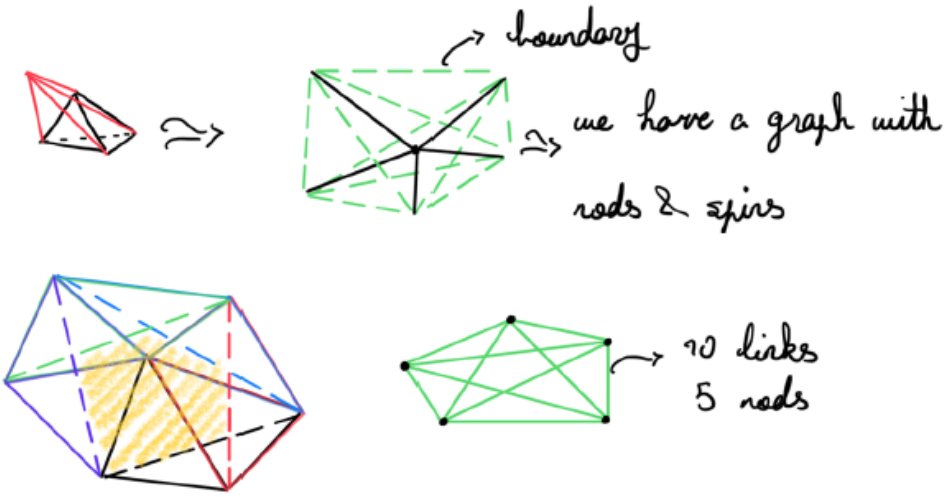

Figure 16.5: **4-dimensional region of spacetime** with a **3-dimensional boundary**, which is discretized by tetrahedra. These tetrahedra serve as the nodes, connected by links, forming a **spin network**. The spin network is composed of 10 links and 5 nodes.

Therefore, on the boundary we have **quantum states of gravity** represented by $SU(2)$ states, while in the bulk we require the full $SL(2,\mathbb{C})$ invariance of the theory to be preserved.

We start with the vertex amplitude defined as:

$$A_v(h) = \int_{SL(2,\mathbb{C})} \prod_{e\ni v} \mathrm{d}g_{ve} \prod_{f\ni v} \sum_j \mathrm{Tr}_j(h_{vf}\, g_{ve}\, g_{v'e'})\,, \tag{16.20}$$

where $h_{vf} \in SU(2)$ and $g_{ve} \in SL(2,\mathbb{C})$. Here, j denotes the spin of the $SU(2)$ representation.

Since we have $SL(2,\mathbb{C})$ group elements but spins j associated with $SU(2)$, we need a way to map $SL(2,\mathbb{C})$ representations onto $SU(2)$. This mapping is implemented by the γ-map:

$$\gamma_\gamma : \mathcal{H}_j \longrightarrow \mathcal{H}^{(p,k)}, \qquad p = \gamma j, \quad k = j. \tag{16.21}$$

The partition function is then written as

$$Z = \int_{SU(2)} \prod_f \mathrm{d}h_{vf}\; \delta(h_{v_1 f}, \ldots, h_{v_n f}) \prod_v A_v(h_{vf}), \tag{16.22}$$

where the vertex amplitude A_v is defined through the $SL(2,\mathbb{C})$ integration:

$$A_v(h) = \int_{SL(2,\mathbb{C})} \prod_{e\ni v} \mathrm{d}g_{ve} \prod_{f\ni v} \sum_j \mathrm{Tr}_j\big(h_{vf}\, \gamma^\dagger g_{ve} g_{v'e'} \gamma\big)\,. \tag{16.23}$$

This expression represents the **amplitude of Loop Quantum Gravity (LQG)**. The object

$$D^{(p,k)}_{jmnm'}(g_{ve} g_{v'e'})$$

denotes the Wigner representation matrix element of $SL(2,\mathbb{C})$, which is unitary and encodes the dynamics of the quantum geometry.

We are ultimately interested in computing the **transition amplitude**—that is, the inner product between boundary spin network states evolved through this spin foam model.

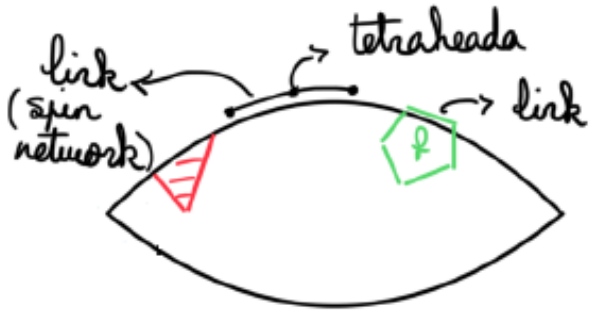

Figure 16.6: The space-time boundary. We are interested in computing the inner product between boundary spin network states evolved through the spin foam model. The 4-simplex is bounded by the boundary with one of its tetrahedrons sitting on the surface of the boundary.

For each link, we define:

$$Z = \int \mathrm{d}h_{vf} \prod_f \delta(h_{v_1 f}, \ldots, h_{v_n f}) \prod_v A_v(h), \tag{16.24}$$

where some of the group elements h_{vf} sit on the boundary. The ones on the boundary will *not* be integrated over. Thus, we define the amplitude that depends explicitly on the boundary variables as:

$$Z_\Delta(h_\ell) = \int \mathrm{d}h_{vf} \prod_f \delta(h_{v_1 f}, \ldots, h_{v_n f}) \prod_v A_v(h), \tag{16.25}$$

where Δ denotes the chosen triangulation. This formula therefore, gives an object that depends on the $SU(2)$ variables of the boundary. Consequently, $Z_\Delta(h_\ell)$ is a function defined on:

$$L^2\left[SU(2)^L\right], \tag{16.26}$$

where L is the total number of boundary links.

We can further describe the boundary Hilbert space as:

$$\mathcal{H}_{\text{boundary}} = \mathcal{H}_{\text{up}} \otimes \mathcal{H}_{\text{down}}. \tag{16.27}$$

Therefore, given any boundary state ψ_{up} or ψ_{down}, we can contract them with $Z_\Delta(h_\ell)$ to obtain the corresponding transition amplitude.

If we have one of the spin network states denoted by ψ_{down} and another by ψ_{up}, then we have the quantum state of geometry $\psi_{\text{up}} \otimes \psi_{\text{down}}$. We can compute the amplitude describing the transition from one to the other.

Is this the final amplitude of Loop Quantum Gravity (LQG)? Keep in mind that Z depends on the triangulation, and this dependence is a truncation artifact. If the Gods are kind, then as we refine the triangulation, the amplitude Z should converge to a fixed number — similar to what happens in lattice QCD.

We have not yet discussed how Einstein's equations and classical General Relativity (GR) emerge from this construction. There exist theorems showing that, under suitable semiclassical conditions, Z contains classical GR within it.

The vertex amplitude is given by:

$$A_v(h) = \int_{SL(2,\mathbb{C})} \prod_{e \ni v} \mathrm{d}g_{ve} \prod_{f \ni v} \sum_j \mathrm{Tr}_j\big(h_{vf}\, \gamma^\dagger g_{ve} g_{v'e'} \gamma\big)\,. \tag{16.28}$$

For each vertex, there are five $SL(2,\mathbb{C})$ integrations corresponding to the five edges emanating from that vertex. After integrating over four $SL(2,\mathbb{C})$ group elements, the resulting expression becomes independent of the fifth. If one integrates over all five, the integral diverges:

$$\int \mathrm{d}g = \infty. \tag{16.29}$$

Hence, we integrate only over four of them to obtain a finite amplitude.

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17 Why Isn't There Any Critical Parameter in LQG?

We have now completed the definition of the theory. Let us summarize it first.

(1) Kinematic Part

We work in Hilbert spaces associated to a graph:

$$\mathcal{H}_\Gamma = L_2\big[\, SU(2)^L/SU(2)^N \,\big]_\Gamma \supset \psi(h_\ell) \tag{17.1}$$

The generators satisfy, for the same link,

$$\big[\, L^i_{\ell'},\, L^j_\ell \,\big] = \delta_{\ell\ell'}\, \epsilon^{ij}{}_k\, L^k_\ell. \tag{17.2}$$

(2) Dynamic Part

We define the edge amplitude:

$$W_e(h_\ell) = \int_{SU(2)} \mathrm{d}h_{vf} \prod_f \delta\left(\prod_v h_{vf}\right) \prod_v A_v(h_{vf})\,, \tag{17.3}$$

where the various vertices surround the face, and the complex is a 2-complex.
The vertex amplitude is

$$A(h_f) = \int_{SL(2,\mathbb{C})} \mathrm{d}g_e \prod_f \left[\sum_j (2j+1)\,\mathrm{Tr}_j\big(h_f\, \gamma^\dagger g_e^\dagger\, g_{e'}\, \gamma\big)\right]. \tag{17.4}$$

Here, the edges come out of the vertex, and a 2-complex is a collection of vertices with 5 edges.

Let's work more closely on the meaning of the indices.

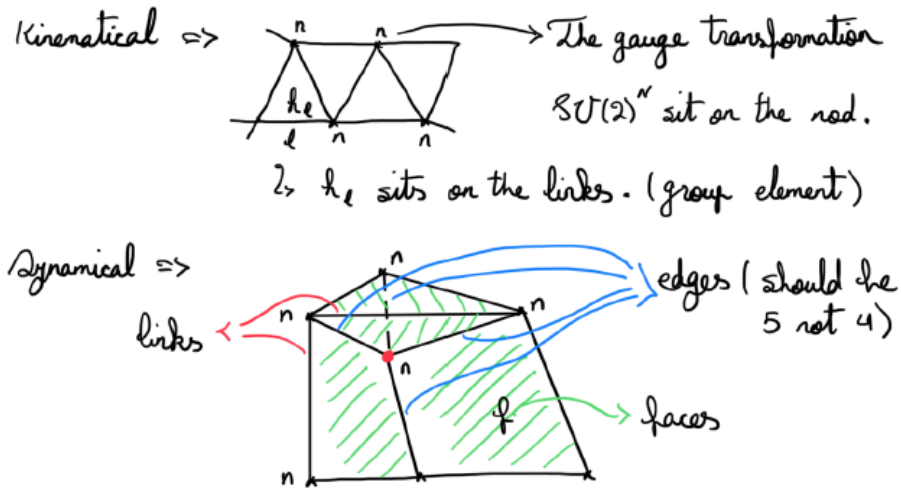

Figure 17.1: Each vertex has edges and faces around itself.

