Correspondence of boundary theories between internal and crystalline symmetry protected topological phases

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Abstract

Symmetry protected topological phases protected by crystalline symmetries and internal symmetries are shown to enjoy fascinating one-to-one correspondence in classification. Here we investigate the physics content behind the abstract correspondence in three or higher-dimensional systems. We show correspondence between anomalous boundary states, which provides a new way to explore the quantum anomaly of symmetry from its crystalline equivalent counterpart. We show such a correspondence directly in two scenarios, including the anomalous symmetry-enriched topological orders (SET) and critical boundary states. (1) First of all, for the surface SET correspondence, we demonstrate it by considering examples involving time-reversal symmetry and mirror symmetry. We show that one 2D topological order can carry the time reversal anomaly as long as it can carry the mirror anomaly and vice versa, by directly establishing the mapping of the time reversal anomaly indicators and mirror anomaly indicators. Besides, we also consider other cases involving continuous symmetry, which leads us to introduce some new anomaly indicators for symmetry from its counterpart. (2) Furthermore, we also build up direct correspondence for (near) critical boundaries. In this perspective, we first consider the edge-corner correspondence between edge theory as 1+1D conformal field theory of internal fermionic SPT and the 0+1D corner modes of (higher-order) crystalline fermionic SPT. By viewing the corner modes on 1D boundary as perturbed CFT is crucial insight for the correspondence, but also help to discover the boundary theory of some intrinsically interacting fermionic SPT, whcih are challenging. Next, we discuss this correspondence bosonic systems, again taking topological phases protected by time reversal and mirror symmetry as examples, the direct correspondence of their (near) critical boundaries can be built up by coupled chain construction that was first proposed by Senthil and Fisher. The examples of critical boundary correspondence we consider in this paper can be understood in a unified framework that is related to *hierarchy structure* of topological $O(n)$ nonlinear sigma model (NLσM), that generalizes the Haldane's derivation of $O(3)$ sigma model from spin one-half system. To our best of knowledge, this hierarchy structure of the $O(n)$ NLσM is first exposed, which inspires us to explore more interesting connection between different theories in the light of the boundary correspondence.

1   Introduction

Symmetry-Protected Topological (SPT) phases, as short-range entangled states of matter, have been the focus of intensive research over the past decades [1]. They represent a new paradigm in the realm of quantum matter, extending beyond Landau's symmetry-breaking framework. A well-known example of an SPT phase in fermionic systems is the topological insulator, which is protected by both time-reversal and charge conservation symmetries [2,3]. Bulk properties of SPT phases have been systematically constructed and classified for interacting bosonic and fermionic systems using various methods [4–17]. Remarkably, one perspective for investigating these intriguing quantum states is the study of anomalous boundary theories associated with SPT phases. This has led to the development of the bulk-boundary correspondence (BBC) theory for SPT phases.

The boundaries of SPT phases can exhibit various intriguing characteristics. That is, they may be gapless, symmetry broken [18], or topologically ordered in three or higher dimensions [19], but they cannot have a unique symmetric gapped ground state. Extensive studies have been conducted on the anomalous boundaries of the SPT [14,18–32]. A celebrated example is the two-dimensional topological superconductor, which is protected by time reversal symmetry with $\mathcal{T}^2 = -1$. This phase is notable for hosting helical Majorana edge modes that are robust and protected as long as time-reversal symmetry is preserved [33]. Generally, one fundamental question is: What types of boundaries can a SPT phase have? Given a SPT, its boundary can be realized many by different phases. It turns out that even though being different as a first glance, all such boundaries have the same topological nature to some sense, that is the 't Hooft anomalies associated with the corresponding symmetries [34,35]. To be more precise, an nD boundary, which can be either gapless or gapped, can be realized with specific 't Hooft anomalies on the boundary of the corresponding $n+1$D SPT phase. This motivates the study of 't Hooft anomalies in nD quantum theories, including the conformal field theory [20,31,32,36], symmetric gapped phases, such as 2D Symmetry-Enriched Topological Order, etc [23,37–42]. Very recently, the study of 't Hooft anomaly are formulated into a more general framework of generalized symmetry. For example, the 't Hooft anomaly corresponding to 2D SPT is shown to be equivalent to the generalized symmetry, described by the unitary fusion category whose objects and fusion are the group elements and their group multiplication, and the associated 3-cocycle is the F symbol [43, 44]. One great significance of 't Hooft anomaly is that it is an invariance under the renormalization group and then provides important non-perturbative knowledge of IR physics even though the corresponding many-body Hamiltonian may be too hard to be attached [45]. For example, the LSM anomaly, one of the 't Hooft anomalies, will forbid the group state from being symmetric uniquely gapped from the only knowledge of each unit cell no matter what Hamiltonian it takes as long as it preserves the symmetry [46–51].

Recently, there has been extensive theoretical [52–77] and experimental [78–81] interest in symmetry-protected topological (SPT) phases protected by crystalline symmetries. Distinguished from internal SPT phases, the boundaries of d-dimensional crystalline SPT phases are typically gapped, yet they exhibit protected lower-dimensional gapless modes at the hinges or corners. This type of topological phase is dubbed higher-order topological phases [82–91], topological crystalline phases or crystalline SPT. The robustness of the hinge or corner modes against symmetry perturbations reflects the nontrivial topology of the bulk phases, resulting in the bulk-boundary correspondence (BBC) of these topological crystalline phases.

The study of topological crystalline phases has been predominantly explored in free-fermion systems. Due to analytical limitations, a few representative examples of strongly correlated topological crystalline phases have been analytically understood through coupled-wire constructions [90,91]. An established real-space construction of topological crystals has been proposed to construct and classify topological crystalline phases in interacting systems [63–68].

This construction encompasses both bulk and boundary properties, achieved by decorating suitable lower-dimensional SPT or invertible topological phases on the corresponding blocks of the topological crystals, often referred to as "decorated block states" or "decorated topological crystals."

Very interestingly, it was pointed out that there are profound relationships between the crystalline and internal SPT phases. In Ref. [57], the so-called "*crystalline equivalence principle*" (CEP) was conjectured: the classification of crystalline SPT phases with space group G is identical to that of SPT phases protected by internal symmetry in a subtle and abstract way. Therefore, the crystalline SPT (in Euclidean space) and internal SPT protected by the groups that have the symmetry in the same group structure are one-to-one correspondence. The crystalline symmetry G and internal symmetry correspond to each other by relating the spacial orientation reversing symmetry (such as mirror symmetry) to anti-unitary internal symmetry (such as time-reversal symmetry). For fermionic systems, it has been conjectured and justified by enumerations that the crystalline equivalence principle should be applied in a twisted way: spinless (spin-1/2) fermions should be mapped into spin-1/2 (spinless) fermions [64–67,92]. Nevertheless, the crystalline equivalence principle is performed in a rather formal term, and a more physical understanding of this principle is very desired.

In this paper, we focus on the physical properties, especially on the boundary, of crystalline topological phases and SPT phases following the crystalline equivalence principle. Inspired by the bulk correspondence, it is natural to postulate that the Bulk-Boundary Correspondence (BBC) should also adhere to the crystalline equivalence principle. In this paper, we demonstrate the universal existence of such a correspondence relation in various situations of SPT phases, which we can dub *Crystalline-Equivalent Bulk-Boundary Correspondence* (CEBBC). More specifically, we consider the examples in 2+1D fermionic SPT, and 3+1D and higher bosonic SPT. The boundary of 2+1D fermionic SPT are typically symmetrically gapless theories while for 3D and higher-dimensional SPT phases, except being symmetric gapless, they can be symmetry-enriched topological orders (SET). So we will investigate the CEBBC in both fermionic and bosonic systems, which mainly consists of two scenarios: surface SET correspondence and critical boundary correspondence.

1. **Fermionic SPT** We will construct the correspondence between the boundaries of 2+1D fermionic SPT phases protected by on-site symmetry and 2+1D crystalline fermionic SPT phases with point group symmetry. The correspondence can be established by treating the corner modes of 2+1D crystalline fermionic SPT phases as "crystalline symmetric" domain walls of 1D modes. For illustrating examples, we build the correspondence between reflection-symmetric topological superconductor (TSC) and the time reversal symmetric TSC with $\mathcal{T}^2 = -1$, and also between C_2-symmetric TSC and TSC protected by unitary on-site $\mathbb{Z}_2$ [93–95, 95–97], whose bulks follow the crystalline equivalence principle. Furthermore, as an application of the crystalline equivalent BBC, we discover the new boundary theory of one intrinsically interacting fSPT protected by non-Abelian $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$ symmetry through its crystalline partner D_4-symmetric TSC.

2. **Bosonic SPT** Now we generalize the CEBBC to bosonic systems. Now we focus on the 3+1D or higher dimensional bosonic SPT. One the one hand, we focus on the surface SET correspondence between internal SPT phases and their crystalline SPT counterparts. We establish that if a particular topological order can reside on the boundary of an internal SPT phase, it can similarly exist on the boundary of the corresponding crystalline SPT phase, and vice versa. To rigorously establish this surface SET correspondence, we leverage the utility of anomaly indicators, which are fundamental in quantifying the BBC for topological phases. Following this correspondence, we introduce, to the best of our knowledge, new anomaly indicators [such as Eqs. (51) and (52)] for $SO(N) \times Z_2^{\mathcal{M}}$ sym-

139   metry indicated by those for the crystalline counterparts $SO(N) \times Z_2^{\mathcal{T}}$ that were obtained
140   by computing the 3+1d partition function on certain manifolds in Ref. [42].

141   On the other hand, we delve into scenario where the boundaries are either in or near a
142   critical state. For this purpose, we harness the description of the nonlinear sigma model
143   for 3+1D or higher dimensional bosonic SPT phases, which is a powerful framework
144   for phases of matter in or near criticality. In the pursuit of establishing direct corre-
145   spondences in these (near) critical surface scenarios, we unveil an intriguing "hierarchy
146   structure" of the $O(n)$ sigma model, which provide a unified understanding of the sur-
147   face criticality correspondence discussed in this paper. This structure can be traced back
148   to Haldane's derivation of the $O(3)$ nonlinear sigma model, originating from the study
149   of spin-half systems on lattice sites. To the best of our knowledge, such a hierarchy
150   structure of topological nonlinear sigma model are for the first-time exposure. In this
151   paper, our illustrative cases primarily revolve around mirror symmetry and time-reversal
152   symmetry, though we also explore rotational systems to offer a comprehensive under-
153   standing of the correspondence between crystalline and SPT phases.

154   Through these few examples, we demonstrated that not just formal establish of the CEBBC,
155   it is a powerful concept and tool to investigate the new boundary theory and relation between
156   two seemingly unrelated theories. For example, the correspondence between anomalous SET
157   is also utilized in Ref. [98] to construct gapped boundary of 4+1d bosonic SPT beyond group
158   cohomology.

159   The rest of the paper is organized as follows: In Sec. 2, we sketch the main idea of the
160   anomalous boundary correspondence of internal SPT and crystalline SPT in Sec. 2.1 and
161   the main result of surface SET correspondence and surface criticality correspondence in Sec.
162   2.2 and in Sec. 2.3. In Sec. 4, we will mainly discuss the surface SET correspondence in
163   details for time reversal and mirror SPT and also generalize to discuss the cases with additional
164   continuous symmetry in Sec. 4.3. In Sec. 5 and Sec. 6, we discuss the surface criticality
165   correspondence of the mirror system and rotation system respectively. We discuss the hierarchy
166   structure of the $O(n)$ sigma model in Sec. 5.3 that provides a unified understanding of the
167   surface criticality correspondence discussed in this paper. We summarize and outlook in Sec.
168   7.

2 Overview

2.1 Generality of anomalous boundary correspondence

171   It was conjectured that the topological action that describes the bosonic crystalline SPT with
172   crystalline symmetry G_s in Euclidean space R^4 is largely classified by [55, 57]

$$H^{d+1}(BG_s, U(1)_r) \tag{1}$$

173   where BG_s is the classifying space for the group G_s and the subscript r means that for spatial
174   orientation reversing group element, it will act as a complex conjugate on the $U(1)$ coefficient.
175   Recall that the bosonic SPT protected by internal symmetry G_{int} is also largely classified by
176   $H^{d+1}(BG_{int}, U(1)_T)$ where the subscript T means the complex conjugate action on the $U(1)$
177   coefficient when the group element is anti-unitary [7]. Therefore, it was observed that the
178   classification of the crystalline SPT by G_s is the same as that of internal SPT by G_{int} which is
179   the same group as G_s by identifying the spatial orientation reversing group element in G_s as an
180   anti-unitary element in G_{int}. On physical ground, when coupling to an external probe field,
181   either conventional gauge field or spatial gauge field, the topological response is controlled by

the same topological class, for example, in $\mathcal{H}^{d+1}(BG, U(1)_\sigma)$ where σ conjugate the coefficient if it reverses the space-time orientation. Such an observation can also be generalized to the beyond group cohomology SPT and crystalline SPT, and also to the fermionic SPT. Now this classification correspondence is phrased as the so-called *crystalline equivalence principle* [57]. So in Euclidean space, the crystalline SPT and internal SPT are one-to-one correspondence. This result is also shown by the decorated block state or topological crystal construction for crystalline topological phases [55].

It is well-known that when putting on the open manifold, the internal SPT phase can have a very interesting boundary phase, which carries a symmetry anomaly that can be canceled by the bulk phase [34, 35]. Usually, the quantum anomaly of symmetry is called the 't Hooft anomaly, which is defined by obstruction to gauging the symmetry. To carry such an anomaly, the boundary phase can be degenerate or gapless but not symmetric unique gapped. Such a phenomenon is called *boundary-bulk correspondence*.

For crystalline SPT, there is also such a phenomenon, such as the hing mode or corner modes of some crystalline SPT. This will make the nontrivial topology of the bulk phase manifest, leading to the boundary-bulk correspondence for crystalline SPT. We may also call such nontrivial boundary carries quantum anomaly of the symmetry. Even though the obstruction of gauging crystalline symmetry is still rarely discussed in the literature, we can still justify such a statement by adopting a modern definition of the quantum anomaly of symmetry as an obstruction to have a symmetric unique gapped ground state.

The decorated block state construction is very helpful for understanding the existence of boundary-bulk correspondence of crystalline topological phase in a strongly correlated system. To see this, we first choose one boundary that is called $standard$ such that its cell decomposition is one-to-one correspondence to that of the bulk. In fact, such a one-to-one correspondence can be easily established if all the cells in the bulk cell decomposition terminate on the chosen boundary. Then from the decorated block state construction, the decorated nontrivial invertible phase on some lower dimensional cell will terminate on the boundary with nontrivial feature, being gapless or having degenerate ground states. Yet if there might be some low dimensional cells that do not terminate on the chosen boundary, which we call *nonstandard*, then the crystalline topological phase constructed from by decorating these cells would have no nontrivial feature on the boundary due to the decorated invertible phase is not exposed to the boundary. For example, in the two-dimensional C_N rotation system on a disk geometry, there are nontrivial phases by decorating the rotation center with Z_N charge which is at the origin that does not terminate with the boundary that now can be symmetric unique gapped.

