Report on the manuscript "Universality in Anderson localization on random graphs with varying connectivity" by Piotr Sierant, Maciej Lewenstein and Antonello Scardicchio

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The submitted article deals with the Anderson transition on random graphs, a subject that has attracted much attention recently due to its analogy with the MBL transition. The article follows an important debate on the nature of the delocalized phase of this transition, namely whether it is non-ergodic, i.e. multifractal, or not. After a number of studies, there is now a consensus on the ergodic nature of the delocalized phase on random graphs of infinite effective dimensionality, without boundary but with loops. The result of this debate has been the interesting discovery of ensembles of random matrices having a non-ergodic delocalized phase, as well as the finite size Cayley tree. Now that this question has gained consensus, the question arises as to why it is so difficult to answer the ergodic–non-ergodic question. The authors, based on finite-size scaling of different models of random graphs, propose that in the delocalized phase there is not one but two characteristic lengths, associated with two different critical exponents. The delocalization transition would be controlled by a first length diverging algebraically with an exponent $\nu = 1/2$, while the ergodic behavior would be reached beyond another length associated with an exponent $\nu = 1$. As the second length diverges faster than the first, this would account for a very large (diverging at the transition) range of system sizes where the behavior would be de facto non-ergodic.

This study challenges the theoretical approach of Fyodorov and Mirlin and Tikhonov and Mirlin, for this transition, which predicts a unique $1/2$ exponent. This unique $1/2$ exponent has been numerically verified in a number of studies. The interest of the present study is that it explains why some of these studies observe this exponent $1/2$ when in fact there is, according to the authors, another exponent 1. Moreover, they explain how to generalize the determination of the critical disorder W_c for the Bethe lattice and random regular graphs, to the different random graph models they consider. This is often the main source of uncertainty on the critical properties so that this is a key point of their study. The authors seem to find the same type of critical behavior in the different types of random graphs they consider, therefore their results seem to be universal. All these points make this study a priori interesting.

I say a priori because unfortunately I disagree with a number of analysis carried out in this work which lead me to think that the proposed new exponent $\nu = 1$ is not well demonstrated. To be more precise, the authors having kindly shared their data with me, I will show later in this report that their data are compatible with a single critical exponent $\nu = 1/2$. It should be noted, at this stage, that their work directly contradicts several papers we published (Refs. [80,81] of the manuscript) or submitted (arXiv:2209.04337) recently. They use the same type of scaling approach, some observables we considered, and even the same smallworld model. It is therefore important to compare the two analyses, so as to see which of the two is right (or at least more accurate).

The authors mainly consider a very popular, although rather imprecise, observable of localization: the average gap ratio, which I note here $\langle r \rangle$ (they denote it $\overline{r}$). This observable tends towards a value $r_P \approx 0.386$ in the localized phase and towards $r_{GOE} \approx 0.53$ in the ergodic phase.

a. Ergodic crossover scale $W^T(L)$.– The proof by the authors of the new critical exponent $\nu = 1$ consists of two steps: first they study the behavior of the disorder $W^T(L)$ below which a system of size L has a $\langle r \rangle > r_{GOE} - p_r$ where p_r is a small threshold, see figures 5-7. $W^T(L)$ controls the crossover to the ergodic ergodic regime. The authors claim that their data follow the following behavior at large L: $W^T(L) - W^T_\infty \sim 1/L$ with $W^T_\infty \approx W_c$ where W_c is the critical disorder they determine independently. To understand what this could mean, it is better to consider the corresponding characteristic size $L^T(W)$ (inverse of $W_T(L)$): for $L > L^T(W)$, a system of size L has an ergodic value $\langle r \rangle = r_{GOE}$. If $W^T(L) - W^T_\infty \sim 1/L$ with $W^T_\infty = W_c$, then $L_T(W) \sim (W_c - W)^{-1}$. This is one indication of a critical exponent $\nu = 1$ controlling the crossover to the ergodic regime.