On the boundary of the 2–complex there is a graph such that the face is either bounded by the internal edges ($e \in \mathrm{d}g_e$) or by a link of the graph. The edges are bounded by the node. Keep in mind that the edges originate from the primary node while the links connect all the other nodes together. The states

$$\varphi(h_\ell) = \langle h_\ell \mid \varphi \rangle \tag{17.5}$$

are written on the basis of the group element.

For the spin network basis:

$$|\Gamma, h_\ell\rangle \longleftrightarrow |\Gamma, j_\ell, v_n\rangle\,, \tag{17.6}$$

for some basis in the intertwiner space, such as one that diagonalizes the volume.

$$\langle h_\ell \mid j_\ell, v_n\rangle = \bigotimes_\ell D^{j_\ell}_{\times\times\times}(h_\ell)\, i \;\otimes\; v_n^{\times\times\times} \tag{17.7}$$

The product of all the representations on the links is contracted with all the intertwiners at the nodes. The three lower indices $\times\times\times$ contract with the upper intertwiner indices.

One can change basis from $|\Gamma, h_\ell\rangle$ to $|\Gamma, j_\ell, v_n\rangle$ analogously to:

$$|x\rangle\,,\ |p\rangle \quad \Rightarrow \quad \langle x \mid p\rangle = e^{ipx}. \tag{17.8}$$

So $W_e(h_\ell)$ can be written entirely in a spin network basis. We get an expression entirely on spin networks and intertwiners without group elements. Try to do it; it is in the book. What you get is:

$$W(j_\ell, i_n) = \sum_{j_f,\, i_e} \left(\prod_j (2j+1)\right) \left(\prod_v A_v(j_\ell, v_n)\right), \tag{17.9}$$

where the j_f are the spins associated to faces and the i_e are the intertwiners associated to edges.

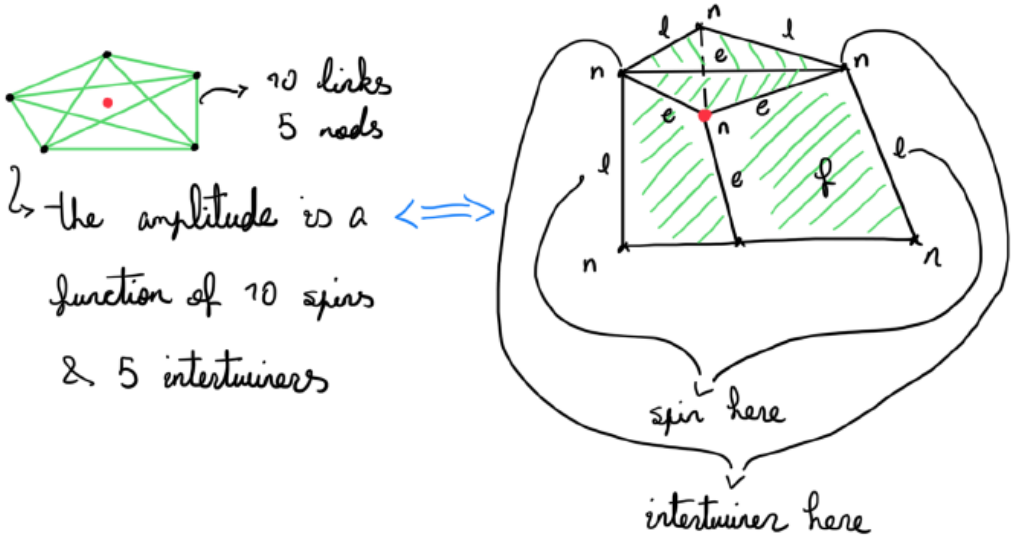

Figure 17.2: The intertwiners sit on the nodes while the spins sit on the edges. The amplitude is a function of 10 spins and 5 intertwiners.

For $A_v(j_\ell, v_n)$ we know the properties for large j (large spin), but we are still working on small spin.

$$A(h_f) = \int_{SL(2,\mathbb{C})} dg_e \prod_f \left[\sum_j (2j+1)\,\mathrm{Tr}_j\left(h_f\, \gamma^\dagger g_e^\dagger\, g_{e'}\, \gamma\right)\right] \tag{17.10}$$

The state around the vertex is a function of $SU(2)$, ψ. With a γ-map you can map it into a function of $SL(2,\mathbb{C})$. Once it is a function of $SL(2,\mathbb{C})$ it is an $SL(2,\mathbb{C})$ network, which is not invariant on the nodes because it started from something which was $SU(2)$ and knows nothing about $SL(2,\mathbb{C})$. If we integrate on the action of $SL(2,\mathbb{C})$, on the nodes you project it down to the $SL(2,\mathbb{C})$ invariant spin network. So you can write the integral of $A(h_f)$ as a projection of the $SL(2,\mathbb{C})$ invariant part.

$$A(\varphi) = \left(P'_{SL(2,\mathbb{C})}\,\gamma\,\psi\right)\cdot\mathbb{1} \Rightarrow \text{If you compute its value on the identity, you get back } A(h_f) \text{ exactly.} \tag{17.11}$$

The core of GR is to take an $SU(2)$ spin network and map it to an $SL(2,\mathbb{C})$ spin network, make it $SL(2,\mathbb{C})$ invariant, and compute its amplitude. The above equation is going to give its action in some limit (Einstein–Hilbert action).

What we have in the theory:

- **Locality**

The fields do not live in a space–time.

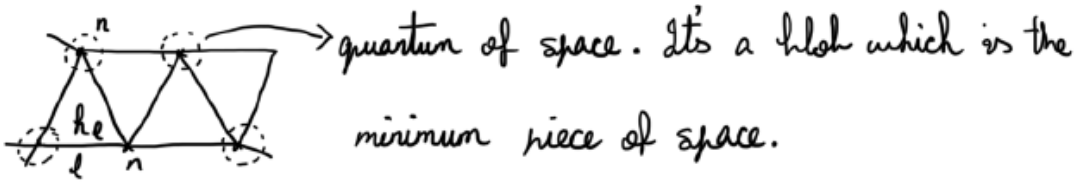

Figure 17.3: The discretization of space further allows us to associate a 3-geometry to the data of the spin network.

The total amplitude $W(j_e, i_n)$ is just a sum over the geometries of the configurations, which is a sum over the local products of amplitudes, something you expect from locality. Things happen locally, each interacting with the next one. There is no long-range interaction. What we expect is:

$$Z = \int \mathcal{D}g\ \exp\left(\frac{i}{\hbar}\int \mathcal{L}\,d^4x\right), \tag{17.12}$$

where g denotes some field and $\mathcal{L}$ is the Lagrangian (so that $\int \mathcal{L}\,d^4x$ is the action).

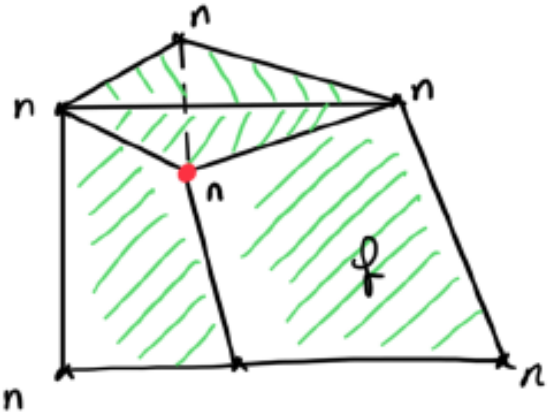

Figure 17.4: Discretization of space-time allows us to associate a 4-geometry to the data of the spin foam.

- **Lorentz invariance**

There is explicit Lorentz invariance induced by Lorentz transformations, and on the boundary we have $SU(2)$ states written in the time gauge. We can go to a fully manifest Lorentz-invariant formalism on the boundary by having a proper $SL(2,\mathbb{C})$ spin network on the boundary, but this does not achieve anything.

- **Ultra Violet Finite (UVF)**

In the sum over j_f and i_e in the amplitude $W(j_e, i_n)$ we are summing over the dimensions of things, and we sum over arbitrarily large dimensions, but we stop at a small scale because j starts at $1/2$. Therefore this cannot reproduce Feynman diagrams for high momentum, because there is no small scale and it is discrete. The entire reason for this formulation is to have discreteness and Lorentz invariance together. The amplitudes do not have a time variable as they evolve, because there is not supposed to be one.

Theorems related to GR (will be shown in the next lectures)

- n-points of GR
- Friedmann equations ...
- Bekenstein–Hawking entropy
- Compute tunneling probability of $BH \to WH$

There are different variants of the theory. The triangulation is only one of these variants. There is a variant introduced by a Polish team where, in the kinematics, the node has more than four links and therefore cannot be explained by the triangulation as discussed.

$$W(j_e, i_n) = \sum_{j_f,\, i_e} \left(\prod_j (2j+1) \right) \left(\prod_v A_v(j_e, v_n) \right). \tag{17.13}$$

What is striking in the above equation is that we have not discussed a parameter to take the critical point. Everybody who does QCD on the lattice and approximations of a quantum theory with a discrete structure knows that you should tune a parameter to a critical point in order to go to the continuous limit. We have not talked about this yet. This is how invariance and lack of a background came into the theory. We go back to Feynman and start from the beginning.