Recalling the bulk phases of internal SPT and crystalline SPT are one-to-one correspondence, while the nontrivial boundary states can also manifest the nontrivial topology of the bulk, it is very natural to expect that there is also correspondence between their nontrivial boundary states.

The main idea of anomalous boundary correspondence or crystalline equivalent BBC is summarized in Fig.1. We will make a few remarks as below.

1. **Exact anomaly** The quantum anomaly of symmetry is invariant under renormalization flow. So the anomaly specified at the UV limit will always be present in the low energy limit (IR limit). There might be an emergent anomaly of symmetry [99] and even anomaly of emergent symmetry (in contrast to the exact symmetry that is defined in UV limit), but we do not consider these cases in this paper by assuming the quantum anomaly of exact (not emergent) symmetry is exact, i.e., present in the UV limit. In this paper by symmetry we always mean the exact symmetry.

2. **Symmetry-preserving operation** In the UV limit, the quantum anomaly of symmetry is invariant under symmetry preserving operation (such as tuning parameters of surface Hamil-
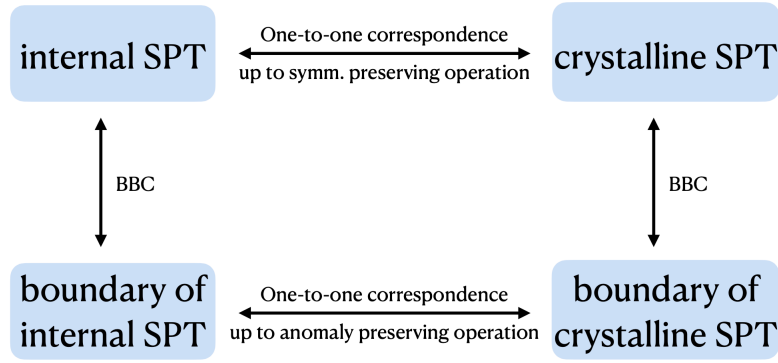
Figure 1: The anomalous boundary correspondence.

tonian) even though phase transition happens in the ground state[1]. That means two boundary phases with the same quantum anomaly could transfer into each other under by tuning some parameter of the surface Hamiltonian. More generally, one can conjecture that two arbitrary boundary phases with the same quantum anomaly can transform into each other by symmetry allowed operation possibly in an indirect way. One general scheme for such a transformation is through one path of a series of Hamiltonian parameterized by $\alpha \in [0, 1]$: $H(\alpha) = \alpha H_0 + (1 - \alpha)H_1$ where H_0 and H_1 realize the two target phases respectively. As both H_0 and H_1 is symmetric, then the whole path are symmetric, which means the symmetry anomaly in the UV limit is also present. While at the two ends of the path are the two target boundary phases, it is possible along the path some intermediate phase(s) occur.

3. **Topological class of anomaly** For orientation preserving symmetry, one way to see the quantum anomaly of boundary theories is to use the anomaly in-flow approach which is cancelled by the presence of the topological bulk[2]. For orientation changing symmetry, there is alternative way to detect the anomaly that is determined by the bulk topology. In other words, the boundary quantum anomaly of symmetry is classified by the topological class of their bulk [34, 35]. We say two quantum anomaly of symmetry are equivalent if they correspond to the same topological class, such as $\mathcal{H}^{d+2}(BG_s, Z)$ for bosonic SPT.

4. **Anomaly-preserving operation** The boundary correspondence is assumed to be in the sense of quantum anomaly, namely the boundary correspondence is defined up to anomaly-preserving operation. The anomaly-preserving operation is defined to the transformation that can not change the topological class of the quantum anomaly of symmetry. In fact, the two symmetry anomalies in the boundary correspondence here we discuss are equivalent because the two bulks are subject to the crystalline equivalence principle and controlled by the same topological classes. So the boundary correspondence is nothing but the two boundary theories carrying the same topological class of anomaly can transform into each other by anomaly-preserving operation.

5. **One-to-one correspondence** Together with the above mentioned points that any two theories on the SPT can be transformed into each other by symmetry-preserving operation, then the one-to-one correspondence can be made if we can make a connection (up to some

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[1]In fact, the anomaly of symmetry is already defined if we have defined the symmetry realization no matter what types of Hamiltonian we take. See Ref [43] for more details about this point.

[2]That is, one can couple the boundary to external probe field, either conventional gauge field or lattice gauge field, which itself is not gauge invariant but becomes invariant when the presence of the bulk. The bulk and boundary together couple to the external probe field, and the gauge variant part of the boundary will be cancelled by the gauge variant of topological terms terminating on the boundary.

proper anomaly-preserving operation) between arbitrary one boundary theory of the internal SPT and arbitrary one boundary of crystalline SPT.

6. **Application** One important application of the boundary correspondence is that we can explore the quantum anomaly of symmetry by studying its crystalline partner. Recalling that for a phase or theory (such the underlyding topological order of SET), whether it can carry quantum anomaly of certain internal symmetry is equivalent to say whether it can be put on the boundary of SPT protected by that internal symmetry. Then by the boundary correspondence, whether it can be put on the boundary of the SPT is equivalent to whether it can be put on the boundary of the crystalline partner of the SPT. On the other hand, we can also study whether a phase/theory be put on the crystalline SPT by studying whether it can be put on the boundary of internal SPT. Another important application is the guidence for exploring the relation between theories, even for theories live on different dimensions, such as the wire/layer construction for high dimension theory from lower dimensional one (See the examples in Sec.5.3).

2.2 Surface SET correspondence

To elucidate the concept of boundary correspondence, we begin by examining scenario where the surfaces of three-dimensional topological phases exhibit topological order with anomalous symmetry, specifically focusing on SET with symmetry anomalies.

Surface correspondence implies that a surface's topological order, denoted as $\mathcal{C}$, can exist on the boundary of topological phases protected by internal symmetry as long as it can also reside on the boundary of the corresponding crystalline topological phases while maintaining the crystalline symmetry, and vice versa. In general, for a given topological order, we can enumerate its symmetry enrichment data, including permutations, symmetry fractionalization, and stacking SPT, for both internal and crystalline symmetries. We then investigate whether there exists at least one anomalous symmetry-enriched pattern for both internal and crystalline symmetries that can be applied to the boundaries of internal SPT and the corresponding crystalline SPT.

To delve into this concept further, we focus our analysis on the surface SET of time-reversal SPT and mirror SPT. It becomes evident that the topological order $\mathcal{C}$ can exhibit a time-reversal anomaly if and only if it can manifest a mirror anomaly. This is demonstrated by establishing that when the topological order $\mathcal{C}$ possesses a nontrivial time-reversal anomaly indicator, denoted as $\eta_{\mathcal{T}} = -1$, it can also exhibit a nontrivial mirror anomaly indicator, $\eta_{\mathcal{M}} = -1$, and vice versa. (For detailed proofs and additional examples involving $U(1)$, $SU(2)$, and $SO(3)$ or $SO(N)$ symmetries, please refer to Sec.4.)

Additionally, we extend the concept of surface correspondence to another intriguing scenario where both surface theories are either at or near criticality.

2.3 Surface criticality correspondence

Now, let's delve into the study of correspondence within the context of edge or surface theories that reside in or near critical states.

2.3.1 Edge-corner correspondence of 2+1D fermionic SPT

We first consider the gapless edge correspondence of 2+1D fermionic SPT. The edge theory of 2+1D SPT phases protected by internal symmetries are 1+1D conformal field theores (CFT) when preserving the symmetry while that of 2+1D higher order SPT are usually corner modes, such as majorana corner modes. A very crucial insight is that the corner modes can be viewed as 1+1D CFT perturbed by some crystalline symmetric terms which will gap out all the fields

but leaving some zero modes that becomes the corner modes of the higher-order SPT. This understanding of the corner modes will directly lead to a very benificial idea of what 1+1D CFT can be put on the boundary of the 2+1D fermionic SPT protected by internal symmetries, which in general are difficult to find.

To directly show the CEBBC is to prove that the same 1+1D CFT can be put on the boundaries of both SPT protected by crystalline and internal symmetries, but with different symmetry properties—one is internal and the other is spatial. We explicitly consider three different examples: (1) the spinless fermion with reflection symmetry $Z_2 \times Z_2^f$ and spin-1/2 fermion with time reversal symmetry Z_4^{Tf}, (2) spin-1/2 fermion with two fold rotation C_2 symmetry $C^2 = P_f$ and spinless fermion with Z_2 symmetry $Z_2 \times Z_2^f$ and (3) spinless fermion with D_4 symmetry and spin-1/2 fermion with $Z_4 \rtimes Z_2^T$. We note that spin-1/2 fermion with $Z_4 \rtimes Z_2^T$ means that the symmetry operators satisfy the relations $A^4 = P_f$, $\mathcal{T}^2 = P_f$ and $\mathcal{T}A\mathcal{T}^{-1}A = 1$ where A and $\mathcal{T}$ are the generators of Z_4 and Z_2^T, respectively. For the former two examples, we know both the edge theories of internal SPT and higher order SPT. So what we contribute is the directly build the connection between their theories, confirming the above perturbed-CFT views of the corner modes and the same CFT can also be live on the boundary of internal SPT.

Remarkably, for the third example, we only know the corner modes of the higher order SPT. The edge theory of the corresponding SPT protected by a non-Abelian symmetry is challenging to obtain. One specific challenging point is that we do not know a priori which CFT can potentially live on the boundary of the corresponding SPT. In other words, given a specific 't Hooft anomaly, we do not know a priori which CFT can carry it. So, without further priori knowledge, we need to enumerate or conjecture which CFT could carry the given 't Hooft anomaly. Back to our specific example here, using the perturbed-CFT views of these corner modes, now we know which CFT are potentially be put on the boundary of the corresponding $Z_4 \rtimes Z_2^T$ FSPT. With this crucial information, we finally also find out the proper symmetry realization in the CFT that has the proper anomaly.

2.3.2 Surface-line correspondence of 3+1D bosonic SPT and beyond

To further illustrate this, we will still use time reversal and mirror bosonic SPT states as examples. As previously mentioned, the surface-SET correspondence discussed above establishes a clear one-to-one relationship between the surfaces of time reversal and mirror SPT states. This correspondence demonstrates that all surface theories belonging to the same SPT class can be transformed into one another through appropriate symmetric operations. These operations may involve adjustments to the surface Hamiltonian parameters or coupling to new degrees of freedom. However, our curiosity extends beyond this, as we are keenly interested in establishing a direct connection between the (near) critical surfaces of time reversal, mirror SPT states.

To explore the behavior of (near) critical surfaces, we adopt an alternative description of the bulk SPT theory, one based on the nonlinear sigma model [19, 100]. This approach is particularly potent when the theory resides near criticality.

In the context of 3D time reversal SPT (within the framework of group cohomology), it is described by the $O(5)$ nonlinear sigma model, featuring a topological theta term with $\Theta = 2\pi$. What makes the topological sigma model especially convenient is that its surface theory corresponds to the $O(5)$ nonlinear sigma model with a level one Wess-Zumino-Witten (WZW) term, which exhibits fascinating critical behavior [19].

In the case of the 3D mirror bulk, it is constructed by decorating the mirror plane with the 2D Levin-Gu state. This state can be effectively characterized using the $O(4)$ nonlinear sigma model, with a topological theta term of $\Theta = 2\pi$ [100]. When the mirror plane terminates at the boundary, a protected gapless mode emerges on the termination line of the mirror plane.

This mode is aptly described by the $O(4)$ nonlinear sigma model featuring a level-one WZW term. It is worth noting that the $O(4)$ nonlinear sigma model with a level one WZW term is renowned for exhibiting 1+1D $SU(2)_1$ CFT behavior at low energies.

To directly illustrate the surface correspondence in this scenario, we must establish a direct connection between the time reversal surface state and the mirror surface state. The time reversal surface state is described as the 2+1D $O(5)$ sigma model with a level one WZW term, while the mirror surface state corresponds to the 1+1D $O(4)$ sigma model localized on the mirror termination line, with the rest of the surface remaining trivial.

To transition from the time reversal surface state to the mirror surface state, we introduce a mirror-symmetric magnetic field, denoted as $g(x)\vec{n}_5(x)$, where the function $g(x)$ exhibits mirror symmetry, meaning $g(x) = -g(-x)$ together with mirror action on $\mathcal{M} : \vec{n}(x) \to -\vec{n}(-x)$. This magnetic field effectively creates a domain wall along the mirror termination line. This domain wall formation is achieved by selecting a specific profile for $g(x)$ such that $g(0) = 0$ and $g(x) \gg 1$ for $x \neq 0$. It can be shown that on this domain wall, the 2+1D $O(5)$ sigma model with a level one WZW term reduces to the 1+1D $O(4)$ sigma model featuring a level-one WZW term.

Transitioning from the mirror surface state to the time reversal surface state presents a more complex challenge. One naive approach involves regarding the 1+1D $O(4)$ theory on the mirror termination line as a reduced description of the 2+1D $O(5)$ theory, utilizing the specific $g(x)$ profile discussed earlier. To recover the 2+1D $O(5)$ theory, one might need to carefully adjust the $g(x)$ profile back to $g(x) = 0$. However, it's crucial to emphasize that this approach has its limitations due to the non-time reversal symmetric nature of the magnetic term $g(x)n_5(x)$. Consequently, it may not provide a entirely valid method for transitioning between the mirror and time reversal surface states while maintaining the required symmetry properties. More sophisticated techniques may be necessary to establish a robust and symmetry-preserving correspondence between these states.

A valid approach to bridging the mirror to time reversal surface states involves the use of a coupled chain construction. Specifically, we can construct the 2+1D $O(5)$ model with a level one Wess-Zumino-Witten (WZW) term by coupling an infinite number of 1+1D $O(4)$ models, each equipped with a level one WZW term, preserving the time reversal symmetry. This construction can be traced back to the pioneering work of Senthil and Fisher [101], where they considered coupled 1+1D $O(4)$ chains with a level one WZW term to construct the 2+1D $O(4)$ sigma model featuring a topological term $\Theta = \pi$, all while preserving the crucial time reversal symmetry.

To complete this construction and establish a direct connection between the mirror and time reversal surface states, it is essential to view the 2+1D $O(4)$ topological sigma model as the 2+1D $O(5)$ sigma model with a level-one WZW term, supplemented by an additional large anisotropic term, such as $(n_5)^2$. Importantly, this anisotropic term maintains time reversal symmetry, ensuring the preservation of symmetry throughout the transition.

The coupled chain construction, stemming from Haldane's work, originally generated the 1+1D $O(3)$ sigma model with a topological term $\Theta = \pi$ by coupling an infinite array of 0+1D spin one-half systems. This construction methodology can be extended to diverse $O(n)$ situations, enabling the creation of an nD $O(n+1)$ sigma model with a topological term $\Theta = \pi$ through the coupling of an infinite number of $(n-1)D$ $O(n+1)$ sigma models, each endowed with a level one WZW term. By introducing a similar supplemented anisotropic term, the resultant nD $O(n+1)$ model with a topological term $\Theta = \pi$ can be regarded as an anisotropic variant of an nD $O(n+2)$ topological sigma model featuring a level one WZW term. This versatile construction framework, which we call it the *hierarchy structure* of $O(n)$ sigma model, can be applied to establish surface criticality correspondences in various scenarios.

