Report on the paper "Mirror Symmetry for Five-Parameter Hulek-Verrill Manifolds" by Philip Candelas, Xenia de la Ossa, Pyry Kuusela, and Joseph McGovern

Hulek and Verrill studied a certain family of Calabi-Yau manifolds which comes from Laurent polynomials with its Newton polytope being given by the root lattice A_4. A general member of this family is a smooth Calabi-Yau threefold, and is called a Hulek-Verrill manifold. In this paper, by finding a toric description of the family and applying the Batyrev-Borisov mirror construction to the family, the authors describe mirror symmetry of Hulek-Verrill manifolds to the complete intersection Calabi-Yau manifolds $X_{1^5,1^5} := (1,1,1,1,1) \cap (1,1,1,1,1)$ in $(\mathbb{P}^1)^5$. As an application of mirror symmetry, the authors calculate Gromov-Witten invariants of genus zero and one, and find a certain infinite symmetry among these invariants which originates from the birational geometry of $X_{1^5,1^5}$, in particular, the birational automorphism group of $X_{1^5,1^5}$. The appearance of such symmetry is known in a previous work (the reference [31] of this paper) where mirror symmetry of similar complete intersection Calabi-Yau manifolds was studied. To calculate Gromov-Witten invariants, Picard-Fuchs differential equations, period integrals, and analytic continuations of the period integrals are also studied in detail.

In general, most calculations related to mirror symmetry become formidable if Calabi-Yau manifolds have deformation spaces of large dimensions. Since this paper adds a good example of Calabi-Yau manifolds to a list of interesting and "calculable" Calabi-Yau manifolds of large deformation spaces, I would recommend publishing this if the following points (1) and (2) are clarified or addressed appropriately.

(1) In the reference [1], Hulek and Verrill introduced the Hulek-Verrill Calabi-Yau threefolds by making small resolutions of $X_{\mathbf{a}} \subset \tilde{P}$ with general parameters $\mathbf{a} = [a_1, \cdots, a_5, a_6]$, which have 30 nodal singularities in $X_{\mathbf{a}} \setminus (\mathbb{C}^*)^4$. Precisely, the small resolutions $\overline{X}_{\mathbf{a}}$ are obtained by blowing $X_{\mathbf{a}}$ up along certain surfaces $S^{ij}_{\mathbf{a}} \subset X_{\mathbf{a}} (\subset \tilde{P})$. In the present paper under review, the toric variety $\tilde{P}$ is given by a MPCP (maximally projective crepant partial) resolution of the toric variety $\mathbb{P}_{\hat{\Delta}^*}$

In Sect.2.3 of this paper, it seems that the authors claim that the small resolution $\overline{X}_{\mathbf{a}}$ above is obtained by Batyrev-Borisov toric construction as the complete intersection $HV_{\mathbf{a}} := \{P_{\Delta_1} = P_{\Delta_2} = 0\} \subset \mathbb{P}_{\Delta^*}$ (precisely the complete intersection in a MCPC resolution of $\mathbb{P}_{\Delta^*}$) by finding a five dimensional polytope which admits the decomposition $\Delta = \Delta_1 + \Delta_2$. It is not clear, however, from the description in Sect.2.3 why $\overline{X}_{\mathbf{a}}$ and $HV_{\mathbf{a}}$ are isomorphic (or bi-holomorphic). Of course, these two are smooth Calabi-Yau threefolds which are birational as described in the text; and this fact of being birational might be sufficient to study mirror symmetry. However, if the authors claim the isomorphism $\overline{X}_{\mathbf{a}} \simeq HV_{\mathbf{a}}$, then a proof of it should be provided.

(2) If we replace the Hulek-Verrill manifold $\overline{X}_{\mathbf{a}}$ by the above toric description $HV_{\mathbf{a}}$, then it is straightforward to apply the Batyrev-Borisov mirror construction to it. Also, it should be noted and addressed to the reader that the framework given in [25] based on the Gel'fand-Kapranov-Zelevinski (GKZ) hypergeometric series directly applies to this case to calculate Gromov-Witten invariants of genus zero and one. For example, the period integral (3.5) is exactly a typical GKZ hypergeometric series studied in general in [25]. As described by the selected examples in [25], the Picard-Fuchs differential operators which determine the period integrals are given by an irreducible part of the GKZ system, which we obtain by finding suitable factorizations of the GKZ differential operators.

Below are minor points which I noticed:

1. p.18,l-3. "a desingularization $\mathbb{P}_{\Delta^*}$" should be "a partial desingularization $\mathbb{P}_{\Delta^*}$".
2. p.19,l+1. "dim$\mathcal{M} \geq 3$" should be "dim$\mathcal{M} \leq 3$"
3. p.25,l-6. "Thus we identify ... resolutions of the singular manifolds $\widehat{\mathrm{HV}}$." As describe in (1) above, it is not clear why this complete intersection gives a "resolution" of the singular manifolds $\widehat{\mathrm{HV}}$.