General Four-Loop Beta Function for Scalar-Fermion Theories in Three Dimensions

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Abstract

We present general four-loop template β-functions and anomalous field dimensions for renormalisable scalar-fermion theories in three dimensions. By imposing $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry, we obtain relations between the template RGE coefficients, valid in any renormalisation scheme. Directly in $d=3$, we identify a new theory with a non-trivial IR fixed point that is under perturbative control in a large-N limit. We provide up-to-date numerical results for all required massive tadpole master integrals up to four loops and complement them with analytic expressions where available.

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1 Introduction

Studying quantum systems in three dimensions is of strategic interest for many aspects of modern physics. For instance, continuous phase transitions in condensed-matter systems can be described as critical points in the systems three-dimensional continuum limit. Prominent examples include scalar $O(N)$ models [1–15], and gapless Dirac electrons characterized by the generalized Gross–Neveu universality class [16–24]. At high temperatures, four-dimensional quantum systems reduce effectively to three dimensions once the heavy Matsubara modes are integrated out [25–28]. The resulting three-dimensional effective field theory governs the infrared dynamics, making its renormalisation-group flow essential for high-quality resummations [29, 30]. Moreover, non-perturbative phenomena such as confinement and dynamic symmetry breaking, which are complicated to capture in four dimensions, can be modelled in three-dimensional theories, see for instance [31–36]. Furthermore, theories in three dimensions are interrelated by a web of dualities [37–41]. Such dualities include strong-weak dualities, which make non-perturbative effects tractable.

After half a century of intensive work, three-dimensional quantum field theories (QFTs) have remained less well-understood than their four-dimensional counterparts in regard to their critical phenomena. A key reason is that in four dimensions, the renormalisation group (RG) flow is known for each interaction of any renormalisable QFT in the perturbative region. While perturbation theory certainly has its limits, it also enables a strong theoretical understanding whenever it is applicable. In four dimensions, the perturbative renormalisation group equations (RGEs) of any renormalisable QFT are readily available to high orders via template expressions [42–69].

This approach of template expressions has proven useful for two main reasons. For one, template RGEs are model independent results, which separate the problem of computing RG flows from the difficulty of loop calculations. When implemented in software packages such as [70–75], high-loop RGEs become available even with minimal know-how or computational power.

The second advantage is that templates allow for an exhaustive and systematic study of perturbative RG flows. This allows to formulate theorems valid for the entirety of renormalisable QFTs. Moreover, it can help to identify QFTs where the RG flow has very rare or unique properties, and which are otherwise difficult to identify. An example in four dimensions would be the Litim–Sannino model [76, 77], which realises a perturbatively accessible UV fixed point. In contrast, in this work, the template method will enable us to identify a three-dimensional theory that features a novel IR fixed point under perturbative control at large N.

The main goal of this work is to bring the advantages of template RGEs into the realm of three-dimensional QFTs. For the time being, we target gaugeless theories and a precision up to four-loop order. Thus, the work can be understood as an extension of the studies of [78, 79], and a three-dimensional equivalent of [65, 66, 68]. We also aim to describe known critical phenomena in a unified framework, namely the UV fixed points in purely scalar [80–84] and purely fermionic theories [85–89].

This work is structured as follows: We introduce our notation and methodology in Sec. 2 and collect our main results in Sec. 3. As an important cross-check, we discuss the emergence of supersymmetry in Sec. 4. In Sec. 5, we analyse the RG flow and find hitherto unknown fixed points. A small outlook follows in Sec. 6. The App. A holds technical details of three-dimensional four-loop integrals, which are key ingredients for the calculations in this work.

2 Scalar-Fermionic Theory

We consider the most general three-dimensional, renormalisable QFT of scalars and spin-1/2 fermions with marginal interactions in three-dimensional Minkowski space

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + \frac{i}{2}\overline{\psi}_i\gamma^\mu\partial_\mu\psi_i - \frac{1}{4}Y^{abij}\phi_a\phi_b\overline{\psi}_i\psi_j - \frac{1}{6!}\eta^{abcdef}\phi_a\phi_b\phi_c\phi_d\phi_e\phi_f\,, \tag{1}$$

where ϕ^a are real scalar fields with $a = 1, .., N_s$. In three dimensions, fermions are real rank-two spinors given a suitable representation of γ-matrices where $(\gamma^0)^{\mathrm{T}} = -\gamma^0$, and $(\gamma^{1,2})^{\mathrm{T}} = \gamma^{1,2}$, which can be furnished via the Pauli matrices. The fermions ψ_i and $\overline{\psi}_i = \psi_i^\dagger\gamma^0$ are enumerated by flavour indices $i = 1, .., N_f$ and spinor indices are kept implicit. We only include the marginal interactions η and Y, which are totally symmetric in their indices $\eta^{abcdef} = \eta^{(abcdef)}$ and $Y^{abij} = Y^{(ab)(ij)}$. In the following, we condense our notation of fermion chains via

$$Y^{a_1b_1i_1i_2}Y^{a_2b_2i_2i_3}\dots Y^{a_nb_ni_ni_{n+1}} = Y^{a_1b_1a_2b_2\dots a_nb_n}\,, \tag{2}$$

where the fermion indices are suppressed altogether. This formalism allows us to compute RGEs up to four-loop order in a very general manner. We utilise dimensional regularisation to $d = 3-2\epsilon$ [90,91], and the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme [92,93]. The technique of infrared rearrangement with a common mass parameter [94] allows us to separate UV from IR divergences while only computing tadpole integrals. The computation is conducted via the `MaRTIn` framework [95–97], which we have extended to four loop-order. It leverages the `FMFT` package [98] for efficient reductions of integrals to a basis of masters given in [99]. These master integrals are computed numerically using the algorithm of [100] to high precision. Where possible, analytic results are extracted using the integer relation algorithm `PSLQ` [101, 102]. This is sufficient to retrieve all template RGEs in analytic form. Details are relegated to App. A, which serves as an update to [103]. As a result, the template RGEs are obtained as contractions of the coupling tensors Y^{ab} and η^{abcdef}, which at this stage are fully model independent but may be cumbersome to interpret in terms of a specific model. For this purpose, we implement them in our in-house software `FoRGEr` [75], which we plan to make openly available in the future.

3 Template RGEs

In this section, we collect the template expressions for scalar and fermionic field anomalous dimensions

$$\gamma_{\phi,\psi} = -\frac{\mathrm{d}\log\sqrt{Z}_{\phi,\psi}}{\mathrm{d}\log\mu} = \sum_{\ell=1}^{\infty}\frac{\gamma_{\phi,\psi}^{(2\ell)}}{(4\pi)^{2\ell}}\,, \tag{3}$$

where $\phi_a^{\text{bare}} = (\sqrt{Z}_\phi)_{ab}\phi_b$ and $\psi_i^{\text{bare}} = (\sqrt{Z}_\psi)_{ij}\psi_j$ as well as β-functions for quartic Yukawa and scalar sextic couplings

$$\beta_Y^{ab} = \frac{\mathrm{d}Y^{ab}}{\mathrm{d}\log\mu} = \gamma_\psi Y^{ab} + Y^{ab}\gamma_\psi^{\mathrm{T}} + \gamma_\phi^{ac}Y^{bc} + \gamma_\phi^{bc}Y^{ac} + \hat{\beta}_Y^{ab}\,, \tag{4}$$

$$\beta_\eta^{abcdef} = \frac{\mathrm{d}\eta^{abcdef}}{\mathrm{d}\log\mu} = \mathcal{S}_6\,\gamma_\phi^{ag}\eta^{gbcdef} + \hat{\beta}_\eta^{abcdef}\,, \tag{5}$$

in terms of their respective vertex corrections

$$\hat{\beta}_X = \sum_{\ell=1}^{\infty}\frac{\hat{\beta}_X^{(2\ell)}}{(4\pi)^{2\ell}} \tag{6}$$

up to four loops ($\ell = 2$). This ansatz incorporates the fact that odd-dimensional RGEs only receive contributions at even loop orders, starting at two loops. For ease of notation, we make heavy use of the symmetric permutation symbol $\mathcal{S}_n$, which indicates that n inequivalent permutations of the external indices (both scalar and fermionic) are to be added. For instance, in (5) $\mathcal{S}_6$ indicates that the leg correction consists of six terms, where a is exchanged with the other external indices:

$$\mathcal{S}_6\,\gamma_\phi^{ag}\eta^{gbcdef} = \gamma_\phi^{ag}\eta^{gbcdef} + \gamma_\phi^{bg}\eta^{gacdef} + \gamma_\phi^{cg}\eta^{gabdef} + \gamma_\phi^{dg}\eta^{gabcef} + \gamma_\phi^{eg}\eta^{gabcdf} + \gamma_\phi^{fg}\eta^{gabcde}\,, \tag{7}$$

whereas other permutations among the indices b–f in the expression on the left-hand-side are not considered as they do not generate new terms owing to the symmetry of η.

Note that the results on β_Y and β_η presented here also allow for the automatised extraction of the RGEs of the operators $\phi\bar{\psi}\psi$ and $\bar{\psi}\psi$ (via β_Y) as well as ϕ^5, ϕ^4, ϕ^3, ϕ^2 and ϕ^1 (via β_η) by utilising the dummy-field technique [47, 60, 104]. Also, these RGEs are implemented in `FoRGEr` [75].

3.1 Anomalous Field Dimensions

The general shape of the two- and four-loop scalar field anomalous dimensions are

$$\gamma_\phi^{ab\,(2)} = \gamma_{\phi,1}^{(2)}\,\mathrm{tr}\left(Y^{acbc}\right)\,, \tag{8}$$

$$\begin{aligned}\gamma_\phi^{ab\,(4)} = {}& \gamma_{\phi,1}^{(4)}\,\eta^{acdefg}\eta^{bcdefg} + \gamma_{\phi,2}^{(4)}\,\mathrm{tr}\left(Y^{acde}\right)\mathrm{tr}\left(Y^{bcde}\right) + \gamma_{\phi,3}^{(4)}\,\mathrm{tr}\left(Y^{acde}\right)\mathrm{tr}\left(Y^{bdce}\right)\\ &+ \gamma_{\phi,4}^{(4)}\,\mathrm{tr}\left(Y^{acbd}\right)\mathrm{tr}\left(Y^{cede}\right) + \gamma_{\phi,5}^{(4)}\,\mathrm{tr}\left(Y^{acbcdede}\right) + \gamma_{\phi,6}^{(4)}\,\mathrm{tr}\left(Y^{acbdcede}\right)\\ &+ \gamma_{\phi,7}^{(4)}\,\mathrm{tr}\left(Y^{acbddece}\right) + \gamma_{\phi,8}^{(4)}\,\mathrm{tr}\left(Y^{acdebcde}\right) + \gamma_{\phi,9}^{(5)}\,\mathrm{tr}\left(Y^{accebdde}\right)\,.\end{aligned} \tag{9}$$

Here we have listed all possible contractions excluding contributions from tadpole diagrams as they do not contribute to the anomalous dimensions. In the $\overline{\mathrm{MS}}$ scheme, we find the explicit coefficients

$$\begin{aligned}&\gamma_{\phi,1}^{(2)} = \tfrac{1}{12}\,, && \gamma_{\phi,1}^{(4)} = \tfrac{1}{1440}\,, && \gamma_{\phi,2}^{(4)} = -\tfrac{\pi^2}{192}\,, && \gamma_{\phi,3}^{(4)} = -\tfrac{1}{144}\,, && \gamma_{\phi,4}^{(4)} = -\tfrac{11}{432}\,,\\ &\gamma_{\phi,5}^{(4)} = -\tfrac{5}{216}\,, && \gamma_{\phi,6}^{(4)} = -\tfrac{1}{18}\,, && \gamma_{\phi,7}^{(4)} = -\tfrac{\pi^2}{96}\,, && \gamma_{\phi,8}^{(4)} = \tfrac{1}{72}\,, && \gamma_{\phi,9}^{(4)} = -\tfrac{1}{72}\,.\end{aligned} \tag{10}$$

Analogously, the fermion anomalous dimension includes the contractions

$$\gamma_\psi^{(2)} = \gamma_{\psi,1}^{(2)}\,Y^{abab}\,, \tag{11}$$

$$\begin{aligned}\gamma_\psi^{(4)} = {}& \gamma_{\psi,1}^{(4)}\,\mathrm{tr}\left(Y^{abcd}\right)Y^{acbd} + \gamma_{\psi,2}^{(4)}\,\mathrm{tr}\left(Y^{abcd}\right)Y^{abcd} + \gamma_{\psi,3}^{(4)}\,\mathrm{tr}\left(Y^{abac}\right)Y^{bdcd}\\ &+ \gamma_{\psi,4}^{(4)}\,Y^{abcdcdab} + \gamma_{\psi,5}^{(4)}\,Y^{abbccdda} + \gamma_{\psi,6}^{(4)}\,Y^{abcdbdac} + \gamma_{\psi,7}^{(4)}\,Y^{abcdabcd}\\ &+ \gamma_{\psi,8}^{(4)}\,Y^{abbcaddc}\end{aligned} \tag{12}$$

up to four-loop order, which take the explicit values

$$\begin{aligned}&\gamma_{\psi,1}^{(2)} = \tfrac{1}{12}\,, && \gamma_{\psi,1}^{(4)} = -\tfrac{1}{36}\,, && \gamma_{\psi,2}^{(4)} = -\tfrac{\pi^2}{192}\,, && \gamma_{\psi,3}^{(4)} = -\tfrac{1}{27}\,, && \gamma_{\psi,4}^{(4)} = -\tfrac{1}{216}\,,\\ &\gamma_{\psi,5}^{(4)} = -\tfrac{\pi^2}{96}\,, && \gamma_{\psi,6}^{(4)} = \tfrac{1}{36}\,, && \gamma_{\psi,7}^{(4)} = \tfrac{1}{72}\,, && \gamma_{\psi,8}^{(4)} = -\tfrac{1}{72}\end{aligned} \tag{13}$$

in the $\overline{\mathrm{MS}}$ scheme.

It is worth noting that both anomalous dimensions in (9) and (12) are manifestly symmetric under exchange of their external indices. In general, anomalous dimension might also contain non-symmetric contributions, inherited from antisymmetric parts of the field strength renormalisations $\sqrt{Z}_{\phi,\psi}$. The scalar and fermion kinetic terms

$$\mathcal{L} = \tfrac{1}{2} Z_\phi^{ab} \partial_\mu \phi_a \partial^\mu \phi_b + \tfrac{i}{2} Z_\phi^{ij} \overline{\psi}_i \gamma^\mu \partial_\mu \psi_j + \dots \tag{14}$$

have renormalisation constants Z_ϕ^{ab} and Z_ψ^{ij} that are symmetric under $(a \leftrightarrow b)$ and $(i \leftrightarrow j)$, respectively, and fixed by the renormalisation of two-point functions. However, there is an ambiguity in defining the field strength tensors $\sqrt{Z}_{\phi,\psi}$ such that $Z_{\phi,\psi} = \sqrt{Z}_{\phi,\psi} \cdot \sqrt{Z}_{\phi,\psi}$, which is related to their antisymmetric part. This ambiguity has also been encountered in four dimensional theories, see, e.g. [50, 67, 68, 105, 106].

The ambiguity leads to subtleties in theories with hidden flavour symmetries. Such kind of QFTs contain several scalars or fermions that have identical quantum numbers, yet do not transform together under a global symmetry. In this case a rotation among the fields is encoded in the theory, but involves a redefinition of couplings. As a result, fixed points in these couplings do not possess conformal symmetry. Instead, β-functions require a shift proportional to antisymmetric leg corrections in order to preserve conformality [105, 107, 108]. In four dimensions, such corrections start occurring from three-loop order. We eventually expect the same phenomena to also occur in three dimensions, though they are still absent at four loops.

3.2 Quartic Yukawa Vertex

We now turn to the vertex corrections of the quartic Yukawa coupling. At two-loop order there are only a few contractions

$$\begin{aligned}\hat{\beta}_Y^{(2)ab} &= \beta_{Y,1}^{(2)} \operatorname{tr}\left(Y^{abcd}\right) Y^{cd} + \beta_{Y,2}^{(2)} \operatorname{tr}\left(Y^{acbd}\right) Y^{cd} + \beta_{Y,3}^{(2)} Y^{cdabcd} + \beta_{Y,4}^{(2)} \mathcal{S}_2 Y^{accddb} \\ &\quad + \beta_{Y,5}^{(2)} \mathcal{S}_4 Y^{acbdcd} \,,\end{aligned} \tag{15}$$

which do not feature the sextic interactions η at all. We find the explicit values

$$\beta_{Y,1}^{(2)} = 0\,, \qquad \beta_{Y,2}^{(2)} = \tfrac{1}{2}\,, \qquad \beta_{Y,3}^{(2)} = \tfrac{1}{2}\,, \qquad \beta_{Y,4}^{(2)} = 0\,, \qquad \beta_{Y,5}^{(2)} = \tfrac{1}{2}\,. \tag{16}$$

At four-loop order, we find a more complicated shape with a total number of 132 coefficients, which we have categorised by the number of sextic interactions and fermion traces

$$\hat{\beta}_Y^{(4)ab} = \hat{\beta}_{Y,\eta^2}^{(4)ab} + \hat{\beta}_{Y,\eta^1}^{(4)ab} + \hat{\beta}_{Y,(2,2,1)}^{(4)ab} + \hat{\beta}_{Y,(4,1)}^{(4)ab} + \hat{\beta}_{Y,(3,2)}^{(4)ab} + \hat{\beta}_{Y,(2,3)}^{(4)ab} + \hat{\beta}_{Y,Y^5}^{(4)ab} \,. \tag{17}$$

The first two terms include contractions with the scalar sextic coupling

$$\hat{\beta}_{Y,\eta^2}^{(4)ab} = \beta_{Y,1}^{(4)} \eta^{acefgh} \eta^{bdefgh} Y^{cd} + \beta_{Y,2}^{(4)} \eta^{abefgh} \eta^{cdefgh} Y^{cd} \,, \tag{18}$$

$$\begin{aligned}\hat{\beta}_{Y,\eta^1}^{(4)ab} &= \beta_{Y,3}^{(4)} \eta^{abcdef} Y^{cgdefg} + \beta_{Y,4}^{(4)} \mathcal{S}_2 \eta^{abcdef} Y^{cdegfg} + \beta_{Y,5}^{(4)} \mathcal{S}_2 \eta^{acdefg} Y^{cdbefg} \\ &\quad + \beta_{Y,6}^{(4)} \mathcal{S}_4 \eta^{acdefg} Y^{bcdefg} + \beta_{Y,7}^{(4)} \eta^{abcdef} \operatorname{tr}\left(Y^{egfg}\right) Y^{cd} \\ &\quad + \beta_{Y,8}^{(4)} \eta^{abcefg} \operatorname{tr}\left(Y^{defg}\right) Y^{cd} + \beta_{Y,9}^{(4)} \mathcal{S}_2 \eta^{acdefg} \operatorname{tr}\left(Y^{befg}\right) Y^{cd} \,.\end{aligned} \tag{19}$$