3 Gapless edge correspondence of 2+1D fermionic topological phases

The CEP for fermionc SPT shows a one-to-one correspondence between SPT phases with crystalline symmetry and on-site symmetry with a twist of spin of fermions where spinless (spin-1/2) fermions should be mapped into spin-1/2 (spinless) fermions, by an explicit lower-dimensional block-state constructions of 2D crystalline fSPT phases, which is called *topological crystals* [65, 66]. In this section, we consider three examples of crystalline symmetries—reflection symmetry, C_2 rotation symmetry and D_4 symmetry. For the first two examples, both the edge theories of crystalline and on-site fermionic SPT are known, we connect the edge theories of the two sides by directly establishing the correspondence. Remarkably, in the light of the proposed CEBBC, we can investigate and discover the unknown CFT boundary theory of the fermionic SPT protected by $Z_4 \rtimes Z_2^T$ in spin-1/2 systems through that of its crystalline counterpart—spinless D_4 FSPT.

3.1 Spinless fermion with reflection symmetry and spin-1/2 fermion with time reversal symmetry

Firstly we study the simplest case of spinless fermions with reflection symmetry. From topological crystals and explicit model construction [66], we know that for a 2D reflection-symmetric system with spinless fermions, there is a nontrivial higher-order topological phase with two Majorana corner modes ξ_1 and ξ_2, see Fig. 2.

An alternative understanding of these Majorana corner modes is provided: For ξ_1, consider two branches of gapless Majorana modes $\gamma_\uparrow$ and $\gamma_\downarrow$ ($\uparrow$ and $\downarrow$ are virtual indices) on the boundary that move oppositely, with a mass term $m(x)$:

$$H = \int \mathrm{d}x \cdot \gamma^T \left[i\sigma^3 \partial_x + m(x)\sigma^2 \right] \gamma \tag{2}$$

where $\gamma(x) = \left(\gamma_\uparrow(x), \gamma_\downarrow(x) \right)^T$. The reflection symmetry $\boldsymbol{M}$ is defined as:

$$\boldsymbol{M} : \ \gamma_\uparrow(x) \leftrightarrow \gamma_\downarrow(-x) \tag{3}$$

It is easy to verify that H is invariant under $\boldsymbol{M}$ if $m(-x) = -m(x)$, which means the mass $m(x)$ has a domain-wall structure. For simplicity, take $m(x) \sim x$, then the Hamiltonian H reduces to:

$$H = \int \mathrm{d}x \cdot \gamma^T \mathcal{H}(x)\gamma, \ \ \mathcal{H}(x) = i\sigma^3 \partial_x + x\sigma^2 \tag{4}$$

There is a Majorana zero mode localized at $x = 0$ as a Gaussian wavepacket (See Appendix A):

$$|0\rangle = \mathcal{A} e^{-x^2/2} (1,1)^T \tag{5}$$

where $\mathcal{A}$ is a normalization factor. Equivalently, the Majorana corner modes of nontrivial higher-order topological phase in 2D reflection-symmetric system with spinless fermions can be treated as a domain-wall at the corner of the system. In particular, this domain wall is reflection symmetric. To see this, denote $H_m(x) = 2im(x)\gamma_\uparrow(x)\gamma_\downarrow(x)$, under M, $H_m(x)$ maps to $H_m(-x)$. The whole domain wall is symmetric under reflection. In other words, the domain wall is carrying neutral reflcetion quantum number.

Subsequently, we define a effective "time-reversal symmetry" $\mathcal{T}$ in γ-basis and treat $\uparrow$ and $\downarrow$ as effective "spin indices": $\mathcal{T} = i\sigma^2 K$, $\mathcal{T}^2 = -1$, where K is the complex conjugate operator. It is easy to verify that the kinetic term is symmetric under $\mathcal{T}$:

$$\mathcal{T}^{-1}(i\sigma^3 \partial_x)\mathcal{T} = i\sigma^3 \partial_x \tag{6}$$
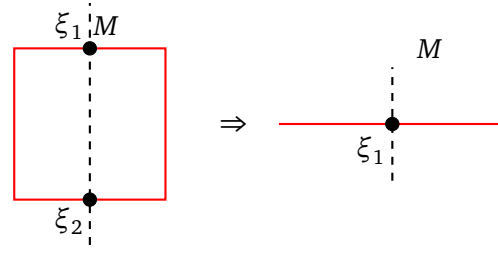
Figure 2: Edge modes of 2D reflection-symmetric system with spinless fermions. Right panel is the zoom-in of the Majorana corner modes ξ_1. Reflection axis is depicted by dashed line.
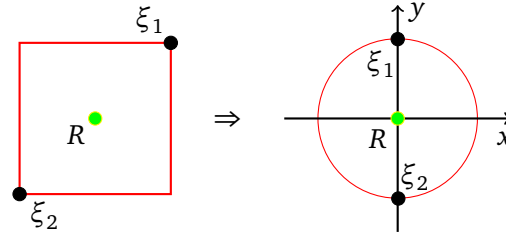
Figure 3: Edge modes of 2D C_2-symmetric system with spin-1/2 fermions. Green dot represents the center of C_2.

But the mass term breaks $\mathcal{T}$:

$$\mathcal{T}^{-1}\left[m(x)\sigma^2\right]\mathcal{T} = -m(x)\sigma^2 \tag{7}$$

i.e., the domain wall structure is time reversal broken.

Furthermore, we investigate the symmetry properties of zero mode (5) under reflection symmetry and "time-reversal symmetry". According to Eq. (3), it is easy to see that this zero mode is reflection symmetric. On the other hand, under $\mathcal{T}$, the zero mode (5) transforms as:

$$\mathcal{T}|0\rangle = (i\sigma^2 K)|0\rangle = \mathcal{A}e^{-x^2/2}(-1,1)^T \tag{8}$$

i.e., this zero mode breaks $\mathcal{T}$. However, this zero mode carries $\mathcal{T}^2 = -1$.

To arrive at the helical edge theory of 2D time-reversal-invariant TSC [93], there are two ways: one can turn off the time reversal broken domain wall, leaving only the kinetic term which is "time-reversal-invariant" [cf. Eq. (6)], or proliferate this domain wall that traps Majorana zero mode [102]. One can also go from the helical edge theory of 2D time-reversal-invariant TSC to obtain the Majorana corner zero modes of refelection symmetric TSC by adding the the reflection symmetric domain wall as in (2) and realizing reflection on helical majorana fermions as (3). This just establishes the "crystalline equivalence principle" of BBC between time reversal and reflection symmetric TSC.

3.2 Spin-1/2 fermion with C_2 symmetry and spinless fermion with Z_2 symmetry

Repeatedly from topological crystals and explicit model construction [64, 66], we know that for a 2-fold rotational-invariant 2D system with spin-1/2 fermions, there is a nontrivial higher-order topological phase with two Majorana corner modes ξ_1 and ξ_2 on the boundary of the system, see Fig. 3. We introduce polar coordinates $(x, y) = r(\cos\theta, \sin\theta)$ to describe the Majorana corner modes at the north/south poles. Under the 2-fold rotation, the two zero modes ξ_1 and ξ_2 exchange.

Consider two branches of gapless Majorana modes γ_A and γ_B that move oppositely, with a mass term $m(\theta) \sim \cos\theta$:

$$H = \int d\theta \cdot \gamma^T \left(-i\sigma^3 \partial_\theta + \cos\theta \cdot \sigma^2 \right) \gamma \tag{9}$$

where $\gamma = (\gamma_A, \gamma_B)^T$, and the C_2 property of γ is ($\mathrm{sgn}(x) = 1$ for $x > 0$, $\mathrm{sgn}(x) = -1$ for $x < 0$):

$$\mathbf{R}: \ \gamma(\theta) \mapsto \mathrm{sgn}(\theta - \pi)(\gamma_A(\theta + \pi), \gamma_B(\theta + \pi))^T \tag{10}$$

It is easy to verify that H is C_2-symmetric. The mass term $\cos(\theta)$ has a domain wall structure, which is C_2-symmetric. Near northpole/southpole, the total Hamiltonian can be approximately expressed as:

$$\begin{aligned}
H^N = \int dx \cdot \gamma^T \mathcal{H}^N \gamma, \ \ \mathcal{H}^N = i\sigma^3 \partial_x + x\sigma^2 \\
H^S = \int dx \cdot \gamma^T \mathcal{H}^S \gamma, \ \ \mathcal{H}^S = -i\sigma^3 \partial_x - x\sigma^2
\end{aligned} \tag{11}$$

We concentrate on the physics near northpole, and the physics near southpole can be obtained from northpole by a 2-fold rotation. The Hamiltonian H^N has a zero mode:

$$|0\rangle = \mathcal{A}e^{-x^2/2}(1,1)^T \tag{12}$$

Equivalently, the Majorana corner mode ξ_1 of nontrivial higher-order fSPT phase in 2D C_2-symmetric system with spin-1/2 fermions can be treated as a domain-wall at the northpole of a spherical geometry as illustrated in Fig. 3. Similar for ξ_2 at the southpole. The C_2 symmetric domain wall structure promises that ξ_1 and ξ_2 exchange under $\mathbf{R}$.

Subsequently, we define an effective "$\mathbb{Z}_2$ on-site symmetry" O in γ-basis: $O = \sigma^3$, the kinetic term is symmetric under O:

$$O^{-1}(-i\sigma^3\partial_\theta)O = -i\sigma^3\partial_\theta \tag{13}$$

But the mass term breaks this "$\mathbb{Z}_2$ symmetry":

$$O^{-1}\left(\cos\theta \cdot \sigma^2\right)O = -\cos\theta \cdot \sigma^2 \tag{14}$$

Furthermore, under O, the zero mode (10) transforms as:

$$O|0\rangle = \sigma^3|0\rangle = \mathcal{A}e^{-x^2/2}(1,-1)^T \tag{15}$$

i.e., this zero mode breaks O, however it has $O^2 = 1$.

Similarly to time reversal TSC, we have two ways to obtain the gapless majorana edge modes of $\mathbb{Z}_2$ TSC: turn off the $\mathbb{Z}_2$-broken mass term $\cos(\theta)$ [94–97], or proliferate the domain wall that traps majorana zero modes [102]. One can also begin with gapless majorana edge theory of $\mathbb{Z}_2$ TSC to construct the boundary corner modes of C_2-symmetric TSC by adding mass term as in (9) and realizing the C_2 symmetry $\mathbf{R}$ as (10). Then the crystalline equivalent BBC between unitary $\mathbb{Z}_2$ TSC and C_2 symmetric TSC is just established.
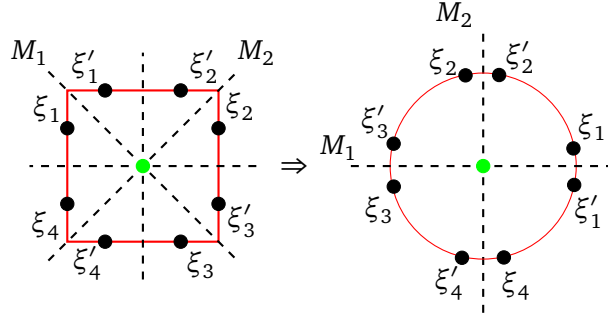
Figure 4: Edge modes of 2D D_4-symmetric system with spinless fermions. All dashed lines are reflection axes and green dot is the center of 4-fold rotation symmetry C_4.

3.3   Spinless fermion with D_4 symmetry and spin-1/2 fermion with $Z_4 \rtimes Z_2^T$

Topological crystals and explicit model construction [65,103] show that for a 2D D_4-symmetric system with spinless fermions (D_4 is 4-fold dihedral symmetry $D_4 = C_4 \rtimes \mathbb{Z}_2^M$ with two generators $R \in C_4$ as a 4-fold rotation and $M_1 \in \mathbb{Z}_2^M$ as a reflection), there is a nontrivial higher-order fSPT phase, with 8 localized Majorana corner modes ξ_j and ξ_j' ($j = 1, 2, 3, 4$), see Fig. 4 [103]. Similar to the C_2-symmetric case, we introduce polar coordinates $(x, y) = r(\cos\theta, \sin\theta)$ to describe the Majorana corner modes at poles (northpole, southpole, westpole and eastpole).

We introduce 4 branches of itinerating Majorana modes $\gamma = (\gamma_1^\uparrow, \gamma_1^\downarrow, \gamma_2^\uparrow, \gamma_2^\downarrow)^T$ on the boundary, where the Majorana modes with $\uparrow$ and $\downarrow$ indices move in opposite directions:

$$H_0 = \int d\theta \cdot \gamma^T \left[i(\tau^3 \otimes \sigma^3)\partial_\theta \right] \gamma \tag{16}$$

where $\tau^{1,2,3}$ are Pauli matrices characterizing the pseudo-spin indices. These Majorana modes have the following symmetry properties:

$$M_1 : \begin{cases} \theta \mapsto 2\pi - \theta \\ \left(\gamma_\uparrow^1, \gamma_\downarrow^1, \gamma_\uparrow^2, \gamma_\downarrow^2 \right) \mapsto \left(\gamma_\uparrow^2, \gamma_\downarrow^2, \gamma_\uparrow^1, \gamma_\downarrow^1 \right) \end{cases}$$
$$R : \begin{cases} \theta \mapsto \theta + \pi/2 \\ \left(\gamma_\uparrow^1, \gamma_\downarrow^1, \gamma_\uparrow^2, \gamma_\downarrow^2 \right) \mapsto \left(\gamma_\uparrow^1, \gamma_\downarrow^1, \gamma_\uparrow^2, \gamma_\downarrow^2 \right) \end{cases} \tag{17}$$

That satisfy $R^4 = M_1^2 = 1$. It is easy to verify that H_0 is invariant under D_4 generators (17). We further consider a D_4-symmetric "mass term" with a spatial-dependent mass $m_j(\theta)$:

$$H_m = \int d\theta \cdot \gamma^T \left[\sin(2\theta) \cdot (\tau^3 \otimes \sigma^2) \right] \gamma \tag{18}$$

And we can straightforwardly confirm that H_m is invariant under D_4 generators (17). After investigating the symmetry properties of the total Hamiltonian $H = H_0 + H_m$, we can concentrate on the Hamiltonian near each pole (intersection between the boundary of system and reflection axes along the vertical/horizontal directions, see right panel of Fig. 4). Near eastpole, from $\theta \sim 0$ we obtain $\sin(2\theta) \sim 0$, the low-energy physics near the eastpole can be described by the following Hamiltonians:

$$H^E = \int dy \cdot \gamma^T \left[i(\tau^3 \otimes \sigma^3)\partial_y + y(\tau^3 \otimes \sigma^2) \right] \gamma \tag{19}$$