However, I will propose below an alternative analysis of the data in this regime which supports instead a critical exponent $\nu = 1/2$. The starting point is the observation that $\langle r \rangle$ close to r_{GOE} follows a volumic and not a linear scaling (the fact that these two types of scaling are distinct and play an important role was shown in [80,81]):

$$\langle r \rangle = \mathcal{F}(N/\Lambda(W)) \,, \qquad (1)$$

where N is the total number of sites and $\Lambda(W)$ is the correlation volume. We predicted this volumic scaling in Ref. [80] for the multifractal properties and confirmed it for the spectral statistics in our recent arXiv:2209.04337 (see Fig. 17 of that paper for the SWN with $p = 0.06$).

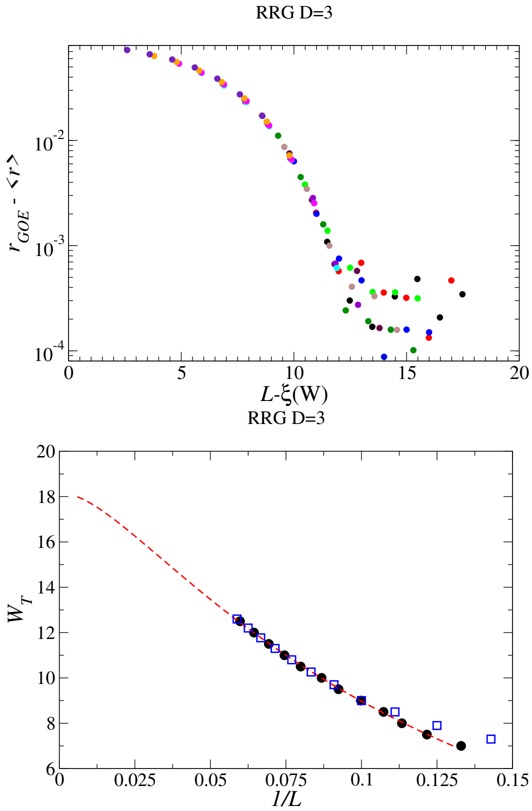

FIG. 1. Upper panel: Volumic scaling of $\langle r\rangle$ close to the ergodic crossover $\langle r\rangle \approx r_{GOE}$. The different data for the RRG D=3 model with $7 \leq W \leq 12.5$ (system sizes $10 \leq L \leq 16$) collapse onto a single scaling curve when plotted as a function of $L-\xi(W)$ where the scaling length $\xi(W)$ depends only on the disorder strength W. Lower panel: $W^T(L)$ (or equivalently $L_T(W)$) is determined by the value of disorder for which a system of size L has $\langle r\rangle = 0.52 \approx r_{GOE}$. $\xi(W)$ is equal to $L_T(W)$ up to an irrelevant additive constant. We fit L_T by the behavior Eq. (3) with the critical exponent $\nu = 1/2$ and $W_c = 18.17$ (determined by the authors) fixed, and a_0 and a_1 two fitting parameters. We find an excellent agreement with the data. We choose to represent the data as W^T as a function of $1/L$ as in Fig. 5 of the authors. This figure shows that W_T is well described by a critical exponent $\nu = 1/2$.

Furthermore, we showed that the correlation volume can be fitted accurately by:

$$\ln \Lambda = a_0 + a_1 (W_c - W)^{-\nu} \,, \tag{2}$$

with an exponent $\nu \approx 0.5$. This form corresponds to $\Lambda = \alpha_0 \exp(a_1 (W_c - W)^{-\nu})$ with $a_0 = \ln \alpha_0$ which should not be omitted, especially far from the transition point where $(W_c - W)^{-\nu}$ is not necessarily large.

In figure 1, I show that these properties are also observed for the data of the authors for the RRG $D = 3$ model. The upper panel of Fig. 1 demonstrates the volumic scaling of $\langle r\rangle = f(L - \xi(W))$ where $\xi(W) = \ln \Lambda(W)$ is the correlation length associated with the correlation volume $\Lambda(W)$. The scaling length $\xi(W)$ (black dots in the lower panel) is determined up to an additive constant that we can fix to reproduce L_T (open blue square dots in the lower panel). This suggests that L_T can be fitted as:

$$L_T = a_0 + a_1 (W_c - W)^{-\nu} \,. \tag{3}$$

This is what I have done here, taking $W_c = 18.17$ determined by the authors and $\nu = 1/2$ fixed while a_0 and a_1 are two fitting parameters. I find a very good agreement with the data in the whole range of disorder values close to the ergodic crossover. In the lower panel of Fig. 1, I show the corresponding W_T as a function of $1/L$. This figure should be compared with Fig. 5 (b) of the authors. The behavior (3) associated with the critical exponent $\nu = 1/2$ works perfectly well for all the range of sizes corresponding to the ergodic regime. Thus, the data for the ergodic crossover in the RRG model with $D = 3$ are compatible with a critical exponent $\nu = 1/2$.

