This paper is devoted to the question of how to make sense of ideal hydrodynamics as a quantum field theory. The main new result is the identification, in 2 spatial dimensions, of a class of gapless states in the spectrum with quasiparticle properties, which the authors refer to as "vortons". Along the way, the authors introduce techniques borrowed from the classical and quantum mechanics of particles on group manifolds, and provide extensive pedagogical reviews of background material. While the ultimate physical implications of the results are not yet realized, the paper presents many novel ideas and methods which are interesting in their own right. I therefore recommend that the paper be published in SciPost. In what follows, I will provide a summary of the motivations for this work, the methods employed, and the main results. I conclude with a short list of points in the presentation that I found to be slightly cryptic, and which the authors may wish to consider elaborating in revisions to the manuscript.

The starting point for the analysis is a high-energy particle theorist's formulation of classical relativistic hydrodynamics, viewed as a non-linear sigma model of fields that map spacetime to a target space of fluid configurations. The latter corresponds to volume preserving diffeomorphisms ("SDiff's") acting on an initial spatial reference manifold of fluid points. A static fluid background breaks the spacetime $\times$ SDiff symmetries of the action down to a diagonal subgroup, resulting in three Goldstone modes which dominate the long distance dynamics. These modes include both longitudinal sound waves (phonons) with a standard linear dispersion, as well as two transverse vortex modes with a degenerate $\omega(k) = 0$ dispersion relation, an exact consequence of the non-linearly realized SDiffs.

While the quantization of the longitudinal modes leads to a standard spectrum of phonon Fock states, there is no stable vacuum state for the transverse modes. Rather the wave functional of states with definite vorticity completely delocalizes under time evolution, behaving as an infinite collection of non-relativistic quantum particles. As a result, it is not clear how to obtain well defined observables in the resulting quantum field theory. For example, by introducing a finite vortex speed of sound $c_T \neq 0$ as a regulator, it becomes possible to define an S-matrix for phonon/vortex scattering, as done by one of the present authors in ref. [20]. However, as [20] found, such scattering amplitudes become singular in the $c_T \to 0$ limit in which the SDiff invariant is recovered. Similarly, it is not clear how to calculate correlation functions of "physical" composite operators (eg conserved currents) in the $c_T \to 0$ QFT, as there does not seem to be a suitable vacuum state in which to define them. It is then tempting to conjecture that these obstacles to obtaining a sensible quantum theory might be the reason why in nature, ordinary matter at $T = 0$ appears to roughly organize itself in the superfluid or solid phase but not into a perfect fluid one with one longitudinal and two transverse fluctuations?

Ultimately, the question of wether zero temperature perfect fluids exist boils down to the fate of the underlying SDiff invariance in the quantum theory. Is SDiff explicitly broken at the quantum level or is it exact? If the latter, does it (A) imply the existence of a huge degeneracy of additional degrees of freedom beyond the classical observables (like the fluid velocity or density) or is it (B) "gauged" in some way, so that only SDiff invariant states are physical. In this paper, evidence is presented that by focusing on the action of SDiff_R (the copy of the diff group acting on the Eulerian coordinates of the fluid), option (A) as well as a variant of (B) exhibit interesting dynamics at sufficiently long distances. In order to zero in on the relevant issues, the authors consider fluids on a fixed compact spatial manifold, at distances much larger than the volume where the phonons decouple. They also focus on the case of two spatial dimensions in which vorticity is a scalar and the fluid Hamiltonian takes on a simple but non-trivial quadratic form in the charges of SDiff_R. For concreteness, the authors choose to study the fluid on a ''bathtub" shaped like a 2D torus, so the relevant symmetry group is $\text{SDiff}(T^2)$.

The discussion in the paper follows two logically independent lines of reasoning, but leading to a consistent picture. One is mainly group theoretical, where the spectrum is dictated by the representation theory of $\text{SDiff}(T^2)$. To yield finite results, the authors exploit the well known fact that the $\text{SDiff}(T^2)$ algebra is contained in a certain $N \to \infty$ limit of the much more familiar $SU(N)$ one. This allows the authors to proceed by analogy with a generalized rigid body, i.e a degree of freedom moving on a compact finite dimensional Lie group configuration space. By keeping both N and the volume finite, the quantum theory can be interpreted as living on a discrete lattice of points both in momentum and in configuration space. They find that as $N \to \infty$, only a $\sim \sqrt{N}$ dimensional subspace of the fundamental and anti-fundamental representations of $SU(N)$ correspond to hydrodynamics states, identifiable as gapped vortices in the continuum $N \to \infty$ limit. Similar conclusions hold for higher rank complex representations. On the other hand, the adjoint representation is ungapped, with a dispersion relation $E \sim \mathbf{p}^2$ in the continuum limit. These states are therefore interpretable as quasiparticles of the quantum fluid, the famous vortons, as $N \to \infty$. From group theoretic considerations, the authors deduce that tensor products of many single-vorton wave packets behave like a multi-particle Fock space, and if the wave packets are widely separated in space, can be regarded as asymptotic states for scattering processes in the continuum limit.

