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Title: Line Operators in 4d Chern-Simons Theory and Cherkis Bows
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This paper relates a) Bow varieties, b) Line operators in "four-dimensional Chern-Simons theory", and c) Integrable spin chains. The bulk of the paper involves

1. Relating Ω-deformed 4d $\mathcal{N} = 4$ super-Yang-Mills (with 1/2 BPS impurity walls) to a 2d BF theory, and
2. Computing the equations defining vacua in the 4d $\mathcal{N} = 4$ super-Yang-Mills (with 1/2 BPS impurity walls) and identifying them with the equations defining the bow varieties.

Conceptually, it appears that it is the Section 5 that is the most significant part of this work. In it the authors conjecture the relation between (the quantum deformation of the algebra of functions on) three different classes of bow varieties and T, Q and L operators. This section is rather short and schematic, implicitly assuming all of [13] and [20]. And its main content are three concrete examples.

The conciseness of Section 5 is in sharp contrast with sections 3 and 4, which repeat many of the calculations (that already appeared in literature) in detail. The latter appears to be a general trend in the field, making papers self-contained. One only wishes Section 5 to follow the same style, since it is the key section of this paper.

Some of the examples of some well established results that are rederived in this paper are: a) the derivation of the bow equations from the 4d super-gauge theory with defects (which appeared in [COS11] and other papers), and b) the (incorrectly stated) boundary conditions on p. 25, Eqs. (4.58) and (4.59) appeared in the original Nahm's work and were derived in [Hur89] and [HM89]. (Importantly, the canonical form of the subleading and of the diagonal terms in the Nahm data are different for the full Nahm formulation and for the holomorphic formulation, since the gauge group function class is different.)

Even though each of the three relations (a-b, a-c, and b-c) appeared in the literature, this paper presents an interesting synthesis of all three. It states an interesting conjecture with concrete examples. It is likely to stimulate further research along this direction. This is an interesting and stimulating paper. We recommend it for publication once the authors address the following concerns.

Main Comments:

1. The main statement of this paper involves line operators in 4d Chern-Simons theory, yet, such operators are described only in terms of string theory brane configurations. To discuss *line operators in a gauge theory*, they should be defined in terms of *that very gauge theory*. The names for the line operator examples in the last section (such as Wilson lines, etc.) are suggestive, however, the operators are still left undefined.

2. Pages 5-6 and 15: There is an issue with two different interpretations of the Ω-twisted brane configuration. Note, that Ω-twist is simply a statement of that the string ambient space is. Namely, Eq. (4.1) implies that (4,5,6,7) directions are cotangent to the (0,1,2,3) directions. This interpretation follows from the choice of the Ω-twist and is essential for the whole reasoning of this paper.

   Alas, it is in contrast with the geometry presented in the beginning of the paper, where (3,4) directions are cotangent to (0,7), while (1,2) and (5,6) form the Taub-NUT space. This latter was used to have a 4d Chern-Simons formulation. The authors should reconcile these two formulations. If this can only be dome when $\Sigma = \mathbb{C}$ and $C = \mathbb{R}^2$, then there is no use of introducing Σ and C, which suggest that these results could be applied to more general setup.

3. P. 21, above Eq. (4.46), "we find that once Ω-deformation is turned on": This is a sudden big step. In contrast to all the details in the preceding sections, Eq. (4.46) appears with little justification.

Minor Issues:

Page 2, paragraph 2: Coupling a gauge theory with gauge group G to a system with symmetry H requires specifying the action of G on that

system, thus if G is a subgroup of H, ($G \subseteq H$) it is straightforward. The inclusion the authors state, however, is in the other direction with $G \supseteq H$. It is far from clear how a system with lower symmetry can be coupled to a gauge theory, without first reducing the gauge group (which, in quantum field theory, requires a lot of extra care).
This is probably a superficial issue, since the coupling is described at the end of Sec. 2.1. However, some clear language in the introduction is in order.

P.4, line 3: "in principle" instead of "in principal".

P.5, after Eq. (2.1): $\vec{\omega}$ is a vector *field* on $\mathbb{R}^3 \setminus$ (a line). This vector field is NOT globally defined of all on $\mathbb{R}^3$, nor is the circle fiber coordinate θ.

P.5, par. -3: the 1d TQM statement is not justified. Please provide a reference or an explanation.

P.6, par. 1: state that K_i and L_i are integer-valued.

P. 15, title of Sec. 4.1: "as a 2d B-model" instead of "as A 2d B-model".

P. 18, Sec. 4.1.2, "the following inner product": where is this inner product valued? Why is (4.25) finite?

P. 22, line 2 after Fig. 4: "nth D5 brane" instead of "ith D5 brane." Also, one line above K_n instead of K_i.

Same par: "under" instead of "udner."

P. 35: top par: Is this statement that turning on the twisted masses corresponds to this particular quantization of the universal enveloping algebra a conjecture? If not, what justifies it? Can one derive the corresponding deformed structure relations?

P. 37, Remark 5.3: "If we unsymmetrize ..." this is unclear. What is unsymmetrization? What is this unsymmetrization applied to? It seems that it could not be the Higgs branchs, so is this an operation on a direct sum of the corresponding algebras? This remark is not clear.

References

[COS11] Sergey A. Cherkis, Clare O'Hara, and Christian Saemann. Super Yang-Mills Theory with Impurity Walls and Instanton Moduli Spaces. *Phys. Rev. D*, 83:126009, 2011.

[HM89] Jacques Hurtubise and Michael K. Murray. On the construction of monopoles for the classical groups. *Comm. Math. Phys.*, 122(1):35–89, 1989.

[Hur89] Jacques Hurtubise. The classification of monopoles for the classical groups. *Comm. Math. Phys.*, 120(4):613–641, 1989.