I would like to stress that this very good agreement comes as a surprise because by definition, W^T or L^T are quantities defined very far from the critical regime, where we can usually expect significant non-linear corrections to the algebraic behavior of the correlation length $\xi \sim (W - W_c)^{-\nu}$. So that this should not be thought as a controlled determination of the critical exponent ν, but rather as a compatibility check with $\nu = 1/2$ which works surprisingly well.

I would suggest that the authors test the behavior (3) with their data for the other models they have considered (I have done that for RRG D=4 and it works very well also) and compare the goodness of fit with the behavior they propose $L_T(W) \sim (W_c - W)^{-1}$. Moreover, it seems to me that their estimation of W_∞^T depends crucially on the range of system sizes where they make a linear fit of $W^T(L) - W_\infty^T$ as a function of $1/L$. Could the authors quantify that uncertainty?

b. Finite-size scaling close to the transition point.– The second argument of the authors in favor of a critical exponent $\nu = 1$ in the delocalized phase mainly lies in the scaling hypothesis Eq. (11) together with figure 8 which aims to validate this scaling hypothesis. One of the key elements of the scaling hypothesis (11) is given by the critical behavior of $\langle r\rangle$ at the transition, $W = W_c$, described as an algebraic convergence with L to its Poisson value:

$$\langle r\rangle(W_c) - r_P \sim L^{-\omega} \,. \tag{4}$$

This behavior was already discussed in Refs. [27,81] and has been discussed in some details in our recent arXiv:2209.04337. One important new point of the argumentation of the authors is that ω should be equal to 2. In the insets of figure 8, they show that the numerical data seem to follow this trend at sufficiently large system sizes. The value of $\omega = 2$, together with $\nu = 1$, is crucial to explain the observation of an "effective" critical exponent $\nu_{\text{eff}} = 1/2$ for the crossover to delocalization observed in several references, see e.g. [30] and

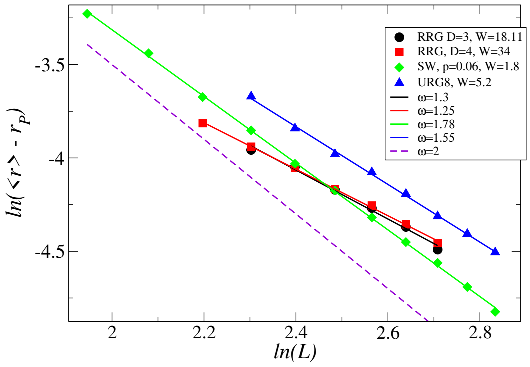

FIG. 2. Critical behavior of $\langle r\rangle$ at $W = W_c$ (or the closest W available in the set of data of the authors) as a function of L, confirming the algebraic decay toward Poisson, Eq. (4). The fitted ω exponents do not seem to be universal, and are smaller than 2. The large system sizes do not show a different behavior as compared to small ones. This figure should be compared with the insets of Fig. 8 of the authors.

arXiv:1810.07545. I say effective because, according to the analysis of the authors, the true critical exponent is $\nu = 1$.

I don't understand the claim of the authors that the data follow the trend with $\omega = 2$, even at large system sizes. In Fig. 2, I plot directly $\ln(\langle r\rangle(W_c) - r_P)$ as a function of $\ln L$. I observe clearly a linear behavior, consistent with Eq. (4) with however $\omega < 2$ and not universal. In particular, I do not observe a different trend of the data at "large" system sizes as compared to small ones. Why do the authors plot in the insets of Fig. 8 their data as $\langle r\rangle$ as a function of $1/L^2$?

