REPORT ON "RANK ONE HCIZ AT HIGH TEMP..." BY P. MERGNY AND M. POTTERS

The paper under review is concerned with the asymptotics of the β-HCIZ integral, $I_N^{(\beta)}(A,B) = I_N^{(\beta)}(\mathbf{a},\mathbf{b})$, in the case where one of the matrix inputs has rank one, or equivalently one of the vector inputs has only one nonzero entry. The motivation for this study is that in the regime where $N \to \infty$ and $\beta = \beta_N \sim \frac{2c}{N}$, which is the large N limit in the high temperature regime where the inverse temperature parameter β decays linearly in N, one arrives at a certain special function which points the way to a binary operation on probability measures which interpolates between classical convolution ($c = 0$) and free convolution ($c \gg 0$). However, as the paper explains, while the two classical convolutions take two probability measures as input and return a new probability measure as output, for the c-dependent convolution defined by the authors it remains unclear whether the output is in fact a positive measure, and the paper under review leaves this natural question unanswered. Nevertheless, the authors argue that, even as a purely formal construction, their c-convolution has value, as it highlights a new algebraic/structural scheme interpolating between classical and free convolution. I for one find this argument convincing.

The paper itself is horribly written, being riddled with issues of spelling and grammar, awkward and confusing diction, unfortunate typesetting, and an overarching inattentiveness to any and all issues of presentation and readability which may try the patience of many who would otherwise be eager to understand the authors' work. It seems that the authors did not even bother to run their work through a spell checker prior to submission. I am not sure what system the authors are using for their bibliography; references are listed neither in alphabetical order by author (the convention in mathematics), nor in the order in which they appear in the body of the paper (the convention in physics), nor in chronological order of publication. It looks like reference [1] might simply be the authors' favorite among the works they cite. All of these irregularities compound, making the article harder to follow than most papers, since it is so far below the usual standard of readability. Nevertheless, if the reader is willing to overlook this assault on her or his senses, the mathematical content of the article is really quite interesting, and ultimately this takes precedence over issues of presentation. The authors have included many detailed calculations in their paper, so that if one focuses primarily on the formulas and is willing to do the extra work of puzzling out how these computations are connected to one another, the article is elucidating and worthwhile, and one can learn something new and valuable from it. To the best of my understanding, all computations in the paper are complete and correct, and the numerical simulations and examples included are helpful in guiding intuition.

In summary, I think that the questions of stability and positivity of this new c-convolution, as well as its intriguing relationship to other nonstandard convolutions such as finite free convolution, are likely to attract the attention of many researchers, and inspire future work and subsequent developments. Therefore, I think that the

paper under review makes an important, original, and useful contribution to the study of generalized convolutions, and **I recommend the paper under review for publication in SciPost.**