The Hamiltonian H^E has two zero modes ($x = r$):

$$
\begin{aligned}
|0\rangle_1 &= \mathcal{A}e^{-y^2/2}(1,1,0,0)^T \\
|0\rangle_{1'} &= \mathcal{A}e^{-y^2/2}(0,0,1,1)^T
\end{aligned}
\tag{20}
$$

Equivalently, the Majorana corner modes γ_1 and γ_1' of nontrivial higher-order topological phase in 2D D_4-symmetric systems with spinless fermions can be treated as two domain walls near the eastpole of the spherical geometry in Fig. 4. Similar for Majorana corner modes at other poles with $\theta \sim \pi/2, \pi, 3\pi/2$ at which $\sin(2\theta) \sim 0$. Near westpole ($x = -r$):

$$
\begin{aligned}
|0\rangle_3 &= \mathcal{A}e^{-y^2/2}(1,1,0,0)^T \\
|0\rangle_{3'} &= \mathcal{A}e^{-y^2/2}(0,0,1,1)^T
\end{aligned}
\tag{21}
$$

Near north and south poles ($y = \pm r$):

$$
\begin{aligned}
|0\rangle_{2,4} &= \mathcal{A}e^{-x^2/2}(1,1,0,0)^T \\
|0\rangle_{2',4'} &= \mathcal{A}e^{-x^2/2}(0,0,1,1)^T
\end{aligned}
\tag{22}
$$

Subsequently we define an effective "$\mathbb{Z}_4$ on-site symmetry" A (with $A^4 = -1$):

$$
A = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_{2\times 2} & -i\sigma^2 \\ -i\sigma^2 & \mathbb{1}_{2\times 2} \end{pmatrix}
\tag{23}
$$

H_0 is symmetric under A, mass term H_m breaks A:

$$
\begin{aligned}
A^{-1}i(\tau^3 \otimes \sigma^3)A &= i(\tau^3 \otimes \sigma^3) \\
A^{-1}(\tau^3 \otimes \sigma^2)A &\neq \tau^3 \otimes \sigma^2
\end{aligned}
\tag{24}
$$

Then we define an effective "time-reversal symmetry" $\mathcal{T}$:

$$
\mathcal{T} = i(\tau^2 \otimes \mathbb{1}_{2\times 2})K, \quad \mathcal{T}^2 = -1
\tag{25}
$$

It is easy to verify that H_0 and H_m preserve $\mathcal{T}$:

$$
\begin{aligned}
\mathcal{T}^{-1}i(\tau^3 \otimes \sigma^3)\mathcal{T} &= i(\tau^3 \otimes \sigma^3) \\
\mathcal{T}^{-1}(\tau^3 \otimes \sigma^2)\mathcal{T} &= \tau^3 \otimes \sigma^2
\end{aligned}
\tag{26}
$$

All zero modes [cf. Eqs. (20)-(22)] are symmetric under D_4 and also $\mathcal{T}$ symmetry, but not symmetric under A (See Appendix B).

Nevertheless, if we "release" the Majorana corner modes ξ_j and ξ_j' (by removing the domain walls, $j = 1, 2, 3, 4$), the Hamiltonian H reduces to H_0 which is "$\mathbb{Z}_4$-symmetric" [cf. Eq. (23)] and "time-reversal symmetric" [cf. Eq. (25)]. Hence the Hamiltonian H_0 describes the edge theory of 2D ($\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$)-symmetric systems with spin-1/2 fermions. Furthermore, we study if these edge modes can not be gapped in a symmetric way. We bosonize the edge theory H_0 in terms of $\gamma_\sigma^{1,2}$ ($\sigma = \uparrow, \downarrow$):

$$
e^{i\phi_1} = \gamma_1^\uparrow + i\gamma_2^\downarrow, \quad e^{i\phi_2} = \gamma_1^\downarrow + i\gamma_2^\uparrow
\tag{27}
$$

And the topological edge theory H_0 can be rephrased in terms of bosonic fields $\Phi = (\phi_1, \phi_2)^T$:

$$
\mathcal{L}_{\text{edge}} = \frac{K_{IJ}}{4\pi}(\partial_x \Phi^I)(\partial_t \Phi^J) + \frac{V_{IJ}}{8\pi}(\partial_x \Phi^I)(\partial_x \Phi^J)
\tag{28}
$$

521 where $K = \sigma^z$ as the K-matrix characterizing the topology of $\mathcal{L}_{\text{edge}}$. Under A and $\mathcal{T}$, the field
522 of edge modes Φ transforms as [32]:

$$A : \Phi \mapsto W^A \Phi + \delta\Phi^A, \quad \mathcal{T} : \Phi \mapsto W^{\mathcal{T}} \Phi + \delta\Phi^{\mathcal{T}} \tag{29}$$

523 where

$$\begin{cases} W^A = \mathbb{1}_{2\times2}, \\ W^{\mathcal{T}} = \sigma^x \end{cases}, \quad \begin{cases} \delta\Phi^A = \pi(1/4, -1/4)^T \\ \delta\Phi^{\mathcal{T}} = \pi(1/2, 1/2)^T \end{cases} \tag{30}$$

524 Also the fermion parity will map $\gamma_i^{\uparrow,\downarrow}$ to $-\gamma_i^{\uparrow,\downarrow}$, so according to Eq.(27), we should have

$$W^{P_f} = \mathbb{1}_{2\times2}, \quad \delta\phi^{P_f} = (\pi, \pi)^T. \tag{31}$$

525 We now try to construct interaction terms that gap out the edge without breaking the A and $\mathcal{T}$
526 symmetries, either explicitly or spontaneously. Consider the backscattering terms of the form:

$$U = \sum_j U(\Lambda_j) = \sum_j U(x) \cos\left[\Lambda_j^T K \Phi - \alpha(x)\right] \tag{32}$$

527 The backscattering term (32) can gap out the edge as long as the vectors $\{\Lambda_j\}$ satisfy the
528 "null-vector" conditions [104] for $\forall i, j$:

$$\Lambda_i^T K \Lambda_j = 0 \tag{33}$$

529 The simplest term is $\Lambda_1 = (1, 1)$ or $\Lambda_2 = (1, -1)$. However, Λ_1 breaks $\mathcal{T}$ symmetry and Λ_2 break
530 A symmetry. We turn to the next simplest term $\Lambda_3 = (2, 2)$ or $\Lambda_4 = (2, -2)$. Λ_3 preserve all
531 the symmetry but leads to spontaneously symmetry breaking, i.e., $\langle\phi_-\rangle$ where $\phi_- := \phi_1 - \phi_2$,
532 has two energy vacca: 0 and π (take $\alpha(x) = 0$ for simplicity) which transform into each other
533 by A^2. Similar analysis show Λ_4 would spontaneously break $\mathcal{T}$. So it seems to be no way to
534 symmetrically gap out the edge fields. One may guess it is possible to stack trivial edge fields
535 to gap them out together symmetrically. We argue that in fact it is impossible by gauging the
536 fermion parity symmetry. It turns out that the fermion parity flux carries projective represen-
537 tation of $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$, which reflects the anomalous nature of the edge theory of the corresponding
538 FSPT. Equivalently, H_0 or $\mathcal{L}_{\text{edge}}$ with (30) characterizes the nontrivial topological edge theory
539 for 2D ($\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$)-symmetric fSPT phase with spin-1/2 fermions.

540 Now we show that fermion parity flux carries projective representation of $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$ for the
541 symmetry properties (30) of H_0. Following the method in Ref. [32], we can gauge the fermion
542 parity symmetry and the fermion parity flux are represented by the "fractionalized" vertext
543 operators $e^{i\frac{\phi}{2}}$ where $\phi = \phi_1 + \phi_2$. Now we study the symmetry properties of the remaining
544 symmetry $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$. In fact, under symmetry, the fermion parity flux form a doublet, which is
545 represented by $(e^{i\frac{\phi}{2}}, e^{-i\frac{\phi}{2}})^T$. The components of the doublet differ by attaching local fermions.
546 Under A and $\mathcal{T}$,

$$A : \begin{pmatrix} e^{i\frac{\phi}{2}} \\ e^{-i\frac{\phi}{2}} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\frac{\phi}{2}} \\ e^{-i\frac{\phi}{2}} \end{pmatrix} \tag{34}$$

$$\mathcal{T} : \begin{pmatrix} e^{i\frac{\phi}{2}} \\ e^{-i\frac{\phi}{2}} \end{pmatrix} \to \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} e^{i\frac{\phi}{2}} \\ e^{-i\frac{\phi}{2}} \end{pmatrix} \tag{35}$$

547 Namely, acting on the fermion parity flux doublet, $U_A = \mathbb{1}_{2\times2}$ and $U_{\mathcal{T}} = -\sigma_y K$. Recalling that
548 the group relation $\mathcal{T}A\mathcal{T}^{-1} = A^{-1}$, namely A and $\mathcal{T}$ do not commute, such a realization of $A, \mathcal{T}$
549 is indeed projective. One can compute the invariants for this projective representation, that is,

$$\mathcal{I}_1 = n_2(A^2\mathcal{T}, A^2\mathcal{T}) = -1 \tag{36}$$

$$\mathcal{I}_2 = n_2(A^3\mathcal{T}, A^3\mathcal{T}) = -1 \tag{37}$$

where n_2 is the 2-cocycle corresponding to this projective representation [105]. We note that the corresponding 2-cocycle with such invariants are in fact nontrivial in $\mathcal{H}^2\left[\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T, U(1)\right]$, and hence nontrivial in $\mathcal{H}^2(\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T, \mathbb{Z}_2)$ which is the true symmetry fractionalization classification of fermion parity flux. In fact, the symmetry fractionalization class $(-1)^{n_2}$ of fermion parity flux corresponds to the complex fermion decoration in the supercohomology theory for fermionic SPT [106]. In the classification of fermionic SPT, the complex fermion decoration with $n_2 = w_2$ is trivialized [12]. [The meaning of w_2 is that it defines the extension that characterizes the fermionic symmetry group, see Eq. (C.7)] Such projective representation U_A and $U_{\mathcal{T}}$, labeled by n_2, differs from the ω_2 in Eq. (C.7) since they have different invariants [see Eq. (C.9)]. Therefore, the edge theory indeed corresponds to a nontrivial fermionic SPT.

4 Surface SET correspondence for 3D topological phases

Here we will show that the surface SET on the surface of SPT protected by time reversal symmetry and mirror symmetry are one-to-one correspondence, namely the same topological order $\mathcal{C}$ can or cannot carry time reversal anomaly and mirror anomaly simultaneously. We also discuss the cases with additional continuous symmetry such as $U(1)$, $SO(N)$ and $SU(2)$.

4.1 Bulk description of time reversal and mirror topological phases

The 3D bosonic SPT protected by time reversal is classified by $Z_2 \times Z_2$ [19]. The first root is group cohomology type that is characterized by the effective action [107]

$$S = \frac{1}{2} \int_M w_1 \cup w_1 \cup w_1 \cup w_1 \tag{38}$$

where $w_i \in \mathcal{H}^i(M, Z_2)$ is the ith-order Stiefel-Whitney class. The physics of the this phase is to see that on the codimension-2 time reversal defect, which is a line defect, there is Haldane phase protected by time reversal, whose effective Lagrangian is $\frac{1}{2} w_1 \cup w_1$. There is another equivalent description of this time reversal SPT state as discussed in Sec.2.3. The second root state is beyond group cohomolgy, whose effective action is [108]

$$S = \frac{1}{2} \int_M w_2 \cup w_2 \tag{39}$$

The physics of this state is on the time reversal domain wall (surface defect), there is E_8 state.

The classification of 3D mirror SPT in bosonic system on Euclidean space is also $Z_2 \times Z_2$, compatible with the crystalline equivalence principle [55]. From the view of decorated topological crystal, the mirror SPT can be constructed by decorating two dimensional SPT protected by $\mathbb{Z}_2$ symmetry and the invertible E_8 state at the mirror plane.

4.2 Surface SET and their correspondence

On the surface of 3D SPT phases, it can develop topological order but with anomalous symmetry realization, called anomalous symmetry enriched topological order (SET). Here we will discuss the surface SET correspondence of 3D topological phases protected by time reversal and mirror symmetry. More exmaples see Sec.4.3

One convenient way to see the surface SET correspondence is to use the anomaly indicator of the anomalous SET. To characterize the SET with G symmetry anomaly, one need two set of data [109]. The first set is the intrinsic data that is the topological quantities that characterize the topological order when ignoring the symmetry, such as the anyon types $a, b, c...$, quantum
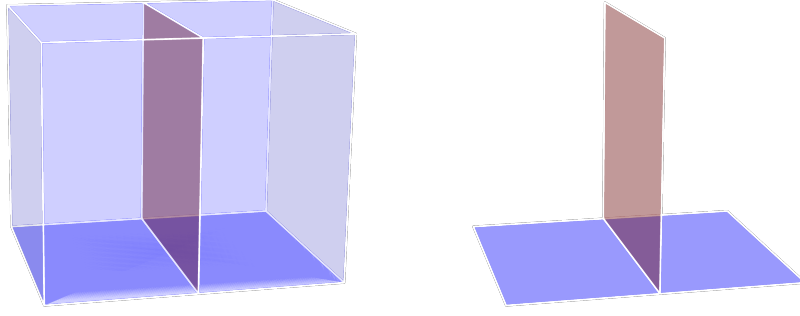
Figure 5: Mirror plane decoration for 3D mirror SPT. Applying some local unitary transformation, we can also remove the entanglement of left and right wing of the mirror plane (orange color plane) so that it is in trivial product state (as illustrated in (b)). Then the nontrivial mirror SPT is to decorate the mirror plane with (1) E_8 state or (2) Z_2 internal SPT state. The bottom purple plane terminates the mirror plane is the chosen boundary of the mirror SPT which can be gapless along this mirror line (the line of mirror plane terminating on the boundary) or be (anomalous) SET.

dimension $d_a, d_b, d_c...$, topological spin of the anyon $\theta_a, \theta_b, \theta_c$ and so on. The other set is the extrinsic data that characterize the symmetry enrichment of the SET, including how the symmetry permute the anyon, G symmetry quantum number carrying by the anyon, etc. The set of anomaly indicator is a set of expression in terms of these two set of data that can uniquely determine surface of which SPT with the same global symmetry G it can be put on. In other words, the anomaly indicator is just the *quantitative* boundary-bulk correspondence.

