A. Overview and general comment

Two quantum integrable models are considered: the quasi-periodic (twist ϕ) spin-1/2 XXZ chain, and its spin-1 generalization - the Zamolodchikov-Fateev model. For root of unity values of the deformation parameter q, it is conjectured that the Hamiltonian enjoys a hidden non-Abelian symmetry associated with (a quotient of, strictly speaking) the Onsager algebra. The conjectures are proposed based on the construction of transfer matrices for a semi-cyclic auxiliary space representation following [29].

This article contains promising results, extending further Vernier-O'Brien-Fendley recent results [15]. In particular, it suggests to investigate above models from the point of view of the Onsager algebra. This might open the possibility of investigating correlation functions and out-of-equilibrium dynamics in these models, see e.g. [16]. For these reasons, it deserves to be published in SciPost Physics after some revisions.

The revisions mainly aim to improve the presentation and clarify few points. Details are given below.

Nota bene: Strictly speaking, the paper deals with different quotients of the Onsager algebra. For what is concerned in the paper, this point is not considered as essential. However, it may become for a deeper analysis of the model considered (see e.g. reference [R0] below).

B. Suggestions

0. In the literature, the Onsager algebra O admits different types of presentations through generators and relations. The historical one [1] is given in terms of generators A_m,G_n with relations (2.1). Another presentation by Dolan-Grady [8] is given in terms of generators A_0,A_1 with (2.3). Now, in [15] a new algebra with relations (2.6)-(2.9) is introduced, with generators $A_m^{0},A_m^{\pm}$. To avoid any confusion, let's call this algebra O'. Following [15], the author gives an embedding $O \subset O'$ in (2.5). Thus, O may be - from this information only - a subalgebra of O', not necessarely equivalent to O. However, as written in the text of the present form, it is not clear for the reader how O' and O are related: are they isomorphic? is one a non-trivial subalgebra of the other? Clarifying that is not just a mathematical question: it is essential, as the key results in further sections are various representations for which $[H,O']=0$. Indeed, for instance without further clarification it may happen that there is no map $O' \rightarrow O$ (O may be smaller than O'). In that case, the real symmetry (characterizing the fine structure of the model) would be O', not O. Then, by (2.5) $[H,O]=0$ would not be essential. But that would contradict the claim that O is the most interesting symmetry of the model. To clarify this issue, one needs to check that there exists an inverse map $O' \rightarrow O$. Looking at [15], this map can be

constructed from the relations below (1.2) of [15]: each generator of O' can be written solely in terms of A_0,A_1 of O.

According to previous comments, O' and O (provided (2.5) holds) are indeed equivalent. So, as this point is crucial although no comments are given in the present form of the paper, I suggest the author to improve the sentence

From (2.1), it is easy to observe...

above (2.13) as follows (for instance):

`From [15], it is known that all generators $A_m^{0},A_m^{\pm}$ of (2.6)-(2.9) can be written as polynomials in A_0,A_1. Thus, from

$$[H,A_m^r]=0 \ , \ r \in \{0,+,-\}, \ m \in {\mathbb Z}\ ,$$

and (2.5) one finds (2.12).'

1. Below (3.5), the notation ${\bf L}^{\textsf{sc}}_{aj}(u,s,\beta)$ is introduced. Please define s (later on it is mentionned below (3.9), but that should be done below (3.5)). Also, to be self-contained and because ${\bf L}^{\textsf{sc}}_{aj}(u,s,\beta)$ plays a crucial role in the following analysis, the definition of ${\bf L}^{\textsf{sc}}_{aj}(u,s,\beta)$ should be clearly stated/improved. Also, it is written `The transfer matrix is therefore denoted as ${\bf L}^{\textsf{sc}}$. I think it should be `The Lax operator is therefore denoted as $L^{\textsf{sc}}_{aj}(u,s,\beta)$'.

2. Below (3.5), it is said that semi-cyclic representations of $U_q(sl_2)$ are condidered. Please add a precise reference (ref with eqs. number) where they are described explicitly. What are the expressions of ${\bf K}_a,{\bf S}^\pm_a$ in this case? Please add it somewhere in the text.

3. Above (3.6), it is written 'As proven in [29],...'. However, in [29] I see (3.3) but can't find a proof of the claim. So the sentence should be modified. If it is not proven in the paper, a reference for the proof should be given.

4. The proof that (3.7) solves (3.6) is not given. It is expected to be a corollary of (3.6), but for a non-expert reader, a reference is welcome.

5. As a corollary of (3.6), it is expected that (3.8) are mutually commuting for arbitrary values of u. For a non-expert reader, a sentence and a reference are welcome.

