REPORT ON THE PAPER "ANTAGONISTIC INTERACTIONS CAN STABILIZE FIXED POINTS IN RANDOMLY COUPLED, LINEAR DYNAMICAL SYSTEMS WITH INHOMOGENEOUS GROWTH RATES", BY CURÉ AND NERI

In this paper, the authors study the stability of a system of randomly coupled linear equations

$$\partial_t x_j(t) = \sum_{i=1}^{n} A_{jk} x_k(t) \ ,$$

where matrix A is a large random non-hermitian matrix whose entries are given by

$$A_{jk} = \frac{J_{jk}}{\sqrt{n}}(1-\delta_{j,k}) + D_j \delta_{j,k} \, .$$

Here $J = (J_{jk})$ is an elliptic matrix whose off-diagonal entries J_{jk} and J_{kj} have a prescribed correlation $\tau \in [-1,1]$ and $D = (D_j)$ is a random diagonal matrix.

For such a matrix A, it is theoretically possible to describe the limiting spectrum (by the cavity method for example), and thus to locate the dominant eigenvalue (eigenvalue with the largest real part) which drives the stability of the linear system: if the dominant eigenvalue has a negative real part, then the system is stable, and unstable otherwise.

The main contribution of the paper is to analyze the sensitivity of the dominant eigenvalue with respect to the correlation τ (and the standard deviation σ of the entries) of the interaction matrix J. Various distributions for matrix D are considered, some of them leading to rather explicit formulas. Among the interesting results provided in the paper: (1) the system might be stable even if the support of the distribution of D's entries intersects $\mathbb{R}^+$; (2) for a negative correlation τ, increasing the s-d. σ may stabilize the system.

Overall the paper is interesting with non-trivial conclusions and nice simulations. The writing of Part 3 (exposition of the cavity method) may be improved for, as written, it does not completely unveil the mystery around the cavity method for newcomers. But maybe this mystery is inherent to the cavity method...

I am willing to support the publication of this paper if the writing is improved along the lines of this report.

(1) The title and abstract are over precise. Please shorten the title and simplify the abstract.
(2) page 3, line 23: "in the model given by Eq. (2), ..." I do not understand this sentence: what does "in isolation" means?
(3) p3, l8 from bottom: to weak $\rightarrow$ too weak
(4) p3, l4 fb: "cavity method which is the mathematical method": I do not want to be fussy about terminology but it rather seems to me that cavity method is a highly efficient methods from theoretical physics to derive limiting spectral distributions for non-hermitian models, without bothering with the math details (such as the highly difficult control of the smallest sigular value, etc.). Anyway, this is just a remark.
(5) p4, l9-13: it is a bit misleading to write that the leading eigenvalue only depends on the moments of (J_{ij}, J_{ji}), and then to write that it depends on the distribution p_D. Please fix.
(6) p4,l9 fb: "finite set" the uniform distribution is supported on a bounded or compact set. But the set $[d^-, d^+]$ is not finite.
(7) p4, last sentence: "when p_D has finite support on positive values $d > 0$". Unclear, please fix.
(8) p6, correct typos: correctiosn - the he - two typeS

(9) p8, equation (22): I do not understand the status of $\delta(y)$ on the r.h.s. Also it will be nice to precise where in [27] we can find this formula.
(10) p10, l8 fb: the the boundary $\to$ the boundary. Please fix.
(11) p12, caption of Fig 4: Is really $\tau = 1$ in the lowest curve? It seems to me that it should be $\tau = -1$. In this case, the system is always (asymptotically) stabilized. It $\tau > -1$, then as some point $\sigma > \sigma^*(D)$, the system becomes unstable. The limiting case $\tau = -1$ should probably correspond to a vertical spectrum centered on the mean of p_D? Am I delusional? Anyhow, the third curve (diamonds) should be more commented.
(12) bibliography: you may want to cite
    - the paper "Properties of networks with partially structured and partially random connectivity" by Ahmadian, Fumarolla, Miller, Phys. Rev. E (2015), about non-hemitian matrix models for neuroscience applications,
    - the paper "Positive Solutions for Large Random Linear systems, Proc. A.M.S., 2021", by P. Bizeul, J. Najim, which describes feasibility and stability for large Lotka-Volterra systems.

    Beside these papers devoted to applications, many papers on non-hermitian matrices (from a math merspective) may deserve to be cited:
    - "Low rank perturbations of large elliptic random matrices" by Sean O'Rourke, David Renfrew , EJP 2014. on the elliptic law,
    - V. L. Girko, Elliptic law, Theory of Probability and Its Applications, Vol. 30, No. 4 (1985). landmark paper on elliptic law
    - "Non-Hermitian random matrices with a variance profile (I): deterministic equivalents and limiting ESDs" by Cook et al. Electron. J. Probab. 23: 1-61 (2018)