The time reversal anomaly indicators of SET on the surface of bosonic time reversal SPT is given by [110]

$$\eta_{\mathcal{T}} = \frac{1}{D_{\mathcal{C}}} \sum_{a \in \mathcal{C}} d_a \theta_a \mathcal{T}_a^2. \tag{40}$$

Here $\mathcal{T}_a^2 = \pm 1, 0$. If the anyon a is invariant under time reversal, i.e., $\rho_{\mathcal{T}}(a) = a$, $\mathcal{T}_a^2 = \mathcal{T}_{\bar{a}}^2 = \pm 1$ correspond to whether a is locally Kramer degenerate or not. In particular, $\mathcal{T}_1^2 = 1$. If $\rho_{\mathcal{T}}(a) \neq a$, it is not well-defined to talk about whether a carries Kramer degeneracy or not, then $\mathcal{T}_a^2 = 0$. When $\rho_{\mathcal{T}}(a) = a$, $\theta_{\mathcal{T}(a)} = -\theta_a$, then $\theta_a = 0, \pi$. Further, if $\rho_{\mathcal{T}}(a) = (\neq)a$, then $\rho_{\mathcal{T}}(\bar{a}) = (\neq)\bar{a}$. When the time reversal anomaly $\eta_{\mathcal{T}}$ takes -1, it is anomalous and can only live on the surface of time reversal SPT.

Meanwhile the mirror anomaly indicator for SET on the surface (See Fig.5) of bosonic topological crystalline phase protected by mirror symmetry is given by [111]

$$\eta_{\mathcal{M}} = \frac{1}{D_{\mathcal{C}}} \sum_{a \in \mathcal{C}} d_a \theta_a \mu_a. \tag{41}$$

Here μ_a is the mirror eigenvalue related to the anyon a. If $\rho_{\mathcal{M}}(a) = \bar{a}$, $\mu_a = \mu_{\bar{a}} = \pm 1$ while $\rho_{\mathcal{M}}(a) \neq \bar{a}$, $\mu_a = \mu_{\bar{a}} = 0$. In particular, $\mu_1 = 1$. When $\rho_{\mathcal{M}}(a) = (\neq)\bar{a}$, $\rho_{\mathcal{M}}(\bar{a}) = (\neq)a$. The indicator $\eta_{\mathcal{M}}$ can only takes values ± 1 and the value -1 means it is anomalous and can only live on the surface of mirror SPT.

Now we show that for a topological order $\mathcal{C}$, when $\eta_{\mathcal{T}} = -1$, then we can also have $\eta_{\mathcal{M}} = -1$ for proper mirror symmetry realization. For this, we first divide the anyon of $\mathcal{C}$ into two sets: $A = \{a | \bar{a} = a, a \in \mathcal{C}\}$ and $B = \{a | \bar{a} \neq a, a \in \mathcal{C}\}$. In other words, A consists of anyon that are self-conjugate and B consists of non-self-conjugate one. For example, all the anyons in toric code are self-conjugate but the nontrivial anyons in $\nu = 1/3$ Laughlin state is

non-self-conjugate. We can further pick one of each conjugate pair $(a, \bar{a})$ in B to consist of B_1. Then the above two anomaly indicators can be expressed into

$$\eta_{\mathcal{T}} = \frac{1}{D_{\mathcal{C}}}\left(\sum_{a \in A} d_a \theta_a \mathcal{T}_a^2 + \sum_{a \in B_1} 2 d_a \theta_a \mathcal{T}_a^2 \right) \tag{42}$$

$$\eta_{\mathcal{M}} = \frac{1}{D_{\mathcal{C}}}\left(\sum_{a \in A} d_a \theta_a \mu_a + \sum_{a \in B_1} 2 d_a \theta_a \mu_a \right) \tag{43}$$

where we have used $d_a \theta_a = d_{\bar{a}} \theta_{\bar{a}}$ and $\mu_a = \mu_{\bar{a}}$ and $\mathcal{T}_a^2 = \mathcal{T}_{\bar{a}}^2$. To make connection between the symmetry properties, we first relate them as

$$\rho_{\mathcal{M}} = \rho_{\mathcal{K}} \rho_{\mathcal{T}} \tag{44}$$

$$\mathcal{T}_a^2 = \mu_a \tag{45}$$

where $\rho_{\mathcal{K}}$ is the charge conjugate operation of a topological order that maps any anyon a into its conjugate $\bar{a}$ while keeping all the topological quantities invariant. We note that such a connection between the time reversal and mirror symmetry properties was also utilized in Ref. [112]. If we have a symmetry permutation $\rho_{\mathcal{T}}$ and $\mathcal{T}_a^2$ for the topological order $\mathcal{C}$, which leads to $\eta_{\mathcal{T}} = -1$, then through Eq.(44) and (45), we have the corresponding properties for mirror symmetry, i.e., $\{\rho_{\mathcal{M}}, \mu_a\}$, which eventually leads to $\eta_{\mathcal{M}} = -1$. Therefore, if we have $\eta_{\mathcal{T}} = -1$ for a topological order, we can also have proper mirror symmetry realization such that $\eta_{\mathcal{M}} = -1$ for the same topological order and vice versa. In other words, a topological order can carry the time reversal anomaly if and only if it can carry the mirror anomaly.

Besides the above time reversal anomaly and mirror anomaly which correspond the group cohomology class, there is still the beyond group cohomology class. For this case, the surface anomalous SET correspondence is more obvious by checking the anomaly indicator for these states [113]

$$\eta_1 = \frac{1}{D_{\mathcal{C}}} \sum_{a \in \mathcal{C}} d_a^2 \theta_a. \tag{46}$$

This anomaly indicator is for both time reversal and mirror SPT that are beyond group cohomology. As the time reversal and mirror properties do not explicitly enter in the expression of this indicator.

4.3 Other examples

4.3.1 Topological insulator and topological crystalline insulator protected by mirror symmetry

For 3D bosonic topological insulators with $U(1) \times Z_2^{\mathcal{T}}$ and their crystalline counterparts—topological crystalline insulator with $U(1) \times Z_2^{\mathcal{M}}$, their classification are both $(Z_2)^4$ [19,114]. Two of the roots are protected by only time reversal or mirror symmetry, which can be detected by the above mentioned two anomaly indicators η_1 and $\eta_{\mathcal{T}}$ or $\eta_{\mathcal{M}}$. The other two roots need the protection jointly by $U(1)$ and time reversal or mirror. For $U(1) \times Z_2^{\mathcal{T}}$, these two roots can be detected by the following two anomaly indicators [115]

$$\eta_{3,\mathcal{T}} = \frac{1}{D_{\mathcal{C}}} \sum_{a \in \mathcal{C}} d_a^2 \theta_a e^{i2\pi q_{\mathcal{T}}(a)} \tag{47}$$

$$\eta_{4,\mathcal{T}} = \frac{1}{D_{\mathcal{C}}} \sum_{a \in \mathcal{C}} d_a \theta_a e^{i2\pi q_{\mathcal{T}}(a)} \mathcal{T}_a^2 \tag{48}$$

641 where $q_\mathcal{T}(a)$ is the fractional $U(1)$ charge carried by anyon a, that satisfies $q_\mathcal{T}(a)+q_\mathcal{T}(b) = q_\mathcal{T}(c)$
642 mod 1 if N_{ab}^c nonzero. We put the subscript $\mathcal{T}$ here to imply this is defined under the present of
643 time reversal symmetry. In particular, $q_\mathcal{T}(a)+q_\mathcal{T}(\bar{a}) = 0$ mod 1 since $N_{a\bar{a}}^1 = 1$. Even though the
644 time reversal does not explicitly enter in the expression of η_3, time reversal indeed has effect
645 on η_3 by enforcing further constraint on the fractional charge $q_\mathcal{T}(a)$. As the group structure
646 $U(1) \times Z_2^\mathcal{T}$ indicates $\hat{Q}\mathcal{T} = -\mathcal{T}\hat{Q}$, we have $q_\mathcal{T}(\rho_\mathcal{T}(a)) = -q_\mathcal{T}(a)$. So for $\rho_\mathcal{T}(a) = a$, we have
647 $q_\mathcal{T}(a) = q_\mathcal{T}(\bar{a}) = 0, \frac{1}{2}$ mod 1.
648   For $U(1) \times Z_2^\mathcal{M}$, the corresponding two indicators are [116]

$$\eta_{3,\mathcal{M}} = \frac{1}{D_\mathcal{C}} \sum_{a \in \mathcal{C}} d_a^2 \theta_a e^{i2\pi q_\mathcal{M}(a)} \tag{49}$$

$$\eta_{4,\mathcal{M}} = \frac{1}{D_\mathcal{C}} \sum_{a \in \mathcal{C}} d_a \theta_a e^{i2\pi q_\mathcal{M}(a)} \mu_a \tag{50}$$

649 where $q_\mathcal{M}(a)$ also satisfies $q_\mathcal{M}(a)+q_\mathcal{M}(b) = q_\mathcal{M}(c)$ mod 1 if N_{ab}^c nonzero and then $q_\mathcal{M}(a)+q_\mathcal{M}(\bar{a}) = 0$
650 mod 1. Similarly, the mirror symmetry enforces additional constraint on the $q_\mathcal{M}(a)$ that is
651 $q_\mathcal{M}(\rho_\mathcal{M}(a)) = q_\mathcal{M}(a)$. Therefore, for $\rho_\mathcal{M}(a) = \bar{a}$, we have $q_\mathcal{M}(a) = q_\mathcal{M}(\bar{a}) = 0, \frac{1}{2}$.
652   Now we can see that we can map $\eta_{3,\mathcal{T}}$ and $\eta_{4,\mathcal{T}}$ to $\eta_{3,\mathcal{M}}$ and $\eta_{4,\mathcal{M}}$ respectively by identifica-
653 tion $\rho_\mathcal{M} = \rho_\mathcal{K}\rho_\mathcal{T}$, $\mu_a = \mathcal{T}_a^2$ as discussed in Sec. 4.2 and also $q_\mathcal{M}(a) = q_\mathcal{T}(a)$ for all $a \in \mathcal{C}$. The
654 last identification can be checked by $q_\mathcal{M}(\rho_\mathcal{M}(a)) = q_\mathcal{T}(\overline{\rho_\mathcal{T}(a)}) = -q_\mathcal{T}(\rho_\mathcal{T}(a)) = q_\mathcal{T}(a) = q_\mathcal{M}(a)$
655 mod 1. The first equality uses the identification $\rho_\mathcal{M} = \rho_\mathcal{K}\rho_\mathcal{T}$. The second equality use
656 $q_\mathcal{T}(a) + q_\mathcal{T}(\bar{a}) = 0$ mod 1. The third equality uses $q_\mathcal{T}(\rho_\mathcal{T}(a)) = -q_\mathcal{T}(a)$.
657   Therefore, we can conclude that if a topological order with time-reversal symmetry can be
658 put on the boundary of the 3D topological phases protected by $U(1) \times Z_2^\mathcal{T}$, it can also be put
659 on the boundary of the 3D topological phases protected by $U(1) \times Z_2^\mathcal{M}$ and vice versa. This
660 just builds up the surface anomalous SET correspondence to a bosonic topological insulator
661 by $U(1) \times Z_2^\mathcal{T}$ and topological crystalline insulator by $U(1) \times Z_2^\mathcal{M}$. The results for $U(1) \rtimes Z_2^\mathcal{T}$
662 and $U(1) \rtimes Z_2^\mathcal{M}$ can be discussed similarly.

663 4.3.2 Topological phase with $SU(2) \times Z_2^\mathcal{T}$ and topological crystalline phase with $SU(2) \times Z_2^\mathcal{M}$
664

665 The classification for 3D bosonic topological phase protected by $SU(2) \times Z_2^\mathcal{T}$ and $SU(2) \times Z_2^\mathcal{M}$
666 is $(Z_2)^3$. The first two roots do not need the protection of $SU(2)$ and have already been
667 discussed above. So we focus on the third root here. As the phase is still protected by breaking
668 the $SU(2)$ down to $U(1)_z = \{e^{i\alpha\hat{S}_z} | \alpha \in [0, 4\pi]\}$ where $\hat{S}_z = \frac{\hat{\sigma}_z}{2}$. So we can utilize the above set
669 of indicators for $U(1)$ case. As it was shown in Ref. [116] we can use $\tilde{\eta}_{3,\mathcal{M}} = \eta_1 \eta_{3,\mathcal{M}}$ to detect
670 to third root state, which always takes the trivial value $\tilde{\eta}_3 = 1$ for all possible topological
671 order. That leads to the result that there is the so-called "symmetry enforced gaplessness"
672 phenomenon on the surface of the topological phase protected jointly by $SU(2) \times Z_2^\mathcal{M}$. On the
673 other hand, it is also shown that there is also "symmetry enforced gaplessness" on the surface
674 of topological phase protected jointly by $SU(2) \times Z_2^\mathcal{T}$. This is consistent with the surface SET
675 correspondence: no SET can carry this anomaly.

676 4.3.3 Topological phase with $SO(N) \times Z_2^\mathcal{T}$ and topological crystalline phase with $SO(N) \times Z_2^\mathcal{M}$
677

678 The classification for 3D topological phases protected by $SO(3) \times Z_2^{\mathcal{T}(\mathcal{M})}$ is classified by $(Z_2)^4$.
679 Two roots of them are protected only by time reversal. And the other two ones are one-to-one
680 correspondence to the ones by breaking $SO(3)$ down to $U(1)_z = \{e^{i\alpha\hat{S}_z} | \alpha \in [0, 2\pi]\}$ where $\hat{S}_z$ is

681   the integer spin z-component operator. It was shown in Ref. [116] by replacing $q(a) \to s_a^z$, we
682   have two anomaly indicators for these two root phases for $SO(3) \times Z_2^{\mathcal{M}}$

$$\eta_{3,\mathcal{M}} = \frac{1}{D_\mathcal{C}} \sum_{a \in \mathcal{C}} d_a^2 \theta_a e^{i2\pi s_a} \tag{51}$$

$$\eta_{4,\mathcal{M}} = \frac{1}{D_\mathcal{C}} \sum_{a \in \mathcal{C}} d_a \theta_a e^{i2\pi s_a} \mu_a \tag{52}$$

683   where we used $s_a^z = s_a \bmod 1$ and s_a is the $SO(3)$ spin carried by anyon a.
684   Following similar reasoning above, we can conjecture the two corresponding anomaly in-
685   dicators for 3D topological phases protected by $SO(3) \times Z_2^{\mathcal{T}}$ are

$$\eta_{3,\mathcal{T}} = \frac{1}{D_\mathcal{C}} \sum_{a \in \mathcal{C}} d_a^2 \theta_a e^{i2\pi s_a} \tag{53}$$

$$\eta_{4,\mathcal{T}} = \frac{1}{D_\mathcal{C}} \sum_{a \in \mathcal{C}} d_a \theta_a e^{i2\pi s_a} \mathcal{T}_a^2. \tag{54}$$

686   Interestingly, these were proved using a different approach in Ref. [42] where these two for-
687   mulas are also shown to be the same for general $SO(N) \times Z_2^{\mathcal{T}}$ [3]. So in turn, we can conjecture
688   the two anomaly indicators (51) and (52) are also the same for $SO(N) \times Z_2^{\mathcal{M}}$ which might be
689   derived using the folding trick in Ref. [116].
690   Therefore, the topological order $\mathcal{C}$ can or cannot be put on the surface of 3D topological
691   phases protected by $SO(N) \times Z_2^{\mathcal{T}}$ and $SO(N) \times Z_2^{\mathcal{M}}$ [4]. This builds up the surface SET corre-
692   spondence of them.

