Review: Reconstructing the gluon

In this work the authors propose a novel approach for investigating the structure of the Landau gauge gluon spectral function. By imposing a series of analytic consistency criteria, the authors select an appropriate functional basis which allows them to reconstruct the form of the spectral function given numerical data of the Euclidean propagator.

This is an interesting study, and it's particularly nice to see that the authors are attempting to develop a different numerical approach for extracting information about the spectral content of the gluon. However, there are a few issues which I feel require further clarification, in particular with regards to the analytic relation derived in Eq. (6) of the paper.

In Eq. (1) the authors write down the Euclidean spectral representation of the propagator. The representation they write down is correct except that the spectral function must depend on λ^2 and not just λ. In other words, the representation has the form: $G(p_0^2) = \int_0^\infty \frac{d\lambda}{\pi} \frac{\lambda\rho(\lambda^2)}{\lambda^2+p_0^2}$. Since $\rho(\lambda^2)$ is non-vanishing only for $\lambda^2 \in [0,\infty)$, Eq. (2) is rather confusing since the spectral function itself is not defined for negative values of λ^2. What exactly is meant by the spectral function being anti-symmetric? If one ignores this issue and takes the derivative with respect to p_0, it is argued that one has the relation

$$\partial_{p_0} G(p_0^2) = -\int_{-\infty}^{\infty} \frac{d\lambda}{\pi} \lambda p_0 \frac{\rho(\lambda^2)}{(\lambda^2+p_0^2)^2}$$

which after taking the limit $p_0 \to 0^+$ on both sides, and using the Poisson kernel representation of the Dirac delta, gives

$$\lim_{p_0\to 0^+} \partial_{p_0} G(p_0^2) = \frac{1}{2}\int_{-\infty}^{\infty} d\lambda\, \frac{d\delta(\lambda)}{d\lambda}\rho(\lambda^2) = -\frac{1}{2}\int_{-\infty}^{\infty} d\lambda\, \delta(\lambda)\frac{d\rho(\lambda^2)}{d\lambda} = -\frac{1}{2}\lim_{\omega\to 0^+} \partial_\omega \rho(\omega^2)$$

Now the dependence of ρ on ω^2 becomes important because the derivative is with respect to ω and not ω^2. By changing variables to $s = \omega^2$, the above relation takes the form

$$\lim_{p_0\to 0^+} \partial_{p_0} G(p_0^2) = -\lim_{s\to 0^+} \sqrt{s}\,\partial_s \rho(s)$$

Assuming that $\partial_s \rho(s)$ is well-behaved at $s = 0$, it follows that $\lim_{s\to 0^+} \sqrt{s}\partial_s\rho(s) = 0$, and therefore Eq. (6) no longer appears to provide a non-trivial connection between the asymptotic behaviour of the propagator and its corresponding spectral function. It may well be that the statements made above are invalidated under certain conditions on $\rho(s)$, and a non-trivial connection between the low-frequency spectral function and infra-red propagator does indeed exist, in which case the authors need to specify precisely what these conditions are, and whether they have any bearing on their results.

Here are some more general comments and optional points to take into consideration:

- In Eq. (11) it is assumed that the spectral function has a sequence of (complex) poles and a continuous contribution $\rho_A(\lambda)$. Is $\rho_A(\lambda)$ completely arbitrary, or are there certain conditions imposed on this component? For example, does $\rho_A(\lambda)$ vanish at $\lambda^2 = M_j^2$? There is also a typo in Eq. (11): the LHS should presumably read $G_A(p_0)$, not $G_A(\omega)$.
- It is stated that Eq. (20) follows from Eq. (6). As with Eq. (6), it would be appreciated if the authors could provide at least a sketch of this argument.
- In Fig. 6 the systematic error on the gluon spectral function is given. How exactly is this estimated? Also, given this error on the spectral function, is it possible to translate this into an error band on the reconstructed propagator? It would perhaps be interesting to see how this uncertainty translates.
- The authors plot the Schwinger function $\Delta(t)$ in Fig 8. As another check of superconvergence the authors might also consider plotting $\dot{\Delta}(t)$, which is proportional to the integral of the spectral function at $t = 0$ (see e.g. the discussion in section III of 1310.7897).

In summary, I find this study interesting but I cannot recommend publication in SciPost Physics until the specific points I've raised, in particular with regards to Eq. (6), have been addressed.