The scaling hypothesis (11) is then checked by the authors in the main panels of Fig. 8. Taking the values $\omega = 2$, W_c determined independently and $\nu = 1$, they manage, with only a single fitting parameter A, to put the data for different system sizes and different values of the disorder into a single scaling function. This scaling function describes the flow towards delocalization for $W < W_c$. The fact that this works with a single fitting parameter A, independent of W, is remarkable.

I have nevertheless several questions here: The data in the scaling plots reach at $W \to W_c$ the value r_P. However the authors have also data for $W > W_c$, in the localized regime. How do these data scale? They have values lower than r_P? How to understand that? The authors suggest a modified scaling assumption, Eq. (10) to describe this regime, but how do they justify its form and how precisely this works in the localized regime? Another question is why the authors consider a limited range of system sizes in their scaling analysis? They have for the SWN with $p = 0.06$ data for $7 \le L \le 16$. Could the authors show the collapse of the data for the whole range of system sizes? This is particularly important as the critical behavior with $\omega = 2$ is clearly not valid for small system sizes, such that one could expect to observe significant deviations. My final question is the limited range of W values shown in Fig. 8. In particular, the authors use this scaling behavior to recover the behavior of the boundary of the ergodic region $W^T(L)$, see Eq. (12). Therefore, their scaling hypothesis Eq. (11) should be valid up to the ergodic regime, i.e. for small values of W far from the transition point W_c. Could the authors show this scaling behavior in this regime?

As discussed by the authors, we recently analyzed the scaling behavior of $\langle r\rangle$ in SWN near the transition, see [81] and arXiv:2209.04337. We found that our data are consistent with another scaling hypothesis:

$$\frac{\langle r\rangle - r_P}{\langle r\rangle(W_c) - r_P} = F_{\rm lin}(L/\xi(W)) \, , \tag{5}$$

with a scaling length $\xi(W)$ which depends only on W. Our approach did not make any assumption on the behavior of the scaling function F or on $\xi(W)$. W_c was determined by a best collapse argument (see arXiv:2209.04337) and is found close to the value predicted by the authors for $p = 0.06$. We found a very good collapse of our data onto a single scaling function for $0.8 \le W \le 2.4$ values both in the delocalized and localized regimes, and for all system sizes $10 \le L \le 18$, see Fig. 11 of arXiv:2209.04337. The scaling length $\xi(W)$ is found to diverge at W_c as $\xi(W) \sim |W - W_c|^{-\nu}$ with $\nu \approx 1/2$. We checked these scaling properties for different values of the p parameter of SWN. I want to stress that this scaling behavior is valid in the delocalized regime sufficiently close to the transition, i.e. not in the ergodic regime. In fact the ergodic regime $\langle r\rangle \approx r_{GOE}$, is rather described by a volumic scaling Eq. (1). We have proposed a possible scaling hypothesis which could describe the two regimes, critical and ergodic as:

$$\langle r\rangle = [r_P + (\langle r\rangle(W_c) - r_P)F_{\rm lin}(L/\xi)]\, F_{\rm vol}(N/\Lambda(W)) + r_{GOE}(1 - F_{\rm vol}(N/\Lambda)) \, , \tag{6}$$

with the volumic scaling function $F_{\rm vol}(N/\Lambda) \to 1$ for $N \ll \Lambda$ while $F_{\rm vol}(N/\Lambda) \to 0$ for $N \gg \Lambda$. This accounts for the two types of scaling observed in both critical and ergodic regimes. It is very difficult to demonstrate the validity of this latter scaling hypothesis, because it is a two-parameter scaling hypothesis. In this description, the finite-size properties are controlled by two scaling parameters (similarly to what propose the authors), a length ξ and a volume Λ, but both of them are associated with the same critical exponent $\nu = 1/2$.

It is interesting to note the similarity between this hypothesis and Eq. (10) of the authors. Indeed, $\langle r\rangle(W_c) - r_P$ is compatible with $\sim L^{-\omega}$ (with $\omega < 2$, see Fig. 8 of arXiv:2209.04337). The scaling function f of Eq. (11) of the authors is replaced by the volumic scaling function $F_{\rm vol}$ and f_1 corresponds to $F_{\rm lin}$. In the critical regime $F_{\rm vol} \to 0$, so that the linear scaling described by $F_{\rm lin}$ is observed, and we recover Eq. (5).

