Supplemental material for report on Ref. [1]

In this note, we aim at analytically investigate the relation between the OTOC fluctuations and the stabilizer Rényi entropy outlined in Ref. [1]. Let us give some quick definitions:

$$\mathrm{OTOC}(U)=\frac{1}{d}\,\mathrm{tr}(U^{\dagger}AUBU^{\dagger}AUB) \tag{1}$$

where A,B are two non-identity Pauli operators such that $[A,B]=0$ (i.e. the setup discussed in Ref. [1]). Let $U=C_1VC_2$, where V is a generic unitary operator. Define the average over C_1,C_2 as

$$\mathbb{E}_C\mathrm{OTOC}(U):=\int \mathrm{d}C_1\mathrm{d}C_2\mathrm{OTOC}(U) \tag{2}$$

and define the flucutations around the average

$$\delta_{\mathrm{OTOC}}(U):=\mathbb{E}_C\mathrm{OTOC}^2(U)-[\mathbb{E}_C\mathrm{OTOC}(U)]^2 \tag{3}$$

Lemma 1. *Let $M_2(|V\rangle)$ the stabilizer entropy of the Choi state [2] $|V\rangle$ associated to V, see Ref. [3] for the definition, then*

$$\mathbb{E}_C\delta_{OTOC}(U)=\left(\frac{d^2}{d^2-1}\right)^2\times 2^{-M_2(|V\rangle)}-2\frac{d^2}{(d^2-1)^2} \tag{4}$$

Proof. See Sec. I. □

Remark 1. *Notice that the above equation can be employed to draw conclusions regarding the efficiency of measuring $M_2(|V\rangle)$ through OTOC fluctuations.*

Thanks to the above equation, we can analytically compute the average δ_{OTOC} for t-doped Clifford circuits and compare the numerics done in Ref. [1] with the actual analytic behavior.

Corollary 1. *The average over random t-doped Clifford circuits V hereby denoted as $\mathbb{E}_{C_t}$, the one investigated in Ref. [1], reads*

$$\begin{aligned}\mathbb{E}_{C_t}\delta_{OTOC}(C_t) &= \frac{d^4}{(d^2-1)^2}\Big[\frac{4(6-9d^2+d^4)}{d^4(d^2-9)} \\ &+ \frac{(d^2-1)}{d^2}\left(\frac{(d+2)(d+4)f_+^t}{6d(d+3)}+\frac{(d-2)(d-4)f_-^t}{6d(d-3)}+2\frac{(d^2-4)\left(\frac{(f_++f_-)}{2}\right)^t}{3d^2}\right)\Big] \\ &- 2\frac{d^2}{(d^2-1)^2}\end{aligned} \tag{5}$$

where $f_\pm=\frac{3d^2\mp 3d-4}{4(d^2-1)}$. In the large d limit it reads

$$\mathbb{E}_{C_t}\delta_{OTOC}(C_t)=\left(\frac{3}{4}\right)^t+O(d^{-2}) \tag{6}$$

Proof. See Eq. (C26) of Ref. [3]. □

In Ref. [4], the average value of stabilizer entropy over t-doped Clifford circuits is (see Eq.(13) of Ref. [4])

$$-\log\left(\frac{4+(d-1)f_{+}^{t}}{3+d}\right) \leq \mathbb{E}_{C_t} M_2(C_t\left|0\right\rangle) \leq \begin{cases} t, & t<n-1 \\ n-1 \end{cases} . \tag{7}$$

i.e. for $t \ll n$ it grows linearly with t which is different from the exponential decay in Eq. (5).

Now, let us put the actual number used by the authors of Ref. [1], i.e. $d = 2^{12}$ and $t \in [0, 26]$ we obtain the following plot

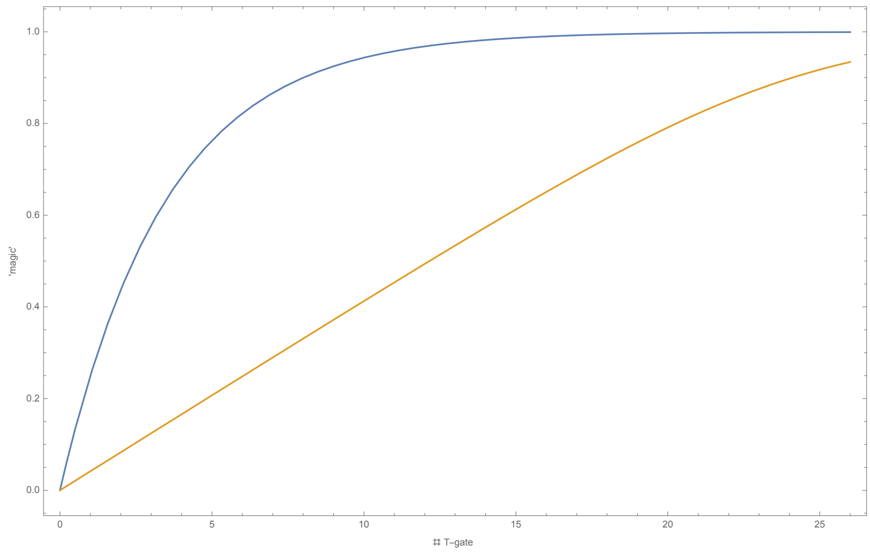

Figure 1: Plot of Eq. (5) as a function of the number t of T-gates that is on the x axis. On the y axis the blue curve corresponds to the plot of Eq. (5), i.e. $1 - \delta_{\text{OTOC}}$, and the orange curve correspond to the lower bound of Eq. (7) normalized to the Haar value, i.e. $\log_2(d+3) - 2$.

