Report

The manuscript formulates an effective field theory of a vortex lattice in a bosonic compressible superfluid using boson-vortex duality. Based on the proposed effective Lagrangian, the authors obtained the following results: 1) re-deriving the spectrum of the Tkachenko mode at long wavelengths; 2) suggesting the absence of the Hall viscosity at zero frequency and momentum in a vortex lattice.

Boson-vortex duality is a powerful tool to study vortex dynamics. As far as I am aware, the authors applied it, for the first time, to a vortex lattice configuration. As the authors presented, from the boson-vortex duality perspective, it is quite natural to write down an effective Lagrangian of a vortex lattice system following the gauge principle. More importantly the manuscript raises several very interesting questions, especially those related to the Hall viscosity, motivating the future research.

However I have several suggestions and comments that the authors may want to consider.

1. To make the connection between the proposed Lagrangian for a uniform system and trapped cold atomic gases in experiments more transparent, it could be useful to remind the readers that the system described by Eq.(1) can be realized when the condition $\Omega \simeq \omega_\perp$ is fulfilled. Here $\omega_\perp$ is the transverse trapping frequency.

2. As far as I understood, the proposed effective field theory is valid in the vortex lattice regime (the intervortex spacing is much larger than the vortex core size) but is not applicable to the quantum Hall regime. If it is indeed the case, this should be clarified more explicitly at the very beginning of the paper.

3. It could be helpful to add Anthony Zee's book "Quantum Field Theory in a Nutshell" as a reference about the boson-vortex duality.

4. In Eq.(1), since the current $j_s^\mu = \epsilon^{\mu\nu\rho}\partial_\nu a_\rho$ is neutral, the source field $\mathcal{A}_\mu$ should be a gauge field associated with an arbitrary external rotation, is it ? It could be helpful if the authors can clarify this.

5. On page 6, I don't quite understand the last sentence of the paragraph just below Eq.(5). The angular velocity $\mathbf{\Omega}$ should change sign under P or T operation, no ?

6. In the leading order theory, Galilean symmetry is broken. What are the consequences of this ? Is it a problem ? Explanations are needed here.

7. In the leading order theory, the Tkachenko mode that has quadratic dispersion relation is obtained. However it is also shown by the authors that to this order the vortex lattice is incompressible, namely $\partial_i u^i = 0$ (above Eq.(10)). Seems to me that there is an inconsistency here. In Ref.[15], the compressibility of the vortex lattice plays a crucial role to obtain the Tkachenko mode of frequency that is quadratic in k. Also, intuitively, how does an incompressible vortex lattice support soft collective modes (of course no edge physics is considered here) ?

8. Eq.(12) should hold for small k. However I could

not find this constraint/condition from the relevant derivations on page 8.

9. Could the proposed theory cover the physics in the incompressible limit ($c_s \to \infty$), where the Tkachenko frequency should be linear in k ?

10. Relevant references about the spin connection need to be added after the first sentence of section VI.

11. A general remark: as also mentioned by the authors in the introduction, if the rotation rate Ω is very high such that the number of vortices is comparable to the number of particles, the system enters a strong correlated vortex liquid phase which can be understood as a bosonic analogy of the fractional quantum Hall sate of electrons. I would expect that this bosonic analogy exhibits a finite Hall viscosity. If the Hall viscosity is zero in a vortex lattice, as claimed by this manuscript, it could be really interesting to see how the Hall viscosity emerges as increasing the rotation rate Ω if there is any.

Once the points above are properly addressed, the manuscript will be ready for publication in the journal SciPost physics.