The next terms have both external fermion lines connected to the same Yukawa, while also featuring two closed fermion traces

$$\begin{aligned}\hat{\beta}_{Y,(2,2,1)}^{(4)ab} &= \beta_{Y,10}^{(4)} \operatorname{tr}\left(Y^{abef}\right) \operatorname{tr}\left(Y^{cdef}\right) Y^{cd} + \beta_{Y,11}^{(4)} \operatorname{tr}\left(Y^{abef}\right) \operatorname{tr}\left(Y^{cedf}\right) Y^{cd} \\ &\quad + \beta_{Y,12}^{(4)} \operatorname{tr}\left(Y^{abce}\right) \operatorname{tr}\left(Y^{dfef}\right) Y^{cd} + \beta_{Y,13}^{(4)} \operatorname{tr}\left(Y^{aebf}\right) \operatorname{tr}\left(Y^{cdef}\right) Y^{cd}\end{aligned}$$

$$+\beta^{(4)}_{Y,14}\,\mathrm{tr}\left(Y^{aebf}\right)\mathrm{tr}\left(Y^{cedf}\right)Y^{cd}+\beta^{(4)}_{Y,15}\,\mathcal{S}_2\,\mathrm{tr}\left(Y^{acbe}\right)\mathrm{tr}\left(Y^{dfef}\right)Y^{cd}$$
$$+\beta^{(4)}_{Y,16}\,\mathrm{tr}\left(Y^{acef}\right)\mathrm{tr}\left(Y^{bdef}\right)Y^{cd}+\beta^{(4)}_{Y,17}\,\mathrm{tr}\left(Y^{aecf}\right)\mathrm{tr}\left(Y^{bfde}\right)Y^{cd}$$
$$+\beta^{(4)}_{Y,18}\,\mathrm{tr}\left(Y^{aecf}\right)\mathrm{tr}\left(Y^{bfde}\right)Y^{cd}+\beta^{(4)}_{Y,19}\,\mathcal{S}_2\mathrm{tr}\left(Y^{acef}\right)\mathrm{tr}\left(Y^{bedf}\right)Y^{cd}\,, \tag{20}$$

or one such trace

$$\begin{aligned}\hat{\beta}^{(4)ab}_{Y,(4,1)}=&\,\beta^{(4)}_{Y,20}\,\mathrm{tr}\left(Y^{abefcdef}\right)Y^{cd}+\beta^{(4)}_{Y,21}\,\mathrm{tr}\left(Y^{abcdefef}\right)Y^{cd}+\beta^{(4)}_{Y,22}\,\mathrm{tr}\left(Y^{abceeffd}\right)Y^{cd}\\&+\beta^{(4)}_{Y,23}\,\mathrm{tr}\left(Y^{abcedfef}\right)Y^{cd}+\beta^{(4)}_{Y,24}\,\mathrm{tr}\left(Y^{aebecfdf}\right)Y^{cd}+\beta^{(4)}_{Y,25}\,\mathcal{S}_2\,\mathrm{tr}\left(Y^{aebfcdef}\right)Y^{cd}\\&+\beta^{(4)}_{Y,26}\,\mathrm{tr}\left(Y^{aebfcedf}\right)Y^{cd}+\beta^{(4)}_{Y,27}\,\mathrm{tr}\left(Y^{aebfcfde}\right)Y^{cd}+\beta^{(4)}_{Y,28}\,\mathcal{S}_2\,\mathrm{tr}\left(Y^{acbedfef}\right)Y^{cd}\\&+\beta^{(4)}_{Y,29}\,\mathcal{S}_2\,\mathrm{tr}\left(Y^{acbeefdf}\right)Y^{cd}+\beta^{(4)}_{Y,30}\,\mathrm{tr}\left(Y^{acbdefef}\right)Y^{cd}+\beta^{(4)}_{Y,31}\,\mathrm{tr}\left(Y^{aecdbfef}\right)Y^{cd}\\&+\beta^{(4)}_{Y,32}\,\mathrm{tr}\left(Y^{acefbdef}\right)Y^{cd}+\beta^{(4)}_{Y,33}\,\mathrm{tr}\left(Y^{aecfbedf}\right)Y^{cd}+\beta^{(4)}_{Y,34}\,\mathrm{tr}\left(Y^{aecebfdf}\right)Y^{cd}\\&+\beta^{(4)}_{Y,35}\,\mathcal{S}_2\,\mathrm{tr}\left(Y^{acdebfef}\right)Y^{cd}\,.\end{aligned} \tag{21}$$

Note that contributions $\propto \mathrm{tr}\left(Y^3\right)$ or higher odd numbers of Yukawas are not a priori excluded in three dimensions. Such terms emerge in diagrams that contain traces over an odd number of γ matrices, which, in contrast to four dimensions, do not vanish in three dimensions. Rather, they give expressions containing fully antisymmetric Levi–Civita tensors, e.g.

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho)=-2i\,\epsilon^{\mu\nu\rho}\,. \tag{22}$$

This excludes many possible contractions. After integration of the fermion loop with an odd number of γ matrices, one of the loop momenta is spent and the remaining three need to be contracted with the Levi–Civita tensor for it not to vanish due to the antisymmetry. Thus, only the contractions

$$\begin{aligned}\hat{\beta}^{(4)ab}_{Y,(3,2)}=&\,\beta^{(4)}_{Y,36}\,\mathrm{tr}\left(Y^{abcdef}\right)Y^{cdef}+\beta^{(4)}_{Y,37}\,\mathrm{tr}\left(Y^{abcedf}\right)Y^{cdef}+\beta^{(4)}_{Y,38}\,\mathrm{tr}\left(Y^{acbedf}\right)Y^{cdef}\\&+\beta^{(4)}_{Y,39}\,\mathcal{S}_2\,\mathrm{tr}\left(Y^{acbdef}\right)Y^{cdef}+\beta^{(4)}_{Y,40}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{acdfef}\right)Y^{bcde}\\&+\beta^{(4)}_{Y,41}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{adcfef}\right)Y^{bcde}+\beta^{(4)}_{Y,42}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{afcdef}\right)Y^{bcde}\\&+\beta^{(4)}_{Y,43}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{afdecf}\right)Y^{bcde}\end{aligned} \tag{23}$$

do not vanish. There are further contractions with a single fermion loop

$$\begin{aligned}\hat{\beta}^{(4)ab}_{Y,(2,3)}=&\,\beta^{(4)}_{Y,44}\,\mathrm{tr}\left(Y^{abcd}\right)Y^{efcdef}+\beta^{(4)}_{Y,45}\,\mathrm{tr}\left(Y^{abcd}\right)Y^{ceefdf}+\beta^{(4)}_{Y,46}\,\mathcal{S}_2\,\mathrm{tr}\left(Y^{abcd}\right)Y^{cedfef}\\&+\beta^{(4)}_{Y,47}\,\mathrm{tr}\left(Y^{acbd}\right)Y^{efcdef}+\beta^{(4)}_{Y,48}\,\mathcal{S}_2\,\mathrm{tr}\left(Y^{acbd}\right)Y^{ceefdf}+\beta^{(4)}_{Y,49}\,\mathcal{S}_2\,\mathrm{tr}\left(Y^{acbd}\right)Y^{cedfef}\\&+\beta^{(4)}_{Y,50}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{acde}\right)Y^{bcdfef}+\beta^{(4)}_{Y,51}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{acde}\right)Y^{bdcfef}\\&+\beta^{(4)}_{Y,52}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{acde}\right)Y^{bfcdef}+\beta^{(4)}_{Y,53}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{acde}\right)Y^{bfcfde}\\&+\beta^{(4)}_{Y,54}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{acde}\right)Y^{bdefcf}+\beta^{(4)}_{Y,55}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{acde}\right)Y^{bfefcd}\\&+\beta^{(4)}_{Y,56}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{acde}\right)Y^{bfdecf}+\beta^{(4)}_{Y,57}\,\mathcal{S}_2\,\mathrm{tr}\left(Y^{acde}\right)Y^{dfbcef}\\&+\beta^{(4)}_{Y,58}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{acde}\right)Y^{dfbfce}+\beta^{(4)}_{Y,59}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{acde}\right)Y^{debfef}\\&+\beta^{(4)}_{Y,60}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{acde}\right)Y^{dfbecf}+\beta^{(4)}_{Y,61}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{cdef}\right)Y^{acbdef}\\&+\beta^{(4)}_{Y,62}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{cdef}\right)Y^{acbedf}+\beta^{(4)}_{Y,63}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{cede}\right)Y^{afbfcd}\\&+\beta^{(4)}_{Y,64}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{cede}\right)Y^{acbfdf}+\beta^{(4)}_{Y,65}\,\mathcal{S}_4\,\mathrm{tr}\left(Y^{cede}\right)Y^{afbcdf}\end{aligned}$$

$$
\begin{aligned}
&+\beta^{(4)}_{Y,66}\,\mathcal{S}_2\,\mathrm{tr}\big(Y^{cdef}\big)\,Y^{acdebf}+\beta^{(4)}_{Y,67}\,\mathcal{S}_2\,\mathrm{tr}\big(Y^{cdef}\big)\,Y^{acefbd}\\
&+\beta^{(4)}_{Y,68}\,\mathcal{S}_2\,\mathrm{tr}\big(Y^{cede}\big)\,Y^{afcdbf}+\beta^{(4)}_{Y,69}\,\mathcal{S}_4\,\mathrm{tr}\big(Y^{cede}\big)\,Y^{acdfbf}\\
&+\beta^{(4)}_{Y,70}\,\mathrm{tr}\big(Y^{cdef}\big)\,Y^{cdabef}+\beta^{(4)}_{Y,71}\,\mathrm{tr}\big(Y^{cdef}\big)\,Y^{ceabdf}\\
&+\beta^{(4)}_{Y,72}\,\mathrm{tr}\big(Y^{cede}\big)\,Y^{cfabdf}\,,
\end{aligned}
\tag{24}
$$

as well as a final family with no closed fermion loop at all

$$
\begin{aligned}
\hat{\beta}^{(4)ab}_{Y,Y^5}=\;&\beta^{(4)}_{Y,73}\,\mathcal{S}_4\,Y^{acbdcedfef}+\beta^{(4)}_{Y,74}\,\mathcal{S}_4\,Y^{acbddecfef}+\beta^{(4)}_{Y,75}\,\mathcal{S}_4\,Y^{acbdceefdf}\\
&+\beta^{(4)}_{Y,76}\,\mathcal{S}_4\,Y^{acbddeefcf}+\beta^{(4)}_{Y,77}\,\mathcal{S}_4\,Y^{acbdefcedf}+\beta^{(4)}_{Y,78}\,\mathcal{S}_4\,Y^{acbdefdecf}\\
&+\beta^{(4)}_{Y,79}\,\mathcal{S}_4\,Y^{acbdefcdef}+\beta^{(4)}_{Y,80}\,\mathcal{S}_4\,Y^{acbdefefcd}+\beta^{(4)}_{Y,81}\,\mathcal{S}_4\,Y^{acdebcdfef}\\
&+\beta^{(4)}_{Y,82}\,\mathcal{S}_4\,Y^{accdbedfef}+\beta^{(4)}_{Y,83}\,\mathcal{S}_4\,Y^{accdbeefdf}+\beta^{(4)}_{Y,84}\,\mathcal{S}_4\,Y^{acdebdcfef}\\
&+\beta^{(4)}_{Y,85}\,\mathcal{S}_4\,Y^{acdebdefcf}+\beta^{(4)}_{Y,86}\,\mathcal{S}_4\,Y^{acdebfcdef}+\beta^{(4)}_{Y,87}\,\mathcal{S}_4\,Y^{acdebffdce}\\
&+\beta^{(4)}_{Y,88}\,\mathcal{S}_4\,Y^{acdebfdecf}+\beta^{(4)}_{Y,89}\,\mathcal{S}_4\,Y^{acdebfcfde}+\beta^{(4)}_{Y,90}\,\mathcal{S}_4\,Y^{accddebfef}\\
&+\beta^{(4)}_{Y,91}\,\mathcal{S}_4\,Y^{accdefbdef}+\beta^{(4)}_{Y,92}\,\mathcal{S}_4\,Y^{accdefbedf}+\beta^{(4)}_{Y,93}\,\mathcal{S}_4\,Y^{acdecfbdef}\\
&+\beta^{(4)}_{Y,94}\,\mathcal{S}_4\,Y^{acdecdbfef}+\beta^{(4)}_{Y,95}\,\mathcal{S}_4\,Y^{acdecfbfde}+\beta^{(4)}_{Y,96}\,\mathcal{S}_4\,Y^{acdedfbcef}\\
&+\beta^{(4)}_{Y,97}\,\mathcal{S}_4\,Y^{acdeefbdcf}+\beta^{(4)}_{Y,98}\,\mathcal{S}_4\,Y^{acdedfbfce}+\beta^{(4)}_{Y,99}\,\mathcal{S}_4\,Y^{acdedebfcf}\\
&+\beta^{(4)}_{Y,100}\,\mathcal{S}_2\,Y^{accddeefbf}+\beta^{(4)}_{Y,101}\,\mathcal{S}_2\,Y^{acdeefcfbd}+\beta^{(4)}_{Y,102}\,\mathcal{S}_2\,Y^{acdecfdebf}\\
&+\beta^{(4)}_{Y,103}\,\mathcal{S}_4\,Y^{accdefdfbe}+\beta^{(4)}_{Y,104}\,\mathcal{S}_4\,Y^{acdecfefbd}+\beta^{(4)}_{Y,105}\,\mathcal{S}_4\,Y^{accdefefbd}\\
&+\beta^{(4)}_{Y,106}\,\mathcal{S}_2\,Y^{acdedfefbc}+\beta^{(4)}_{Y,107}\,\mathcal{S}_4\,Y^{cdacbedfef}+\beta^{(4)}_{Y,108}\,\mathcal{S}_4\,Y^{cdacbeefdf}\\
&+\beta^{(4)}_{Y,109}\,\mathcal{S}_4\,Y^{cdaebcdfef}+\beta^{(4)}_{Y,110}\,\mathcal{S}_4\,Y^{cdaebcefdf}+\beta^{(4)}_{Y,111}\,\mathcal{S}_4\,Y^{cdaebecfdf}\\
&+\beta^{(4)}_{Y,112}\,\mathcal{S}_4\,Y^{cdaebfcdef}+\beta^{(4)}_{Y,113}\,\mathcal{S}_4\,Y^{cdaebfcedf}+\beta^{(4)}_{Y,114}\,\mathcal{S}_4\,Y^{cdaebfcfde}\\
&+\beta^{(4)}_{Y,115}\,\mathcal{S}_4\,Y^{cdaebfefcd}+\beta^{(4)}_{Y,116}\,\mathcal{S}_2\,Y^{cdacdebfef}+\beta^{(4)}_{Y,117}\,\mathcal{S}_4\,Y^{cdacefbdef}\\
&+\beta^{(4)}_{Y,118}\,\mathcal{S}_4\,Y^{cdacefbedf}+\beta^{(4)}_{Y,119}\,\mathcal{S}_2\,Y^{cdaecfbdef}+\beta^{(4)}_{Y,120}\,\mathcal{S}_2\,Y^{cdaecfbedf}\\
&+\beta^{(4)}_{Y,121}\,\mathcal{S}_4\,Y^{cdaeefbcdf}+\beta^{(4)}_{Y,122}\,\mathcal{S}_2\,Y^{cdaeefbfcd}+\beta^{(4)}_{Y,123}\,\mathcal{S}_2\,Y^{cdabcedfef}\\
&+\beta^{(4)}_{Y,124}\,\mathcal{S}_2\,Y^{cdabceefdf}+\beta^{(4)}_{Y,125}\,\mathcal{S}_2\,Y^{cdabefcedf}+\beta^{(4)}_{Y,126}\,\mathcal{S}_2\,Y^{cdabefcdef}\\
&+\beta^{(4)}_{Y,127}\,\mathcal{S}_2\,Y^{cdabefefcd}+\beta^{(4)}_{Y,128}\,Y^{cdefabefcd}+\beta^{(4)}_{Y,129}\,Y^{cdefabcdef}\\
&+\beta^{(4)}_{Y,130}\,Y^{cdefabcedf}+\beta^{(4)}_{Y,131}\,Y^{cedeabcfdf}+\beta^{(4)}_{Y,132}\,Y^{cedeabdfcf}\,.
\end{aligned}
\tag{25}
$$

In the $\overline{\text{MS}}$ scheme, we find the explicit coefficients

$$
\begin{aligned}
&\beta^{(4)}_{Y,1}=\tfrac{1}{48}\,, && \beta^{(4)}_{Y,2}=0\,, && \beta^{(4)}_{Y,3}=-\tfrac{1}{2}\,, && \beta^{(4)}_{Y,4}=-\tfrac{1}{4}\,, && \beta^{(4)}_{Y,5}=-\tfrac{1}{4}\,,\\
&\beta^{(4)}_{Y,6}=-\tfrac{1}{8}\,, && \beta^{(4)}_{Y,7}=0\,, && \beta^{(4)}_{Y,8}=0\,, && \beta^{(4)}_{Y,9}=0\,, && \beta^{(4)}_{Y,10}=0\,,\\
&\beta^{(4)}_{Y,11}=0\,, && \beta^{(4)}_{Y,12}=0\,, && \beta^{(4)}_{Y,13}=0\,, && \beta^{(4)}_{Y,14}=\tfrac{1}{8}\,, && \beta^{(4)}_{Y,15}=-\tfrac{1}{12}\,,\\
&\beta^{(4)}_{Y,16}=-\tfrac{\pi^2}{32}\,, && \beta^{(4)}_{Y,17}=\tfrac{1}{8}\,, && \beta^{(4)}_{Y,18}=0\,, && \beta^{(4)}_{Y,19}=-\tfrac{1}{4}\,, && \beta^{(4)}_{Y,20}=0\,,\\
&\beta^{(4)}_{Y,21}=0\,, && \beta^{(4)}_{Y,22}=0\,, && \beta^{(4)}_{Y,23}=-\tfrac{\pi^2}{8}\,, && \beta^{(4)}_{Y,24}=-\tfrac{1}{2}\,, && \beta^{(4)}_{Y,25}=0\,,\\
&\beta^{(4)}_{Y,26}=-\tfrac{\pi^2}{8}\,, && \beta^{(4)}_{Y,27}=-\tfrac{\pi^2}{16}\,, && \beta^{(4)}_{Y,28}=-\tfrac{3}{4}\,, && \beta^{(4)}_{Y,29}=-\tfrac{\pi^2}{16}\,, && \beta^{(4)}_{Y,30}=-\tfrac{1}{4}\,,\\
&\beta^{(4)}_{Y,31}=0\,, && \beta^{(4)}_{Y,32}=-\tfrac{1}{4}\,, && \beta^{(4)}_{Y,33}=-\tfrac{1}{2}\,, && \beta^{(4)}_{Y,34}=-\tfrac{1}{4}\,, && \beta^{(4)}_{Y,35}=-\tfrac{3}{4}\,,
\end{aligned}
$$