Remark 2. *Notice that in Eq. (4), it appears the 2-stabilizer entropy of the Choi state* $\left|V\right\rangle$ *associated to the unitary* V *and not the second stabilizer entropy* $M_2(V\left|0\right\rangle)$*. This is another issue (although minor) with the authors work. Indeed notice that* $M_2(V\left|0\right\rangle)$ *is inherently different from* $M_2(\left|V\right\rangle)$ *because* $\left|V\right\rangle$ *is a ket defined on two copies of the Hilbert space* $\mathcal{H}$ *via the Choi isomorphism*

$$\mathcal{U}(\mathcal{H}) \ni V \mapsto \left|V\right\rangle \equiv \mathbb{1} \otimes V \left|I\right\rangle \in \mathcal{H}^{\otimes 2} \tag{8}$$

where $\mathcal{U}(\mathcal{H})$ *is the unitary group defined on* $\mathcal{H}$ *and* $\left|I\right\rangle \equiv 2^{-n/2}\sum_{i=1}^{2^n}\left|i\right\rangle \otimes \left|i\right\rangle \in \mathcal{H}^{\otimes 2}$*. However, for* V *being a random* t*-doped Clifford circuit it holds that*

$$\mathbb{E}_{C_t} 2^{-M_2(\left|C_t\right\rangle)} = \mathbb{E}_{C_t} 2^{-M_2(C_t\left|0\right\rangle)} + O(d^{-1}) \tag{9}$$

where the proof is straighforward and comes from Eq. (5) and Eq. (7). Therefore, in the case of a t*-doped Clifford circuit, there is no distinction between the stabilizer entropy of* $V\left|0\right\rangle$ *and* $\left|V\right\rangle$*, rendering the previously mentioned issue of minor importance.*

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[2] M.-D. Choi, Linear Algebra and its Applications **10**, 285 (1975).
[3] L. Leone, S. F. E. Oliviero, and A. Hamma, Phys. Rev. A **107**, 022429 (2023).
[4] L. Leone, S. F. E. Oliviero, and A. Hamma, Phys. Rev. Lett. **128**, 050402 (2022).
[5] L. Leone, S. F. E. Oliviero, Y. Zhou, and A. Hamma, Quantum **5**, 453 (2021).
[6] D. A. Roberts and B. Yoshida, JHEP **2017** (4), 121.

I. PROOF OF LEMMA 1

Proof. Let us start by looking at the first term of δ_{OTOC}. By simple algebra, one can obtain

$$\mathbb{E}_C \text{OTOC}^2(U) = \int \mathrm{d}C_1 \mathrm{d}C_2 \frac{1}{d^2} \operatorname{tr}\left(T_{(12)(34)} V^{\dagger\otimes 4} C_1^{\dagger\otimes 4} A^{\otimes 4} C_1^{\otimes 4} V^{\otimes 4} C_2^{\otimes 4} B^{\otimes 4} C_2^{\dagger\otimes 4}\right) \tag{10}$$

The average over C_1 reads

$$\int \mathrm{d}C_1 C_1^{\dagger\otimes 4} A^{\otimes 4} C_1^{\otimes 4} = \frac{1}{d^2-1} \sum_{P\in\mathbb{P}_n\setminus\{\mathbb{1}\}} P^{\otimes 4} \tag{11}$$

since the average over Clifford circuits returns a flat distribution over the Pauli group $\mathbb{P}_n$ but the identity. Defining $Q := d^{-2}\sum_{P\in\mathbb{P}_n} P^{\otimes 4}$, one can rewrite the above average as

$$\int \mathrm{d}C_1 C_1^{\dagger\otimes 4} A^{\otimes 4} C_1^{\otimes 4} = \frac{d^2}{d^2-1} Q - \frac{1}{d^2-1} \mathbb{1}^{\otimes 4} \tag{12}$$

one obtains the analogous result for the average over C_2 on the non-identity Pauli operator B. Plugging everything into Eq. (10) one obtains

$$\mathbb{E}_C \text{OTOC}^2(U) = \frac{1}{d^2}\left(\frac{d^2}{d^2-1}\right)^2 \operatorname{tr}(Q V^{\otimes 4} Q V^{\dagger\otimes 4}) - \frac{2d^2-1}{(d^2-1)^2} \tag{13}$$

where we used the fact that $\operatorname{tr}(\mathcal{O} Q T_{(12)(34)}) = \operatorname{tr}(\mathcal{O} Q)$ for every $\mathcal{O}$ (see Eq. (130) in Ref. [5]). The average $\mathbb{E}_C \text{OTOC}(U)$ is straightforward because Clifford is a 2-design and thus $\mathbb{E}_C \text{OTOC}(U) = -(d^2-1)^{-1}$ (see Eq. (119) in Ref. [6]). We finally obtain

$$\delta_{\text{OTOC}} = \left(\frac{d^2}{d^2-1}\right)^2 \times 2^{-M_2(|V\rangle)} - 2\frac{d^2}{(d^2-1)^2} \tag{14}$$

where we defined (see Ref. [3]) the stabilizer 2-Rényi entropy of the Choi state [2] of the unitary V as

$$M_2(|V\rangle) = -\log \frac{1}{d^2} \sum_{P,P'} \operatorname{tr}^4(P V P' V^{\dagger}) = -\log \frac{1}{d^2} \operatorname{tr}(Q V^{\otimes 4} Q V^{\dagger\otimes 4}) \tag{15}$$

□