I would like to address a problematic part of the definitions of the Lax-operators. In section 7 the authors give an iterative definitions for higher loop Lax operator (7.6), (7.8)

$$\mathcal{L}_{a_1\dots a_n,i}(u) = P_{a_1,i}\dots P_{a_n,i}\check{\mathcal{L}}_{a_1\dots a_n,i}(u) \tag{1}$$

where

$$\check{\mathcal{L}}_{a_1\dots a_n,i}(u) = \check{\mathcal{L}}_{a_1\dots a_n}(u) + g^{2n}\mathcal{A}_{a_1\dots a_n,i}(u) \tag{2}$$

where $\check{\mathcal{L}}_{a_1\dots a_n}(u)$ is the g^{2n-2} order Lax:

$$\mathcal{L}_{a_1\dots a_{n-1},i}(u) = P_{a_1,i}\dots P_{a_{n-1},i}\check{\mathcal{L}}_{a_1\dots a_{n-1},i}(u) \tag{3}$$

The integrability requires the RLL-relation. The RLL-relation for the Lax operator $\mathcal{L}_{a_1\dots a_{n-1},i}(u)$ is

$$\mathcal{R}_{a_1\dots a_{n-1};b_1\dots b_{n-1}}(u)\mathcal{L}_{a_1\dots a_{n-1},i}(u)\mathcal{L}_{b_1\dots b_{n-1},i}(v) = \mathcal{L}_{b_1\dots b_{n-1},i}(v)\mathcal{L}_{a_1\dots a_{n-1},i}(u)\mathcal{R}_{a_1\dots a_{n-1};b_1\dots b_{n-1}}(u) + \mathcal{O}(g^{2n}) \tag{4}$$

and for the Lax operator $\mathcal{L}_{a_1\dots a_n,i}(u)$ is

$$\mathcal{R}_{a_1\dots a_n;b_1\dots b_n}(u)\mathcal{L}_{a_1\dots a_n,i}(u)\mathcal{L}_{b_1\dots b_n,i}(v) = \mathcal{L}_{b_1\dots b_n,i}(v)\mathcal{L}_{a_1\dots a_n,i}(u)\mathcal{R}_{a_1\dots a_n;b_1\dots b_n}(u) + \mathcal{O}(g^{2n+2}) \tag{5}$$

But we can truncate this equation in order $\mathcal{O}(g^{2n})$ as

$$\mathcal{R}_{a_1\dots a_n;b_1\dots b_n}(u)\mathcal{L}_{a_1\dots a_n,i}(u)\mathcal{L}_{b_1\dots b_n,i}(v) = \mathcal{L}_{b_1\dots b_n,i}(v)\mathcal{L}_{a_1\dots a_n,i}(u)\mathcal{R}_{a_1\dots a_n;b_1\dots b_n}(u) + \mathcal{O}(g^{2n}), \tag{6}$$

which obviously do not contain $\mathcal{A}_{a_1\dots a_n,i}(u)$, it contains only $\check{\mathcal{L}}_{a_1\dots a_{n-1},i}(u)$. My question is: what guaranties that if (4) has a solution $\check{\mathcal{L}}_{a_1\dots a_{n-1},i}(u)$ then this Lax solves the second equation (6), too. Without this proof the recursive definition (2) is not consistent.

There is an other related question. Let us expand the R-matrix in the similar way:

$$\mathcal{R}_{a_1\dots a_n;b_1\dots b_n}(u) = \mathcal{R}^0_{a_1\dots a_n;b_1\dots b_n}(u) + g^{2n}\mathcal{B}_{a_1\dots a_n;b_1\dots b_n}(u), \tag{7}$$

where $\mathcal{R}^0_{a_1\dots a_n;b_1\dots b_n} = \mathcal{O}(g^{2g-2})$. Clearly the equation (4) contains $\check{\mathcal{L}}_{a_1\dots a_{n-1},i}(u)$ and $\mathcal{R}_{a_1\dots a_{n-1};b_1\dots b_{n-1}}(u)$, the equation (6) contains $\check{\mathcal{L}}_{a_1\dots a_{n-1},i}(u)$ and $\mathcal{R}^0_{a_1\dots a_n;b_1\dots b_n}(u)$ therefore the matrices $\mathcal{R}_{a_1\dots a_{n-1};b_1\dots b_{n-1}}(u)$ and $\mathcal{R}^0_{a_1\dots a_n;b_1\dots b_n}(u)$ should be connected somehow. My second question is: what is the connection between these matrices.

I also found two typos.

- In eq (3.6) $V^{-1} \to V^{-1}(u)$.
- In eq (4.7) $P_{2,1} \to P_{a_2,a_1}$.