$$
\begin{array}{lllll}
\beta^{(4)}_{Y,36}=0\,, & \beta^{(4)}_{Y,37}=0\,, & \beta^{(4)}_{Y,38}=0\,, & \beta^{(4)}_{Y,39}=0\,, & \beta^{(4)}_{Y,40}=0\,,\\
\beta^{(4)}_{Y,41}=0\,, & \beta^{(4)}_{Y,42}=0\,, & \beta^{(4)}_{Y,43}=0\,, & \beta^{(4)}_{Y,44}=0\,, & \beta^{(4)}_{Y,45}=0\,,\\
\beta^{(4)}_{Y,46}=0\,, & \beta^{(4)}_{Y,47}=0\,, & \beta^{(4)}_{Y,48}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,49}=0\,, & \beta^{(4)}_{Y,50}=-\frac{1}{4}\,,\\
\beta^{(4)}_{Y,51}=-\frac{1}{8}\,, & \beta^{(4)}_{Y,52}=-\frac{3}{8}\,, & \beta^{(4)}_{Y,53}=0\,, & \beta^{(4)}_{Y,54}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,55}=0\,,\\
\beta^{(4)}_{Y,56}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,57}=-\frac{1}{2}\,, & \beta^{(4)}_{Y,58}=-\frac{1}{4}\,, & \beta^{(4)}_{Y,59}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,60}=-\frac{\pi^2}{16}\,,\\
\beta^{(4)}_{Y,61}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,62}=-\frac{1}{8}\,, & \beta^{(4)}_{Y,63}=0\,, & \beta^{(4)}_{Y,64}=-\frac{1}{12}\,, & \beta^{(4)}_{Y,65}=-\frac{1}{8}\,,\\
\beta^{(4)}_{Y,66}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,67}=0\,, & \beta^{(4)}_{Y,68}=0\,, & \beta^{(4)}_{Y,69}=0\,, & \beta^{(4)}_{Y,70}=-\frac{\pi^2}{32}\,,\\
\beta^{(4)}_{Y,71}=0\,, & \beta^{(4)}_{Y,72}=-\frac{1}{6}\,, & \beta^{(4)}_{Y,73}=-\frac{1}{8}\,, & \beta^{(4)}_{Y,74}=0\,, & \beta^{(4)}_{Y,75}=-\frac{\pi^2}{32}\,,\\
\beta^{(4)}_{Y,76}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,77}=\frac{1}{8}\,, & \beta^{(4)}_{Y,78}=0\,, & \beta^{(4)}_{Y,79}=\frac{1}{8}\,, & \beta^{(4)}_{Y,80}=-\frac{1}{24}\,,\\
\beta^{(4)}_{Y,81}=-\frac{1}{4}\,, & \beta^{(4)}_{Y,82}=0\,, & \beta^{(4)}_{Y,83}=0\,, & \beta^{(4)}_{Y,84}=\frac{1}{8}\,, & \beta^{(4)}_{Y,85}=-\frac{1}{4}\,,\\
\beta^{(4)}_{Y,86}=\frac{1}{2}-\frac{\pi^2}{16}\,, & \beta^{(4)}_{Y,87}=-\frac{1}{8}\,, & \beta^{(4)}_{Y,88}=-\frac{1}{8}\,, & \beta^{(4)}_{Y,89}=\frac{1}{8}\,, & \beta^{(4)}_{Y,90}=0\,,\\
\beta^{(4)}_{Y,91}=0\,, & \beta^{(4)}_{Y,92}=0\,, & \beta^{(4)}_{Y,93}=-\frac{\pi^2}{16}\,, & \beta^{(4)}_{Y,94}=-\frac{1}{8}\,, & \beta^{(4)}_{Y,95}=-\frac{\pi^2}{32}\,,\\
\beta^{(4)}_{Y,96}=-\frac{1}{4}\,, & \beta^{(4)}_{Y,97}=-\frac{3}{8}\,, & \beta^{(4)}_{Y,98}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,99}=-\frac{1}{12}\,, & \beta^{(4)}_{Y,100}=0\,,\\
\beta^{(4)}_{Y,101}=0\,, & \beta^{(4)}_{Y,102}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,103}=0\,, & \beta^{(4)}_{Y,104}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,105}=0\,,\\
\beta^{(4)}_{Y,106}=0\,, & \beta^{(4)}_{Y,107}=0\,, & \beta^{(4)}_{Y,108}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,109}=\frac{1}{2}-\frac{\pi^2}{16}\,, & \beta^{(4)}_{Y,110}=-\frac{\pi^2}{16}\,,\\
\beta^{(4)}_{Y,111}=-\frac{1}{4}\,, & \beta^{(4)}_{Y,112}=\frac{1}{2}-\frac{\pi^2}{16}\,, & \beta^{(4)}_{Y,113}=1-\frac{\pi^2}{6}\,, & \beta^{(4)}_{Y,114}=\frac{1}{2}-\frac{\pi^2}{16}\,, & \beta^{(4)}_{Y,115}=-\frac{1}{4}\,,\\
\beta^{(4)}_{Y,116}=0\,, & \beta^{(4)}_{Y,117}=0\,, & \beta^{(4)}_{Y,118}=-\frac{1}{4}\,, & \beta^{(4)}_{Y,119}=1-\frac{\pi^2}{6}\,, & \beta^{(4)}_{Y,120}=-1\,,\\
\beta^{(4)}_{Y,121}=-\frac{\pi^2}{16}\,, & \beta^{(4)}_{Y,122}=-\frac{\pi^2}{32}\,, & \beta^{(4)}_{Y,123}=0\,, & \beta^{(4)}_{Y,124}=-\frac{\pi^2}{16}\,, & \beta^{(4)}_{Y,125}=0\,,\\
\beta^{(4)}_{Y,126}=0\,, & \beta^{(4)}_{Y,127}=-\frac{1}{12}\,, & \beta^{(4)}_{Y,128}=-\frac{1}{4}\,, & \beta^{(4)}_{Y,129}=\frac{1}{4}\,, & \beta^{(4)}_{Y,130}=1-\frac{\pi^2}{8}\,,\\
\beta^{(4)}_{Y,131}=\frac{1}{4}\,, & \beta^{(4)}_{Y,132}=-\frac{\pi^2}{16}\,. & & &
\end{array}
\tag{26}
$$

Especially noteworthy is the vanishing of the coefficients $\beta^{(4)}_{Y,36}$–$\beta^{(4)}_{Y,43}$, which are the ones featuring traces over an odd number of γ matrices. Their vanishing implies that the results are insensitive to complications arising from the naïve dimensional regularisation scheme, which clashes with the manifestly three-dimensional Levi–Civita tensor. In fact, the four-loop Yukawa vertex corrections are the only place where such traces may have occurred in this work.

3.3 Sextic Vertex

Finally, we turn towards the scalar sextic vertex corrections. The two-loop expression

$$
\begin{aligned}
\hat{\beta}^{(2)abcdef}_{\eta} = {} & \beta^{(2)}_{\eta,1}\,\mathcal{S}_{10}\,\eta^{abcghi}\eta^{defghi} + \beta^{(2)}_{\eta,2}\,\mathcal{S}_{15}\,\mathrm{tr}\left(Y^{abgh}\right)\eta^{cdefgh} \\
& + \beta^{(2)}_{\eta,3}\,\mathcal{S}_{15}\,\mathrm{tr}\left(Y^{agbh}\right)\eta^{cdefgh} + \beta^{(2)}_{\eta,4}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcdegfg}\right) \\
& + \beta^{(2)}_{\eta,5}\,\mathcal{S}_{45}\,\mathrm{tr}\left(Y^{abcgdefg}\right)
\end{aligned}
\tag{27}
$$

with the explicit coefficients

$$
\beta^{(2)}_{\eta,1}=\frac{1}{6}\,, \qquad \beta^{(2)}_{\eta,2}=0\,, \qquad \beta^{(2)}_{\eta,3}=\frac{1}{2}\,, \qquad \beta^{(2)}_{\eta,4}=-1\,, \qquad \beta^{(2)}_{\eta,5}=-2\,. \tag{28}
$$

The four-loop corrections consist of several terms

$$
\hat{\beta}^{(4)abcdef}_{\eta} = \hat{\beta}^{(4)abcdef}_{\eta,\eta^3} + \hat{\beta}^{(4)abcdef}_{\eta,\eta^3} + \hat{\beta}^{(4)abcdef}_{\eta,\eta^2Y^2} + \hat{\beta}^{(4)abcdef}_{\eta,\eta Y^2Y^2} + \hat{\beta}^{(4)abcdef}_{\eta,\eta Y^4}
$$

$$+\hat{\beta}^{(4)abcdef}_{\eta,Y^2Y^2Y^2}+\hat{\beta}^{(4)abcdef}_{\eta,Y^4Y^2}+\hat{\beta}^{(4)abcdef}_{\eta,Y^6}\,, \tag{29}$$

which read

$$\begin{aligned}\hat{\beta}^{(4)abcdef}_{\eta,\eta^3} = {}& \beta^{(4)}_{\eta,1}\,\mathcal{S}_{15}\,\eta^{abghij}\eta^{ghijkl}\eta^{cdefkl} + \beta^{(4)}_{\eta,2}\,\mathcal{S}_{15}\,\eta^{aghijk}\eta^{bhijkl}\eta^{cdefkg} \\ & + \beta^{(4)}_{\eta,3}\,\mathcal{S}_{10}\,\eta^{abcghi}\eta^{ghijkl}\eta^{defjkl} + \beta^{(4)}_{\eta,4}\,\mathcal{S}_{60}\,\eta^{aghijk}\eta^{bcghil}\eta^{defjkl} \\ & + \beta^{(4)}_{\eta,5}\,\mathcal{S}_{15}\,\eta^{abghij}\eta^{cdijkl}\eta^{efklgh}\,,\end{aligned} \tag{30}$$

$$\begin{aligned}\hat{\beta}^{(4)abcdef}_{\eta,\eta^2Y^2} = {}& \beta^{(4)}_{\eta,6}\,\mathcal{S}_{10}\,\eta^{abcghk}\eta^{defijl}\mathrm{tr}\left(Y^{ghij}\right) + \beta^{(4)}_{\eta,7}\,\mathcal{S}_{10}\,\eta^{abcghk}\eta^{defijl}\mathrm{tr}\left(Y^{gihj}\right) \\ & + \beta^{(4)}_{\eta,8}\,\mathcal{S}_{10}\,\eta^{abcgij}\eta^{defhij}\mathrm{tr}\left(Y^{gkhk}\right) + \beta^{(4)}_{\eta,9}\,\mathcal{S}_{15}\,\eta^{abghij}\eta^{cdefgh}\mathrm{tr}\left(Y^{ikjk}\right) \\ & + \beta^{(4)}_{\eta,10}\,\mathcal{S}_{15}\,\eta^{abghij}\eta^{cdefgk}\mathrm{tr}\left(Y^{khij}\right) + \beta^{(4)}_{\eta,11}\,\mathcal{S}_{30}\,\eta^{abcdgh}\eta^{eghijk}\mathrm{tr}\left(Y^{eijk}\right) \\ & + \beta^{(4)}_{\eta,12}\,\mathcal{S}_{60}\,\eta^{abcghi}\eta^{deghjk}\mathrm{tr}\left(Y^{fijk}\right) + \beta^{(4)}_{\eta,13}\,\mathcal{S}_{60}\,\eta^{abcghi}\eta^{deghjk}\mathrm{tr}\left(Y^{fkij}\right) \\ & + \beta^{(4)}_{\eta,14}\,\mathcal{S}_{60}\,\eta^{abcghi}\eta^{dghijk}\mathrm{tr}\left(Y^{efjk}\right) + \beta^{(4)}_{\eta,15}\,\mathcal{S}_{60}\,\eta^{abcghi}\eta^{dghijk}\mathrm{tr}\left(Y^{ejfk}\right) \\ & + \beta^{(4)}_{\eta,16}\,\mathcal{S}_{45}\,\eta^{abgijk}\eta^{cdhijk}\mathrm{tr}\left(Y^{efgh}\right) + \beta^{(4)}_{\eta,17}\,\mathcal{S}_{90}\,\eta^{abgijk}\eta^{cdhijk}\mathrm{tr}\left(Y^{egfh}\right)\,,\end{aligned} \tag{31}$$

$$\begin{aligned}\hat{\beta}^{(4)abcdef}_{\eta,\eta Y^2Y^2} = {}& \beta^{(4)}_{\eta,18}\,\mathcal{S}_{45}\,\eta^{abghij}\mathrm{tr}\left(Y^{cdgh}\right)\mathrm{tr}\left(Y^{efij}\right) + \beta^{(4)}_{\eta,19}\,\mathcal{S}_{45}\,\eta^{abcghk}\mathrm{tr}\left(Y^{cgdh}\right)\mathrm{tr}\left(Y^{eifj}\right) \\ & + \beta^{(4)}_{\eta,20}\,\mathcal{S}_{90}\,\eta^{abcghk}\mathrm{tr}\left(Y^{cdgh}\right)\mathrm{tr}\left(Y^{eifj}\right) + \beta^{(4)}_{\eta,21}\,\mathcal{S}_{60}\,\eta^{abcghi}\mathrm{tr}\left(Y^{deij}\right)\mathrm{tr}\left(Y^{fghj}\right) \\ & + \beta^{(4)}_{\eta,22}\,\mathcal{S}_{60}\,\eta^{abcghi}\mathrm{tr}\left(Y^{deij}\right)\mathrm{tr}\left(Y^{fjgh}\right) + \beta^{(4)}_{\eta,24}\,\mathcal{S}_{120}\,\eta^{abcghi}\mathrm{tr}\left(Y^{diej}\right)\mathrm{tr}\left(Y^{fghj}\right) \\ & + \beta^{(4)}_{\eta,24}\,\mathcal{S}_{120}\,\eta^{abcghi}\mathrm{tr}\left(Y^{diej}\right)\mathrm{tr}\left(Y^{fjgh}\right) + \beta^{(4)}_{\eta,25}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{ghij}\right)\mathrm{tr}\left(Y^{efij}\right) \\ & + \beta^{(4)}_{\eta,26}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{gihj}\right)\mathrm{tr}\left(Y^{efij}\right) + \beta^{(4)}_{\eta,27}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{giij}\right)\mathrm{tr}\left(Y^{efhj}\right) \\ & + \beta^{(4)}_{\eta,28}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{ghij}\right)\mathrm{tr}\left(Y^{eifj}\right) + \beta^{(4)}_{\eta,29}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{gihj}\right)\mathrm{tr}\left(Y^{eifj}\right) \\ & + \beta^{(4)}_{\eta,30}\,\mathcal{S}_{30}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{giij}\right)\mathrm{tr}\left(Y^{ehfj}\right) + \beta^{(4)}_{\eta,31}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{eijg}\right)\mathrm{tr}\left(Y^{fijh}\right) \\ & + \beta^{(4)}_{\eta,32}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{eijg}\right)\mathrm{tr}\left(Y^{fjih}\right) + \beta^{(4)}_{\eta,33}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{egij}\right)\mathrm{tr}\left(Y^{fhij}\right) \\ & + \beta^{(4)}_{\eta,34}\,\mathcal{S}_{30}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{eijg}\right)\mathrm{tr}\left(Y^{fhij}\right)\,,\end{aligned} \tag{32}$$

$$\begin{aligned}\hat{\beta}^{(4)abcdef}_{\eta,\eta Y^4} = {}& \beta^{(4)}_{\eta,35}\,\mathcal{S}_{45}\,\eta^{abghij}\mathrm{tr}\left(Y^{cdefghij}\right) + \beta^{(4)}_{\eta,36}\,\mathcal{S}_{45}\,\eta^{abghij}\mathrm{tr}\left(Y^{cdghefij}\right) \\ & + \beta^{(4)}_{\eta,37}\,\mathcal{S}_{90}\,\eta^{abghij}\mathrm{tr}\left(Y^{cdegfhij}\right) + \beta^{(4)}_{\eta,38}\,\mathcal{S}_{180}\,\eta^{abghij}\mathrm{tr}\left(Y^{cgdhefij}\right) \\ & + \beta^{(4)}_{\eta,39}\,\mathcal{S}_{45}\,\eta^{abghij}\mathrm{tr}\left(Y^{cgdheifj}\right) + \beta^{(4)}_{\eta,40}\,\mathcal{S}_{60}\,\eta^{abcghi}\mathrm{tr}\left(Y^{defghjij}\right) \\ & + \beta^{(4)}_{\eta,41}\,\mathcal{S}_{60}\,\eta^{abcghi}\mathrm{tr}\left(Y^{degjfhij}\right) + \beta^{(4)}_{\eta,42}\,\mathcal{S}_{60}\,\eta^{abcghi}\mathrm{tr}\left(Y^{defjghij}\right) \\ & + \beta^{(4)}_{\eta,43}\,\mathcal{S}_{60}\,\eta^{abcghi}\mathrm{tr}\left(Y^{defjgjhi}\right) + \beta^{(4)}_{\eta,44}\,\mathcal{S}_{60}\,\eta^{abcghi}\mathrm{tr}\left(Y^{degjfjhi}\right) \\ & + \beta^{(4)}_{\eta,45}\,\mathcal{S}_{60}\,\eta^{abcghi}\mathrm{tr}\left(Y^{gjdhejfi}\right) + \beta^{(4)}_{\eta,46}\,\mathcal{S}_{120}\,\eta^{abcghi}\mathrm{tr}\left(Y^{gjdjehfi}\right) \\ & + \beta^{(4)}_{\eta,47}\,\mathcal{S}_{60}\,\eta^{abcghi}\mathrm{tr}\left(Y^{ghdjeifj}\right) + \beta^{(4)}_{\eta,48}\,\mathcal{S}_{120}\,\eta^{abcghi}\mathrm{tr}\left(Y^{ghdiejfj}\right) \\ & + \beta^{(4)}_{\eta,49}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{efghijij}\right) + \beta^{(4)}_{\eta,50}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{efijghij}\right) \\ & + \beta^{(4)}_{\eta,51}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{efgihjij}\right) + \beta^{(4)}_{\eta,52}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{efgiijhj}\right)\end{aligned}$$