Feynman had the idea that if you want to compute the propagator, you do a functional integral as follows:

$$W = \int \mathcal{D}q \ \exp\left(\frac{i}{\hbar} \int L(q, \dot{q})\, dt \right) = \lim \int dq_n \ \exp\left(\frac{i}{\hbar} \sum_n \int_n L_n[q_n]\, dt \right). \tag{17.14}$$

Path integrals concretely mean either a way to juggle with perturbation theory or a way to discretize the path. We discretize the time interval T into N steps of size

$$\epsilon = \frac{T}{N}, \tag{17.15}$$

so that

$$\langle \phi_f | e^{iHt} | \phi_i \rangle = \langle \phi_f | e^{iH\epsilon_N} \cdots e^{iH\epsilon_1} | \phi_i \rangle. \tag{17.16}$$

Let us do the calculation for a harmonic oscillator. Its Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2), \qquad \dot{q} \ \longrightarrow \ \frac{q_{n+1} - q_n}{\epsilon}, \tag{17.17}$$

which is what we discretize. The discretized action is then

$$S^N = \sum_n \epsilon \left[\left(\frac{q_{n+1} - q_n}{\epsilon} \right)^2 - \omega^2 q_n^2 \right], \tag{17.18}$$

that is,

$$S^N = \sum_n \left[\frac{(q_{n+1} - q_n)^2}{\epsilon} - \epsilon \omega^2 q_n^2 \right]. \tag{17.19}$$

We then change variables according to

$$q_n = \sqrt{\epsilon}\, q_n^*. \tag{17.20}$$

After the change of variables we have

$$S^N = \sum_n \left[(q^*_{n+1} - q^*_n)^2 - \epsilon^2 \omega^2 (q^*_n)^2 \right]. \tag{17.21}$$

From now on I keep writing q instead of q^*, with the same definition of q^*. Thus

$$W = \lim_{\substack{N\to\infty \\ \omega\epsilon\to 0}} \int \prod_n dq_n \ \exp\left[\frac{i}{\hbar} \sum_{n=1}^{N} \Big((q_{n+1} - q_n)^2 - (\epsilon^2\omega^2)\, q_n^2 \Big) \right]. \tag{17.22}$$

Key point: the path integral gives back a continuous theory provided that a parameter is taken to a critical value. This is exactly what we have in lattice QCD.

Let us do exactly the same thing in a language without the background time. We treat q and t on the same ground and do exactly the same thing: both $q(\tau)$ and $t(\tau)$ evolve with respect to some parameter τ (as we do in GR). The action becomes

$$S = \int d\tau \, \frac{dt}{d\tau} \left[\frac{1}{2} \left(\frac{dq/d\tau}{dt/d\tau} \right)^2 - \omega^2 q^2 \right] = \frac{1}{2} \int d\tau \left(\frac{\dot{q}^2}{\dot{t}} - \omega^2 q^2 \dot{t} \right), \tag{17.23}$$

where dots denote derivatives with respect to τ.

The discretized action becomes

$$S^N = \sum_n \epsilon \left[\frac{\left(\dfrac{q_{n+1} - q_n}{\epsilon} \right)^2}{\dfrac{t_{n+1} - t_n}{\epsilon}} - \omega^2 \, \frac{t_{n+1} - t_n}{\epsilon} \, q_n^2 \right]. \tag{17.24}$$

All the ϵ's cancel out and there is no ϵ inside. This is why Regge discretization of GR has no lattice spacing and why the amplitude of LQG has no parameter inside it. The path integral is

$$W = \lim_{N\to\infty} \int \prod_n dt_n \, dq_n \ \exp\left[\frac{i}{\hbar} \sum_n \mathcal{L}(q_n, t_n) \right], \tag{17.25}$$

so the lattice spacing in τ is irrelevant: it is replaced by an extra integration over dt_n. We are not really simplifying our lives; we are rather making it harder.

References and Further Study

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18 Regge Calculus and Intrinsic Coherent States

The moral of the end part of the last section is that when you discretize and do the path integral for a theory defined in a Newtonian or background time (physical time), the continuous limit is obtained by taking the number of steps to approach infinity and by bringing a constant to its critical value. But when we do the same for a theory written in the time parameter there is no critical value to take, and the limit is just taking the number of variables to go to infinity $N \to \infty$. This is the core of the distinction between LQG and QFT, between spin foams and lattice QCD.

$$W = \lim_{N\to\infty} \int \prod_n dt_n \, dq_n \, \exp\left[\frac{i}{\hbar} \sum_n \mathcal{L}(q_n, t_n)\right]. \tag{18.1}$$

This was a very easy example: the harmonic oscillator discretized with independent $q(\tau)$ and $t(\tau)$ variables.

Now, let us do it for GR without going to the quantum theory.
Regge calculus: it shows how you discretize a theory without a parameter and what the truncation is, Figure 18.1. It is also important because, to do the classical limit of LQG (transition amplitudes), we need these steps. It is the clear intermediate step between LQG and classical geometry.

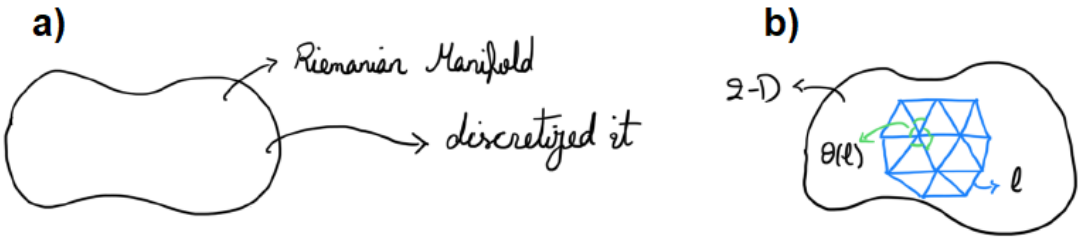

Figure 18.1: a) Regge took a Riemannian manifold and truncated or discretized it. Then, in terms of the discretized variables, he wrote the Einstein–Hilbert action. b) discritizing the surface of a 2-D manifold with triangles.

He did it in the following way. First, assume a 2-D manifold, Figure 18.1 (b). We discretize the surface with flat triangles. His idea was to use lengths l as variables. They do not correspond at all to the links in the discretization. If we have the lengths of each triangle, we can get all the given angles. This is where curvature comes into play: if the sum of the angles is greater or less than 2π, we have curvature.

$$\delta = 2\pi - \sum_n \theta_n, \tag{18.2}$$

which, around one vertex, measures the curvature.

Now in 3-D we have a tetrahedron as follows. From the six lengths, we can calculate the dihedral angles. When we have a number of tetrahedra, we have the same formula. If we know the six lengths we know the geometry of the tetrahedron, and the curvature sits there.

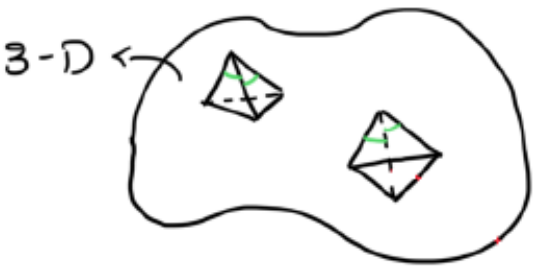

Figure 18.2: In 3-D we have a tetrahedron.

In 4-D we have a 4-simplex. It is bounded by five tetrahedra, and between each pair there is a face. Around a given face there are not only two tetrahedra, but we can make a loop going around the face with multiple tetrahedra. For example, if we have coordinates x, y, z, t, two determine the face and with the other two we go around it. Around the 4-simplex you have two tetrahedra, and these two have two normals in the flat geometry of the 4-simplex. This gives an angle associated to the face between the two tetrahedra, and we have the same formulas. The angles $\theta(\ell)$ are functions of the ten segments which are around a 4-simplex, and the curvature sits on the triangles.

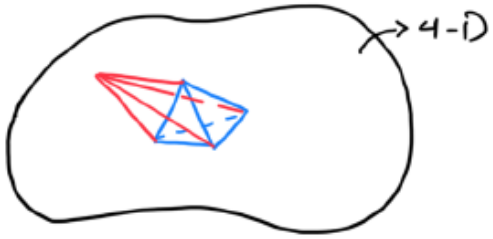

Figure 18.3: In 4-D we have a 4-simplex.

This remarkable construction was done by Regge. If you take the following expression further on, which is the sum over all the triangles in 4-D. For each triangle we take the area A_Δ as a function of all the lengths and multiply it by the deficit angle δ_Δ. We have

$$S_R(\ell) = \sum_\Delta A_\Delta(\ell)\, \delta_\Delta(\ell), \tag{18.3}$$

which gives us a number depending only on the lengths ℓ. If these ℓ's can be approximated as some geometry, $\ell \mapsto g_{\mu\nu}(x)$, this number is computed from that geometry.

What could the number generated out of the geometry be? It has to be something local that depends on the curvature. So we have

$$S_R(\ell) = \sum_\Delta S_\Delta(\ell)\, \delta_\Delta(\ell) = \int d^4x\, \sqrt{g}\, R[g]. \tag{18.4}$$

This is rigorously proven. So, given a manifold with a 4-D metric g and a chosen accuracy ε, you can always find a triangulation such that, when the number of triangles $\Delta \to \infty$, you recover the integral above.

We can think of defining quantum gravity by taking a triangulation, integrating over all the metrics (namely the length variables ℓ), and writing

$$Z = \int \mathrm{d}\ell\ \exp\left(\frac{i}{\hbar} S_R(\ell)\right), \tag{18.5}$$

defined for a fixed triangulation (triangles, tetrahedra, ...). If we take the limit in which the triangulation goes to infinity, we write

$$Z = \lim_{N\to\infty} \int \mathrm{d}\ell\ \exp\left(\frac{i}{\hbar} S_g(\ell)\right). \tag{18.6}$$

We clearly see that we have no parameter here to scale, for the reasons described earlier. The length ℓ does not have to be taken to zero because it is integrated over. So we are not fixing a size, but rather a number of triangulations. It is not a truncation over a length scale, but over the number of degrees of freedom. Therefore it is neither a UV nor an IR cut-off, since we are not dealing with size: we have a manifold, and we approximate a geometry with a finite number of degrees of freedom and we can approximate it with more and more degrees of freedom using the given expression. When we go to LQG we have the same story. The length ℓ can be arbitrarily small or large, and the reason it has a minimum is that it is quantized.