5   Surface criticality correspondence of 3D topological phases: mirror system

695   In this section, we discuss the (near) critical surface of 3D mirror symmetry-protected topolog-
696   ical phase and also its counterpartner—3D time-reversal topological phase—that is classified
697   by group cohomology.

5.1   The near critical bulk description of mirror and time reversal topological phase

700   As mentioned in Sec.4.1, the 3D bosonic mirror SPT can be constructed by decorated the
701   mirror plane with the Levin-Gu state, the Z_2 bosonic SPT. We note that there are a few effective
702   theories to characterize the Levin-Gu state, such as Chern-Simons theory and nonlinear sigma
703   model (NLσM). Here we will adopt the nonlinear sigma model because it is a powerful theory
704   for the phases at or near criticality. In 2D the Levin-Gu state can be characterized by the O(4)
705   NLσM with Θ term [100]

$$\mathcal{L}_{\text{NLSM}} = \frac{1}{g} \left(\partial_\mu \vec{n} \right)^2 + \mathcal{L}_\Theta \tag{55}$$

706   where $\mathcal{L}_\Theta$ is the topological Θ term

$$\mathcal{L}_{2D}^\Theta = \frac{i2\pi}{\Omega_3} \epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d \tag{56}$$

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[3]We thanks Dr.Weicheng Ye to tell us the results in this paper [42].
[4]There is another root state for $N \geq 4$, but they are shown to be "symmetry enforced gapless" [117], so there is not surface SET of these phases

with Ω_3 the area of unit 3-sphere. The on-site $\mathbb{Z}_2$ symmetry acts on the vector

$$\mathbb{Z}_2 : \ \vec{n} \mapsto -\vec{n} \tag{57}$$

The vector field $\vec{n}$ is specified on the interface [5]. As $g \to \infty$, the ground state of the nd NLσM is disordered and symmetric.

Furthermore, the time reversal SPT classified by group cohomology can also described by the NLσM [100]. Recall that the classification of odd spatial dimensional time-reversal SPT by group cohomology is Z_2. The $(n{+}1)$d (odd n) system with time-reversal symmetry $\mathbb{Z}_2^T$ which can be described by an $O(n + 2)$ NLσM with a topological Θ term with $\Theta = 2\pi$. In 1+1d, it is the well-known $O(3)$ NLσM with Θ term which describes the Haldane phase. In 3+1d, the time-reversal bosonic SPT by group cohomology can be characterized $O(5)$ NLσM Eq. (55) with Θ term

$$\mathcal{L}_{3D}^{\Theta} = \frac{i2\pi}{\Omega_4} \epsilon_{abcde} n^a \partial_x n^b \partial_y n^c \partial_z n^d \partial_\tau n^e \tag{58}$$

where Ω_4 is the unit 4-sphere area. The time-reversal symmetry to the $O(5)$-vector $\vec{n}$ is

$$\mathbb{Z}_2^T : \ \vec{n} \mapsto -\vec{n} \tag{59}$$

5.2 Critical surface theory

Now we turn to discuss the (near) critical surface theory of the 3D mirror and time reversal symmetric SPT.

First, we investigate the boundary of the mirror SPT by truncating the block states with an open surface, which is chosen to be perpendicular to the mirror interface. Therefore, the two surface 2 cells are the boundary of the bulk 3 cells and the 1 cell is that of the bulk 2 cell. This surface is to some sense *standard* since its cell decomposition is one-to-one correspondence to the bulk cell decomposition.

Now we are interested in the surface where both the 2-cell are trivially symmetric gapped out but there is a gapless mode on the 1-cell, i.e., the surface mirror interface, that is now described by 1+1d $O(4)$ NLσM with the Wess-Zumino-Witten (WZW) term (see Fig.5)

$$\mathcal{L}_{2D}^{\text{WZW}}[\vec{n}] = \int \mathrm{d}u\mathrm{d}x \frac{i2\pi}{\Omega_3} \epsilon_{abcd} n^a \partial_u n^b \partial_x n^c \partial_\tau n^d. \tag{60}$$

The WZW term involves an extension of $O(n + 1)$ vector $\vec{n}(\boldsymbol{r}, \tau)$ to $\vec{n}(\boldsymbol{r}, \tau, u)$, such that

$$\vec{n}(\boldsymbol{r}, \tau, u = 0) = (0, \cdots, 1), \ \vec{n}(\boldsymbol{r}, \tau, u = 1) = \vec{n}(\boldsymbol{r}, \tau) \tag{61}$$

In other words, the topological boundary theory of the mirror-symmetric SPT phase is the 1+1d $O(4)$ NLσM with WZW term on the mirror interface of the boundary of the system. The 1+1d NLσM with WZW term (if the O(4) symmetry does not break) describes the so-called $SU(2)_1$ CFT in 1+1d, so typically there are protected gapless modes along the surface mirror domain wall. We note that such a boundary theory is derived from the viewpoint of a topological crystal where away from the mirror interface it is disentangled. However, in reality, the nontrivial entanglement can spread over the whole system, not just confine at the interface.

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[5]One can imagine that away from the interface, these vector fields are trivially gapped with a large enough energy mass in a mirror-symmetric way.

Then we turn to the surface of 3D time-reversal invariant SPT phases which is well-known and can be described by the 2+1d $O(5)$ NLσM with level one WZW term

$$\mathcal{L}_{3D}^{WZW} = \int du dx dy \frac{i2\pi}{\Omega_4} \epsilon_{abcde} n^a \partial_u n^b \partial_x n^c \partial_y n^d \partial_\tau n^e \tag{62}$$

With preserving time-reversal, we have $\langle n_5 \rangle = 0$ which can be treated as zero so that we can integrate the n_5 component and obtain the $O(4)$ NLσM with topological Θ term (56) but now $\Theta = \pi$ in contrast to 2π, with which values symmetric gapped unique ground state is possible. However, in fact, the $\Theta = \pi$ prevents the boundary from being featureless, i.e., having a symmetric gapped unique ground state.

Now we discuss the correspondence between surface criticalities. The surface of time reversal SPT is described by the 2+1d $O(5)$ level-one WZW term with time-reversal symmetry (59) while the surface of mirror SPT is $O(4)$ level-one WZW term (i.e., 1+1d $SU(2)_1$ CFT) with $\mathbb{Z}_2$ symmetry (57) on the mirror interface. The correspondence between these two surfaces means that they can be connected to each other under proper operation that are allowed by symmetry.

The above correspondence implies, on the one hand, we can obtain 1+1d $SU(2)_1$ CFT at the mirror interface from the 2+1d $O(5)$ NLσM with WZW term in a mirror symmetric way. In fact, this is easily done by adding a magnetic field $g(x)n_5(x)$ with $g(x) = -g(-x)$ and choosing the profile of $g(x) : g(0) = 0$ and $|g(x)| \geq 1$ for $x \neq 0$. On the other hand, it implies that we can construct the 2+1d $O(5)$ NLσM with WZW term in a time-reversal invariant manner according to (59) from the $SU(2)_1$ CFT which are located at the mirror interface. Indeed, such construction does exist which was done in Ref. [101]. Roughly speaking, the authors started from the infinite coupled spin-1/2 chain described by the $SU(2)_1$ CFT which in fact has an emergent symmetry $O(4) \cong SU(2)_L \times SU(2)_R$, and turned on the interchain coupling which is also $O(4)$ invariant and followed the way similar to Haldane's well-known derivation for $O(3)$ Θ term in 1+1d spin chain, they obtained the 2+1d $O(4)$ NLσM with Θ term where $\Theta = \pi$, which is equivalent to the $O(5)$ NLσM with WZW term when preserving the time-reversal symmetry. Below we will review such construction in more detail and then generalize to general cases.

5.3   Coupled chain approach for $O(5)$ WZW and generalization to $O(n)$ case

To see the Senthil-Fisher's coupled chain construction $O(5)$ WZW [101], we will first review Haldane's derivation for 1+1d O(3) Θ term from WZW term that can generalized straightforwardly to $O(n)$ case. Then $n = 5$ gives us the Senthil-Fisher's construction.

5.3.1   Review on Haldane's derivation

To begin with, we first review that effective description of 0+1d spin one-half system that is the 0+1d $O(3)$ nonlinear sigma model with level one WZW term

$$S_0 = \int d\tau \frac{1}{g}(\partial_\tau \vec{n})^2 + 2\pi k \Gamma[\vec{n}(u, \tau)] \tag{63}$$

We have level $k = 1$ and $\Gamma[\vec{n}(u, \tau)]$ is the 0+1D WZW

$$\Gamma[\vec{n}] = \frac{1}{4\pi} \int du d\tau \vec{n} \cdot (\partial_u \vec{n} \times \partial_\tau \vec{n}) \tag{64}$$

where we have extended the vector field $\vec{n}$ from τ dependent to (u, τ) with $u \in [0, 1]$ such that $\vec{n}(0, \tau) = \hat{z}$ and $\vec{n}(1, \tau) = \vec{n}(\tau)$. The extension of $\vec{n}(u, \tau)$ is a choice of the convention as

775 one can use the south pole instead of the north one at the position of $u = 0$. It is easy to see
776 that $\Gamma[-\vec{n}] = 4\pi - \Gamma[\vec{n}] = -\Gamma[\vec{n}]$ mod 4π. We note that the 4π here is nothing but the volume
777 of unit S^2.

778   Now we consider a chain consisting of an infinite number of spin one half $\vec{n}_i$ and turn on
779 the antiferromagnetic coupling between two nearest neighborhood spin one half. Then the
780 theory is effectively given by

$$S_1 = \int d\tau \sum_i S_0[\vec{n}_i(\tau)] + u \int d\tau \sum_i \vec{n}_i(\tau) \cdot \vec{n}_{i+1}(\tau) \tag{65}$$

781 with $u > 0$. The antiferromagnetic coupling drives the nearest neighboring spin to polarize in
782 the opposite direction, namely

$$\vec{n}_i = (-1)^i \vec{n}'_i \tag{66}$$

783 where $\vec{n}'$ are smooth varying fields. Substitute this ansatz into the intersite coupling u, we
784 have

$$u \int d\tau \sum_i \vec{n}_i \cdot \vec{n}_{i+1} \simeq \frac{au}{2} \int dx d\tau (\partial_x \vec{n}'(x,\tau))^2 \tag{67}$$

785   Now we turn to the term $S_0[\vec{n}_i]$. The first term in S_0 combined with the inter-site coupling
786 just discussed above, we arrive at

$$S_1^0 = \frac{1}{ag} \int dx d\tau \; \frac{1}{v_1^2}(\partial_x \vec{n}')^2 + (\partial_\tau \vec{n}')^2 \tag{68}$$

787 where $v_1 = 1/(a\sqrt{ug})$.

788   Now we turn to the WZW terms in S_0. We follow Haldane's approach to derive the topo-
789 logical Theta term from the summed alternative WZW terms. Due to the ansatz Eq.(66) and
790 $\Gamma[-\vec{n}] = -\Gamma[\vec{n}]$ mod 4π, the summed WZW terms becomes

$$\sum_i \Gamma[n'_{2i}] - \Gamma[n'_{2i-1}]$$

$$= \frac{1}{2} \int \frac{dx}{a} \frac{\delta\Gamma[\vec{n}'(x,\tau)]}{\delta\vec{n}'} \cdot \partial_x \vec{n}' a$$

$$= \frac{1}{8\pi} \int dx d\tau \; \vec{n}' \cdot (\partial_\tau \vec{n}' \times \partial_x \vec{n}')$$

$$= \frac{1}{2}\mathcal{I}[\vec{n}] \tag{69}$$

791 where we have used [6]

$$\delta\Gamma[\vec{n}] = \frac{1}{4\pi} \int d\tau \delta\vec{n} \cdot (\vec{n} \times \partial_\tau \vec{n}). \tag{70}$$

792 $\mathcal{I}[\vec{n}]$ is the winding number from S^2 spanned by (x,t) to S^2 spanned by $\vec{n}$. We note that in the
793 continuum limit, $\Gamma[n'_{2i}] - \Gamma[n'_{2i-1}] = \Gamma[n'_{2i-1}] - \Gamma[n'_{2i-2}]$, which lead to the factor $\frac{1}{2}$. Therefore,
794 the second term of summed $S_0[n_i]$ gives the $O(3)$ topological theta term with $\Theta = \pi$.

795   Now we give a more intuitive derivation of Eq.(69). Recall that the physical meaning of
796 $\Gamma[\vec{n}'_i]$ is the surface angle on the sphere surrounded by the trajectory of physical vector $\vec{n}'_i(\tau)$

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[6]See Eq.(10.34) in [118].
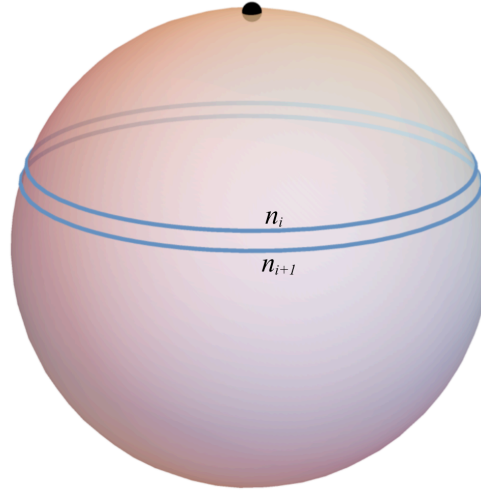
Figure 6: The n-Sphere angles traced by two configurations of $n_i(\vec{x}, \tau)$ and $n_{i+1}(\vec{x}, \tau)$ sweeping from $\tau = 0$ to $\tau = \beta$. (Here we use n_i instead of n'_i.) The shaded "narrow ribbon" is equal to the difference between the sphere angles, namely $\Delta\Gamma_i$.

from $\tau = 0$ to $\tau = \beta$. Then $\Gamma[\vec{n}'_{2i}] - \Gamma[\vec{n}'_{2i-1}]$ is just the difference of surface angle of the two nearest vectors $\vec{n}'_{2i}$ and $\vec{n}'_{2i-1}$. As the field $\vec{n}'$ varies smoothly in space, then the surface angle difference can be represented by a very "narrow ribbon" on the sphere (See Fig.6). For the closed boundary condition, as i starts from the initial site, passes over the chain, and then back to the initial site, the "narrow ribbon" just smoothly moves on the sphere and then back to the original position. In the continuum limit, it will swap around the sphere continuously. Therefore, the summation just gives us the winding number of the mapping $\vec{n}' : S^2 \to S^2$. This geometric derivation is convenient to generalize to higher dimensions.