The authors state that they have used our scaling approach to analyse their data for RRG $D = 3$ and

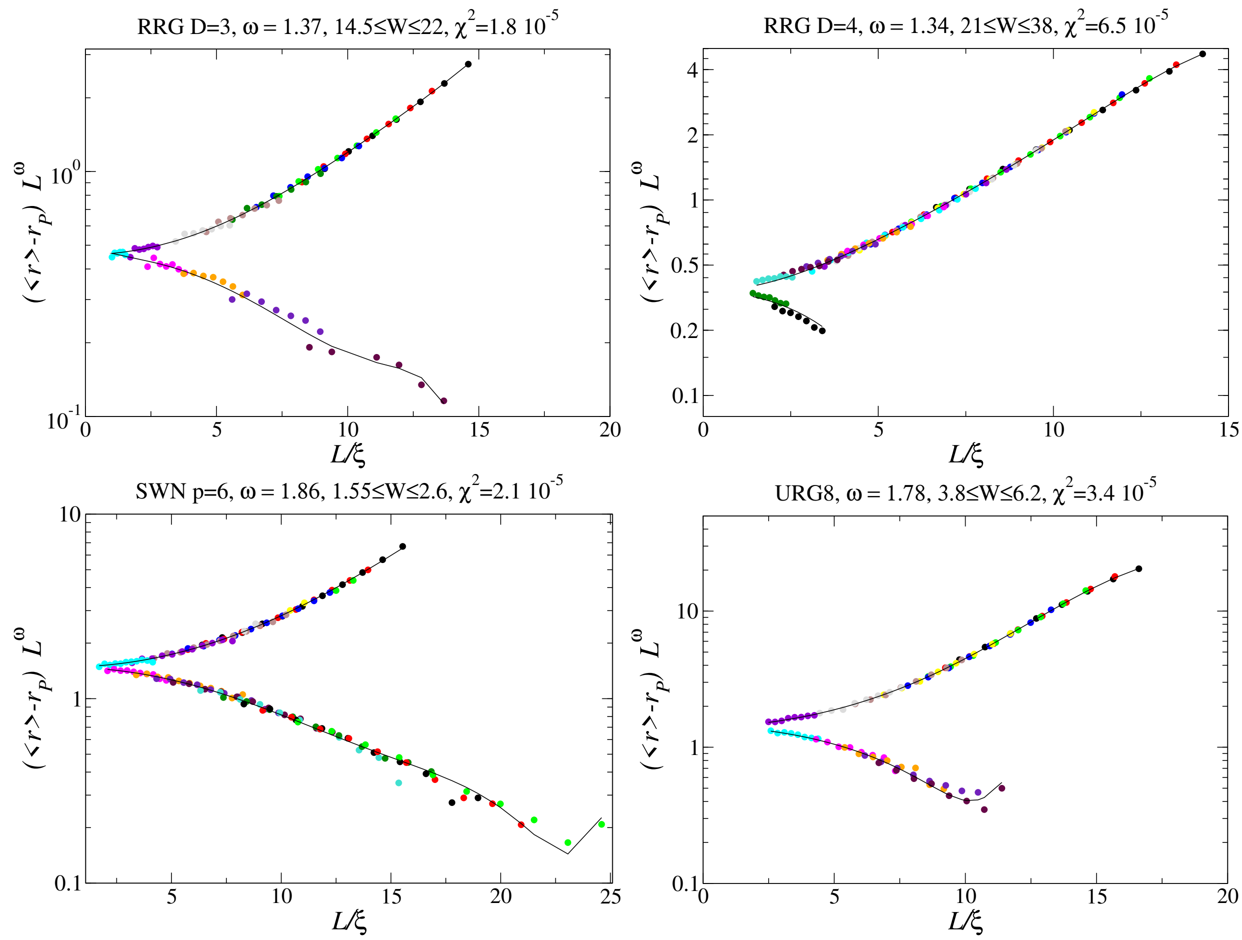

FIG. 3. Finite-size scaling of the $\langle r\rangle$ data close to the transition W_c. The data for the different models are accurately fitted by Eq. (7) with W_c given by the authors in the Table 1 of their paper, and $\nu = 1/2$ fixed. ω is a fitting parameter and its obtained value is given for each model considered. It agrees well with that found in Fig. 4. Non-linear corrections are small but necessary to describe accurately the data. In the localized phase far from the transition, they lead to non-monotonous behavior of the scaling function which is an artifact of the Taylor expansion of the scaling function and parameters. An important note is that all L and W data in the W range indicated have been used and considered.