There is another way to write $S_R(\ell)$ by rearranging the terms. The angle $\delta_\Delta(\ell)$ has a sum of

contributions from each simplex. So the Regge action can be written as

$$S_R(\ell) \sim \sum_{\text{simplexes}} \sum_{\Delta \subset \text{simplex}} S_\Delta(\ell)\, \theta_\Delta(\ell), \tag{18.7}$$

and therefore

$$e^{\frac{i}{\hbar} S_R(\ell)} = \prod_{\text{simplexes}} \exp\left[\frac{i}{\hbar} \sum_{\Delta \subset \text{simplex}} S_\Delta(\ell)\, \theta_\Delta(\ell)\right], \tag{18.8}$$

which can be interpreted as the action of a single simplex.

Remember that in the spin-network amplitude the total amplitude is the product of the action for a single simplex. So how is the amplitude $A(j_\ell, i_n)$, written on the boundary for a simplex, related to $e^{\frac{i}{\hbar} S_R}$?

$$e^{\frac{i}{\hbar} S_R} = \prod_{\text{simplex}} \exp\left[\frac{i}{\hbar} \sum_{\Delta \subset \text{simplex}} S_\Delta\, \theta_\Delta\right] \quad \longleftrightarrow \quad A(j_\ell, i_n)\ ? \tag{18.9}$$

So if the two are related, we have the amplitude being related to the Regge action over a simplex. When we put them together we get the Regge action in full over many simplexes, and when $\Delta \to \infty$ for $S_R(\ell)$ we obtain GR in some continuous limit. Thus we have the spin-foam amplitude related to GR. Now the question is whether the vertex amplitude $A(j_\ell, i_n)$ is related to $e^{\frac{i}{\hbar} S_R}$ or not.

So let us prove it now. We go back to the spin networks. The states

$$|\Gamma, j_\ell, v_n\rangle \tag{18.10}$$

two variables give geometrical information about the individual tetrahedron in terms of the areas, and one quantity gives the information about the volume (or the quantity diagonalized in intertwiner space). However, these states do not have a direct geometrical interpretation because:

$$\varphi(x) \ \longleftrightarrow\ |x\rangle\,, \tag{18.11}$$

states that diagonalize position. Similarly, $|p\rangle$ are states that diagonalize momentum. But both kinds of states are completely spread in the conjugate variable.

Now, do we have states in which a particle can be represented by a given position x_0 and a given momentum p_0 such that the spread is minimal? Yes: we have the wave packets as shown in the Figure below.

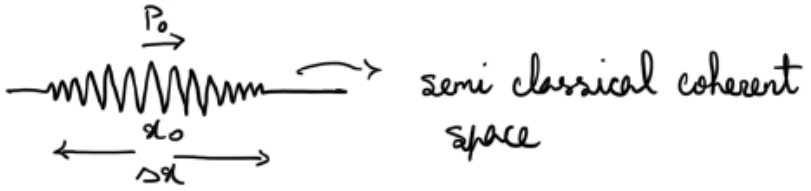

Figure 18.4: The spread in the quantum wave packet.

The same idea might apply to the intertwiner v_n in the states

$$|\Gamma, j_e, v_n\rangle\,. \tag{18.12}$$

If we look at this state, we only have five numbers, while the geometry of the tetrahedron is given by six numbers. Therefore some quantity is completely spread. Can we write a coherent state for the tetrahedron? Can we write a linear combination of $|\Gamma, j_e, v_n\rangle$ which gives us a given shape of a tetrahedron (regular or non-regular) and minimizes the spread in various quantities? Yes, it can be done.

So let us see how this is done. We want a state for the tetrahedron that, for example, keeps the four areas sharp. The intertwiner describes the given geometry of a tetrahedron as well as possible. The

intertwiner is an element of the invariant part:

$$I \in \mathrm{Inv}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_4)\,. \tag{18.13}$$

We write the coherent states in this space and later project to the invariant part. The operator $\vec{L}$ lives in this space; it is the collection of four perpendicular vectors normal to the faces of a tetrahedron as shown in the following Figure.

Figure 18.5: The four operators $\vec{L}$ coming out of the four faces of the tetrahedron.

So we try to make the operator $\vec{L}$ as semi-classical as possible. The operator $\vec{L}$ has three components, and we cannot diagonalize them all simultaneously, but we can minimize their spread. Then we can take over the invariant part of the state that we have obtained by integrating over the rotations. This means that we consider the four directions (the normals in green) only up to common rotations.
The single-edge Hilbert space contains the usual spin-j basis,

$$\mathcal{H}_j \ni |j,m\rangle\,, \tag{18.14}$$

and the state $|j,m\rangle$ is associated with the components of the angular-momentum operator,

$$|j,m\rangle \;\longrightarrow\; \big(L^1,\, L^2,\, L^3\big). \tag{18.15}$$

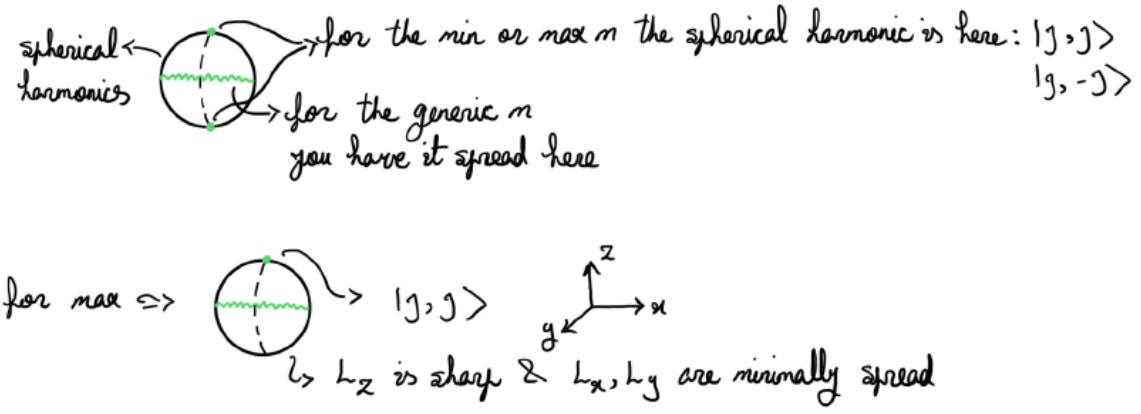

Figure 18.6: Spherical harmonics with the generic spread of m at the center and the max and min values at the poles. It is analogous to the angular momentum vector at quantum mechanics for when L_z is sharp L_X and L_y are minimally spread.

So we take $|j,j\rangle$ as the semiclassical state in which we at least know L_z pointing sharply up and the other two components are minimally spread. We do not want to be attached to the z-direction, and we want to be able to turn and rotate this state. We know how to rotate it because $\mathcal{H}_j$ carries a representation of $SU(2)$, and rotations act naturally on it.

Given any unit vector $\hat{n}$ we can always choose a rotation matrix n that turns the z-direction into it:

$$D^j(n)\;|j,j\rangle = |j,\hat{n}\rangle\,. \tag{18.16}$$

Then we take $|j,j\rangle$ and rotate it with $D^j(n)$; in the $|j,m\rangle$ basis we have

$$\langle j,m|j,\hat{n}\rangle = D^j_{mj}(n). \tag{18.17}$$

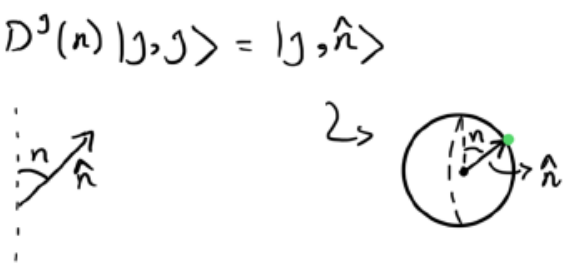

Figure 18.7: We can use the Wigner matrix to rotate the vector along the $\hat{n}$ direction.

So we now have a state corresponding to a vector pointing in a given direction. Now we want four of them, so we consider

$$|j_1,\hat{n}_1\rangle \otimes \cdots \otimes |j_4,\hat{n}_4\rangle\,, \tag{18.18}$$

which depends on four unit vectors. Recall that the intertwiner lives in

$$I \in \mathrm{Inv}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_4})\,, \tag{18.19}$$

and the states (18.18) sit in this tensor product space.

Now we want the invariant part. We rotate the state by a common rotation and integrate over all possible rotations. The result is invariant under the common action of $SU(2)$, because we are rotating all four legs together, and it no longer depends on the actual four directions $\hat{n}_a$ and keeps the angle invariant.

The averaging over rotations keeps the angles invariant. It depends on the vectors only up to a common rotation, and therefore it really depends on the geometry of the tetrahedron.

$$\int_{SU(2)} \mathrm{d}h\ |j_1,h\hat{n}_1\rangle \otimes \cdots \otimes |j_4,h\hat{n}_4\rangle = |j_1,j_2,j_3,j_4;\hat{n}_1,\hat{n}_2,\hat{n}_3,\hat{n}_4\rangle\,, \tag{18.20}$$

where h is an arbitrary $SU(2)$ element acting on each $\hat{n}_a$. This final state does not really depend on the individual four directions; it depends only on their relative orientation, i.e. on the direction up to a common rotation. Thus, it depends on the angles between the four normals (the four green vectors sketched on the tetrahedron earlier in Fig. 18.5). The state

$$|j_1,j_2,j_3,j_4;\hat{n}_1,\hat{n}_2,\hat{n}_3,\hat{n}_4\rangle \tag{18.21}$$

is a coherent intertwiner that corresponds to the given geometry of a tetrahedron.