$$
\begin{aligned}
&+\beta^{(4)}_{\eta,53}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{egijfhij}\right)+\beta^{(4)}_{\eta,54}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{eigjfihj}\right)\\
&+\beta^{(4)}_{\eta,55}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{eiijfjgh}\right)+\beta^{(4)}_{\eta,56}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{eigifjhj}\right)\\
&+\beta^{(4)}_{\eta,57}\,\mathcal{S}_{30}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{eghifjij}\right)+\beta^{(4)}_{\eta,58}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{egfhijij}\right)\\
&+\beta^{(4)}_{\eta,59}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{eifigjhj}\right)+\beta^{(4)}_{\eta,60}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{eifjgihj}\right)\\
&+\beta^{(4)}_{\eta,61}\,\mathcal{S}_{15}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{eifjgjhi}\right)+\beta^{(4)}_{\eta,62}\,\mathcal{S}_{30}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{eifjgjhi}\right)\\
&+\beta^{(4)}_{\eta,63}\,\mathcal{S}_{30}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{egfihjij}\right)+\beta^{(4)}_{\eta,64}\,\mathcal{S}_{30}\,\eta^{abcdgh}\mathrm{tr}\left(Y^{egfiijhj}\right)\,,
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\hat\beta^{(4)abcdef}_{\eta,Y^2Y^2Y^2} &= \beta^{(4)}_{\eta,65}\,\mathcal{S}_{15}\,\mathrm{tr}\left(Y^{abgh}\right)\mathrm{tr}\left(Y^{cdhi}\right)\mathrm{tr}\left(Y^{efgi}\right)\\
&+\beta^{(4)}_{\eta,66}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{agbh}\right)\mathrm{tr}\left(Y^{cdhi}\right)\mathrm{tr}\left(Y^{efgi}\right)\\
&+\beta^{(4)}_{\eta,67}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abgh}\right)\mathrm{tr}\left(Y^{chdi}\right)\mathrm{tr}\left(Y^{egfi}\right)\\
&+\beta^{(4)}_{\eta,68}\,\mathcal{S}_{120}\,\mathrm{tr}\left(Y^{agbh}\right)\mathrm{tr}\left(Y^{chdi}\right)\mathrm{tr}\left(Y^{egfi}\right)\,,
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\hat\beta^{(4)abcdef}_{\eta,Y^4Y^2} &= \beta^{(4)}_{\eta,69}\,\mathcal{S}_{45}\,\mathrm{tr}\left(Y^{abgh}\right)\mathrm{tr}\left(Y^{cdgiefhi}\right)+\beta^{(4)}_{\eta,70}\,\mathcal{S}_{45}\,\mathrm{tr}\left(Y^{abgh}\right)\mathrm{tr}\left(Y^{cdefgihi}\right)\\
&+\beta^{(4)}_{\eta,71}\,\mathcal{S}_{45}\,\mathrm{tr}\left(Y^{agbh}\right)\mathrm{tr}\left(Y^{cdgiefhi}\right)+\beta^{(4)}_{\eta,72}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{agbh}\right)\mathrm{tr}\left(Y^{cdefgihi}\right)\\
&+\beta^{(4)}_{\eta,73}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abgh}\right)\mathrm{tr}\left(Y^{cdeighfi}\right)+\beta^{(4)}_{\eta,74}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abgh}\right)\mathrm{tr}\left(Y^{cdeigifh}\right)\\
&+\beta^{(4)}_{\eta,75}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abgh}\right)\mathrm{tr}\left(Y^{cdeifigh}\right)+\beta^{(4)}_{\eta,76}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abgh}\right)\mathrm{tr}\left(Y^{cdeifghi}\right)\\
&+\beta^{(4)}_{\eta,77}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abgh}\right)\mathrm{tr}\left(Y^{cdegfihi}\right)+\beta^{(4)}_{\eta,78}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{agbh}\right)\mathrm{tr}\left(Y^{cdeighfi}\right)\\
&+\beta^{(4)}_{\eta,79}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{agbh}\right)\mathrm{tr}\left(Y^{cdeghifi}\right)+\beta^{(4)}_{\eta,80}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{agbh}\right)\mathrm{tr}\left(Y^{cdeifigh}\right)\\
&+\beta^{(4)}_{\eta,81}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{agbh}\right)\mathrm{tr}\left(Y^{cdeifghi}\right)+\beta^{(4)}_{\eta,82}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{agbh}\right)\mathrm{tr}\left(Y^{cdegfihi}\right)\\
&+\beta^{(4)}_{\eta,83}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abgh}\right)\mathrm{tr}\left(Y^{cgdiehfi}\right)+\beta^{(4)}_{\eta,84}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abgh}\right)\mathrm{tr}\left(Y^{cgdheifi}\right)\\
&+\beta^{(4)}_{\eta,85}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{agbh}\right)\mathrm{tr}\left(Y^{cgdiehfi}\right)+\beta^{(4)}_{\eta,86}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{agbh}\right)\mathrm{tr}\left(Y^{cgdheifi}\right)\\
&+\beta^{(4)}_{\eta,87}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{aghi}\right)\mathrm{tr}\left(Y^{bcdhegfi}\right)+\beta^{(4)}_{\eta,88}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{aghi}\right)\mathrm{tr}\left(Y^{bcdgehfi}\right)\\
&+\beta^{(4)}_{\eta,89}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{aghi}\right)\mathrm{tr}\left(Y^{bcdgefhi}\right)+\beta^{(4)}_{\eta,90}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{aghi}\right)\mathrm{tr}\left(Y^{bcefdghi}\right)\\
&+\beta^{(4)}_{\eta,91}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{aghi}\right)\mathrm{tr}\left(Y^{bcdhefgi}\right)+\beta^{(4)}_{\eta,92}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{aghi}\right)\mathrm{tr}\left(Y^{bcefdhgi}\right)\\
&+\beta^{(4)}_{\eta,93}\,\mathcal{S}_{45}\,\mathrm{tr}\left(Y^{gihi}\right)\mathrm{tr}\left(Y^{abcgdefh}\right)+\beta^{(4)}_{\eta,94}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{gihi}\right)\mathrm{tr}\left(Y^{abcdegfh}\right)\\
&+\beta^{(4)}_{\eta,95}\,\mathcal{S}_{45}\,\mathrm{tr}\left(Y^{gihi}\right)\mathrm{tr}\left(Y^{abcdefgh}\right)\,,
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
\hat\beta^{(4)abcdef}_{\eta,Y^6} &= \beta^{(4)}_{\eta,96}\,\mathcal{S}_{45}\,\mathrm{tr}\left(Y^{abcdefghhiig}\right)+\beta^{(4)}_{\eta,97}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcdghefhiig}\right)\\
&+\beta^{(4)}_{\eta,98}\,\mathcal{S}_{15}\,\mathrm{tr}\left(Y^{abghcdhiefig}\right)+\beta^{(4)}_{\eta,99}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcdegfhgihi}\right)\\
&+\beta^{(4)}_{\eta,100}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcdegfhhigi}\right)+\beta^{(4)}_{\eta,101}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcdegfghihi}\right)\\
&+\beta^{(4)}_{\eta,102}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcdeggifhhi}\right)+\beta^{(4)}_{\eta,103}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcdeghifhgi}\right)\\
&+\beta^{(4)}_{\eta,104}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcdeghifghi}\right)+\beta^{(4)}_{\eta,105}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcdeggihifh}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\beta^{(4)}_{\eta,106}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcdeghigifh}\right)+\beta^{(4)}_{\eta,107}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcdeghihifg}\right)\\
&+\beta^{(4)}_{\eta,108}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcdgiegfhhi}\right)+\beta^{(4)}_{\eta,109}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcdgiehfghi}\right)\\
&+\beta^{(4)}_{\eta,110}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcdhiegfghi}\right)+\beta^{(4)}_{\eta,111}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcgdefhhigi}\right)\\
&+\beta^{(4)}_{\eta,112}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcgdefhgihi}\right)+\beta^{(4)}_{\eta,113}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcgdefghihi}\right)\\
&+\beta^{(4)}_{\eta,114}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcgdehifghi}\right)+\beta^{(4)}_{\eta,115}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcgdegifhhi}\right)\\
&+\beta^{(4)}_{\eta,116}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abghcdeifigh}\right)+\beta^{(4)}_{\eta,117}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abghcdegfihi}\right)\\
&+\beta^{(4)}_{\eta,118}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abghcdeifghi}\right)+\beta^{(4)}_{\eta,119}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abghcdeighfi}\right)\\
&+\beta^{(4)}_{\eta,120}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abghcdeghifi}\right)+\beta^{(4)}_{\eta,121}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcgdgefhihi}\right)\\
&+\beta^{(4)}_{\eta,122}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcgdhefgihi}\right)+\beta^{(4)}_{\eta,123}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abcgdhefhigi}\right)\\
&+\beta^{(4)}_{\eta,124}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abghcidefigh}\right)+\beta^{(4)}_{\eta,125}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abghcgdefihi}\right)\\
&+\beta^{(4)}_{\eta,126}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abghcidefghi}\right)+\beta^{(4)}_{\eta,127}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abghcideghfi}\right)\\
&+\beta^{(4)}_{\eta,128}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abghcgdehifi}\right)+\beta^{(4)}_{\eta,129}\,\mathcal{S}_{90}\,\mathrm{tr}\left(Y^{abghcidegifh}\right)\\
&+\beta^{(4)}_{\eta,130}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abghcgdheifi}\right)+\beta^{(4)}_{\eta,131}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abghcgdiehfi}\right)\\
&+\beta^{(4)}_{\eta,132}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abghcidgehfi}\right)+\beta^{(4)}_{\eta,133}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abghcgdieifh}\right)\\
&+\beta^{(4)}_{\eta,134}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abghcidgeifh}\right)+\beta^{(4)}_{\eta,135}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abghcidiegfh}\right)\\
&+\beta^{(4)}_{\eta,136}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abcgghdheifi}\right)+\beta^{(4)}_{\eta,137}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abcgghdiehfi}\right)\\
&+\beta^{(4)}_{\eta,138}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abcgghdieifh}\right)+\beta^{(4)}_{\eta,139}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abcghidgehfi}\right)\\
&+\beta^{(4)}_{\eta,140}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abcghidhegfi}\right)+\beta^{(4)}_{\eta,141}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abcghidheifg}\right)\\
&+\beta^{(4)}_{\eta,142}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcgdhgiehfi}\right)+\beta^{(4)}_{\eta,143}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{abcgdhhieifg}\right)\\
&+\beta^{(4)}_{\eta,144}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abcgdhgieifh}\right)+\beta^{(4)}_{\eta,145}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{abcgdhgheifi}\right)\\
&+\beta^{(4)}_{\eta,146}\,\mathcal{S}_{120}\,\mathrm{tr}\left(Y^{agbgchdheifi}\right)+\beta^{(4)}_{\eta,147}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{agbgchdieifh}\right)\\
&+\beta^{(4)}_{\eta,148}\,\mathcal{S}_{360}\,\mathrm{tr}\left(Y^{agbgchdiehfi}\right)+\beta^{(4)}_{\eta,149}\,\mathcal{S}_{180}\,\mathrm{tr}\left(Y^{agbhcgdiehfi}\right)\\
&+\beta^{(4)}_{\eta,150}\,\mathcal{S}_{60}\,\mathrm{tr}\left(Y^{agbhcidgehfi}\right)\,.
\end{aligned}
\tag{36}
$$

The 150 coefficients contributing to the four-loop sextic vertex correction are found to be

$$
\begin{aligned}
&\beta^{(4)}_{\eta,1}=0\,, && \beta^{(4)}_{\eta,2}=\tfrac{1}{48}\,, && \beta^{(4)}_{\eta,3}=0\,, && \beta^{(4)}_{\eta,4}=-\tfrac{1}{12}\,, && \beta^{(4)}_{\eta,5}=-\tfrac{\pi^2}{32}\,,\\
&\beta^{(4)}_{\eta,6}=-\tfrac{\pi^2}{32}\,, && \beta^{(4)}_{\eta,7}=0\,, && \beta^{(4)}_{\eta,8}=-\tfrac{1}{12}\,, && \beta^{(4)}_{\eta,9}=0\,, && \beta^{(4)}_{\eta,10}=0\,,\\
&\beta^{(4)}_{\eta,11}=0\,, && \beta^{(4)}_{\eta,12}=-\tfrac{\pi^2}{32}\,, && \beta^{(4)}_{\eta,13}=-\tfrac{1}{4}\,, && \beta^{(4)}_{\eta,14}=0\,, && \beta^{(4)}_{\eta,15}=0\,,\\
&\beta^{(4)}_{\eta,16}=0\,, && \beta^{(4)}_{\eta,17}=-\tfrac{1}{12}\,, && \beta^{(4)}_{\eta,18}=0\,, && \beta^{(4)}_{\eta,19}=0\,, && \beta^{(4)}_{\eta,20}=0\,,\\
&\beta^{(4)}_{\eta,21}=0\,, && \beta^{(4)}_{\eta,22}=0\,, && \beta^{(4)}_{\eta,23}=-\tfrac{1}{4}\,, && \beta^{(4)}_{\eta,24}=-\tfrac{\pi^2}{32}\,, && \beta^{(4)}_{\eta,25}=0\,,\\
&\beta^{(4)}_{\eta,26}=0\,, && \beta^{(4)}_{\eta,27}=0\,, && \beta^{(4)}_{\eta,28}=0\,, && \beta^{(4)}_{\eta,29}=\tfrac{1}{8}\,, && \beta^{(4)}_{\eta,30}=-\tfrac{1}{12}\,,\\
&\beta^{(4)}_{\eta,31}=0\,, && \beta^{(4)}_{\eta,32}=\tfrac{1}{8}\,, && \beta^{(4)}_{\eta,33}=-\tfrac{\pi^2}{32}\,, && \beta^{(4)}_{\eta,34}=-\tfrac{1}{4}\,, && \beta^{(4)}_{\eta,35}=\tfrac{\pi^2}{16}\,,\\
&\beta^{(4)}_{\eta,36}=\tfrac{\pi^2}{16}\,, && \beta^{(4)}_{\eta,37}=1\,, && \beta^{(4)}_{\eta,38}=\tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,39}=\tfrac{\pi^2}{6}\,, && \beta^{(4)}_{\eta,40}=0\,,\\
&\beta^{(4)}_{\eta,41}=1-\tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,42}=0\,, && \beta^{(4)}_{\eta,43}=0\,, && \beta^{(4)}_{\eta,44}=\tfrac{1}{2}\,, && \beta^{(4)}_{\eta,45}=2-\tfrac{\pi^2}{3}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\beta^{(4)}_{\eta,46} = 1-\tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,47} = 1-\tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,48} = 0\,, && \beta^{(4)}_{\eta,49} = 0\,, && \beta^{(4)}_{\eta,50} = 0\,,\\
&\beta^{(4)}_{\eta,51} = -\tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,52} = 0\,, && \beta^{(4)}_{\eta,53} = -\tfrac{1}{4}\,, && \beta^{(4)}_{\eta,54} = -\tfrac{1}{2}\,, && \beta^{(4)}_{\eta,55} = 0\,,\\
&\beta^{(4)}_{\eta,56} = -\tfrac{1}{4}\,, && \beta^{(4)}_{\eta,57} = -\tfrac{3}{4}\,, && \beta^{(4)}_{\eta,58} = -\tfrac{1}{4}\,, && \beta^{(4)}_{\eta,59} = -\tfrac{1}{2}\,, && \beta^{(4)}_{\eta,60} = -\tfrac{\pi^2}{8}\,,\\
&\beta^{(4)}_{\eta,61} = -\tfrac{\pi^2}{16}\,, && \beta^{(4)}_{\eta,62} = 0\,, && \beta^{(4)}_{\eta,63} = -\tfrac{3}{4}\,, && \beta^{(4)}_{\eta,64} = -\tfrac{\pi^2}{16}\,, && \beta^{(4)}_{\eta,65} = 0\,,\\
&\beta^{(4)}_{\eta,66} = 0\,, && \beta^{(4)}_{\eta,67} = 0\,, && \beta^{(4)}_{\eta,68} = -\tfrac{\pi^2}{32}\,, && \beta^{(4)}_{\eta,69} = 0\,, && \beta^{(4)}_{\eta,70} = 0\,,\\
&\beta^{(4)}_{\eta,71} = 1\,, && \beta^{(4)}_{\eta,72} = \tfrac{1}{4}\,, && \beta^{(4)}_{\eta,73} = 0\,, && \beta^{(4)}_{\eta,74} = 0\,, && \beta^{(4)}_{\eta,75} = 0\,,\\
&\beta^{(4)}_{\eta,76} = 0\,, && \beta^{(4)}_{\eta,77} = 0\,, && \beta^{(4)}_{\eta,78} = 0\,, && \beta^{(4)}_{\eta,79} = 0\,, && \beta^{(4)}_{\eta,80} = 0\,,\\
&\beta^{(4)}_{\eta,81} = 0\,, && \beta^{(4)}_{\eta,82} = \tfrac{1}{2}\,, && \beta^{(4)}_{\eta,83} = 0\,, && \beta^{(4)}_{\eta,84} = 0\,, && \beta^{(4)}_{\eta,85} = 1-\tfrac{\pi^2}{8}\,,\\
&\beta^{(4)}_{\eta,86} = 0\,, && \beta^{(4)}_{\eta,87} = 1\,, && \beta^{(4)}_{\eta,88} = \tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,89} = \tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,90} = \tfrac{\pi^2}{16}\,,\\
&\beta^{(4)}_{\eta,91} = \tfrac{1}{2}\,, && \beta^{(4)}_{\eta,92} = \tfrac{3}{4}\,, && \beta^{(4)}_{\eta,93} = \tfrac{1}{3}\,, && \beta^{(4)}_{\eta,94} = \tfrac{1}{4}\,, && \beta^{(4)}_{\eta,95} = 0\,,\\
&\beta^{(4)}_{\eta,96} = 0\,, && \beta^{(4)}_{\eta,97} = \tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,98} = \tfrac{\pi^2}{2}\,, && \beta^{(4)}_{\eta,99} = \tfrac{3}{4}\,, && \beta^{(4)}_{\eta,100} = \tfrac{\pi^2}{16}\,,\\
&\beta^{(4)}_{\eta,101} = \tfrac{1}{6}\,, && \beta^{(4)}_{\eta,102} = \tfrac{1}{2}\,, && \beta^{(4)}_{\eta,103} = \tfrac{1}{4}\,, && \beta^{(4)}_{\eta,104} = \tfrac{1}{4}\,, && \beta^{(4)}_{\eta,105} = \tfrac{\pi^2}{16}\,,\\
&\beta^{(4)}_{\eta,106} = 0\,, && \beta^{(4)}_{\eta,107} = \tfrac{1}{12}\,, && \beta^{(4)}_{\eta,108} = \tfrac{\pi^2}{16}\,, && \beta^{(4)}_{\eta,109} = \tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,110} = \tfrac{1}{4}\,,\\
&\beta^{(4)}_{\eta,111} = \tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,112} = 0\,, && \beta^{(4)}_{\eta,113} = \tfrac{1}{3}\,, && \beta^{(4)}_{\eta,114} = 1-\tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,115} = 1\,,\\
&\beta^{(4)}_{\eta,116} = \tfrac{1}{2}\,, && \beta^{(4)}_{\eta,117} = \tfrac{\pi^2}{4}\,, && \beta^{(4)}_{\eta,118} = \tfrac{\pi^2}{4}\,, && \beta^{(4)}_{\eta,119} = 1\,, && \beta^{(4)}_{\eta,120} = 1\,,\\
&\beta^{(4)}_{\eta,121} = \tfrac{1}{6}\,, && \beta^{(4)}_{\eta,122} = 0\,, && \beta^{(4)}_{\eta,123} = \tfrac{\pi^2}{4}\,, && \beta^{(4)}_{\eta,124} = 1\,, && \beta^{(4)}_{\eta,125} = \tfrac{\pi^2}{8}\,,\\
&\beta^{(4)}_{\eta,126} = -2+\tfrac{\pi^2}{4}\,, && \beta^{(4)}_{\eta,127} = 1-\tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,128} = \tfrac{1}{2}\,, && \beta^{(4)}_{\eta,129} = 2\,, && \beta^{(4)}_{\eta,130} = \tfrac{1}{2}\,,\\
&\beta^{(4)}_{\eta,131} = \tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,132} = \tfrac{\pi^2}{12}\,, && \beta^{(4)}_{\eta,133} = \tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,134} = \tfrac{\pi^2}{4}\,, && \beta^{(4)}_{\eta,135} = 1\,,\\
&\beta^{(4)}_{\eta,136} = \tfrac{\pi^2}{16}\,, && \beta^{(4)}_{\eta,137} = \tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,138} = \tfrac{1}{2}\,, && \beta^{(4)}_{\eta,139} = -2+\tfrac{\pi^2}{3}\,, && \beta^{(4)}_{\eta,140} = -1+\tfrac{\pi^2}{8}\,,\\
&\beta^{(4)}_{\eta,141} = 1\,, && \beta^{(4)}_{\eta,142} = 2\,, && \beta^{(4)}_{\eta,143} = \tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,144} = \tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,145} = \tfrac{1}{2}\,,\\
&\beta^{(4)}_{\eta,146} = \tfrac{\pi^2}{16}\,, && \beta^{(4)}_{\eta,147} = \tfrac{1}{4}\,, && \beta^{(4)}_{\eta,148} = \tfrac{\pi^2}{8}\,, && \beta^{(4)}_{\eta,149} = \tfrac{\pi^2}{6}\,, && \beta^{(4)}_{\eta,150} = 0\,. \qquad (37)
\end{aligned}
$$

Our results agree with the two-loop and partial four-loop terms of [79]. As for the four-loop leg and Yukawa vertex corrections, we find large agreement but also minor discrepancies with [78].[1] Note that in [78] the number of coefficients for $\hat{\beta}_Y^{(4)}$ is smaller; the additional ones quoted in the present work all vanish. Both [78, 79] were cross-checked against older publications such as [82, 109–112].

Another important cross-check is the emergence of $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry, to which the next section is dedicated.

In passing, we observe that the RGE coefficients are purely rational at two-loop order, and contain $\zeta_2 \equiv \pi^2/6$ at four loops. The number content appears to get more complicated at higher loops—a phenomenon also found in four dimensions. There, all coefficients are rational up to two loops, while three- and four-loop corrections introduce ζ_3 and ζ_4, respectively [66–68], and five and six loops even feature ζ_5, ζ_6, ζ_7 and $\zeta_{3,5}$ [15, 113–116]. In contrast, the three-dimensional six-loop RGEs of [83] predict the appearance of ln 2, Catalan's constant, and the Dirichlet beta function. This is a surprising result, which merits an

[1] We thank I. Jack for helpful discussions and for revisiting the relevant expressions in private correspondence. An ongoing re-examination indicates minor inconsistencies in [78], which are plausibly the reason for the remaining discrepancies.

independent verification.

4 Supersymmetry

We now impose $\mathcal{N}=1$ supersymmetry on the Lagrangian (1). It is generated via two supercharges Q_α that fulfil

$$\{Q_\alpha, \overline{Q}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu \,. \tag{38}$$

These supercharges and the corresponding supersymmetry generators can be defined via superspace coordinates θ^α, which are real, two-component Grassmann numbers

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\gamma^\mu\theta_\alpha\partial_\mu \,, \qquad D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\gamma^\mu\theta_\alpha\partial_\mu \,. \tag{39}$$

Thus, each matter superfield

$$\Phi_A = \phi_A + \overline{\theta}\psi_A + \tfrac{1}{2}\overline{\theta}\theta\, F_A \tag{40}$$

consists of real scalar fields ϕ_A, two-component Majorana fields ψ_A and real auxiliary fields F_A, again using $\overline{\theta} = \theta^T\gamma^0$. A marginal superpotential takes the general form

$$W(\Phi) = \tfrac{1}{4!}\lambda^{ABCD}\Phi_A\Phi_B\Phi_C\Phi_D \,, \tag{41}$$

with $\lambda^{ABCD} = \lambda^{(ABCD)}$ the real superquartic coupling tensor.