$D = 4$ and find critical exponents $\nu \approx 0.64$ and 0.67, and that they find deviations from our scaling for data with $\langle r\rangle \geq 0.4$ which is quite small and could indicate that our scaling behavior Eq. (5) would have for these models a very limited range of validity. I am surprised by these observations because I found I am able to fit accurately the data of the authors for these models with our assumption Eq. (5), using the critical disorder determined by the authors and the critical exponent taken as $\nu = 1/2$. More precisely, I fit the data with

$$\langle r\rangle - r_P = L^{-\omega} F(L^{1/\nu} w)\ , \qquad (7)$$

equivalent to (5), with $w = (W - W_c) + A_2(W - W_c)^2 + A_3(W - W_c)^3$ and $F(X) = \sum_{k=0}^{5} B_k X^k$. In this analysis, the fitting parameters are the A_ks, B_ks and ω, whereas W_c and $\nu = 1/2$ are fixed. All curves for different W, in a range that I indicate for each model, are fitted simultaneously. The data that the authors kindly gave to me did not have error bars such that the goodness of fit cannot be evaluated, but I indicate the value of the χ^2 defined as:

$$\chi^2 = \sum_{W,L} \left[(\langle r\rangle - r_P) - L^{-\omega} F(L^{1/\nu} w) \right]^2\ . \qquad (8)$$

The results are shown in figure 3. The agreement with the data is very good as shown by the $\chi^2 \approx 10^{-5}$. The fitted ω values correspond well to that found in Fig. 4. Note that the non-linear corrections are quite small but nevertheless have to be taken into account to describe accurately the data. An important final note is that I have considered all system sizes available, and indicated clearly the range of W values considered for the fit. The restriction is rather in the delocalized side where too far from the transition the data deviate from linear scaling and crossover to a volumic scaling as shown in Fig. 1. Note that the minimal disorder considered corresponds to rather large values of $\langle r \rangle$: RRG D=3 $\langle r \rangle_{\max} = 0.45$, RRG D=4 $\langle r \rangle_{\max} = 0.51$, SWN p=0.06 $\langle r \rangle_{\max} = 0.44$, URG8 $\langle r \rangle_{\max} = 0.52$. In the localized side far from the transition, the non-monotonous behavior of the fitted scaling function is an artifact of the Taylor expansion and related to the fluctuations of the data at very small $\langle r \rangle$.

This figure 3 shows quantitatively that the data of the authors close to the transition are also compatible with a critical exponent $\nu = 1/2$. I think the authors should compare the χ^2 they obtain from their fit with the χ^2 I have indicated, taking into account all system sizes in the range of W considered. After all, the scaling considered here is L/ξ and one should allow for L to vary in the largest range to have a significant determination of the relevant scaling function and critical exponent.

c. Conclusion of the report.– In this long and detailed report, I have offered an alternative analysis to that of the authors of their data. I first showed that the characteristic scale $W_T(L)$ of the crossover to the ergodic regime $\langle r \rangle \approx r_{GOE}$ was perfectly compatible with a critical exponent $\nu = 1/2$. Eq. (3) takes into account the authors' prediction for the critical disorder W_c with only two fitting parameters, and describes all the data for the different accessible system sizes corresponding to the ergodic regime crossover. Also, I showed that the data in the vicinity of the transition were also perfectly compatible with a critical exponent $\nu = 1/2$ and the linear scaling assumption, Eq. (7).

I think the authors' data are precise enough to determine quantitatively which of the two scenarios, mainly $\nu = 1/2$ or $\nu = 1$ and $\omega = 2$ is more likely. I therefore invite the authors to make this quantitative comparison.