Take a given 4-simplex as the following Figure, in which we have five tetrahedra.

Figure 18.8: A 4-simplex with 5 tetrahedra shown in green.

Now we fix the geometry of the 4-simplex. The geometry of the 4-simplex determines the geometry of each tetrahedron and determines the area of each face. So we can write a state having all the areas j_1,j_2,j_3,j_4 sharp and all the geometries $\hat{n}_1,\hat{n}_2,\hat{n}_3,\hat{n}_4$ fixed, by putting an intertwiner on each of the five

nodes of the boundary graph. On each node we put one of the coherent intertwiners

$$|j_1, j_2, j_3, j_4; \hat{n}_1, \hat{n}_2, \hat{n}_3, \hat{n}_4\rangle\,. \tag{18.22}$$

Thus we have a boundary state that depends on the geometry of the 4-simplex, which we denote schematically by

$$\varphi\big(\text{geometry of 4-simplex}\big). \tag{18.23}$$

Now we can associate an amplitude to this state,

$$A\big[\,\varphi\big(\text{geometry of 4-simplex}\big)\big] \;\longrightarrow\; \exp\left(\frac{i}{\hbar}\sum_{\Delta} A_\Delta\,\Theta_\Delta\right), \tag{18.24}$$

where the sum is over all triangles Δ of the 4-simplex, the A_Δ are their areas, and the Θ_Δ are the corresponding dihedral angles of the geometry discussed.

This means that the amplitude, which at first sight seems to be defined purely by group combinatorics and not to have anything to do with geometry or with Einstein's equations in the large-scale limit, actually has the correct classical behaviour. In the large–quantum-number (Bohr classical) limit it gives locally the Regge action, which, in the suitable limit in which the triangulation is refined, reproduces general relativity.

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19 Extrinsic coherent states & Classical limit

In the Hilbert space of LQG the basis is given by the spin network states.

$$|\Gamma, j_\ell, v_n\rangle \Longleftrightarrow \text{diagonalize, make certain variables sharp but not the 3D geometry.}$$

Only some pieces of it are: j_ℓ is the quantum number of area, and v_n is the quantum number of volume. But at fixed volume the dihedral angles are completely spread. So there is a one-parameter family of tetrahedra that all have the same area and volume but are completely different in shape. So these states are completely spread on this one-parameter family of states.

In the previous sections we described a technique to write coherent states, which describe a given geometry. There are 6 classical numbers that define the geometry of a tetrahedron. We have only 5

$$|\gamma, j_\ell, v_n\rangle \Longrightarrow 4 \text{ surfaces \& one volume.} \tag{19.1}$$

The clean way is that instead of j_ℓ, v_n we use $\hat{n}_i, n_i$: the perpendicular vector on each surface, or the normal, which has a direction in space given by θ & ϕ. So we have $4\,\hat{n}_i$ multiplied by θ & ϕ and therefore 8 variables.

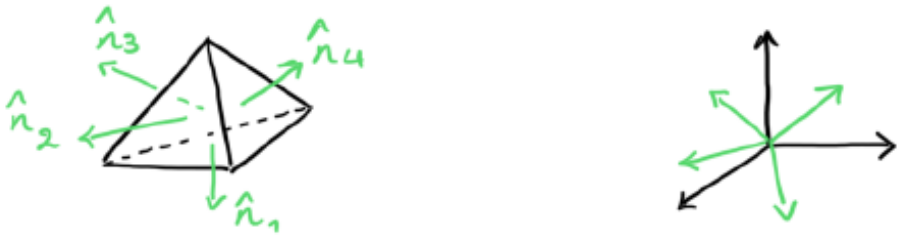

Figure 19.1: Using $\hat{n}_i, n_i$: the perpendicular vector on each surface which has a direction in space given by θ & ϕ instead of j_ℓ, v_n we.

Pay attention that there are a lot of redundancies regarding $\hat{n}_1, \ldots, \hat{n}_4$ because as we rotate the tetrahedron we get the same thing. So we have 3 parameters of rotation which define the same state, and also we know that the sum of the 4 normal vectors along with the area is zero. So we write the following:

$$\sum_{a=1}^{4} A_a\, \hat{n}_a = 0 \quad \Rightarrow \quad 3 \text{ condition(3 free parameters)} + 3 \text{ rotation} \quad \Rightarrow \quad 8 - 6 = 2 \;\; |j_\ell, v_n\rangle\,. \tag{19.2}$$

So given $\hat{n}_1, \ldots, \hat{n}_4$ how do we write the state? We have:

$$|\gamma, j_\ell, (\hat{n}_1, \ldots, \hat{n}_4)\rangle \qquad \text{for each node } \hat{n}_i \text{ there is an intertwiner related to it.} \tag{19.3}$$

$$\text{Inv}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_4}) \ni i^{m_1,\ldots,m_4}_{\hat{n}_1,\ldots,\hat{n}_4} \qquad \text{each one in a representation } j_1, \ldots, j_4. \tag{19.4}$$

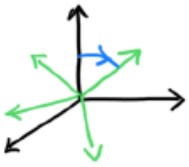

Figure 19.2: Rotating the $|j, j\rangle$ state.

The way we construct it is by taking the $|j,j\rangle$ state and rotating it to the direction we want. This can be seen in Fig. 19.2. We tensor the 4 and then make the whole thing invariant by integrating over SU(2).

$$R^i{}_j(n_a)\,\hat{z}^{\,j} = \hat{n}_a^{\,i} \qquad \text{unit element.} \tag{19.5}$$

where $R^i{}_j$ is the rotation matrix, n_a is the normal vector and $\hat{z}$ is a unit vector in the z direction. Rotating $\hat{z}$ in the direction of $\hat{n}$. There are, of course, plenty of rotations coming to the same result. You can rotate as follows

Figure 19.3: Different ways of rotating the vector.

This arbitrariness reflects in the definition of the coherent state $|\Gamma, j_\ell, (\hat{n}_1,\ldots,\hat{n}_4)\rangle$.

$$\mathrm{Inv}(\mathcal{H}_{j_1}\otimes\cdots\otimes\mathcal{H}_{j_4}) \ni i^{m_1,\ldots,m_4}_{\hat{n}_1,\ldots,\hat{n}_4} = \int dh\; D^{j_1}_{j_1 m_1}(h\,n_1)\cdots D^{j_4}_{j_4 m_4}(h\,n_4). \tag{19.6}$$

These are not yet the full set of coherent states needed to write the geometry.

$\uparrow\uparrow$ Intrinsic coherent states

$\downarrow\downarrow$ Extrinsic coherent states

The different components of L_a^i don't commute with one another, so we cannot make L_a^i all that sharp. But we can make states so that the spread of L_a^i becomes minimum. Pay attention that L_a^i are half of the variables. We have the $\vec{L}$ & h_ℓ operators.

$$\vec{L}_\ell, \qquad h_\ell. \tag{19.7}$$

$\vec{L}_\ell$: the derivative, the left-invariant vector field.

h_ℓ: the diagonal operators in the representation $\Psi(h_\ell)$.

$\vec{L}_\ell, h_\ell \Rightarrow$ in representation, when you diagonalize a part of $\vec{L}_\ell, h_\ell$, it will become complicated. The operators act in a complicated manner. They are like x,p or p,x, depending on what we want to see them.

$$\vec{L}_\ell \;\to\; \begin{matrix} x \\ p \end{matrix}, \qquad h_\ell \;\to\; \begin{matrix} p \\ x \end{matrix}. \tag{19.8}$$

So can we make coherent states for which both $\vec{L}_\ell$ & h_ℓ are sharp?
First, what is the operator h_ℓ? We said it has something to do with the connection when we go from one tetrahedron to the other, associated to the link; however, we have been vague about this connection.

So we want to know to which geometrical property h_ℓ corresponds in general relativity. One is tempted to say it's a 3-dim rotation, but we'll see this isn't the case. Because the spin connection which knows about the 2-dimensional connection, is a function of the intrinsic geometry. So if this were true, h_ℓ would be a function of $\vec{L}_\ell$ and would commute with it. Therefore, it has to do something with the time derivative of the 3-geometry.

The time derivative of the 3-geometry in the 4-dimensional picture is the extrinsic geometry. So if you have a 3-dim geometry in 4-dim, the intrinsic geometry is the one that you get on the surface itself, and the extrinsic geometry is how it is embedded in a 4-dim space.

So let's do it more precisely and write the action as follows:

$$S = \frac{1}{16\pi G}\int \big((e\wedge e)\wedge(F^* + \tfrac{1}{\gamma}F)\big), \qquad \vec{L}_\ell \text{ is what describes the geometry of the 3-D surface} \tag{19.9}$$

$$L^i = \frac{1}{2}\, e^j \wedge e^k\, \epsilon^i{}_{jk} \qquad \text{when it's on a link.} \tag{19.10}$$

$$L_i = \frac{1}{2} \int_\Delta e^j_a \wedge e^k_b\, \epsilon^i{}_{jk}\, dx^a\, dx^b, \tag{19.11}$$

Where $e^j_a \wedge e^k_b$ is the space–space component of the object which is there, and we integrate over the triangle dual to the link.

So now let's look for its conjugate variable. How do we look for the conjugate variable from the action? We take the derivative of the Lagrangian

$$(e \wedge e) \wedge \left(F^* + \tfrac{1}{\gamma} F\right) \tag{19.12}$$

with respect to the time derivative of $(e \wedge e)$.