Integrating the superspace coordinates as well as the auxiliary fields, we arrive at a generalisation of the three-dimensional Wess–Zumino model [117]

$$\begin{aligned}\mathcal{L}_{\text{SUSY}} = \int \mathrm{d}^2\theta \left[\tfrac{1}{2}D_\alpha\Phi_A D^\alpha\Phi_A + W(\Phi)\right] &= \tfrac{1}{2}\partial_\mu\phi_A\partial^\mu\phi_A + \tfrac{i}{2}\overline{\psi}_A\gamma^\mu\partial_\mu\psi_A \\ &- \tfrac{1}{4}\lambda^{ABCD}\phi_A\phi_B\overline{\psi}_C\psi_D - \tfrac{1}{36}\lambda^{ABCG}\lambda^{DEFG}\phi_A\phi_B\phi_C\phi_D\phi_E\phi_F \,.\end{aligned} \tag{42}$$

Comparing this with (1) implies

$$Y^{ABCD} = \lambda^{ABCD} \,, \quad \eta^{ABCDEF} = \mathcal{S}_{10}\lambda^{ABCG}\lambda^{DEFG} \,. \tag{43}$$

Therefore, at the loop level $\mathcal{N}=1$ supersymmetry yields

$$\begin{aligned} \gamma^{AB}_\Phi \equiv \gamma^{AB}_\phi = \gamma^{AB}_\psi \,, \qquad \hat{\beta}^{ABCD}_\lambda \equiv \hat{\beta}^{ABCD}_Y = \hat{\beta}^{(ABCD)}_Y \,, \\ \hat{\beta}^{ABCDEF}_\eta = \mathcal{S}_{20}\lambda^{ABCG}\hat{\beta}^{DEFG}_\lambda + 2\mathcal{S}_{10}\lambda^{ABCG}\gamma^{GH}_\Phi\lambda^{HDEF} \,. \end{aligned} \tag{44}$$

At two-loop order, this leads to

$$\begin{aligned} &\gamma^{(2)}_{\phi,1} = \gamma^{(2)}_{\psi,1} \,, \qquad \beta^{(2)}_{Y,1} = \beta^{(2)}_{Y,4} = 0 \,, \quad \beta^{(2)}_{Y,2} = \beta^{(2)}_{Y,3} = \beta^{(2)}_{Y,5} = 6\gamma^{(2)}_{\phi,1} \,, \\ &\beta^{(2)}_{\eta,2} = 0 \,, \qquad \tfrac{1}{2}\beta^{(2)}_{\eta,1} = \tfrac{1}{6}\beta^{(2)}_{\eta,3} = -\tfrac{1}{12}\beta^{(2)}_{\eta,4} = -\tfrac{1}{24}\beta^{(2)}_{\eta,5} = \gamma^{(2)}_{\phi,1} \,, \end{aligned} \tag{45}$$

which fixes all RGEs up to a single coefficient. At four loops, there are 214 independent relations all of which are compatible with the direct results of this work. We list them in a separate file attached to the arXiv submission of the present work.

With $\mathcal{N}=2$ supersymmetry, there are four supercharges that satisfy the algebra

$$\{Q_{\alpha,i}, \overline{Q}_{\beta,j}\} = 2\gamma^\mu_{\alpha\beta}\delta_{ij}P_\mu \,. \tag{46}$$

Here $i,j = 1,2$ form an $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ R-symmetry. Thus, the algebra can be implemented with complex 2-component superspace coordinates θ. This is analogous to $\mathcal{N}=1$ supersymmetry in four spacetime dimensions. Supercoordinates, superfields and all their components are complex, and we need to carefully distinguish the contractions $\overline{X}Y \equiv X^\dagger\gamma^0 Y$ and $(XY) \equiv X^{\mathrm{T}}\gamma^0 Y$ between two spinors X and Y for the remainder of this section. The superfields

$$\Phi_A = \varphi_A + \sqrt{2}(\theta\chi_A) + (\theta\theta)F_A + i\overline{\theta}\gamma^\mu\theta\partial_\mu\varphi_A - \tfrac{i}{\sqrt{2}}(\theta\theta)\overline{\theta}\gamma^\mu\partial_\mu\chi_A + \tfrac{1}{4}(\theta\theta)(\theta\theta)^*\partial_\mu\partial^\mu\varphi_A \quad (47)$$

consist of complex scalars φ_A and F_A as well as complex two-component Dirac fermions χ_A. The superpotential reads

$$W(\Phi) = \tfrac{1}{4!}\lambda^{ABCD}\Phi_A\Phi_B\Phi_C\Phi_D\,, \quad (48)$$

Note that the R-symmetry enforces a holomorphy of the superpotential, meaning that tensors λ^{ABCD} only couple to superfields Φ_A but not their complex conjugates Φ_A^*. The Lagrangian then reads

$$\begin{aligned}\mathcal{L}_{\text{SUSY}} &= \int \mathrm{d}^2\theta\mathrm{d}^2\theta^*\,\Phi_A^*\Phi_A + \int \mathrm{d}^2\theta\,W(\Phi) + \int \mathrm{d}^2\theta^*\,W^*(\Phi^*) = \partial_\mu\varphi_A^*\partial^\mu\varphi_A + i\overline{\chi}_A\gamma^\mu\partial_\mu\chi_A \\ &\quad - \left[\tfrac{1}{4}\lambda^{ABCD}\varphi_A\varphi_B(\chi_C\chi_D) + \text{h.c.}\right] - \tfrac{1}{36}\lambda^{ABCG}(\lambda^{DEFG})^*\,\varphi_A\varphi_B\varphi_C\varphi_D^*\varphi_E^*\varphi_F^*\,. \end{aligned} \quad (49)$$

In order to map the $(\mathcal{N}=2)$–supersymmetric theory (49) to the language of (1), we decompose the complex matter fields φ and χ into real components ϕ, ψ.

We first repackage the M complex scalar field φ_A and fermions χ_A into $(M\times 2)$-dimensional matrices

$$\tilde{\phi}_A^\alpha = (\varphi_A,\,\varphi_A^*)^\alpha \qquad \text{and} \qquad \tilde{\psi}_A^\alpha = (\chi_A,\,\chi_A^*)^\alpha \quad (50)$$

with $\alpha = 1, 2$. The interaction terms of (49) can then be rewritten as

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{4}\tilde{Y}^{ABCD}_{\alpha\beta\gamma\delta}\,\tilde{\phi}_A^\alpha\tilde{\phi}_B^\beta(\tilde{\psi}_C^\gamma\tilde{\psi}_D^\delta) - \tfrac{1}{36}\tilde{\eta}^{ABCDEF}_{\alpha\beta\gamma\delta\epsilon\zeta}\,\tilde{\phi}_A^\alpha\tilde{\phi}_B^\beta\tilde{\phi}_C^\gamma\tilde{\phi}_D^\delta\tilde{\phi}_E^\epsilon\tilde{\phi}_F^\zeta \quad (51)$$

with

$$\tilde{Y}^{ABCD}_{1111} = \lambda^{ABCD}\,, \qquad \tilde{Y}^{ABCD}_{2222} = (\lambda^{ABCD})^*\,, \qquad \eta^{ABCDEF}_{111222} = \lambda^{ABCG}(\lambda^{DEFG})^*\,. \quad (52)$$

Next, we translate the pseudo-real fields $\tilde{\phi}, \tilde{\psi}$ into real components ϕ, ψ as in (1), using the embeddings

$$\phi_A^\alpha = (\operatorname{Re}\varphi_A,\,\operatorname{Im}\varphi_A)^\alpha \qquad \text{and} \qquad \psi_A^\alpha = (\operatorname{Re}\chi_A,\,\operatorname{Im}\chi_A)^\alpha\,. \quad (53)$$

Note that the scalar $(a, b, c, \dots)$ and fermionic indices $(i, j, \dots)$ running from $1\dots 2M$ in (1) are here split up in two indices A and α with the ranges M and 2. The bases (50) and (53) are related via the rotation

$$\tilde{\phi}_A^\alpha = X^\alpha{}_\beta\phi_A^\beta\,, \qquad \tilde{\psi}_A^\alpha = X^\alpha{}_\beta\psi_A^\beta\,, \qquad \text{where} \qquad X = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}. \quad (54)$$

Finally, we obtain the mapping

$$Y^{ABCD}_{\alpha\beta\gamma\delta} = X^\alpha{}_{\alpha'}X^\beta{}_{\beta'}X^\gamma{}_{\gamma'}X^\delta{}_{\delta'}\tilde{Y}^{ABCD}_{\alpha'\beta'\gamma'\delta'}\,, \qquad \eta^{ABCDEF}_{\alpha\beta\gamma\delta\epsilon\zeta} = X^\alpha{}_{\alpha'}X^\beta{}_{\beta'}X^\gamma{}_{\gamma'}X^\delta{}_{\delta'}X^\epsilon{}_{\epsilon'}X^\zeta{}_{\zeta'}\tilde{\eta}^{ABCDEF}_{\alpha'\beta'\gamma'\delta'\epsilon'\zeta'}\,, \quad (55)$$

which refer the couplings Y and η in (1), again with their original indices being split up

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{4}Y^{ABCD}_{\alpha\beta\gamma\delta}\,\phi_A^\alpha\phi_B^\beta(\psi_C^\gamma\psi_D^\delta) - \tfrac{1}{6!}\eta^{ABCDEF}_{\alpha\beta\gamma\delta\epsilon\zeta}\,\phi_A^\alpha\phi_B^\beta\phi_C^\gamma\phi_D^\delta\phi_E^\epsilon\phi_F^\zeta\,. \quad (56)$$

While these structures appear complicated, their practical evaluation in template RGEs is straightforward: all internal contractions between two coupling tensors include a summation over greek indices of the form

$$Y^{A\dots}_{\alpha\dots}\,Y^{A\dots}_{\alpha\dots} = \tilde{Y}^{A\dots}_{\beta\dots}\,\tilde{Y}^{A\dots}_{\gamma\dots}\,X^\beta{}_\alpha X^\gamma{}_\alpha = \tilde{Y}^{A\dots}_{\beta\dots}\,\tilde{Y}^{A\dots}_{\gamma\dots}\,\sigma_1^{\beta\gamma}\,. \quad (57)$$

The Pauli matrix σ_1 can be understood from the fact that propagators connect fields and their complex conjugates, e.g. φ_A and φ_A^* and vice versa. Inserting (55) everywhere in our template expressions, all internal contractions of greek indices are carried over the metric σ_1. Together with (52), this enforces a holomorphic structure of RGEs: all λ are only contracted with λ^* but no other λ. Many contractions in the template RGEs vanish in $\mathcal{N}=2$ supersymmetry because they are incompatible with this holomorphic structure. In consequence, the superpotential parameters do not receive vertex corrections [118, 119]. Overall, the relations

$$\gamma_\Phi{}^{AB} \equiv \gamma_\phi{}^{AB} = \gamma_\psi{}^{AB}\,, \qquad \hat{\beta}_Y^{ABCD} = 0\,, \qquad \hat{\beta}_\eta{}^{ABCDEF} = 2\lambda^{ABCG}\gamma_\Phi{}^{HG}(\lambda^{HDEF})^* \tag{58}$$

must hold as long as the renormalisation scheme does not break the supersymmetry. At two-loop order this implies

$$\gamma^{(2)}_{\phi,1} = \gamma^{(2)}_{\psi,1}\,, \qquad \beta^{(2)}_{Y,1} = \beta^{(2)}_{Y,4} = 0\,, \qquad \beta^{(2)}_{\eta,1} = -\tfrac{1}{3}\beta^{(2)}_{\eta,3} - \tfrac{1}{3}\beta^{(2)}_{\eta,4} = 2\gamma^{(2)}_{\phi,1}\,, \qquad \beta^{(2)}_{\eta,2} = 0\,. \tag{59}$$

It is obvious that (59) is less restrictive than its $\mathcal{N}=1$ counterpart (45). At four loops, 46 conditions arise, which are all fulfilled for our results and are listed in a separate file along with the arXiv submission of the present work.

5 Fixed Points

In this section, we investigate whether perturbatively accessible fixed points can arise directly in $d=3$ in the context of the renormalisable theories in (1). To this end, we analyse the leading (two-loop) contribution to the quartic Yukawa and scalar sextic β-functions from Sec. 3, which we repeat here for convenience

$$\begin{aligned}(4\pi)^2\beta_Y^{ab} &= \frac{1}{12}\mathcal{S}_2\mathrm{tr}\left(Y^{accd}\right)Y^{bd} + \frac{1}{12}\mathcal{S}_2 Y^{abcdcd}\\ &\quad + \frac{1}{2}\mathrm{tr}\left(Y^{acbd}\right)Y^{cd} + \frac{1}{2}Y^{cdabcd} + \frac{1}{2}\mathcal{S}_4 Y^{acbdcd}\,,\end{aligned} \tag{60}$$

$$\begin{aligned}(4\pi)^2\beta_\eta^{abcdef} &= \frac{1}{6}\mathcal{S}_{10}\eta^{abcghi}\eta^{defghi} + \frac{1}{12}\mathcal{S}_6\mathrm{tr}\left(Y^{aggh}\right)\eta^{bcdefh} + \frac{1}{2}\mathcal{S}_{15}\mathrm{tr}\left(Y^{agbh}\right)\eta^{cdefgh}\\ &\quad - \mathcal{S}_{90}\mathrm{tr}\left(Y^{abcdegfg}\right) - 2\mathcal{S}_{45}\mathrm{tr}\left(Y^{abcgdefg}\right)\,.\end{aligned} \tag{61}$$

Schematically, RGEs take the form

$$\beta(\alpha) = -B\,\alpha^2 + C\,\alpha^3 + \mathcal{O}(\alpha^4) \tag{62}$$

where the first term is determined by (60) or (61). A fixed point α^* arises at $\beta(\alpha^*)=0$ and exhibits a perturbative expansion $\alpha^* = B/C + \mathcal{O}(B^2)$. For perturbation theory to be reliable around the fixed point, the leading coefficient $|B|$ should be small with respect to $|C|$ and higher order coefficients. For instance, the smallness of $|B|$ could be the product of a cancellation of terms or a large-N suppression in (60) and (61).

For purely scalar theories ($Y=0$), the existence of a UV fixed point is long known [80–84], and under perturbative control in a large-N limit. This is in contrast to renormalisable QFTs in $d=4$, where fixed points may only arise in the presence of a non-Abelian gauge sector [120–122].

5.1 General Considerations

In the following, we will search for fixed points involving fermions. The leading-order β functions for Y does not involve η. Thus, a fixed-point coupling Y^* must emerge from Y itself.

Crucially, all terms in (60) enter with positive coefficients, prohibiting any straightforward cancellations among them. As a result, most QFTs exhibit a manifestly positive β_Y, and the emergence of a perturbative fixed point is highly non-trivial.

Nevertheless, there is a window of opportunity, which can be recognised by considering the quantity [120, 122]

$$
\begin{aligned}
(4\pi)^2 \frac{\mathrm{d}}{\mathrm{d}\log\mu} \mathrm{tr}\left(Y^{abab}\right) &= (4\pi)^2 \mathrm{tr}\left(Y^{ab}\beta_Y^{ab}\right) \\
&= \frac{1}{6}\mathrm{tr}\left(Y^{accb}\right)\mathrm{tr}\left(Y^{addb}\right) + \frac{1}{6}\langle Y^{abcd}, Y^{abcd}\rangle \\
&\quad + \frac{1}{2}\langle Y^{ij}Y^{kl}, Y^{kl}Y^{ij}\rangle + \frac{1}{2}\langle Y^{abcd}, Y^{cdab}\rangle + 2\langle Y^{accb}, Y^{bdda}\rangle ,
\end{aligned} \tag{63}
$$

with $\langle A, B\rangle = \sum_{i,j} A_{ij}B_{ij}$, i.e. the sum runs over indices not explicitly written in the brackets $\langle\cdot,\cdot\rangle$. For the first term in the second line we have made the fermion indices explicit but suppressed the scalar ones. For the first term in the second line we have made the fermion indices explicit but suppressed the scalar ones. A requirement for a QFT to contain a perturbative fixed point is that the absolute value of (63) can be made arbitrary small or even vanish. In our case, this is signaled by a change of sign in (63) by any choice of Y.

The first two terms in (63) are manifestly positive while the situation is less clear for the last three. If we fix the fermionic indices for the third and the scalar ones for the fourth and fifth term, we observe that each can be written as a scalar product of the form $\langle AB, BA\rangle$ where A, B are symmetric matrices. In case the product BA is antisymmetric, the last three terms of (63) are negative. Matrices with these properties are part of a Clifford algebra $\{\gamma^A, \gamma^B\} = 2\delta^{AB}\mathbb{1}$.

Thus, a promising candidate for negative contributions to the two-loop β-function is the theory

$$
\mathcal{L} = \partial_\mu\phi_i^*\partial^\mu\phi_i + i\overline{\psi}_i\not{\partial}\psi_i - Y\,(\phi_i^*\gamma^A_{ij}\phi_j)(\overline{\psi}_k\gamma^A_{kl}\psi_l) - \tfrac{1}{36}\eta(\phi_i^*\phi_i)^3 \tag{64}
$$

with complex scalars ϕ_i and Dirac fermions ψ_i both in the spinor representation of SO(n). Here we have the fundamental index $A = 1\dots n$ as well as $i, j, k, l = 1, \dots d_\gamma$ for the spinor reprentation, where $d_\gamma = 2^{\lfloor N/2\rfloor}$ and $\overline{\psi}_i = \psi_i^\dagger\gamma^0$. The corresponding RGE

$$
\beta_Y = -\left[2(n-2) + \frac{4}{3}(n-3)d_\gamma\right]\frac{Y^3}{(4\pi)^2} + \mathcal{O}(4\text{-loop}), \tag{65}
$$

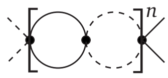
is indeed negative for $n \geq 3$. Thus, we find that negative two-loop β-functions are viable, which motivates the existence of perturbative fixed points.

For any fixed point to be strictly perturbative, it needs to be under the control of a small expansion parameter, ϵ. Since we search for fixed points directly in $d = 3$, ϵ must stem from the parameters of the theory and arise within the leading order RGEs. For instance,

$$
\beta_\alpha = -\epsilon\,\alpha^2 + C\,\alpha^3 + \dots \tag{66}
$$

yields a fixed-point value $\alpha^* = \epsilon/C + \mathcal{O}(\epsilon^2)$, suppressing higher loop contributions by ever-increasing powers of ϵ. A natural source of perturbative control originates from large field multiplicities. Commonly, two realizations can be distinguished. Large-N expansions have $\epsilon \propto N^{-1}$ and become exact in the limit $N \to \infty$. A notable example is found in [80–82, 84]. On the other hand, Veneziano–type expansions [123] feature two large multiplicities $N_{1,2} \to \infty$ with a fixed ratio such that $\epsilon = N_1/N_2 + \text{const}$, see for instance [76, 124]. In the following, we will discuss some examples how large-N limits can be taken for the theories at hand.

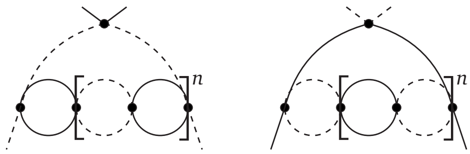
A straightforward idea is to have both scalar and fermions in large multiplicities N_s and N_f, respectively, where $N_f \propto N_s \propto N \to \infty$. In this large-$N$ expansion, the leading-order contributions to β_Y are given by an alternating bubble chain of fermions and scalars


(67)

and appear at every non-trivial loop order. Explicitly, they yield factors $(N_f N_s Y^2)^n$ at $(2n)$-loop, such that a 't Hooft–rescaled coupling [125]

$$\hat{Y}^2 = N^2 Y^2 \tag{68}$$

has a finite β-function for $N \to \infty$. However, each diagram of (67) factorises into subgraphs without logarithmic divergences, thus not contributing to the β-functions.[2] At two and four loops, this class of diagrams corresponds to the vanishing coefficients $\beta^{(2)}_{Y,1}$ and $\beta^{(4)}_{Y,10}$ in (15) and (20), respectively. The dominant contributions therefore stem from "next-to-leading order" diagrams given by inserting (67) between two scalar, or two fermion legs, respectively


(69)

corresponding to the non-vanishing coefficients $\beta^{(2)}_{Y,2}$, $\beta^{(2)}_{Y,5}$ and $\beta^{(4)}_{Y,16}$, $\beta^{(4)}_{Y,70}$. For β-functions to remain finite at $N \to \infty$, we still require the rescaling (68). This, however, produces a factor of $\epsilon = 1/N$ for every dominant contribution (69), suppressing the entire β-function by $1/N$, i.e. $\beta_{\hat{Y}^2} \propto \epsilon$. We stress that taking the strict $N \to \infty$ limit and only then searching for fixed points is inconsistent, since the large-N rescaling renders the full β-function subleading and hence trivially vanishing. The correct procedure is to search for fixed points within $\beta_{\hat{Y}^2}/\epsilon = \sum_{\ell=1}^{\infty} c_{2\ell} \hat{Y}^{2\ell} + \mathcal{O}(\epsilon)$, keeping the leading $1/N$ contributions explicit before sending N to infinity [84].