The second line is a more conventional EFT approach, working directly in the continuum limit and at (nearly) infinite volume. The authors invoke a realization of the vorticity algebra in terms of a complex canonical field Φ. If $\Phi, \Phi^\dagger$ satisfy canonical commutators or anti-commutators, the operator $\Omega = i\nabla\Phi \wedge \nabla\Phi^\dagger$ automatically observes the vorticity algebra in Eq. (66). In terms of such variables the bare (classical) fluid Hamiltonian is purely quartic in gradients of $\Phi, \Phi^\dagger$. The quantum theory can now be formulated as a path integral over $\Phi, \Phi^\dagger$, with a bare action and path integral measure that are manifestly invariant under a new ("emergent"?) area preserving diffeomorphism group, which treats Φ and $\Phi^\dagger$ as coordinates on a complex 1D manifold.

In naive perturbation theory, the Fock states generated by $\Phi^\dagger$ on the vacuum are infinitely degenerate and non-propagating. This degeneracy is a manifestation of the emergent SDiff invariance on Φ, which is however broken by the presence of a UV cutoff Λ introduced by hand in order to make sense of radiative corrections in the bare theory. The radiative corrections break the emergent SDiff, but by power counting, the authors argue that the only relevant operator generated by renormalization is a term $\sim |\nabla\Phi|^2$,with a UV quadratically divergent Wilson coefficient. As a result, the single particle states acquire a non-relativistic dispersion relation $E = p^2/2M$, with $1/M \sim \Lambda^2$ and one finds yet again a gapless spectrum of many-particle asymptotic vorton states. Finally, in order to illustrate the non-trivial dynamics of the theory, the authors compute S-matrix elements for various elementary processes.

The paper provides a concrete and coherent interpretation of the low energy properties of quantized hydrodynamics, and in my opinion is suitable for publication in its present form. However, the authors may want to address the following points:

- It turns out that, in a suitable gauge, the quantum theory of a relativistic 2D membrane is isomorphic to the incompressible non-relativistic fluid in 2D. As a result, there is some overlap in methodology between the literature on quantized membranes from the 1980's and some of the results of this paper. For example, the interpretation of SDiff(T^2) as a large N limit of $SU(N)$ has been discussed in detail by J. Hoppe (1989) in \https://www.worldscientific.com/doi/abs/10.1142/S0217751X89002235 (this paper also provides embeddings of SDiff for other 2D surfaces into infinite rank limits of compact finite dimensional Lie groups). It would therefore perhaps be useful if the authors more explicitly indicated which of the ideas in Secs. 4 and 5 are truly original vs. results already obtained in the literature by other researchers.

- Why is the additional UV regularization necessary in the Hamiltonian of Eq. 120? Does the finite N not already provide a suitable UV cutoff for the theory? If one has to cut off the UV modes in Eq. 120 by hand even at finite N (presumably to restrict to momenta smaller than $\sqrt{N}$) why bother with the lattice theory at all?

- More generally, it is a bit troublesome that many properties of the IR physics seem to depend crucially on the precise way that the UV cutoff is implemented. One example of this is again Eq. 120, where the low energy spectrum of vortons with $E \sim \mathbf{p}^2$ only emerges if one cuts off modes with $k > \sqrt{N}$ by hand. Similarly, in Sec. 6.2, the quasiparticle interpretation of ungapped vortons with quadratic dispersion disappears if the cutoff is sent to infinity. Does that mean that quantum hydrodynamics is not a conventional EFT, where the IR physics is robust against its specific embedding into a UV theory (thereby violating decoupling)?

- At least superficially, some of the qualitative properties of quantum fluid that this paper has grappled with are reminiscent of the exotic "fracton" phases of matter studied in the recent condensed matter literature. These include the infinite degeneracy of states with $\omega(k) = 0$ in the continuum limit, UV/IR mixing, and the presence of global or gauge symmetries which render certain states "immobile" (non-propagating). Is there a connection between the vorton field theory considered here and some of the models exhibiting this type of fractonic behavior, e.g. the 2+1 dimensional continuum QFTs studied in https://arxiv.org/pdf/2003.10466.pdf?

- Could the authors elaborate why they say (at the end of sec. 1.2) that their interpretation is at odds with the approach to quantum hydrodynamics suggested in ref. [28], which claims that vacuum correlators of L-invariant composite operators such as velocity, pressure or vorticity are IR and UV safe?
- Should one expect to find branch cuts due to the vorton continuum states in the 2-pt functions of physical (L-invariant) operators? Do the vortons show up in the response functions associated with any of the conventional conserved currents of hydrodynamics. How would one in principle prepare a multi-vorton asymptotic state?
- Does anything interesting happen on more complicated background 2D manifolds? If the surface has a boundary, does one get gapless states on the edge (as in the quantum Hall effect)?