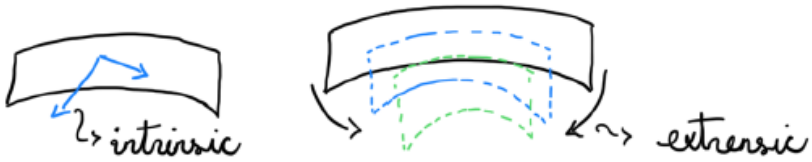

Figure 19.4: The intrinsic and the extrinsic curvature.

$$F = d\omega + \omega \wedge \omega. \tag{19.13}$$

We only want to see the time derivative, and the term $\omega \wedge \omega$ doesn't have any time derivative; therefore we won't concern ourselves with it.

$$S = \frac{1}{16\pi G} \int (e \wedge e)^{IJ} \wedge \left(F^* + \frac{1}{\gamma} F\right)_{IJ}. \tag{19.14}$$

$$= \frac{1}{16\pi G} \int (e \wedge e)^{IJ} \wedge \left(d\omega^* + \frac{1}{\gamma}\, d\omega\right)_{IJ} \qquad \Rightarrow \quad \text{we integrate by parts and write} \tag{19.15}$$

$$= \frac{1}{16\pi G} \int d\Big[(e \wedge e)^{IJ} \wedge \left(\omega^* + \tfrac{1}{\gamma}\omega\right)_{IJ}\Big] - \frac{1}{16\pi G} \int d(e \wedge e)^{IJ} \wedge \left(\omega^* + \tfrac{1}{\gamma}\omega\right)_{IJ}. \tag{19.16}$$

$$= -\frac{1}{16\pi G} \int d_\mu\, e^I_\nu\, e^J_\rho \left(\omega^{KL}_b\, \epsilon_{IJKL} + \tfrac{1}{\gamma}\, \omega^{IJ}_b\right) \epsilon^{\mu\nu\rho b}. \tag{19.17}$$

So where is the time derivative? Only when $\mu = 0$.

$$\Rightarrow \quad S = -\frac{1}{16\pi G} \int d_0\, e^I_\nu\, e^J_\rho \left(\omega^{KL}_b\, \epsilon_{IJKL} + \tfrac{1}{\gamma}\, \omega^{IJ}_b\right) \epsilon^{0\nu\rho b}. \tag{19.18}$$

(Here ν, ρ, b should be the space coordinates.)

$$S = -\frac{1}{16\pi G} \int d_0\, e^I_a\, e^J_b \left(\omega^{KL}_c\, \epsilon_{IJKL} + \tfrac{1}{\gamma}\, \omega^{IJ}_c\right) \epsilon^{0abc}. \tag{19.19}$$

So this is the conjugate.

When we are in a time gauge $e^0_a = 0$ we have:

$$S = -\frac{1}{16\pi G} \int d_0\, e^i_a\, e^j_b \left(\omega^{KL}_c\, \epsilon_{ijKL} + \tfrac{1}{\gamma}\, \omega^{ij}_c\right) \epsilon^{0abc}, \qquad \text{(one should be zero).} \tag{19.20}$$

$$S = -\frac{1}{16\pi G}\int d_0\, e^i_a\, e^j_b \left(\omega^{0k}_c\, \epsilon_{ijk0} + \tfrac{1}{\gamma}\,\omega^{ij}_c\right) \epsilon^{0abc}. \tag{19.21}$$

So the conjugate variable of

$$L^i = \frac{1}{2}\int e^j_a \wedge e^k_b\, \epsilon^i{}_{jk}\, dx^a\, dx^b \tag{19.22}$$

is the sum of $\left(\omega^{0k}_c\, \epsilon_{ijk0} + \frac{1}{\gamma}\,\omega^{ij}_c\right)$, where we have two terms of the connection. One is the connection in the space direction, ω^{ij}_c, which is the 3-D spin connection on the surface itself. It is on the solution of the equation of motion and is determined by $d_0 e^i_a e^j_b$.

$$de^i + \omega^i{}_j \wedge e^j = 0 \qquad \Rightarrow \qquad \text{This equation, given } e, \text{ defines the spin connection.} \tag{19.23}$$

Now the term ω^{0k}_c has a zero component and has nothing to do with the 3-D geometry. This has to do with the 4-D geometry, and it is a piece of the 4-D connection. So we have:

$$E^i \iff \frac{1}{\gamma}\,\omega^{ij} + \omega^{0i} \qquad \Rightarrow \qquad \text{Ashtekar connection } A. \tag{19.24}$$

Here ω^{ij} is the 3-D connection, and ω^{0i} is the extrinsic curvature (the boost component of the 4-D connection), and E^i is the conjugate of L^i (I haven't written the epsilon).
Be careful that h_ℓ is not the holonomy of only ω^{0i}, but it is the holonomy of the entire Ashtekar connection, which knows about the extrinsic curvature.

So where does the extrinsic curvature sit? Let's explain it with the following spin network.

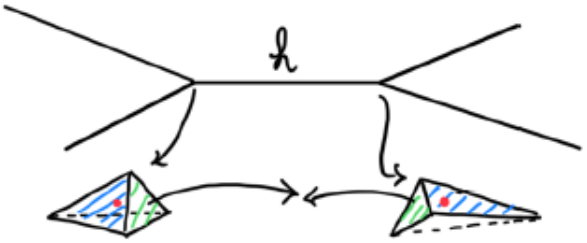

Figure 19.5: The two green triangles are attached to one another and the line goes from one tetrahedron to the other.

We can always orient the reference frame between the 2 tetrahedra so that there is no rotation between the two. Just parallel transport the frame from one to the other. Therefore, ω^{ij} can be chosen to be zero.

$$E^i \iff \frac{1}{\gamma}\,\omega^{ij} + \omega^{0i} \qquad \Rightarrow \tag{19.25}$$

Here ω^{ij} can be chosen to be zero, but ω^{i0} still remains, and it represents part of the 4-D connection between 2 points.

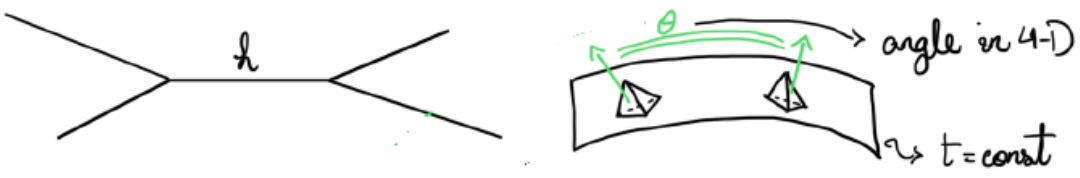

Figure 19.6: Because of the angle between the two normal vectors on the surface, there is an extrinsic curvature. θ is a discrete version of the extrinsic curvature.

The angle is a rotation in the 4-D space. So the conjugate variable is θ. Now let's think about the Regge action.

$$S_{\text{Regge}} = \sum_{\text{4-simplices}} S_{\text{4-simplex}}. \tag{19.26}$$

($S_{\text{4-simplex}}$ is the 4-D angle.)

$$S_{4s}(\ell) = \sum_{\Delta} A_{\Delta}(\ell)\,\theta(\ell). \tag{19.27}$$

The Regge action can be written as a sum over 4-simplices, as an action of a 4-simplex. There are 10 numbers: all the sides of a 4-simplex, which are the 6 sides of a tetrahedron, and then 4 more coming out of it.

Figure 19.7: Each node is a tetrahedron, and between each two there is a triangle.

So we can think about the action of the 4-simplex as a sum of the variables & its dual.

$$S_{4s}(\ell) = \sum_{\Delta} A_{\Delta}(\ell)\,\theta(\ell) \;\sim\; \sum x\,p(x), \tag{19.28}$$

where θ is the angle in green. This is the conjugate variable to the intrinsic geometry, whose main variable is the area.
This is basically the general form of the action:

$$S\big(q_a^{\text{in}}, q_a^f\big) \;\Rightarrow\; \text{satisfies} \;\Rightarrow\; \frac{\partial S}{\partial q_a^{\text{in}}} = P_a\big(q_a^{\text{in}}, q_a^f\big)\,. \tag{19.29}$$

Take a free particle and we have:

$$S = \frac{1}{2} m\,\frac{\big(q^{\text{in}} - q^f\big)^2}{t^t - t^{\text{in}}}. \tag{19.30}$$

$$S = \frac{1}{2} m\,\frac{\big(q^{\text{in}} - q^f\big)^2}{t^t - t^{\text{in}}} = \frac{1}{2} m\left[q^{\text{in}}\frac{q^{\text{in}} - q^f}{t^t - t^{\text{in}}} - q^f\,\frac{q^{\text{in}} - q^f}{t^t - t^{\text{in}}}\right]. \tag{19.31}$$

$$\sim\; q^{\text{in}}\,P\big(q^{\text{in}}, q^f\big) - q^t\,P\big(q^{\text{in}}, q^f\big)\,. \tag{19.32}$$

So we can always think of the Hamilton function as a sum of q and p as a function of q.

$$S = \sum q\,p(q) \qquad \Rightarrow \qquad \text{this is where the information about the dynamics is contained.} \tag{19.33}$$

We said we should consider q & t at the same level. So why have we only considered q and not t? We'll consider it now.

$$t_{\text{in}}\,E_{\text{in}}(q,t) - t_p\,E_p(q,t) = (t_{\text{in}} - t_p)\,E(q,t), \qquad \text{(energy is conserved).} \tag{19.34}$$

$$= (t_{\text{in}} - t_f)\,\frac{1}{2} m\,\frac{(q_{\text{in}} - q_f)^2}{(t_{\text{in}} - t_f)^2} \qquad \Rightarrow \qquad \text{we get back } S\big(q_a^{\text{in}}, q_a^f\big)\,. \tag{19.35}$$

Extrinsic coherent states

Taking a classical particle with a certain velocity and building the quantum state in which the wave packet moves. Similarly, we can take a surface in curved space-time, approximate it by triangulation, and explicitly write a quantum state which is peaked around it, both in terms of intrinsic and extrinsic geometry. The wave packet is written as

$$\langle x \mid x_0, p_0\rangle = e^{-\frac{(x-x_0)^2}{2\delta} + i p_0 x}, \tag{19.36}$$

where σ is the spread and the variable x is centered about x_0 with a certain spread. The variables are x & p, and the wave packet is built on the classical quantity on which the state has been built.