Another popular large-N–expansion is to only assign one large field multiplicity, for instance $N_f \propto N \to \infty$ but not N_s. In this case, the rescaling $\check{Y}^2 = NY^2$ is sufficient. Leading-order contributions do not merely arise through (67), but from many other graphs as well, even at two loops. In the current example, the leading-order β-functions include contributions from both leg corrections $\propto \gamma^{(2)}_{\phi,1}$ and the non-vanishing vertex correction $\propto \beta^{(2)}_{Y,2}$.

5.2 Example: Perturbative IR Fixed Point

Let us choose the explicit example of a QFT with symmetry SO(3) × $U(N)$, where complex fermions and scalars both transform under the spinor representation of SO(3), but fermions are in the fundamental of $U(N)$, while scalars are singlets. Explicitly, the Lagrangian reads

$$\mathcal{L} = \partial_\mu \phi_i^* \partial^\mu \phi_i + i\overline{\psi}_{ia} \partial\!\!\!/ \psi_{ia} - Y\,(\phi_i^* \sigma^A_{ij} \phi_j)(\overline{\psi}_{ak} \sigma^A_{kl} \psi_{al}) - \tfrac{1}{36}\eta(\phi_i^*\phi_i)^3\,, \tag{70}$$

with $a = 1, \ldots, N$ labelling the $U(N)$ flavour, $i, j, k, l = 1, 2$ labelling the SO(3) spinor indices, $\overline{\psi} = \psi^\dagger \gamma^0$, and $A = 1, 2, 3$ denoting the three Pauli matrices of dimension $d_\gamma = 2$. Using the abbreviations

$$\epsilon = \frac{1}{N}\,, \qquad \alpha_Y = \frac{NY^2}{(4\pi)^2}\,, \qquad \text{and} \qquad \alpha_\eta = \frac{\eta}{(4\pi)^2} \tag{71}$$

[2]We remark that the vanishing is renormalisation-scheme independent. Although, e.g. in cutoff-schemes, massive tadpoles exhibit a pole, they do not enter the vertex counterterm. See [84] for a discussion in the scalar ϕ^6 theory.

we find the RGEs

$$\beta_{\alpha_Y} = -4\epsilon\,\alpha_Y^2 + 4\left[6+\pi^2+\mathcal{O}(\epsilon^2)\right]\alpha_Y^3 - \frac{32}{3}\epsilon\,\alpha_\eta\alpha_Y^2 + \frac{4}{9}\,\alpha_\eta^2\alpha_Y + \mathcal{O}(\alpha^4)\,, \tag{72}$$

$$\beta_{\alpha_\eta} = \frac{17}{3}\alpha_\eta^2 + 48\alpha_Y\alpha_\eta - 288\epsilon\,\alpha_Y^2 - \left[288\pi^2+\mathcal{O}(\epsilon)\right]\alpha_Y^3 + \alpha_\eta\mathcal{O}(\alpha^2) + \mathcal{O}(\alpha^4)\,. \tag{73}$$

Note that the two-loop contribution of β_{α_Y} is negative. This is due to fermions transforming under a spinor representation of SO(n), as discussed earlier. However, the fact that the two-loop coefficient is also suppressed $\propto \epsilon$ is a special feature of $n = 3$. This is evident from (65), where dominant large-N contributions $\propto d_\gamma$ vanish at two loops and first reappear in the RGEs at four loops. More precisely, there is a cancellation of terms $\propto \gamma^{(2)}_{\phi,1}$ and $\propto \beta^{(2)}_{Y,2}$. We could have also assigned an additional multiplicity N_s to the scalar field and observe an equivalent cancellation between $\propto \gamma^{(2)}_{\psi,1}$ and $\propto \beta^{(2)}_{Y,5}$. The special constellation of SO(3) spinors yields an IR fixed point under perturbative control

$$\alpha_Y^* = \frac{\epsilon}{6+\pi^2} + \mathcal{O}(\epsilon^2)\,, \qquad \alpha_\eta^* = \frac{12(3+\pi^2)\epsilon^2}{(6+\pi^2)^2} + \mathcal{O}(\epsilon^3)\,. \tag{74}$$

Since $\alpha_\eta^* > 0$, the scalar potential is bounded from below. The two eigenvalues of the corresponding stability matrix $S_{mn} = (\partial\beta_{\alpha_m}/\partial\alpha_n)\big|_{\alpha^*}$ read

$$\vartheta_1 = \frac{4\epsilon^2}{6+\pi^2} + \mathcal{O}(\epsilon^3)\,, \qquad \vartheta_2 = \frac{48\epsilon}{6+\pi^2} + \mathcal{O}(\epsilon^2)\,, \tag{75}$$

and reveal that the fixed point is totally IR attractive. The RG flow around the fixed point is suppressed by $\propto \epsilon^2$ in one of the directions. This slow trajectory also connects the IR fixed point with the Gaussian one ($\alpha_Y^* = \alpha_\eta^* = 0$). The situation is depicted in Fig. 1 for the case $\epsilon = 10^{-1}$, where the slow trajectory is marked red. We also show the purely scalar UV fixed point [80–82, 84] in this setting, which is not under strict perturbative control as the number of scalars is not large. The UV fixed point is not directly connected to the IR one (74), but flows into the Gaussian along $\alpha_Y = 0$, which is coloured violet in Fig. 1. There are hints of even more ultraviolet fixed points, though they lie beyond the perturbative region.

For completeness, we also investigate if our model predicts fixed points in $d = 4$ via dimensional continuation from $d = 3 + \delta$. This is established by evaluating

$$0 = 2\delta\alpha_{Y,\eta} + \beta_{\alpha_{Y,\eta}} \tag{76}$$

using the four-loop RGEs (72) and evaluating fixed points as a power series in δ. As the two-loop coefficient of β_{α_Y} is negative, an interacting UV fixed point emerges at $0 < \delta \ll 1$, which needs to be tracked to $\delta \to 1$. Conversely, there is no fixed-point solution for $\delta < 0$, excluding fixed points in two dimensions. However, such phenomena are predicted in different three-dimensional theories, see for instance [79] in the context of melonic CFTs. Back to the model at hand, we find

$$\alpha_Y^* = \frac{\delta}{2\epsilon} + \mathcal{O}(\delta^2)\,, \tag{77}$$

highlighting that the dimensional continuation is not reliable in the limit $\epsilon \to 0$ where perturbative control is established in $d = 3$, but rather towards the upper bound $\epsilon \to 1$. We find two fixed-point solutions

$$\alpha_Y^\pm = \frac{\delta}{2\epsilon} + \left[\frac{50437}{5202} + \frac{173}{64}\pi^2 \mp \frac{94\sqrt{577}}{289} + \mathcal{O}(1-\epsilon)\right]\delta^2 + \mathcal{O}(\delta^3)\,,$$

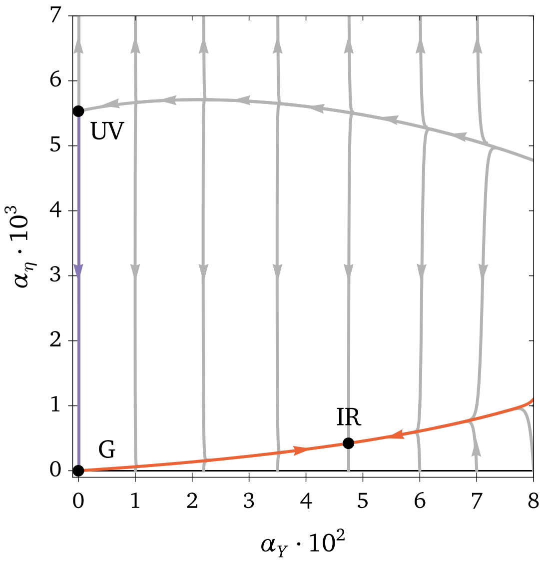


Figure 1: RG flow for the theory (70) with $\epsilon = 10^{-1}$, showing the Gaussian fixed point ($\alpha_Y^* = \alpha_\eta^* = 0$), the infrared fixed point (74), and the purely scalar ultraviolet fixed point ($\alpha_Y^* = 0$) [80–82, 84]. Arrows point from the UV to the IR. The UV fixed point is connected to the Gaussian only through the trajectory $\alpha_Y = 0$ (violet). The RG flow along the red trajectory is slower–around the IR fixed point, it is suppressed by factor $\propto \epsilon^2$ as opposed to $\propto \epsilon^1$.

$$\alpha_\eta^\pm = \left[\frac{3(\pm\sqrt{577} - 13)}{17} + \mathcal{O}(1-\epsilon)\right]\delta + \mathcal{O}(\delta^2)\,, \tag{78}$$

where the coefficient $\propto \delta^2$ for $\alpha_\eta^\pm$ is known but omitted for brevity. Only $\alpha_{Y,\eta}^+$ describes a stable fixed point as the leading coefficient of α_η^- is negative. Computing the eigenvalues of the stability matrix yields only one UV attractive direction

$$\vartheta_1 = -2\,\delta + \left[\frac{100874}{2601} - \frac{376\sqrt{577}}{289} + \frac{173}{16}\pi^2 + \mathcal{O}(1-\epsilon)\right]\delta^2 + \mathcal{O}(\delta^3)\,. \tag{79}$$

A smooth limit $\delta \to 1$ is not possible, given the UV fixed point turns marginal around $\delta \approx 0.0175$, possibly hinting at a fixed-point merger. Although the precision is rather limited, there is no indication that the UV fixed point survives the continuation to four dimensions.

6 Conclusion

In this work, we derived general, renormalisation-scheme-agnostic template expressions at four loops for all β-functions and anomalous dimensions in any three-dimensional renormalisable theory with scalars and fermions. We explicitly computed all coefficients of the template RGEs in the $\overline{\text{MS}}$ scheme and checked them against existing literature.

Our results are compatible with both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry to emerge with general superpotentials. In fact, we have extracted algebraic relations required to hold in any supersymmetry-preserving renormalisation scheme. Both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ can be extended to four dimensions, where they correspond to $\mathcal{N} = \frac{1}{2}$ and $\mathcal{N} = 1$, respectively. Curiously, $\mathcal{N} = \frac{1}{2}$ supersymmetry is violated in the $\overline{\text{MS}}$ at four-loop order [18, 68], while its counterpart in three dimensions is not. The source of the violation are terms stemming from odd spinor traces, which always vanish in four dimensions. At four-loop in $d = 3$, such terms do not contribute, though they might play a role at higher loops. It would be interesting to see if such terms eventually pose a problem for $\mathcal{N} = 1$ supersymmetry in three dimensions.

As a direct application of our results, we have surveyed template RGEs for perturbative fixed points in strictly three dimensions. In doing so, we discovered a systematic way to construct theories with such critical points. This revealed a rare example of a QFT with an interacting IR fixed point under strict perturbative control. Our search for such theories, however, is not exhaustive, and other critical phenomena may yet be discovered.

A natural direction for future work is to search for a non-trivial UV fixed point, which would render the theory asymptotically safe. Such a scenario is not unmotivated: UV fixed points are found in purely scalar [80–82, 84] and purely fermionic [85–89] theories in three dimensions. In contrast, UV completion in four dimensions is a more complicated business: at least a non-Abelian gauge sector is required [120], and even more restrictions apply for non-trivial fixed points [121, 122, 126].

Our work has revealed that the perturbative landscape of critical points in three dimensions is quite rich. This picture will become more complete once the catalogue of template RGEs is extended to include gauge interactions. The gauge sector admits an exactly marginal Chern–Simons term at level k amenable to a controlled large-k expansion. This, in turn, provides a systematic setting for perturbative tests of dualities between fermionic and scalar Chern–Simons theories enjoying rich applications, e.g. in condensed-matter physics [38–41, 127–129]. In contrast, the gauge coupling is classically relevant and requires e.g. large-N expansions to be studied with perturbative methods. A prominent example is QED_3 [31–35, 130], sharing phenomena such as confinement, chiral symmetry breaking, and asymptotic freedom with real-world QCD_4.

Acknowledgements

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A Master integrals at four loops

In this Appendix, we collect the fully massive tadpole master integrals, required to compute the template RGEs in dimensional regularisation and the $\overline{\text{MS}}$ scheme up to four loops. First, in App. A.1, we provide an update on [103] by giving analytical results where available, and complementing with some relations found by applying the integer relation algorithm `PSLQ` [101, 102] to our high-precision numerical expansions. We show the numerical results for all master integrals in App. A.2. Lastly, in App. A.3, we discuss alternative choices of masters, and connect with the basis used in the `FORM` package `FMFT` [98].

While we present analytic and numeric data in an odd dimension $d = 3 - 2\epsilon$ here, let us note that the equivalent information for an even dimension $d = 2 - 2\epsilon$ had been listed in [131]. Taken together, this suffices for expansions around any integer dimension, given the existence of dimensional recurrences [132] that allow to analytically map $d \longleftrightarrow d + 2$.

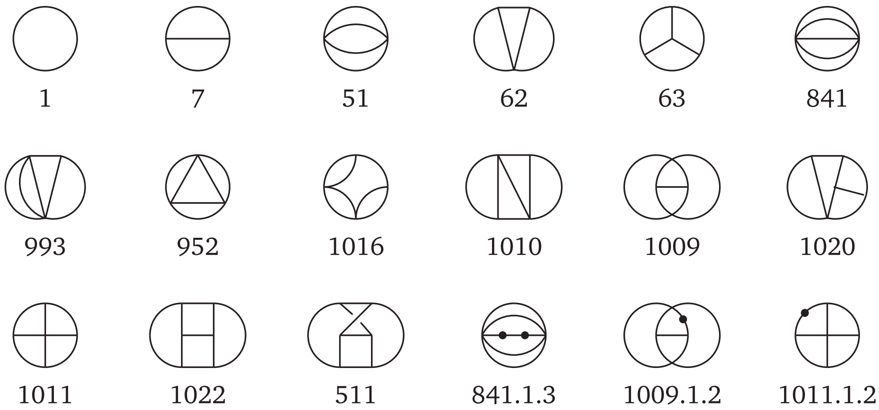


Figure 2: The master integrals up to four loops, labelled by their respective sector identifiers whose binary representation corresponds to the propagators that are present, with momenta from the set $\{k_1, k_2, k_3, k_4, k_1 - k_4, k_2 - k_4, k_3 - k_4, k_1 - k_2, k_1 - k_3, k_1 - k_2 - k_3\}$ in the four-loop case. The dotted masters carry an additional pair of labels indicating which propagator carries a non-unity power. For possible alternative choices of the latter class, see the basis change relations of App. A.3.

A.1 Analytic Results in $d = 3 - 2\epsilon$

We treat here the set of master integrals displayed in Fig. 2. All our propagators $1/(1 + p_i^2)$ are Euclidean with unit mass. We normalise each L-loop tadpole with the (1-loop tadpole)^L, leading to

$$I_1 = 1 \tag{80}$$

for the 1-loop tadpole, exactly. For reference, in any other normalisation, the 1-loop tadpole can be inferred from $\int \mathrm{d}^d p/(1 + p^2) = \pi^{d/2}\,\Gamma(1 - d/2)$.

At two loops, the only master integral I_7 can be represented by a hypergeometric function [133], see also Sec. 6.2 of [134]. For its (fast) expansion around $d = 3 - 2\epsilon$ dimensions, we write

$$I_7 = \frac{1-2\epsilon}{2\epsilon}\left\{\frac{3}{2} - \frac{9}{16}\sum_{j=1}^{\infty}(-\epsilon)^j\, g_j - \frac{\cos(\pi\epsilon)}{\sqrt{\mathrm{sinc}(2\pi\epsilon)}}\exp\left[\epsilon\ln(\tfrac{16}{3}) + 2\epsilon\sum_{j=1}^{\infty}\epsilon^{2j}\,(4^j - 1)\frac{\zeta(2j+1)}{2j+1}\right]\right\},$$
$$g_j \equiv {}_{j+2}F_{j+1}(\tfrac{3}{2}, 1, \ldots, 1; 2, \ldots, 2; \tfrac{3}{4}),$$

in terms of generalized hypergeometric functions g_j (which can be assigned transcendental weight j, and evaluate rapidly), odd zeta values, and with $\mathrm{sinc}(x) \equiv \frac{\sin(x)}{x}$. We note that $g_1 = \frac{16}{3}\ln(\frac{4}{3})$, $g_2 = \frac{16}{3}\mathrm{Li}_2(\frac{1}{4}) - \frac{8}{3}\ln^2(\frac{4}{3})$, $g_3 = {}_5F_4(\frac{3}{2}, 1, 1, 1, 1; 2, 2, 2, 2; \frac{3}{4})$, etc.

Another analytically known result is the three-spoked wheel at $d = 3$. From Broadhurst's work [135, 136] we know the leading term of the fully massive 3d 3-loop tetrahedron I_{63} to be a difference of two Clausen integrals

$$I_{63} = c_{\mathrm{tet}} + \mathcal{O}(\epsilon), \quad \text{with} \;\; c_{\mathrm{tet}} = \tfrac{1}{\sqrt{2}}\,\mathrm{Im}\Big[\mathrm{Li}_2(w^2) - \mathrm{Li}_2(w^4)\Big], \tag{81}$$

where $w = e^{i\arcsin(1/3)} = (\sqrt{8} + i)/3$.

For two other integrals, I_{51} and I_{952} (which in fact constitute the first two instances of a class of necklace integrals), it can be shown that their expansion coefficients contain multiple zeta values (MZVs) only. The constructive proof follows from introducing two distinct masses, and solving a differential equation in $z = M/m$ in terms of harmonic polylogarithms, which at $z = 1$ reduce to MZVs [137–139]. Recently, a recursive construction of the expansion coefficients of the necklace integrals has been given, see App. A of [84].

Some further terms had contributed to specific physics questions and therefore been analytically identified before. See, e.g. [140], where the 4-loop integrals I_{841}, I_{993} and I_{952} were needed up to their constant ϵ^0 coefficients (the ϵ^1 terms of I_{51} and I_7 had entered as well).