5.3.2 Generalization to $(n+1)D$

Let us begin with the nD $O(n+2)$ nonlinear sigma model with level one WZW terms

$$S_n = \int dx^n \frac{1}{g}(\partial_\mu \vec{n})^2 + 2\pi k \Gamma_{n+1}[\vec{n}] \tag{71}$$

where $\vec{n}$ is the $n+2$ component unit vector $k = 1$ is the level and $\Gamma_n[\vec{n}]$ is the WZW term

$$\Gamma_{n+1}[\vec{n}] = \frac{1}{\Omega_{n+1}} \int du dx^n \epsilon_{abc...d} n^a \partial_u n^b \partial_x n^c ... \partial_\tau n^d \tag{72}$$

where Ω_{n+1} is the volume of $n+1$ sphere. We have extended the vector field $\vec{n}(\vec{x}, \tau)$ to $\vec{n}(u, \vec{x}, \tau)$ such that $\vec{n}(u = 0, \vec{x}, \tau) = \hat{x}_{n+2}$ and $\vec{n}(u = 1, \vec{x}, \tau) = \vec{n}(\vec{x}, \tau)$. The meaning of $\Gamma_{n+1}[\vec{n}]$ is then the volume (divide Ω_{n+1}) surrounded by the trajectory of $\vec{n}(\vec{x}, \tau)$ on the $n+1$ sphere. Note that the extension of $\vec{n}(\vec{x}, \tau)$ is a convention, as one can also choose the $-\hat{x}_{n+2}$ are the position with $u = 0$. By reversing the direction of $\vec{n}$, the surrounded volume $\Gamma_{n+1}[-\vec{n}]$ is connected to $\Gamma_{n+1}[\vec{n}]$ by $\Gamma_{n+1}[-\vec{n}] = -\Gamma_n[\vec{n}]$ mod Ω_{n+1}.

Now we consider an infinite copy of these nD theories arranged in the $n+1$ (spatial) direction, whose vector fields are denoted as $\vec{n}_i$. Then we turn on the antiferromagnetic coupling between two nearest neighboring vectors $\vec{n}_i$ and $\vec{n}_{i+1}$. So the total action is given by

$$S_{n+1} = \sum_i S_n[\vec{n}_i] + u \int dx^n \sum_i \vec{n}_i \cdot \vec{n}_{i+1}. \tag{73}$$

with $u > 0$. The antiferromagnetic coupling then drives, similarly to the 1+1D case in Sec. 5.3.1, $\vec{n}_i$ and $\vec{n}_{i+1}$ polarize in opposite direction. So we can take the following ansatz

$$\vec{n}_i = (-)^i \vec{n}'_i. \tag{74}$$

Substitute this ansatz into antiferromagnetic coupling action, very similar to Eq.(67), we have

$$u \int dx^n \sum_i \vec{n}_i \cdot \vec{n}_{i+1} \simeq \frac{au}{2} \int dx^n dz (\partial_z \vec{n}'(x,\tau))^2 \tag{75}$$

Then the first term in S_n combined with the inter-layer coupling just discussed above, we arrive at

$$S^0_{n+1} = \int dx^n dz \; \frac{1}{ag}(\partial_\mu \vec{n}')^2 + \frac{au}{2}(\partial_z \vec{n}')^2. \tag{76}$$

Now we consider the WZW terms in S_n. Noticing the ansatz (74), the summed WZW terms becomes, similarly to 1+1D case,

$$S^1_{n+1} = \sum_i \Gamma_{n+1}[\vec{n}'_{2i}] - \Gamma_{n+1}[\vec{n}'_{2i-1}]$$
$$= \frac{1}{2} \sum_i \Delta\Gamma_{n+1}[\vec{n}'_i] \tag{77}$$

Each term $\Delta\Gamma_{n+1}[\vec{n}'_i]$ is a very "narrow ribbon" on the S_{n+1} sphere (See Fig.6). In the continuum limit, the summation of them gives the winding number $\mathcal{I}[\vec{n}']$ of S_{n+1} by the vector field $\vec{n}'(z, \vec{x}, \tau)$, that is

$$S^1_{n+1} = \frac{1}{2}\mathcal{I}[\vec{n}']$$
$$= \frac{1}{2\Omega_{n+1}} \int dx^{n+1} \epsilon_{abc...d} n'^a \partial_z n'^b \partial_x n'^c ... \partial_\tau n'^d. \tag{78}$$

where $\vec{n}' : S_{n+1} \to S_{n+1}$. This is a topological term so it is invariant under any local continuous coordinate transformation, such as the rescaling of one coordinate. Combined Eqs. (76) and (78), and up to proper rescaling of z coordinate, we arrive at

$$S_{n+1} = \frac{1}{ag} \int dx^{n+1} (\partial_\mu \vec{n})^2$$
$$+ \frac{\pi}{\Omega_{n+1}} \int dx^{n+1} \epsilon_{abc...d} n'^a \partial_z n'^b \partial_x n'^c ... \partial_\tau n'^d \tag{79}$$

which is just the $n+1$D $O(n+2)$ nonlinear sigma model with theta term $\Theta = \pi$. We note that when $n = 2$, it is just the Senthil-Fisher's construction in Ref. [101] as mentioned in Sec. 5.2.

Now we discuss how to obtain the level one WZW term from Θ term with $\theta = \pi$ while preserving the anomaly. Let us begin with the $1 + 1d$ case. As it is known in 1+1D spin 1/2 chain, there are two equivalent descriptions for the IR physics [119]: the $O(3)$ nonlinear sigma model with Θ term $\theta = \pi$, and the $O(4)$ nonlinear sigma model with level one WZW term [7]. Even though the direct proof of the equivalence between these two models is not at all straightforward, we utilize another indirect way to argue that they are indeed equivalent

---

[7]There is some study that shows the two theories may deviate from each other in the critical behavior when beyond some critical temperature [120].
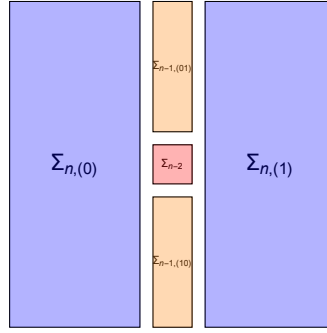
Figure 7: The cell decomposition of the 2-fold rotation-symmetric systems. The nD system with C_2 symmetry can be cell decomposed into two n cells $\Sigma_{n,(0)}$ and $\Sigma_{n,(1)}$, two $n-1$ cells $\Sigma_{n-1,(01)}$ and $\Sigma_{n-1,(10)}$ and one $n-2$ cell Σ_{n-2}. For example, in 2D, there are two 2-cells, two 1-cells, and one 0-cell. Often the (n-2)-cell is also called the rotation center.

(at least in terms of the quantum anomaly of certain symmetry). The way is that we start with the $O(4)$ model, by adding an anisotropic term of one component, says n^4, i.e., $-u(n^4)^2$ with ferromagnetic coupling constant $u > 0$ [8]. This term preserves the symmetry $n^4 \to -n^4$ and has the effect that the expectation value of n^4 is pinned to zero. So we can integrate out the n^4 component in $O(4)$ WZW model which exactly gives us the $O(3)$ nonlinear sigma model with $\theta = \pi$. Such a justification of equivalence between the 1+1D $O(3)$ and $O(4)$ model also hints that the anisotropic coupling is irrelevant in the IR limit for the 1+1D case [119].

For a general nD situation, we can still apply the above anisotropic term argument to relate the $O(n+2)$ $k=1$ WZW model and $O(n+1)$ model with $\theta = \pi$. In terms of focus on the quantum anomaly, as long as the anisotropic term preserves the symmetry, these two theories still have the same anomaly, so such a connection between the two theories is satisfying. We note that while the two theories may maintain the same quantum anomaly, they might not be equivalent in terms of local dynamics in the IR limit. In other words, the possible anisotropic term added may not be irrelevant in the IR limit for general $n > 2$ but it seems to be irrelevant for $n = 1$.

From the above discussion, we see that it is possible to construct the nD $O(n+1)$ nonlinear sigma model with $\Theta = \pi$ (nD $O(n+2)$ $k=1$ WZW model) that usually maintains quantum anomaly of certain symmetry (such as some symmetry with only order-two element) from lower dimension counterparts, and step by step eventually down to the $0+1D$ case. This messages us that we can track the quantum anomaly down to the $0d$ system. In fact, such a construction/scenario is also shown in other situations, such as the fermion surface anomaly [121] and the LSM anomaly.

6 More examples of surface criticality correspondence: rotation system

Consider systems with rotation symmetry C_N. We demonstrate C_2 symmetry as an example.

---

[8]We can also add anisotropic term for another component with special attention to the symmetry requirement which we will discuss soon later.

6.1 Bulk description of rotation systems and their counterpartners

The rotation SPT can be constructed by decorating the rotation center. At the rotation center (see Fig.7), the rotation symmetry acts as a unitary on-site $\mathbb{Z}_2$ symmetry. The rotation symmetric SPT can be constructed by decorating the $(n-2)$D $\mathbb{Z}_2$ SPT at the rotation center. For 0D and 2D, there is one nontrivial $\mathbb{Z}_2$ bosonic SPT (the nontrivial $\mathbb{Z}_2$ charge 0D SPT and 2D Levin-Gu state), while there is none in 1D, which indicates that there is no nontrivial 3D rotation SPT. There is one nontrivial two-fold rotation SPT in 2D and 4D. So here we focus on two cases: 2D and 4D rotation systems. The effective actions of Levin-Gu states are given by Eq. (56) with $\mathbb{Z}_2$ symmetry $\vec{n} \to -\vec{n}$. For 0D Z_2 SPT, the effective action $\mathcal{L}_{0D}^{\theta}$ takes the following form

$$\mathcal{L}_{0D}^{\Theta} = \frac{i2\pi}{\Omega_1} \epsilon^{ab} n_a \partial_\tau n_b \tag{80}$$

which together with the $O(2)$ NLσM dynamical term $\frac{1}{g}(\partial_\tau \vec{n})^2$ guarantees the ground state is in 0D $\mathbb{Z}_2$ SPT state.

On the other hand, we consider an $(n+1)$d system ($n = 2, 3, 4$) with $\mathbb{Z}_2$ on-site symmetry, which is the crystalline equivalence counterpart of two-fold rotation SPT. The effective field theory is still NLσM with topological Θ-term [cf. Eq. (55)], with $\mathbb{Z}_2$ symmetry defined as Eq. (57), which is only well-defined in (2+1)d system as a Levin-Gu state [14] and (4+1)d system with $O(6)$ vector $\vec{n}$ and topological $\Theta = 2\pi$ term

$$\mathcal{L}_{4D}^{\Theta} = \frac{i2\pi}{\Omega_5} \epsilon_{abcdef} n^a \partial_x n^b \partial_y n^c \partial_z n^d \partial_w n^e \partial_\tau n^f . \tag{81}$$

where x, y, z, w are the four physical spatial dimensions while τ is the imaginary time dimension.

The classifications of the $\mathbb{Z}_2$ SPT phases in $(n+1)$D systems are

$$n = 2 : \mathbb{Z}_2; \quad n = 3 : \mathbb{Z}_1; \quad n = 4 : \mathbb{Z}_2 \tag{82}$$

which is one-to-one corresponding to the classifications of the C_2-symmetric SPT phases in $(n+1)$d systems.

We now discuss the (higher) domain defect of SPT protected by internal Z_2. For 2D, there is 0D SPT protected by $\mathbb{Z}_2$ at the codimension-2 domain defect. Meanwhile, for 4D, there is 2D SPT protected by $\mathbb{Z}_2$ at the codimension-2 domain defect. This physics can be easily checked using the NLσM framework. Interestingly, both decorated rotation center and higher codimension domain defect carry similar nontrivial states.

6.2 Critical boundary theories and their correspondence

It is well-known that the topological boundaries of $\mathbb{Z}_2$ SPT are $O(n+2)$ NLσM with level-one WZW term [cf. (60) for 2+1d] [100]. The WZW term for $3+1$d is given by

$$\mathcal{L}_{3D}^{\text{WZW}} = \int_0^1 du \frac{i2\pi}{\Omega_5} \epsilon_{abcdef} n^a \partial_x n^b \partial_y n^c \partial_z n^d \partial_u n^e \partial_\tau n^f . \tag{83}$$

where $O(6)$ vector field $\vec{n}$ are subject to the similar extension as (61).

Now we consider the boundary theory of rotation SPT. The boundary we choose is required to respect the rotation symmetry. For 2D, the systems can naturally be made into a disk D_2 whose boundary is just $S_1 = \partial D_2$. However, such a boundary is *non-standard* since it intersects with only the 2-cells and 1-cells but not the rotation center, which means that CD of S_1 respect rotation symmetry is not one-to-one correspondence with the bulk CD. For such a boundary, we

do not expect there is the surface criticality correspondence. In fact, the boundary theory on S_1 can be symmetrically gapped out without any degeneracy. In contrast, the boundary with the same geometry of the $2 + 1$d internal $\mathbb{Z}_2$ SPT can not be gapped out without degeneracy, i.e., is necessary to be gapless or spontaneously symmetry breaking.

For 3D and 4D systems, we can choose a standard boundary, whose structure of CD is one-to-one correspondence to the bulk one. For 3D, we can take the boundary to be perpendicular to the rotation center axis, which is in the same form as Fig. 7, while the bulk extends into the third direction that is not present. The 4D situation is similar.

Physically, with such chosen boundary geometry, the $(4+1)$d boundary we consider here is trivially symmetrically gapped out almost everywhere but at the boundary rotation center axis there is a gapless mode, i.e., the $(1+1)$d edge theory that is the $(2+1)$d Levin-Gu state, described by the $O(4)$ level-one WZW term. As with the standard boundary we do expect there is surface correspondence for 4D C_2 rotation systems and internal Z_2 SPT.

Now we discuss the correspondence of surface criticality for rotation SPT and $\mathbb{Z}_2$ SPT. We focus on $4 + 1$d. In fact, with the $\mathbb{Z}_2$ symmetry, the six component n_6 can be treated as zero and then integrated so that $3+1$d $O(6)$ level-one WZW term (83) becomes the $3+1$d $O(5)$ topological Θ term with $\theta = \pi$ in contrast to (58) where $\theta = 2\pi$. The surface correspondence for the C_2 rotation system implies that the $1+1$d $SU(2)_1$ CFT and $3+1$d NLσM with topological $\Theta = \pi$ term might have an ultimate connection while preserving symmetry. On the one hand, it is easy to see that the $3+1$d NLσM with topological $\Theta = \pi$ term reduces to the $O(4)$ NLσM with level-one WZW term, which is equivalent to $1+1$d $SU(2)_1$ CFT on the codimension-2 $\mathbb{Z}_2$ domain wall.

On the other hand, we should be able to construct the $O(5)$ NLσM with $\theta = \pi$ term from the $1+1$d $SU(2)_1$ CFT in a $\mathbb{Z}_2$ symmetric manner. We expose the existence of such a construction by recalling the Hierarchy construction of $O(n)$ topological term discussed in Sec.5.3. More explicitly, for the construction, we notice that the $2+1$d $O(4)$ NLσM with $\Theta = \pi$ term is equivalent to $2+1$d $O(5)$ NLσM with level-one WZW term if the symmetry $\mathbb{Z}_2 : \vec{n} \to -\vec{n}$ is preserved without changing the surface anomaly. Then the desired construction comes into two steps: First, we use the Senthil-Fisher's construction to obtain $2+1$d $O(5)$ NLσM with level-1 WZW term for each layer $z \in \mathbb{Z}$ from the coupled $1 + 1$d $SU(2)_1$ CFT; Second, turning on the O(5) invariant coupling which preserves the $\mathbb{Z}_2$ symmetry between different layers, and following the derivation in Sec.5.3, we obtain the $3+1$d $O(5)$ NLσM with $\Theta = \pi$ which is then equivalent to the $3+1$d $O(6)$ NLσM with level-one WZW term while preserving the anomaly.