Similarly, we can build states labeled by:

$$|\text{classical 3-geometry, classical extrinsic geometry}\rangle = \sum_{j_\ell,\, v_n} |j_\ell, v_n, \hat{n}_\ell\rangle\,(\ldots), \tag{19.37}$$

where the classical 3-geometry is the geometry of triangulation, the classical extrinsic geometry is the angle θ, and $(\ldots)$ is the coefficients depending on j_ℓ, v_n which are determined by the extrinsic and intrinsic geometry.

Barrett et al.:
The results are written in terms of the intrinsic state, not the extrinsic state. We consider the 4-simplices, which gives us 10 areas as shown as follow

Figure 19.8: The areas are defined $j_{ab} = j_{ba}$

$$A(j_{ab}, n_{ab}) \Rightarrow \begin{cases} \text{exponentially suppressed unless } n_{ab} = n_{ab}(j_{ab}), \\ \sim n\, \exp[i \textstyle\sum_{ab} j_{ab}\, \theta_{ab}(j_{ab})] + \text{complex conjugate}, \end{cases} \tag{19.38}$$

(n_{ab} is the normals n for all the tetrahedrons).

If we fix the areas, we fix the geometry. If we fix the geometry, we fix the geometry of each tetrahedron, so we fix the n's as a function of j up to the rotation of each one of them. If we also fix the geometry, we fix the angles $\theta(\ell)$. Then we have $n_{ab}(j_{ab})$ and $\theta_{ab}(j_{ab})$. The result says that the amplitude in this case is exponentially suppressed in the large j_{ab} limit unless $n_{ab} = n_{ab}(j_{ab})$. So the quantum dynamics suppresses all the configurations that are not right and don't fit. If the n_{ab} are right, $A(j_{ab}, n_{ab})$ is proportional to the second line above. The calculations are very complicated, but in the end, it shows that GR knows about the geometry, and the geometry knows about GR.

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20 Applications, the End of Black Holes

Where are the places that we can use the theory?

In cases where quantum gravitational phenomena are not negligible. Two typical cases are the early universe and black holes (BHs). There might be others, but these are the two cases where we clearly need quantum gravity. Here we want to see what happens to BHs using the mechanism we have developed so far. Where we get lost is at $r = 0$, where a particle goes inside a BH, crosses the horizon, and goes to $r = 0$. What happens? We don't know. Also, we don't know what happens in the distant future of a BH. The BH shrinks, but what happens then? It can't just turn into nothing.

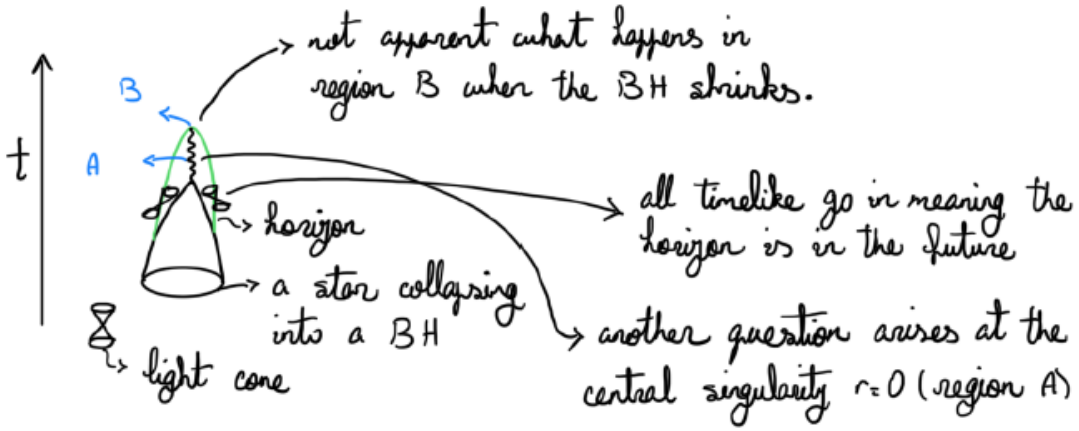

Figure 20.1: The evaporation of a Black Hole

Let's draw this in a more relativistic way, shown in Fig. 20.2.
We put the light rays at 45°. We get the standard picture of a BH forming and a star collapsing. So we have the following:
We don't know what happens when we fall into a singularity, and we don't know what happens at B, which means we need LQG to go there.

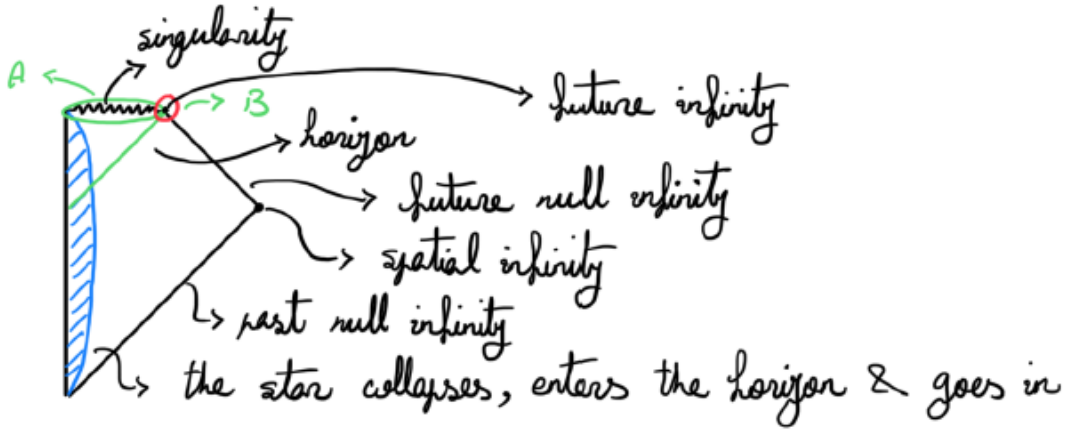

Figure 20.2: The collapse of a star.

So the picture shown shouldn't be taken seriously because it's not yet complete. Another way of viewing this story is by thinking of an end of the story which is a spacetime with a causal structure, and that the story happens in the middle. Due to a conjecture by Penrose called *cosmological censorship conjecture*. We know that every time we have a singularity in classical GR it is always hidden inside a horizon. This conjecture is mostly true. So if it's true, GR is invariant under time reversal, and therefore its opposite is also true. So if you look from the future back into the singularity, it will also be closed by a horizon.

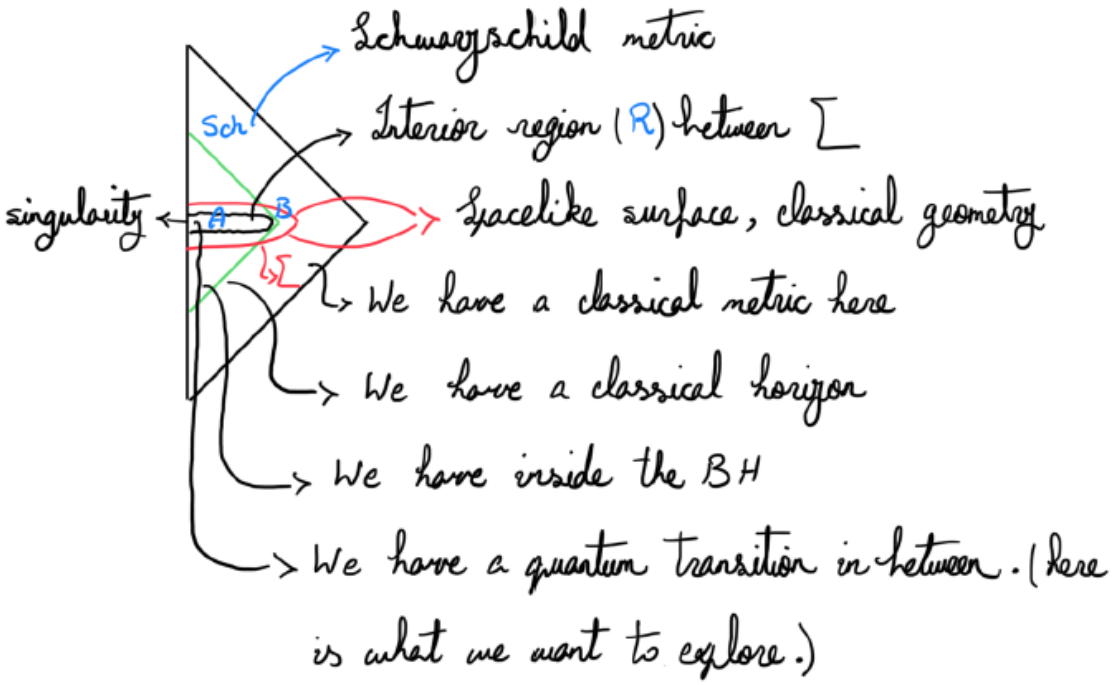

Figure 20.3: Schwarzschild metric and quantum singularity.

Studying the quantum transition we need two things: a classical description of whatever is around the singularity and a quantum description of what happens at the singularity. What we need is what exactly LQG has given us. Based on a Wigner description of a quantum system, we can choose our boundary between the classical system that we want and the probability would be the same. We can always move our boundary out and close the singularity that we have inside the red boundaries of the figure. We go from one classical geometry to another classical geometry and calculate the amplitude for it. This is exactly how LQG has been formulated. To study this phenomenon, the project is:

1. Study the metric outside (everything outside is a 3-D surface).
2. From that we compute the metric of the 3-D surface, Σ. We not only know the metric, but we know the intrinsic & extrinsic geometry of Σ. Now we forget all the rest and compute the probability amplitude of the red boundary. We can use LQG to compute the amplitudes, and to do that, since LQG is formulating truncation, so we must truncate the theory. Personally, what tunnels from down to up? Why tunnelling? Because there is no classical transition from the lower to the upper bound. (We have a singularity in the middle and can't do it classically.)
3. Truncate the theory by doing a discretization of Σ (3-D) and the interior region blue$\mathcal{R}$ (4-D), and then use the quantum theory. So we have the intrinsic and extrinsic geometry of a discretization of redΣ, and we can write the coherent state in the Hilbert space of LQG.
4. Writing the coherent state.
5. Finally we can write the amplitude and see if we can learn something about it.