Using high-precision numerical results (for reference, we show 65 digits and a dozen ϵ-orders in App. A.2) generated by an in-house implementation [141] of Laporta's difference equation method [100], we have attempted to identify some numbers by the integer relation algorithm PSLQ, implemented in Mathematica [142] as `FindIntegerNullVector`. We have mostly searched at transcendental weight 2 (two ϵ^0 terms are found with weight 3, I_{952} and I_{1016}), and our (rather meager) findings are documented below. Note that some dilogarithms with different arguments are related, such as $\mathrm{Li}_2(\frac{1}{4}) + 2\mathrm{Li}_2(\frac{1}{3}) = \zeta(2) - \ln^2(2) - \ln^2(\frac{3}{2})$. In the interest of compact formulae, for some integrals we pull out a rational prefactor that renders the remaining expansion in pure-weight form.

$$\begin{aligned} I_7 = \frac{1-2\epsilon}{4\epsilon} \Big\{ & 1 - 4\ln(\tfrac{3}{2})\,\epsilon + 4\Big[3\mathrm{Li}_2(\tfrac{1}{3}) + 2\ln^2(\tfrac{3}{2}) - \zeta_2\Big]\epsilon^2 \\ & + \Big[\tfrac{9}{8}\,g_3 - \tfrac{1}{3}\ln^3(\tfrac{16}{3}) + 2\ln(\tfrac{16}{3})\zeta_2 - 4\zeta_3\Big]\epsilon^3 \\ & + \Big[-\tfrac{9}{8}\,g_4 - \tfrac{1}{12}\ln^4(\tfrac{16}{3}) + \ln^2(\tfrac{16}{3})\zeta_2 + \tfrac{9}{2}\zeta_4 - 4\ln(\tfrac{16}{3})\zeta_3\Big]\epsilon^4 \\ & + \Big[\tfrac{9}{8}\,g_5 - \tfrac{1}{60}\ln^5(\tfrac{16}{3}) + \tfrac{1}{3}\ln^3(\tfrac{16}{3})\zeta_2 + \tfrac{9}{2}\ln(\tfrac{16}{3})\zeta_4 - 2\ln^2(\tfrac{16}{3})\zeta_3 + 4\zeta_2\zeta_3 - 12\zeta_5\Big]\epsilon^5 \\ & + \ldots \Big\}, \end{aligned} \tag{82}$$

$$\begin{aligned} I_{51} = \frac{(1-2\epsilon)^2}{(1-6\epsilon)\epsilon} \Big\{ & 1 - 4\ln(2)\,\epsilon + 4\Big[\ln^2(2) + \zeta_2\Big]\epsilon^2 + 8\Big[-\tfrac{1}{3}\ln^3(2) - \ln(2)\zeta_2 - \tfrac{19}{4}\zeta_3\Big]\epsilon^3 \\ & + 8\Big[20\mathrm{Li}_4(\tfrac{1}{2}) + \ln^4(2) - 4\ln^2(2)\zeta_2 + 27\ln(2)\zeta_3 + 11\zeta_4\Big]\epsilon^4 \\ & + 8\Big[40\mathrm{Li}_5(\tfrac{1}{2}) - \tfrac{2}{5}\ln^5(2) + \tfrac{8}{3}\ln^3(2)\zeta_2 - 27\ln^2(2)\zeta_3 - 27\zeta_2\zeta_3 - 22\ln(2)\zeta_4 - \tfrac{1445}{8}\zeta_5\Big]\epsilon^5 \\ & + \ldots \Big\}, \end{aligned} \tag{83}$$

$$\begin{aligned} I_{62} = \frac{(1-2\epsilon)^2}{\epsilon^2} \Big\{ & \Big[-2\mathrm{Li}_2(\tfrac{1}{3}) - \tfrac{1}{2}\ln^2(3) + \tfrac{1}{2}\zeta_2\Big]\epsilon^2 \\ & + \Big[-\tfrac{45}{64}\,g_3 + \tfrac{5}{2}\mathrm{Li}_3(\tfrac{1}{3}) + 14\mathrm{Li}_2(\tfrac{1}{3})\ln(2) + 20\ln^3(2) - 4\mathrm{Li}_2(\tfrac{1}{3})\ln(3) - 30\ln^2(2)\ln(3) \\ & + \tfrac{37}{2}\ln(2)\ln^2(3) - \tfrac{85}{24}\ln^3(3) - 11\ln(2)\zeta_2 + \tfrac{19}{4}\ln(3)\zeta_2 + \tfrac{79}{24}\zeta_3\Big]\epsilon^3 + \ldots \Big\}, \end{aligned} \tag{84}$$

$$I_{63} = (c_{\mathrm{tet}}) + \mathcal{O}(\epsilon), \tag{85}$$

$$\begin{aligned} I_{841} = \frac{15}{4\epsilon(1+2\epsilon)(1-6\epsilon)} \Big\{ & \tfrac{3}{4} + \tfrac{10}{3}\ln(\tfrac{2}{5})\,\epsilon + \Big[-6\mathrm{Li}_2(\tfrac{1}{3}) + 6\mathrm{Li}_2(\tfrac{1}{5}) + 6\mathrm{Li}_2(\tfrac{1}{6}) \\ & + 5\zeta_2 - 3\ln^2(5) + 11\ln^2(\tfrac{2}{5}) + 6\ln(2)\ln(3)\Big]\epsilon^2 + \ldots \Big\}, \end{aligned} \tag{86}$$

$$\begin{aligned} I_{993} = \frac{1+12\epsilon^2}{4\epsilon^2} \Big\{ & \tfrac{1}{8} + \ln(\tfrac{2}{3})\,\epsilon + \Big[7\mathrm{Li}_2(\tfrac{1}{3}) + 4\mathrm{Li}_2(\tfrac{1}{5}) - 4\mathrm{Li}_2(\tfrac{1}{6}) \\ & - 5\zeta_2 + 2\ln^2(3) + 2\ln^2(\tfrac{2}{5}) - 4\ln(2)\ln(5)\Big]\epsilon^2 + \ldots \Big\}, \end{aligned} \tag{87}$$

$$I_{952} = \frac{(1-2\epsilon)^3}{\epsilon^3} \Big\{ \tfrac{3}{16}\zeta_2\,\epsilon^2 + \tfrac{3}{8}\Big[2\ln(2)\zeta_2 - 7\zeta_3\Big]\epsilon^3$$

$$+\tfrac{3}{4}\Big[40\text{Li}_4(\tfrac{1}{2})+\tfrac{5}{3}\ln^4(2)-8\ln^2(2)\zeta_2+21\ln(2)\zeta_3-\tfrac{25}{2}\zeta_4\Big]\epsilon^4$$
$$+\Big[-228\text{Li}_5(\tfrac{1}{2})-108\text{Li}_4(\tfrac{1}{2})\ln(2)-\tfrac{13}{5}\ln^5(2)+10\ln^3(2)\zeta_2-\tfrac{63}{4}\ln^2(2)\zeta_3$$
$$-\tfrac{135}{8}\zeta_2\zeta_3-\tfrac{75}{2}\ln(2)\zeta_4+\tfrac{1023}{8}\zeta_5\Big]\epsilon^5+\ldots\Big\}, \tag{88}$$

$$I_{1016}=\frac{1}{\epsilon^3}\Big\{\Big[-3\text{Li}_3(\tfrac{1}{3})+\tfrac{17}{8}\zeta_3-\tfrac{3}{2}\ln(2)\zeta_2+\tfrac{1}{4}\ln^3(3)\Big]\epsilon^3+\ldots\Big\}, \tag{89}$$

$$I_{1010}=(c_{1010})+\mathcal{O}(\epsilon), \tag{90}$$

$$I_{1009}=(c_{1009})+\mathcal{O}(\epsilon), \tag{91}$$

$$I_{1020}=(c_{1020})+\mathcal{O}(\epsilon), \tag{92}$$

$$I_{1011}=(c_{1011})+\mathcal{O}(\epsilon), \tag{93}$$

$$I_{1022}=(c_{1022})+\mathcal{O}(\epsilon), \tag{94}$$

$$I_{511}=(c_{511})+\mathcal{O}(\epsilon), \tag{95}$$

$$I_{841.1.3}=-\frac{1}{2\epsilon(1-4\epsilon)}\Big\{\tfrac{1}{4}+\Big[-\tfrac{1}{4}+\ln(\tfrac{2}{5})\Big]\epsilon+\Big[-\text{Li}_2(\tfrac{1}{3})+\text{Li}_2(\tfrac{1}{5})+\text{Li}_2(\tfrac{1}{6})+\tfrac{3}{2}\zeta_2$$
$$-\tfrac{1}{2}\ln^2(5)+\tfrac{5}{2}\ln^2(\tfrac{2}{5})+\ln(2)\ln(3)\Big]\epsilon^2+\ldots\Big\}, \tag{96}$$

$$I_{1009.1.2}=\frac{1}{12\epsilon^2}\Big\{\Big[2c_{1009}-9\text{Li}_2(\tfrac{1}{3})+4\text{Li}_2(\tfrac{1}{5})+5\text{Li}_2(\tfrac{1}{6})+\zeta_2$$
$$+2\ln^2(5)-3\ln^2(3)+\tfrac{5}{2}\ln^2(2)+8\ln(2)\ln(\tfrac{3}{5})\Big]\epsilon^2+\ldots\Big\}, \tag{97}$$

$$I_{1011.1.2}=\frac{1}{12\epsilon^2}\Big\{\Big[4c_{1011}-10\text{Li}_2(\tfrac{1}{3})+8\text{Li}_2(\tfrac{1}{5})+6\text{Li}_2(\tfrac{1}{6})+\tfrac{1}{2}\zeta_2$$
$$+4\ln^2(5)-5\ln^2(3)+3\ln^2(2)+16\ln(2)\ln(\tfrac{3}{5})\Big]\epsilon^2+\ldots\Big\}. \tag{98}$$

Note that six constants c_s, corresponding to finite (as $d\to 3$) corner integrals of some sectors s have not yet been identified analytically. Two of them (c_{1009} and c_{1011}) appear also in the leading terms of the corresponding dotted master integrals, suggesting they are of weight 2.

A.2 Numerical Results in $d=3-2\epsilon$

The set of integrals shown in Fig. 2, normalised as explained in App. A.1, is given in numerical form (we chose to display 65 digits and a dozen ϵ-orders here) as

$$\begin{aligned}
I_7 = &+0.00000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.25000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&-0.90546510810816438197801311546434913657199042346249419761401432 41\,\epsilon^{0}\\
&+0.59343974708562364963912239028150760346004729555253082509885405908\,\epsilon^{1}\\
&+0.52512566387783544082913378723610992562729317536114660898506286 67\,\epsilon^{2}\\
&-0.27884375060961024071414452925362363585846664527741080006522624 93\,\epsilon^{3}\\
&+0.29260732917420669154099509707396437196279803442550538920627536 60\,\epsilon^{4}\\
&-0.28562643729630591966190897011113916070017525661256850252992594 69\,\epsilon^{5}\\
&+0.28343642078129755533533174869873820697027276887050676640697130 37\,\epsilon^{6}\\
&-0.28231833342486097546852862374738431627013076122425108643014863 10\,\epsilon^{7}\\
&+0.28177755769530668629821513956943569073681027922896519653955157 96\,\epsilon^{8}
\end{aligned}$$

$$
\begin{aligned}
&-0.2815116035637308218683520750434441111060181572146255407809751321\,\epsilon^{9}\\
&+0.2813800985507556971174815239799876449109648586773520170404285604\,\epsilon^{10}\\
&-0.2813148194727843972356391513506495794244403642468997018218251014\,\epsilon^{11}\\
&+0.2812823342607278074357898205544258655554486958992765555900329463\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13}),
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
I_{51} =\;&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+1.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&-0.7725887222397812376689284858327062723020005374410210164827200379\,\epsilon^{0}\\
&+18.9563708785861489692202138002253480991940103363233340654442493 76\,\epsilon^{1}\\
&+12.9539977464503729021763205407863784022118257559019559981807984 80\,\epsilon^{2}\\
&+669.0503477180669373132190318576684031252239270133611444771663138 6\,\epsilon^{3}\\
&+444.8941555932515078520286013774996563051936478385853327723620897 0\,\epsilon^{4}\\
&+24092.0072117187897498468599654625636876948663163600951038642711 84\,\epsilon^{5}\\
&+16009.8107676676041313572252978186414997138398010258011990382953 64\,\epsilon^{6}\\
&+867315.3632990980531830147385157587009715455287606097455666113246 2\,\epsilon^{7}\\
&+576353.5203605310928446245964553834749419036866048369228004655692 1\,\epsilon^{8}\\
&+31223348.5001263513764197967000067348032814489197162899944271721 71\,\epsilon^{9}\\
&+20748736.4744586062861305645709426626614321985439947516237068356 29\,\epsilon^{10}\\
&+1124040529.9957259675717097477275074862039621670166784243500595915\,\epsilon^{11}\\
&+746954536.6754222753177463166644670231289047802546893979117217389 7\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13}),
\end{aligned}
\tag{100}
$$

$$
\begin{aligned}
I_{62} =\;&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&-0.5134339069363047459191745739301958898038835538905682145624149320\,\epsilon^{0}\\
&+5.7095684615899292794862376279588751106060985499584093920508473919\,\epsilon^{1}\\
&-37.648074995706791560142239266650126622503344413012137547297409871\,\epsilon^{2}\\
&+224.09748146261060335877889602790798958489614465031252380780097021\,\epsilon^{3}\\
&-1339.4703001627123300036272073007602941947313396981280010338399715\,\epsilon^{4}\\
&+8034.3777908087848490070116519359134599537706638758754531851698513\,\epsilon^{5}\\
&-48203.858560109269901168864155866002112114048423199772229294827551\,\epsilon^{6}\\
&+289221.32366907318778038693730181824862801656033781087842470320783\,\epsilon^{7}\\
&-1735326.6669460210271940650616318339597050248542806279433969102276\,\epsilon^{8}\\
&+10411959.374170972905026051700333844871033927354194847097605834640\,\epsilon^{9}\\
&-62471756.376952049703149659518733718984948385239468448678541434518\,\epsilon^{10}\\
&+374830539.29383322923380951002583268565046696331749930000979367083\,\epsilon^{11}\\
&-2248983237.8671245141557536382160325513510430577744937226407723089\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13}),
\end{aligned}
\tag{101}
$$

$$
\begin{aligned}
I_{63} =\;&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&-0.0217375763327503428408132521394582459517260422873373496515994940\,\epsilon^{0}\\
&+0.0765336321133093854067600182595029834757335052149621362922698456\,\epsilon^{1}
\end{aligned}
$$

$$
\begin{aligned}
&+0.0579067747736512759094492759848848526298596795634080920929429833\,\epsilon^{2}\\
&-0.4434602064084302101813748581409186969642656195400891131645164005\,\epsilon^{3}\\
&+0.3500576529478710482652721964962749809598095962196122929858767415\,\epsilon^{4}\\
&+0.1773796466935324822091222077716446916954673245127639216491696301\,\epsilon^{5}\\
&-0.4059552861037855521858525807488028207138376797995537234907960572\,\epsilon^{6}\\
&+0.7699218077585947722881008308989432923624213682729622461460099026\,\epsilon^{7}\\
&-1.1760399359044822280385045059917948165800135273220891962552311904\,\epsilon^{8}\\
&+1.6603333219010436448891656413581396272998836796289177706714181052\,\epsilon^{9}\\
&-2.2369416988852873423783274395913879713300015049468803030231520827\,\epsilon^{10}\\
&+2.9257651318163983863358564530958038058286056940021618955507476100\,\epsilon^{11}\\
&-3.7498542228612113820834815642900533679099159525722217434663717634\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13})\,, \qquad (102)
\end{aligned}
$$

$$
\begin{aligned}
I_{841} = &+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+2.8125000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&-0.2036341484269383147940901471001383931262652488532808473995985247\,\epsilon^{0}\\
&+86.830821990825688152407740061694121553645153216845571942924644672\,\epsilon^{1}\\
&+45.522230692605414940981730189798030051319531268529413323820472602\,\epsilon^{2}\\
&+3284.6509518299759688331483772804228891248191247538026964866027370\,\epsilon^{3}\\
&+1707.6271747695301363580569298170696889325884987221322680564534831\,\epsilon^{4}\\
&+120965.23009891368364043141541466445366569800874277602462740376845\,\epsilon^{5}\\
&+62865.525276433278935959062135828719273709630748781527244190128748\,\epsilon^{6}\\
&+4398212.5214601951029983541151813642863087358036782575241682867454\,\epsilon^{7}\\
&+2285474.8672910275047270597406190813581312000598673699090087295818\,\epsilon^{8}\\
&+159031061.91370649969772152140973597247543609306992574712616768782\,\epsilon^{9}\\
&+82634137.312180982765576200986973126994428225019873189834485476525\,\epsilon^{10}\\
&+5736244853.4883876703118317390782206668123548049969920666901823029\,\epsilon^{11}\\
&+2980541539.2935515565764734020184945128433113892475869409749588342\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13})\,, \qquad (103)
\end{aligned}
$$

$$
\begin{aligned}
I_{993} = &+0.0312500000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&-0.1013662770270410954945032788660872841429976058656235494035035810\,\epsilon^{-1}\\
&-1.0957787366268196443963067347565253672951367482995236383840337687\,\epsilon^{0}\\
&+11.988553631020143167816608026973182815225793914343747814688812858\,\epsilon^{1}\\
&-79.151364835440668791480916984517867501349251382810943942658357314\,\epsilon^{2}\\
&+487.83612400927362669344992987484615725647953424090124006374839899\,\epsilon^{3}\\
&-2989.3390233560778834407224104486118851963625262762370320995333034\,\epsilon^{4}\\
&+18204.857886630491581123986636721395745530266353973057516863279084\,\epsilon^{5}\\
&-110310.03517221441552909233975258092609023137581451468591330685173\,\epsilon^{6}\\
&+666189.64569193289904643103435867746709680317986593812128779183593\,\epsilon^{7}\\
&-4014457.4525299073181650930165489878509054857144628008599477613653\,\epsilon^{8}\\
&+24156021.599416591381598834005462533839393530838310452192584817084\,\epsilon^{9}\\
&-145213232.21636294012276746184248200041936436897499198738457431610\,\epsilon^{10}
\end{aligned}
$$

$$
\begin{aligned}
&+872387795.4774221368086895290255048593143767735826610063701860360 7\,\epsilon^{11} \\
&-5238760370.2398729650257527576865438683244611341317955180306061709\,\epsilon^{12} \\
&+\mathcal{O}(\epsilon^{13})\,, \qquad (104)
\end{aligned}
$$

$$
\begin{aligned}
I_{952} =&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2} \\
&+0.3084251375340424568385778437461297229785531064762747070754171680\,\epsilon^{-1} \\
&-4.1508141380160501720998448422000725547169352752279686177416993597\,\epsilon^{0} \\
&+31.549872818438118671926038980481234232483796715337834408322412566\,\epsilon^{1} \\
&-201.94428993188771265169628751954548164802617562239654080941214975\,\epsilon^{2} \\
&+1251.1106249388529983194838151460325056013459097478378746952587105\,\epsilon^{3} \\
&-7679.2285698054877882123267972199537878884542064350913339936505170\,\epsilon^{4} \\
&+46784.291276688988776324235796576700031905651026805724193769803935\,\epsilon^{5} \\
&-283544.17917659161289206354386156257793904030712558899891781280316\,\epsilon^{6} \\
&+1712620.2217882952272440262847646385364810974740862982219830815821\,\epsilon^{7} \\
&-10321136.598061739253152188859812121503633063066634550550560999337\,\epsilon^{8} \\
&+62108469.206188874366033370471042304178155102694369103139890778213\,\epsilon^{9} \\
&-373377395.43947565307165093581883464861416426203157797322166945631\,\epsilon^{10} \\
&+2243170668.0603173448980298981419489016518432903259518811000880576\,\epsilon^{11} \\
&-13470649157.511402519190281333422453026122130453991960070579044314\,\epsilon^{12} \\
&+\mathcal{O}(\epsilon^{13})\,, \qquad (105)
\end{aligned}
$$

$$
\begin{aligned}
I_{1016} =&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2} \\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1} \\
&+0.1291074598213152985210243915418107780934739544429115056519581510\,\epsilon^{0} \\
&-1.4002327916247422804946832553087161948787356659419769886224153744\,\epsilon^{1} \\
&+7.5103317302403489479682855310196107940903895075113622512450558212\,\epsilon^{2} \\
&-30.581203881649301217638270485484486733381394563210204652763544763\,\epsilon^{3} \\
&+119.34371666980758851804197684594154288500290351725983919849159742\,\epsilon^{4} \\
&-473.83200088109162748777101028461860653383739741623327329839716205\,\epsilon^{5} \\
&+1893.0065880004566546212705978896481979625823885252498146564540508\,\epsilon^{6} \\
&-7569.2492900989746620567677098574739383386128210151792211363969055\,\epsilon^{7} \\
&+30274.418687366965929260051380696998154657577652866727730385186415\,\epsilon^{8} \\
&-121095.43259100558044613878373462644261728831481957865334109526317\,\epsilon^{9} \\
&+484380.28642108088705607179109316179595637152946223431261309543764\,\epsilon^{10} \\
&-1937521.1903508295691721694486375489159765845675501291899563752012\,\epsilon^{11} \\
&+7750087.3348811980067177117038958220957816230453915036689522456936\,\epsilon^{12} \\
&+\mathcal{O}(\epsilon^{13})\,, \qquad (106)
\end{aligned}
$$

$$
\begin{aligned}
I_{1010} =&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2} \\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1} \\
&+0.1077181593584969503054046183107323659963422543199052556553408135\,\epsilon^{0} \\
&-1.1383106814611776874204354292450638171490491591319502094309442455\,\epsilon^{1} \\
&+5.9177711577329682890469542805919265891951819355675306389912313068\,\epsilon^{2} \\
&-23.500169366616157711812375508797421753065961900022655168659367602\,\epsilon^{3}
\end{aligned}
$$

