

The article “Mirror Symmetry for Five-Parameter Hulek-Verrill Manifolds” by Candelas, de la Ossa, Kuusela and McGovern gives an in-depth analysis of a highly symmetric type of five-parameter Calabi-Yau threefolds. After a brief introduction, a section is dedicated to the toric construction of the Calabi-Yaus of interest and their mirrors, including a discussion of singular vs smooth geometries and the discrete symmetries that play a central role in the analysis. The next section is concerned with methods to determine the periods and the Picard-Fuchs system. Due to the complexity of the problem, the authors obtain the relevant information by studying certain subloci of the moduli space and by making use of the symmetries which makes it possible to extract the relevant information without having the full Picard-Fuchs system. The information is used to compute genus zero and genus 1 instanton numbers via mirror symmetry. The authors then discuss patterns in these invariants that they name ‘duality webs’ which encode a large symmetry group. The authors formulate several conjectures about the structure of these webs. The next step is to analyse monodromies. The computation again becomes tractable thanks to the highly symmetric nature of the setup. Finally, the authors check the correctness of the invariants by direct counting of rational curves. The explicit results are collected in a very large appendix that makes up almost half of the volume of the paper.

The article is very well written. The content is extremely technical but nevertheless presented in a way that makes it relatively easy to follow.

While the objects that are computed have been studied many times before in the context of mirror symmetry (periods, Picard-Fuchs systems, genus zero and genus one invariants) the complexity of the model requires the use of clever tricks and the symmetry of the system to make calculations feasible. The paper showcases that, while many of the standard methods of mirror symmetry still apply in principle, further tools are needed to tackle the necessary calculations. While the approach used in this article is very much tailored to the problem and it is unclear to what extent this is applicable to other examples, the general ideas should be applicable more broadly.

The article also demonstrates that there is a lot to be discovered about Calabi-Yaus with more than one or two moduli. While this is known to the experts, this does not seem to be widely appreciated, in part due to the highly technical nature of the problem. Thanks to the symmetry of the example, the authors discover an interesting mathematical structure that is reflected in the invariants and leads them to conjectures that may be an interesting task for mathematicians to prove. Since many examples of multi-parameter Calabi-Yaus have additional structures such as discrete symmetries or fibration structures, the interplay between this and the enumerative invariants may lead to further interesting new discoveries and methods for explicit calculation of invariants.

There are connections to recent findings such as new results in the context of the attractor mechanisms and the associated arithmetic, pioneered by the authors and collaborators, or the connection between Calabi-Yaus and certain Feynman integrals. This is further motivation to consider mirror symmetry for more exotic models such as the one presented in the article.

In summary, the article showcases the power of mirror symmetry in a highly non-trivial example, discovers new interesting mathematical structures, and outlines new methods to deal with Calabi-Yaus that have multiple moduli and a large amount of symmetry. Only the following minor modifications are suggested:

1. Reference [14] has incomplete information.

2. In Section 2 a specific triangulations are chosen. The authors argue that the triangulations they consider are compatible with the symmetries, but it is not fully clear if this reduces the number of triangulations to a unique one. Above eq. (2.6) two triangulations are mentioned. Around eq. (2.9) it is only mentioned that a triangulation compatible with the symmetries is chosen. A few further comments as to whether all triangulations are equivalent or if there are further inequivalent ones would be appreciated.
3. There is a typo “genu-1” in the second line of p.49.
4. While the necessity of an outlook section in scientific articles may be disputable, it would be very interesting have one for this article, and to have the authors’ opinion on questions like: feasibility of computing invariants of $g > 1$ and expectations concerning web structures as introduced in the article, applicability of the methods used in the paper to other examples, how/if the results of the paper can be used concretely to gain new insights into attractors and arithmetic on one hand and certain types of amplitudes on the other. The addition of a short conclusions section addressing such potential directions for further research is suggested.

Publication of the article, including the impressive appendix, is recommended after implementation of the minor revisions, as suggested.