6. In (3.13), ${\bf T}_s(u,\phi)$ is introduced without definition. How is it related with ${\bf T}^{sc}_s(u,\beta,\phi)$? ${\bf T}_s(u,\phi)$ should be clearly defined below (3.8).

7. Above (3.13)-(3.14), it is claimed that both relations hold. No assumptions on the parameter ϕ are specified. However, top of page 7 it is written 'Note that ..Y charges satisfy (3.13)-(3.14) only when the twist ϕ is commensurate'. That is confusing. If (3.13)-(3.14) holds only for ϕ commensurate, this sentence top of page 7 should be right above (3.13).

8. In the literature (physics and maths), the terminology `Onsager generators' is standard, and always refers to $A_m,G_n$. In the paper, the author sometimes used the term `Onsager generators' for A_m,G_n, but also for the new generators $A_m^{0},A_m^{\pm}$. Below (2.9), I suggest either to add a sentence explaining that in the text, the terminology `Onsager generators' is also used for $A_m^{0},A_m^{\pm}$ (there may be still some readers for whom that will remain anyway confusing), or to introduce the terminology `Onsager type generators' for $A_m^{0},A_m^{\pm}$.

C. Additional references suggested, typos, cosmetic changes

1. Introduction: "Later Onsager has been used" $\rightarrow$ "Later the Onsager algebra has been used".

2. In the article, the Onsager algebra is generated from studying transfer matrices associated with RLL quantum Yang-Baxter algebras. In the literature, it has been shown recently that the Onsager algebra (and generalizations of [50]) arises from classical non-standard Yang-Baxter algebras [R1]. For completeness, it may be helpful to complete the sentence in the Introduction: "A thorough and comprehensive summary...[14]" by:

"Furthermore, recently an isomorphism between the Onsager algebra and a non-standard classical Yang-Baxter algebra is obtained [R1]".

3. In the text, please replace when appropriate "Onsager algebra" $\rightarrow$ "the Onsager algebra" (ex: beginning of section 2); Below (2.2): "the Dolan-Grady (DG) relation" $\rightarrow$ "the Dolan-Grady (DG) relations" (indeed, one has two relations).

4. In Figure 1: to be consistent with previous notations: "${\bf T}^{sc}$" $\rightarrow$"${\bf T}_a^{sc}(u,\beta,\phi)$".

5. First line of Section 4. It is said "It is well-known that XX model ...possesses Onsager algebra symmetry." Please add a reference.

6. In Section 7, it is written "Despite the credibility of the conjectures, it would be interesting to prove them using quantum integrability".

Actually, it may happen that part of the analysis in the author's paper (and of [29]) share some similarity with the analysis and proofs in a series of papers of Shi-shyr Roan between 2006 and 2012. For instance, see reference [R2] below.

If relevant, a comment may be added and adding few references would make sense.

7. About the last sentence of Section 7. Actually, generalizations of Onsager algebra were first introduced by Uglov-Ivanov (A-type) in [R3], and Date-Usami (D-type) [R4]. I would recommend to add [R3,R4] together with [50]. It makes sense, not only for historical purpose. Indeed, SciPost is a physics journal, so connections between generalized Onsager algebras and integrable models - as pointed out in [R3] - would be helpful to the reader.

8. Typo in Ref. [34]: "...$ofU(qsl2)in$..." $\rightarrow$ "of ...$U_q(sl_2)$... in".

D. References

[R0] B. Davies, Onsager's algebra and superintegrabilityDOI:10.1088/0305-4470/23/12/010Corpus ID: 119898494

[R1] Baseilhac, P., Belliard, S. Cramp\'e, N. FRT presentation of the Onsager algebras. Lett Math Phys 108, 2189–2212 (2018). https://doi.org/10.1007/s11005-018-1068-x

[R2] S-s Roan, The Transfer Matrix of Superintegrable Chiral Potts Model as the Q-operator of Root-of-unity XXZ Chain with Cyclic Representation of $U_q(sl_2)$, J.Stat.Mech.0709:P09021,2007. doi 10.1088/1742-5468/2007/09/P09021

[R3] D.B. Uglov, I.T. Ivanov, $sl(N)$ Onsager's algebra and integrability, J. Stat. Phys. 82 (1996), 87–113.

[R4] E. Date, K. Usami, On an analog of the Onsager algebra of type $D_n^{(1)}$. In: "Kac-Moody Lie algebras and related topics", 43–51, Contemp. Math., 343, Amer. Math. Soc., Providence, RI, 2004.