Now let's go over the scheme and see what we can get from it.

1. If you are a good relativist, you'll argue that there is no geometry possible outside the structure drawn because it seems impossible. But it's not. The metric outside the BH and WH has a Schwarzschild metric, and its Kruskal metric is as follows:

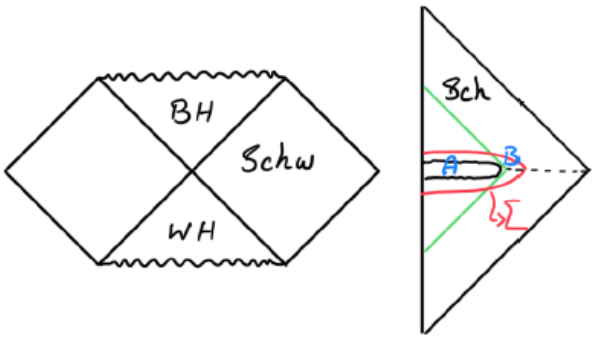

Figure 20.4: We want to build a metric around B that is locally isomorphic to Kruskal, from cutting and pasting the surface appropriately.

There is a metric in which the big picture is as follows. We do the simplest version, which assumes the picture along the dotted line above is time symmetric.

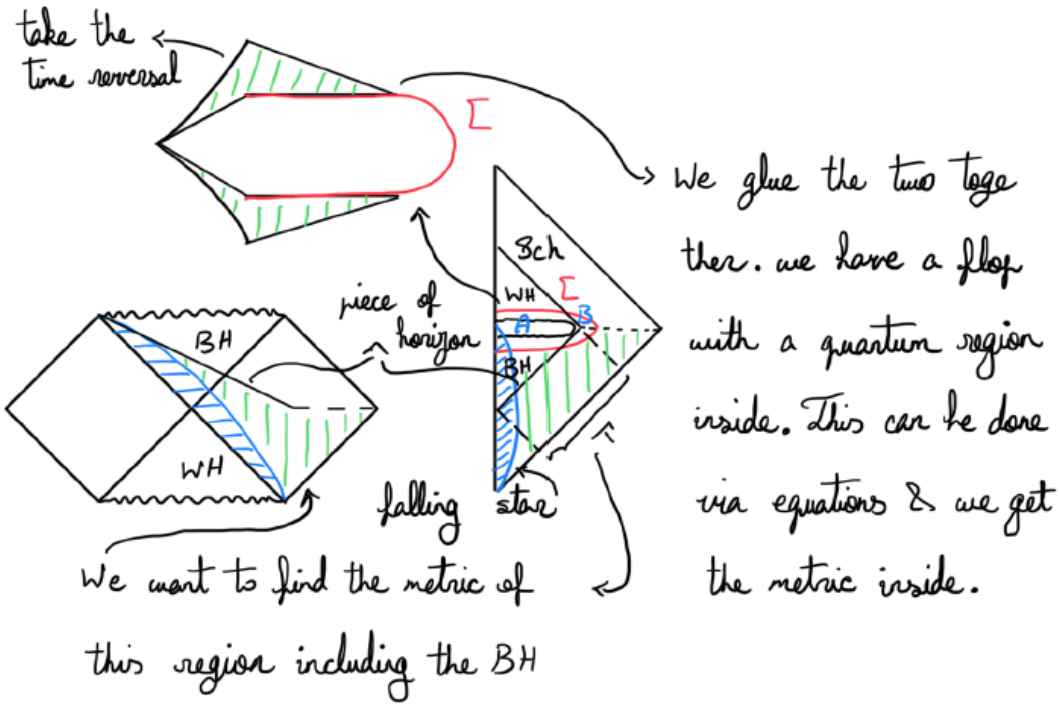

Figure 20.5: Piecing the metric together.

The result of the metric only depends on two parameters: mass m and time T. The mass is the mass of the Schwarzschild BH, and the time is the time that passes, and can be formulated in different ways. It can be written as the time at which the horizon forms and the transition happens. This characterization of time has been shown in the previous figure. The external metric can be written in terms of mass and time, $ds(m, T)$. The mass is just the mass of the star. The quantum theory tells you how you will continue after the BH, and it would be a quantum theory that gives us the time. The quantum theory tells us how much time it takes for a star of mass m to undergo this procedure.

2. The geometry of Σ, both intrinsic & extrinsic, is written as (m, T), which means the amplitude at the end of the day is written as $W(m, T)$.

3. Now we truncate the theory using discretization.

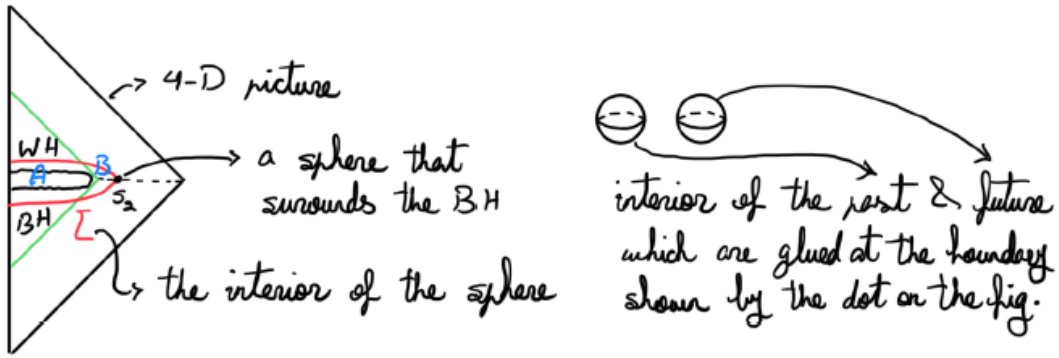

Figure 20.6: Truncating the theory using dicretization.

To discretize a bawl we take the interior of a tetrahedron. This would be too complicated. So we can take a point and break it into 4 tetrahedron.

Figure 20.7: We have 4 tetrahedron attached to each other whose surface is attached to the other 4 tetrahedrons.

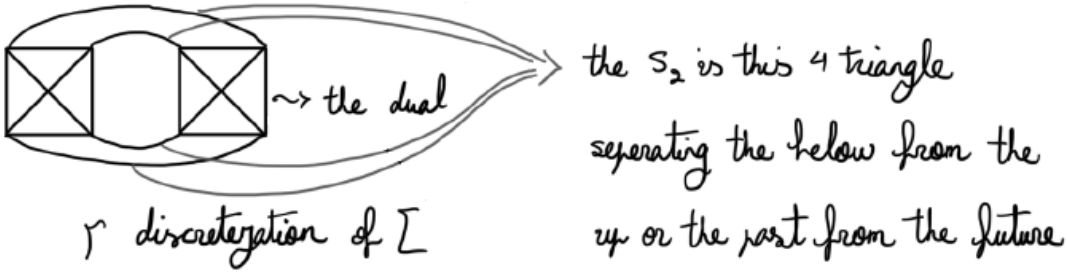

Figure 20.8: Graph of the Spin Network.

So now we have a discrete metric on the 2 pair 4 tetrahedrons and so we explicitly have sizes and angles of the triangles. So we can write a coherent state corresponding to the metric we get from outside. So we end up by writing a state in the Hilbert space of 8 nodes and 16 links. The state depends on the intrinsic and extrinsic geometry, $\psi(m, T)$. Once we have it, we can calculate the amplitude $W(m, T) = W(\psi)$. How to calculate it? We put a 2-complex spin foam in between and use the formulas. We choose the simplest 2-complex. The simplest is a triangulation of the inside, and the easiest is to break the boundary Σ into two simplices: one lower simplex and one upper simplex. The lower 4-simplex and the upper 4-simplex are joined by a tetrahedron. The lower one is bounded by 4 tetrahedrons, and so is the upper one. So we have:

Figure 20.9: Two simplices connected by a tetrahedron.

So the integral corresponds to this explicitly, which is a bunch of integrals over $\mathrm{SL}(2,\mathbb{C})$, a bunch over $\mathrm{SO}(4)$, and a bunch of Wigner matrices. At the end we compute a function of two variables, m and T. This has been done by Fabris and Rovelli. The square of $|W(m,T)|^2$ is proportional to the probability of the transition happening for a mass m in a certain time. The probability turns out to be

$$p \propto \exp\left[-\frac{m^2}{m_{\mathrm{Pl}}^2}\,\beta\right], \tag{20.1}$$

where β is a constant number.

Therefore, for a macroscopic BH in which $m \gg m_{\mathrm{Pl}}$, the exponential is suppressed, so this does not happen. But it will happen for very small BHs, $m \sim m_{\mathrm{Pl}}$. So what comes out at the other end is not nothing, but a White Hole with a huge interior. So the theory tells us we would have a White Hole with a small size but a large interior. So the classic BH image in general relativity is wrong as the BH does not just completely evaporate and vanish.

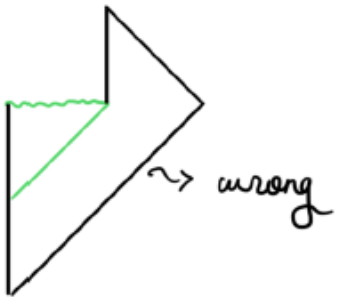

Figure 20.10: Wrong classical schema of BH.

References and further work

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