$$
\begin{aligned}
&+90.9143062910253804712778143823487807219414639945888632269245316 35\,\epsilon^{4}\\
&-360.9939994435411000470014809586656123543390035790777071423612796 4\,\epsilon^{5}\\
&+1443.288958618854137464089580396758497658326662224401960231704098 4\,\epsilon^{6}\\
&-5772.700418157949331146150762518609098898282316386606974064738395 5\,\epsilon^{7}\\
&+23091.17024463570675856529562003359342230263704644381860124113080 6\,\epsilon^{8}\\
&-92365.77871782710956277998790554291639052617644734548183543165511 0\,\epsilon^{9}\\
&+369465.0255603071802564483491303880913893615672395216327266203434 4\,\epsilon^{10}\\
&-1477862.897074606153805734144152303504097517405675001520373775856 6\,\epsilon^{11}\\
&+5911455.353139006659309494042149324289937924638414454459808027458 3\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13})\,,
\end{aligned}
\tag{107}
$$

$$
\begin{aligned}
I_{1009} = &+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&+0.0775372561143257787414875957509475136030747042669114989127188118\,\epsilon^{0}\\
&-0.7850657447681336205546345615905118994345599157639593394166547118\,\epsilon^{1}\\
&+3.872229870921178941702161846918884449535096438582725224152187397 3\,\epsilon^{2}\\
&-14.74451691574359399242072004329981730410970474814734459733087683 7\,\epsilon^{3}\\
&+56.32911378325151796577289170724700817982927072015107120129662567 9\,\epsilon^{4}\\
&-224.0385267193074508560334710501372861417931491433611182447288120 8\,\epsilon^{5}\\
&+897.1352514040778274358112568190385541868949495110462802296824582 4\,\epsilon^{6}\\
&-3590.095805985614188297534642527487695041287146587458936443243775 8\,\epsilon^{7}\\
&+14363.10108773768945714380880701885975615698639612935306104479768 3\,\epsilon^{8}\\
&-57455.94032674131905409020659828132677432225972945447805133223260 8\,\epsilon^{9}\\
&+229827.8732023376438176743254720615124544012239375322346834503965 9\,\epsilon^{10}\\
&-919315.7086416797709595563616859060580283652731506757541910405838 4\,\epsilon^{11}\\
&+3677266.418489072240377928448142835814944626590479833323023215533 5\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13})\,,
\end{aligned}
\tag{108}
$$

$$
\begin{aligned}
I_{1020} = &+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&+0.0078616587307741514524519579553998692962244357731351282244807040\,\epsilon^{0}\\
&-0.0420630557733301531767376865993991145551483376102269432397899465\,\epsilon^{1}\\
&+0.0281445875447985958193247281655960473603619387876491701541532989\,\epsilon^{2}\\
&+0.2003465303315888390661886952499613753047176444211517626682057632\,\epsilon^{3}\\
&-0.3968729517094802727107857201381604256920771041700628254053286817\,\epsilon^{4}\\
&+0.1182562596947772051669820853363639396307120368086953454399942155\,\epsilon^{5}\\
&+0.2365939206518576854693178263124970235570895709756691393698854955\,\epsilon^{6}\\
&-0.3939917897375007515335809037619681215676788745755882021500202668\,\epsilon^{7}\\
&+0.7275037173721455913672060099659930078851570624796151275020549584\,\epsilon^{8}\\
&-1.137253845929603608658476498591937240297698730269077698968810140 8\,\epsilon^{9}\\
&+1.706337990196386071377401858175628382065887724277915237149435500 6\,\epsilon^{10}\\
&-2.489278223486184933616435044539668295228065144681571617982995335 1\,\epsilon^{11}\\
&+3.569970399628106480273075140811300863675053904456357041399033503 0\,\epsilon^{12}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}(\epsilon^{13})\,, && (109)\\
I_{1011} = &+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&+0.0049698068041011492163979479935232913687486244009787674397837111\,\epsilon^{0}\\
&-0.0221871599216739679311204418032839563006731953374356775687891781\,\epsilon^{1}\\
&-0.0104554874014880232655748601819752340362804370224375575296020832\,\epsilon^{2}\\
&+0.1730719623993056502322942758452476334832374486853169923215250325\,\epsilon^{3}\\
&-0.1917151879815380595480336530782280933876667192953262402261544328\,\epsilon^{4}\\
&-0.2053931699093452903325942602701897663344101697595982728435894486\,\epsilon^{5}\\
&+0.5246779153555777418181627380478070116619756080037918559090517679\,3\epsilon^{6}\\
&-0.5545337956637010742099086897697627011871819400496161843674813329\,\epsilon^{7}\\
&+0.6061811472378074412415447883684687933065703366616370491854746376\,\epsilon^{8}\\
&-0.4849138948221826458073564111976296403569150678740896226660036024\,\epsilon^{9}\\
&+0.1765510035929661235124295875476157797303297604193881756940510938\,\epsilon^{10}\\
&+0.4143836233267367414226970576417150244387629019145900728606145804\,\epsilon^{11}\\
&-1.4052442411781122895829493534226111430421991019412378791788684904\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13})\,, && (110)\\
I_{1022} = &+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&+0.0009137653616418094002736626931788099695908996557191386726549905\,\epsilon^{0}\\
&-0.0018954360966446162828353001997718110866304367021524775390047257\,\epsilon^{1}\\
&-0.0111610370205852109677424037081480594931790404185888768789339302\,\epsilon^{2}\\
&+0.0247097124237810080182937710123133941654128465632925095048110995\,\epsilon^{3}\\
&+0.0409629053668332103291869868461450002335192144119277054313161684\,\epsilon^{4}\\
&-0.1047105248579111155379936351124082341509153797401766843986975108\,\epsilon^{5}\\
&-0.0211895768856691406309490143053716581246410403040032880889871551\,\epsilon^{6}\\
&+0.1225861577274033934820248818626847974700299128363948071955439078\,\epsilon^{7}\\
&-0.0810541046563265505633632098547768710852600184275122534971389645\,\epsilon^{8}\\
&+0.0986846249444064804510300279069038848531358154735920399312757952\,\epsilon^{9}\\
&-0.0755812774255241876014006273876997366040860222222252240392019801\,\epsilon^{10}\\
&+0.0404086327599496956553015158739085139114155476373797680019449253\,\epsilon^{11}\\
&+0.0181108246869346685800390875841064878677340480222443554113786474\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13})\,, && (111)\\
I_{511} = &+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&+0.0007652924327752750641180337499004334480825547740395167556522498\,\epsilon^{0}\\
&-0.0014466499930356254239057454215807741287799876415127826228993859\,\epsilon^{1}\\
&-0.0096933643047541598467607892194657353106745005248510009426732767\,\epsilon^{2}\\
&+0.0191185860160505124609009500375183175419573812629700802185490906\,\epsilon^{3}\\
&+0.0386859239613796596681008332932349941003126485299013296093935054\,\epsilon^{4}\\
&-0.0831822741079942849356017478540266825037580457727156448365481056\,\epsilon^{5}
\end{aligned}
$$

$$
\begin{aligned}
&- 0.0352226534095132742579651027313373549102143974217382043839387510\,\epsilon^{6}\\
&+ 0.1063919856752840340832932068029899586735166124141740818936028108\,\epsilon^{7}\\
&- 0.0519182918329414326567615077737025117141420605336303704421997953\,\epsilon^{8}\\
&+ 0.0645974715850955373802690549089026672938473373221210540491044436\,\epsilon^{9}\\
&- 0.0399381145624023517077105230914568694287417489845575678601720904\,\epsilon^{10}\\
&+ 0.0115770193015759899260813518671910205481315225299908764467143430\,\epsilon^{11}\\
&+ 0.0320755141779737283305714138110235068992289879490912429628390464\,\epsilon^{12}\\
&+ \mathcal{O}(\epsilon^{13})\,, \qquad (112)
\end{aligned}
$$

$$
\begin{aligned}
I_{841.1.3} =& + 0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&- 0.1250000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&+ 0.0831453659370775325917636058840055357250506099541312338959839409\,\epsilon^{0}\\
&- 1.6932674288524445117588019822518530483034181177209921975753613853\,\epsilon^{1}\\
&- 3.2873099987095956757969436032513632205869735813843805919151186840\,\epsilon^{2}\\
&- 66.2983285032639600901197481122616556723522599635053967176871449090\,\epsilon^{3}\\
&- 96.6307509992545428853557702633343410351752054421210942056589385800\,\epsilon^{4}\\
&- 2388.4074978095403861182590067087856017068388258287281938603256469\,\epsilon^{5}\\
&- 3527.9159122619659626052440964830539457349635722103171019158907893\,\epsilon^{6}\\
&- 85801.071023612001597007103390458643386194213667744152423047192495\,\epsilon^{7}\\
&- 127732.998897074145804309813919200388745719783018575349988169092 04\,\epsilon^{8}\\
&- 3085942.3716941658254169908206653236880253507731325991666076560171\,\epsilon^{9}\\
&- 4609946.8913558213933828964531599275298079479584314873449846338921\,\epsilon^{10}\\
&- 111047726.5342445890781720996755147780741180101846814415207339384 5\,\epsilon^{11}\\
&- 166142837.5386302367890000663693332713543044432223304527043346393 1\,\epsilon^{12}\\
&+ \mathcal{O}(\epsilon^{13})\,, \qquad (113)
\end{aligned}
$$

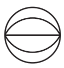
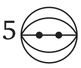
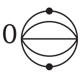
$$
\begin{aligned}
I_{1009.1.2} =& + 0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+ 0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&+ 0.0122708448416023818590539285032346817103352542129458892187383173\,\epsilon^{0}\\
&- 0.0772945419272941552494559081789631104038042676481336126865453872\,\epsilon^{1}\\
&+ 0.1310729016018951094007039791555048446906345110550299042607664414\,\epsilon^{2}\\
&+ 0.0830032872542511616855400933461805257194802909378379914072997302\,\epsilon^{3}\\
&- 0.4111859770696879562995085686276732339664519748331157849692640167\,\epsilon^{4}\\
&+ 0.2917361299698484839244198340902152100296371115094273579909835713\,\epsilon^{5}\\
&- 0.0205038890492164145365517692299900817070599213718630657116308488\,\epsilon^{6}\\
&- 0.0307315741851277041939370700380270957342428390264537014132059621\,\epsilon^{7}\\
&+ 0.1949950842652080071368994870681490446705959741792342456942626159\,\epsilon^{8}\\
&- 0.3613091678052469235941959651868445479905362991134840668607979518\,\epsilon^{9}\\
&+ 0.5699806769206655392130335637412118814029072304169966209653863902\,\epsilon^{10}\\
&- 0.8262326961919464825959457166218723815675068128659463659934200323\,\epsilon^{11}\\
&+ 1.1449693539833461550233201217755443550688994668592872703388835156\,\epsilon^{12}\\
&+ \mathcal{O}(\epsilon^{13})\,, \qquad (114)
\end{aligned}
$$

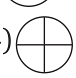
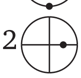
$$
I_{1011.1.2} = + 0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}
$$

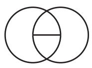
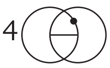
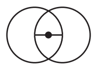
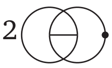
$$
\begin{aligned}
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&+0.0013216583759421658118506617173625472155572471312453246441417419\,\epsilon^{0}\\
&-0.0033507404000985557252229750084916732797827506971271931692182700 4\,\epsilon^{1}\\
&-0.0142926531059601236805432650181191646140667173169650719097336794\,\epsilon^{2}\\
&+0.0413104817431111762739064636103173511549751706483265167763975196\,\epsilon^{3}\\
&+0.0377135785410811595328655073702917287830658903769998836376738191\,\epsilon^{4}\\
&-0.1573556992565570036175903501572181529739317604477866745204907716\,\epsilon^{5}\\
&+0.0414759973036337189114289633954007080383918115298857956550595447\,\epsilon^{6}\\
&+0.1226262841962267645440916235676111767421788867801026236777617409\,\epsilon^{7}\\
&-0.1364222632033876822770604902541541035318635610199698354583333333\,\epsilon^{8}\\
&+0.2031939873871243350447990605419256585523672682057470307937052264\,\epsilon^{9}\\
&-0.2302775006009696797498707121925476230892866400467845530823192245\,\epsilon^{10}\\
&+0.2316650722837236128103411749222982108670696573669340323366752891\,\epsilon^{11}\\
&-0.1869974032359585443277596524323931419741320429847105134435046407\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13})\,. \qquad (115)
\end{aligned}
$$

A.3 Alternative Choices for Dotted Masters

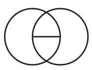
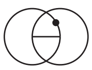
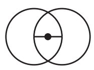
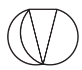
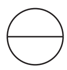
Other possible choices (than our $I_{841.1.3}$, $I_{1009.1.2}$ and $I_{1011.1.2}$ shown in Fig. 2) for the dotted masters can be inferred from linear relations among the integrals. Some relations follow from simple dimensional analysis (by hitting the corner integral with mass derivatives), such as

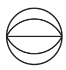
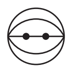
$(d-3)(5-2d)$  $+5$  $+10$  $=0\,,$ (116)

$(d-4)$  $+2$  $+2$  $=0\,,$ (117)

$(2d-7)$  $+4$  $+$  $+2$  $=0\,.$ (118)

Other relations can be derived via IBPs, such as

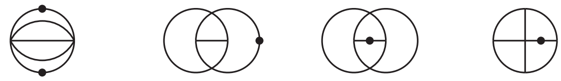
$(d-4)$  $+2$  $+2$  $=\frac{3d-8}{4}$  $-\frac{2(d-3)}{3}$  $+\frac{(d-2)^2}{8(d-3)}$ 

$-\frac{(2d-5)(2d-7)}{20(d-3)}$  $+\frac{5}{4(d-3)}$  $.$ (119)

Together with (118), the latter relation can be used to, e.g. exchange our master integral choice $I_{1009.1.2}$ (cf. Fig. 2) with $I_{1009.3.2}$ (dot on peripheral line) or $I_{1009.4.2}$ (dot on central line). Using (117) one could exchange our master integral choice $I_{1011.1.2}$ (dot on wheel rim) with $I_{1011.3.2}$ (dot on spoke), and this was in fact the choice that had been made in [103]. See Fig. 3 for these alternatives.

Complementing the numerical values given in App. A.2, the set of integrals shown in Fig. 3, normalised as explained in App. A.1, reads

$$
\begin{aligned}
I_{841.1.2.2.2} = &+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.0625000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&-0.6040726829685387662958818029420027678625253049770656169479919704\,\epsilon^{0}\\
&+3.1373605441116099188382190205459542027769621086311522682676003976\,\epsilon^{1}\\
&-15.885416717551890444418348328393253414936556051759548770547048412\,\epsilon^{2}\\
&+93.509375705771447578789720067526519068828346301523273248419193672\,\epsilon^{3}
\end{aligned}
$$



841.1.2.2.2 1009.3.2 1009.4.2 1011.3.2

Figure 3: Alternative choices for dotted masters. Note that the integral $I_{1009.4.2}$ is part of the basis chosen in the package FMFT [98, 99], replacing $I_{1009.1.2}$ of the basis displayed in Fig. 2, with (119) available for an analytic basis change if needed.

$$
\begin{aligned}
&-572.19703031228359037116640617257898326632059721487646153543470002\,\epsilon^{4}\\
&+3480.3990754148449408540368192153171743667570129729798005081543164\,\epsilon^{5}\\
&-21062.986323836129637697215490837708009126049163472340560077645509\,\epsilon^{6}\\
&+127099.61453565029192365687159979514077091358767833659046460858525\,\epsilon^{7}\\
&-765483.58462235532454874866636800968746991956462733860804422070006\,\epsilon^{8}\\
&+4604446.2295570334941617667543539370014334240175354316208367114885\,\epsilon^{9}\\
&-27672859.043230567239071208262871965643678284586875509825534236953\,\epsilon^{10}\\
&+166221885.33565129774414802676815154161852223454430668049440402418\,\epsilon^{11}\\
&-998070242.07861762945540535384139899608977615937233463511828075970\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13})\,,
\end{aligned}
\tag{120}
$$

$$
\begin{aligned}
I_{1009.3.2} = &+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&+0.0070021517316387759710681723384149836640891376613641574589887600\,\epsilon^{0}\\
&-0.0348569502098203174519581609509179307107150912672971844319319560\,\epsilon^{1}\\
&+0.0066782570271397328406301341440248816602066735159673784899475292\,\epsilon^{2}\\
&+0.2176664006631901242938439382471244390241609311923495535494055913\,\epsilon^{3}\\
&-0.3436146816801707440014497167090786998580751462449871159221207082\,\epsilon^{4}\\
&-0.0659377567081129949421115716133928605947320455729317236431903845\,\epsilon^{5}\\
&+0.5398496020239930726778100845766925862193550883471538537922137203\,\epsilon^{6}\\
&-0.7405524537812277214709126164272416892154711344744790725009108792\,\epsilon^{7}\\
&+0.9893079792739452964641686060197536814706119684687792556501604199\,\epsilon^{8}\\
&-1.1108281816246770590945834642794379624655136705782229354643392417\,\epsilon^{9}\\
&+1.0797935182782644554226543577665330590758569047351996620835347067\,\epsilon^{10}\\
&-0.7998052109587316960084105177949934814042124565789472929742073172\,\epsilon^{11}\\
&+0.1451124926213349020905107593334406886159813285453769932904080455\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13})\,,
\end{aligned}
\tag{121}
$$

$$
\begin{aligned}
I_{1009.4.2} = &+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&+0.0144495732846386993631355370611788194335554120923996271197880225\,\epsilon^{0}\\
&-0.0960246521820132496869442239690335419856138455691845241557340034\,\epsilon^{1}\\
&+0.1943187713867845561995474156467677097139053842748334924626077260\,\epsilon^{2}\\
&-0.0229433824022631209419209055032504868955620199524947734501373514\,\epsilon^{3}\\
&-0.3169806080837646907090547580234107010275900806158698163055841856\,\epsilon^{4}
\end{aligned}
$$

$$
\begin{aligned}
&+0.2428594072354530612446395857166714585948493823513205757801371155\,\epsilon^{5}\\
&-0.0165391210030964755320404737440152258881175536050481939706941450\,\epsilon^{6}\\
&+0.0492308350000877054279582616731134676605650817809574261407026963\,\epsilon^{7}\\
&-0.0407325003760186678056982514031275497857980541371802222547916244\,\epsilon^{8}\\
&+0.1309172439091162975945805834185044225790276005445749853299939813\,\epsilon^{9}\\
&-0.3276143718715897541983640342925565315783230168159076865624568021\,\epsilon^{10}\\
&+0.6887088774895604633408062420692657287010516178062228109246240777\,\epsilon^{11}\\
&-1.2861800480195147212729027941862276320096833942216440147381497066\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13})\,, \qquad (122)
\end{aligned}
$$

$$
\begin{aligned}
I_{1011.3.2} = &+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-2}\\
&+0.0000000000000000000000000000000000000000000000000000000000000000\,\epsilon^{-1}\\
&+0.0011632450261084087963483122793990984688170650692440590757501135\,\epsilon^{0}\\
&-0.0027730327566372790239325228232019539837604662964671396524281775\,\epsilon^{1}\\
&-0.0131222505164578558833646068761524087047466965316893844238565403\,\epsilon^{2}\\
&+0.0347700120550536255766658141303312315503631166718944218547629133\,\epsilon^{3}\\
&+0.0395007898674554609254119419358418580063381986606539885707739970\,\epsilon^{4}\\
&-0.1370560736796537010967404330561048235809400437273387021274583855\,\epsilon^{5}\\
&+0.0154697904649096998467904665734445839370760587294752110466098463\,\epsilon^{6}\\
&+0.1247847333277001165325814120255775892839862232330078432290152719\,\epsilon^{7}\\
&-0.1150209588414096713120758053313742010020332106988278243164106808\,\epsilon^{8}\\
&+0.1605302124395917832930675222277283145757455345188452070587676099\,\epsilon^{9}\\
&-0.1663608924247299043012709052312741274024635476176109817366588309\,\epsilon^{10}\\
&+0.1520777429726108814134369414461750810826415540097491797876830949\,\epsilon^{11}\\
&-0.1012410940263608590410179666371974051082046060713183532853150240\,\epsilon^{12}\\
&+\mathcal{O}(\epsilon^{13})\,. \qquad (123)
\end{aligned}
$$

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