7 Summary and outlook

In this work, we established the anomalous surface correspondence for internal SPT and crystalline SPT that are subject to the crystalline equivalence principle in three and higher-dimensional bosonic systems, generalizing the so-called *crystalline equivalent bulk-boundary correspondence* for topological phases. We first discuss the correspondence for the surface Symmetry-Enriched Topological order that can appear on the boundary of three or higher-dimensional SPT. For time reversal and mirror SPT, we explicitly show, by utilizing the anomaly indicators, that if one topological order with anomalous time reversal symmetry can live on the boundary of time reversal SPT, it can also live on the boundary of mirror SPT with assigning appropriate mirror symmetry properties and vice versa. We also discuss the SET correspondence including other symmetries which lead us to propose some new anomaly indicators. On the other hand, we also established the direct correspondence between the surface theory in or near criticality by taking the advantage of topological nonlinear sigma model. In pursuit of the direct connection of the (near) critical surface theories, we find the hierarchy structure

of the topological $O(n)$ sigma model, originally pioneered by Haldane, that provides a unified way to understand the surface criticality correspondence discussed in this paper. Below we discuss several future problems.

1. Generalize to three and higher fermionic dimension

2. Study the surface criticality correspondence for SPT beyond group cohomology

3. Generalize to SPT phases with average symmetry [122–126].

Note − While preparing this manuscript, we notice that the mapping between the time reversal and reflection data in 2D SET are also discussed in Ref. [117].

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A   Majorana zero modes as a domain wall

In the main text, we have concluded that for a 2D higher-order fSPT phase protected by reflection symmetry M with spinless fermions, the "domain-wall" physics near $x = 0$ is described by the Hamiltonian:

$$H = \int \mathrm{d}x \cdot \gamma^T \mathcal{H}(x)\gamma \tag{A.1}$$

where $\gamma = \left(\gamma_\uparrow, \gamma_\downarrow\right)^T$ and

$$\mathcal{H}(x) = i\sigma^3 \partial_x + x\sigma^2 \tag{A.2}$$

To get the zero-energy solution, we define an alternative basis from a unitary transformation on γ:

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma^1 + \sigma^3)\gamma = \frac{1}{\sqrt{2}}\begin{pmatrix} \gamma_\uparrow + \gamma_\downarrow \\ \gamma_\uparrow - \gamma_\downarrow \end{pmatrix} \tag{A.3}$$

Under the basis χ, the Hamiltonian $\mathcal{H}$ will be transformed to:

$$\begin{aligned} \mathcal{H}' &= \frac{1}{\sqrt{2}}(\sigma^1 + \sigma^3)\left[i\sigma^3 \partial_x + x\sigma^2\right]\frac{1}{\sqrt{2}}(\sigma^1 + \sigma^3) \\ &= i\sigma^1 \partial_x - x\sigma^2 \end{aligned} \tag{A.4}$$

971   Define the effective creation/annihilation operators a and $a^\dagger$ in terms of x and ∂_x:

$$\begin{cases} a = \dfrac{1}{\sqrt{2}}(x + \sigma^3 \partial_x) \\ a^\dagger = \dfrac{1}{\sqrt{2}}(x - \sigma^3 \partial_x) \end{cases} \tag{A.5}$$

972   with commutation relation:

$$\left[a, a^\dagger\right] = \frac{1}{2}\left[x + \sigma^3 \partial_x, x - \sigma^3 \partial_x\right] = \sigma^3$$

973   Then we can rephrase the Hamiltonian $\mathcal{H}'^2$ in terms of a and $a^\dagger$ we defined above:

$$\mathcal{H}'^2 = (i\sigma^1 \partial_x - x\sigma^2)^2 = -\partial_x^2 + x^2 + \sigma^3 = 2a^\dagger a \tag{A.6}$$

974   So if $\mathcal{H}'^2$ has a zero mode, so do $\mathcal{H}'$. Suppose $|0\rangle$ is a zero mode of $\mathcal{H}'^2$ that is proportional to
975    $(1, 0)^T$ and satisfying $a|0\rangle = 0$ in χ-basis:

$$a|0\rangle = (x + \partial_x)|0\rangle, \ \Rightarrow \ |0\rangle \propto e^{-x^2/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{A.7}$$

976   that is a Gaussian wavepacket localized near $x = 0$. As the consequence, $\mathcal{H}'^2$ (and thus $\mathcal{H}'$)
977   has zero mode $|0\rangle$ that is localized near $x = 0$. In γ-basis, this zero mode is expressed as:

$$|0\rangle \propto e^{-x^2/2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{A.8}$$

978   # B   Symmetry properties of Majorana corner modes in D_4-symmetric
979   case

980   In the main text, for 2D D_4-symmetric systems with spinless fermions, we have reformulated
981   the Majorana corner modes of the corresponding higher-order fSPT phase in terms of the
982   domain walls on the boundary. In particular, these domain walls can be expressed in terms of
983   the following basis:

$$\gamma = \left(\gamma_\uparrow^1, \gamma_\downarrow^1, \gamma_\uparrow^2, \gamma_\downarrow^2\right)^T$$

984   The Majorana corner modes at other poles can also be formulated in γ-basis:

$$\begin{aligned} |0\rangle_{1,3} &= \mathcal{A}e^{-y^2/2}(1, 1, 0, 0)^T \\ |0\rangle_{1',3'} &= \mathcal{A}e^{-y^2/2}(0, 0, 1, 1)^T \\ |0\rangle_{2,4} &= \mathcal{A}e^{-x^2/2}(1, 1, 0, 0)^T \\ |0\rangle_{2',4'} &= \mathcal{A}e^{-x^2/2}(0, 0, 1, 1)^T \end{aligned} \tag{B.1}$$

985   Under 4-fold rotation $\boldsymbol{R} \in C_4$ and reflection $\boldsymbol{M}_1$, these zero modes transform as:

$$\begin{aligned} \boldsymbol{R} : \ &\left(|0\rangle_1, |0\rangle_{1'}, |0\rangle_2, |0\rangle_{2'}, |0\rangle_3, |0\rangle_{3'}, |0\rangle_4, |0\rangle_{4'}\right) \\ &\mapsto \left(|0\rangle_2, |0\rangle_{2'}, |0\rangle_3, |0\rangle_{3'}, |0\rangle_4, |0\rangle_{4'}, |0\rangle_1, |0\rangle_{1'}\right) \end{aligned} \tag{B.2}$$

986   and

$$\begin{aligned} \boldsymbol{M}_1 : \ &\left(|0\rangle_1, |0\rangle_{1'}, |0\rangle_2, |0\rangle_{2'}, |0\rangle_3, |0\rangle_{3'}, |0\rangle_4, |0\rangle_{4'}\right) \\ &\mapsto \left(|0\rangle_{1'}, |0\rangle_1, |0\rangle_{4'}, |0\rangle_4, |0\rangle_{3'}, |0\rangle_3, |0\rangle_{2'}, |0\rangle_4\right) \end{aligned} \tag{B.3}$$

i.e., all Majorana corner modes are D_4-invariant. Alternatively, we can phrase the D_4 symmetry properties in a more transparent way be redefine the Majorana zero modes:

$$
\begin{aligned}
|z\rangle_{1,3} &= (|0\rangle_{1,3} + |0\rangle_{1',3'})/\sqrt{2} \\
|z\rangle_{1',3'} &= (|0\rangle_{1,3} - |0\rangle_{1',3'})/\sqrt{2} \\
|z\rangle_{2,4} &= (|0\rangle_{2,4} + |0\rangle_{2',4'})/\sqrt{2} \\
|z\rangle_{2',4'} &= (|0\rangle_{2,4} - |0\rangle_{2',4'})/\sqrt{2}
\end{aligned}
\tag{B.4}
$$

Under 4-fold rotation $\boldsymbol{R} \in C_4$ and reflection $\boldsymbol{M}_1$, these zero modes transform as:

$$
\begin{aligned}
\boldsymbol{R} : \;& \big(|z\rangle_1, |z\rangle_{1'}, |z\rangle_2, |z\rangle_{2'}, |z\rangle_3, |z\rangle_{3'}, |z\rangle_4, |z\rangle_{4'}\big) \\
&\mapsto \big(|z\rangle_2, |z\rangle_{2'}, |z\rangle_3, |z\rangle_{3'}, |z\rangle_4, |z\rangle_{4'}, |z\rangle_1, |z\rangle_{1'}\big)
\end{aligned}
\tag{B.5}
$$

and

$$
\begin{aligned}
\boldsymbol{M}_1 : \;& \big(|z\rangle_1, |z\rangle_{1'}, |z\rangle_2, |z\rangle_{2'}, |z\rangle_3, |z\rangle_{3'}, |z\rangle_4, |z\rangle_{4'}\big) \\
&\mapsto \big(|z\rangle_1, -|z\rangle_{1'}, |z\rangle_4, -|z\rangle_{4'}, |z\rangle_3, -|z\rangle_{3'}, |z\rangle_2, -|z\rangle_{2'}\big)
\end{aligned}
\tag{B.6}
$$

i.e., Majorana zero modes $|z\rangle_j \big| j = 1, 2, 3, 4; 1', 2', 3', 4'$ carry charges of reflection generator $\boldsymbol{M}_1$.

In the main text, we have defined an effective "$\mathbb{Z}_4$ on-site symmetry" A which satisfies $A^4 = -1$. Under A, these Majorana corner modes will be transformed as:

$$
\begin{aligned}
|0\rangle_{1,3} &\mapsto \mathcal{A}e^{-y^2/2}(1, 1, -1, 1)^T \\
|0\rangle_{1',3'} &\mapsto \mathcal{A}e^{-y^2/2}(-1, 1, 1, 1)^T \\
|0\rangle_{2,4} &\mapsto \mathcal{A}e^{-x^2/2}(1, 1, -1, 1)^T \\
|0\rangle_{2',4'} &\mapsto \mathcal{A}e^{-x^2/2}(-1, 1, 1, 1)^T
\end{aligned}
\tag{B.7}
$$

i.e., these Majorana corner modes are not invariant under A symmetry. Furthermore, we have defined another effective "time-reversal symmetry" $\mathcal{T}$ which satisfies $\mathcal{T}^2 = -1$. Under $\mathcal{T}$, these Majorana corner modes will be transformed as:

$$
\begin{aligned}
|0\rangle_{1,3} &\mapsto \mathcal{A}e^{-y^2/2}(0, 0, -1, -1)^T = -|0\rangle_{1',3'} \\
|0\rangle_{1',3'} &\mapsto \mathcal{A}e^{-y^2/2}(1, 1, 0, 0)^T = |0\rangle_{1,3} \\
|0\rangle_{2,4} &\mapsto \mathcal{A}e^{-x^2/2}(0, 0, -1, -1)^T = -|0\rangle_{2',4'} \\
|0\rangle_4^{N,S} &\mapsto \mathcal{A}e^{-x^2/2}(1, 1, 0, 0)^T = |0\rangle_{2,4}
\end{aligned}
\tag{B.8}
$$

i.e., there Majorana corner modes are invariant under $\mathcal{T}$ symmetry. Equivalently, these 8 Majorana corner modes are not compatible with the effective $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$ on-site symmetry (with $A^4 = -1$ and $\mathcal{T}^2 = -1$).

C   Representation of $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$ in 2D system with spin-1/2 fermions

In this section, we demonstrate that 4 Majorana fermions γ_σ^j ($\sigma = \uparrow, \downarrow$ and $j = 1, 2$) introduced in the main text, with the following symmetry properties:

$$
A = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_{2\times2} & -i\sigma^2 \\ -i\sigma^2 & \mathbb{1}_{2\times2} \end{pmatrix}
\tag{C.1}
$$

and

$$\mathcal{T} = i(\tau^2 \otimes \mathbb{1}_{2\times 2})K \tag{C.2}$$

realize a reprensentation of $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$ group in 2D system with spin-1/2 fermions, where $A \in \mathbb{Z}_4$ and $\mathcal{T} \in \mathbb{Z}_2^T$ are two generators of the symmetry group $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$. For a fermionic system, there is always a fermion parity symmetry $\mathbb{Z}_2^f = \{1, P_f = (-1)^F\}$, where F is the total number of fermions. The spin of fermions is characterized by the factor system ω_2 of the following short exact sequence:

$$0 \to \mathbb{Z}_2^f \to G_f \to \mathbb{Z}_4 \rtimes \mathbb{Z}_2^T \to 0 \tag{C.3}$$

where G_f depicts the total symmetry group of the system, as a group extension of $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$ and fermion parity $\mathbb{Z}_2^f$. ω_2 is an element of the following group 2-cohomology:

$$\omega_2 \in \mathcal{H}^2\left[\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T, \mathbb{Z}_2^f\right] = \mathbb{Z}_2^3 \tag{C.4}$$

In particular, the spin-1/2 fermions corresponding to the 2-cocycle ω_2 satisfying the following conditions:

$$\begin{cases} A^4 = P_f \\ \mathcal{T}^2 = P_f \\ \mathcal{T}A\mathcal{T}^{-1}A = 1 \end{cases} \tag{C.5}$$

To satisfy these conditions, we consider the 2-cocycle ω_2 as following. For $\forall a_g, b_h \in \mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$ defined as:

$$\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T = \left\{ (a, g) = a_g \middle| 0 \le a \le 3, 0 \le g \le 1 \right\} \tag{C.6}$$

we choose

$$\omega_2(a_g, b_h) = \left\lfloor \frac{\left[(-1)^{g+h}a\right]_{2n} + \left[(-1)^h b\right]_{2n}}{2n} \right\rfloor + (1 - \delta_a)(a+1)h + g \cdot h \tag{C.7}$$

where we define $[x]_n \equiv x \pmod{n}$ with $n = 2$, $\lfloor x \rfloor$ as the greatest integer less than or equal to x, and

$$\delta_a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases} \tag{C.8}$$

It is straightforward to check that A and $\mathcal{T}$ satisfy condition (C.5), hence A and $\mathcal{T}$ are generators of the symmetry group $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$ for spin-1/2 fermions. One can also calculate that the two invariants defined above are both trivial, namely

$$\begin{aligned} \mathcal{I}_1 &= (-1)^{\omega_2(A^2\mathcal{T}, A^2\mathcal{T})} = 1 \\ \mathcal{I}_2 &= (-1)^{\omega_2(A^3\mathcal{T}, A^3\mathcal{T})} = -1. \end{aligned} \tag{C.9}$$

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