Main comments: The authors consider the supergravity solutions corresponding to the dual CFTs realized from 5d $\mathcal{N}=1$ (resp. 3d $\mathcal{N}=4$) linear and balanced quiver gauge theories with $(P-1)$ gauge nodes of unitary gauge groups and $(P-1)$ flavor nodes, i.e. with the product gauge groups $\prod_{j=1}^{P-1} SU(N_j)$ (resp. $\prod_{j=1}^{P-1} U(N_j)$) as well as the product flavor group $\prod_{j=1}^{P-1} SU(F_j)$ with $F_j = 2N_j - N_{j-1} - N_{j+1}$ rotating the fundamental hypermultiplets. As the supergravity solution can be found by solving the 2d Laplace equation, they generalize the equation and discuss that it corresponds to a certain real mass deformation of the gauge theory.

As the proposal is new and the problem is certainly interesting for many experts working on the holography as well as the gauge theories, this work may be worth to be published. However, having read the manuscript, I am confused with several unclear points which I list below. I would like the authors to clarify them more carefully and seriously. I mainly make comments on the case of the 3d $\mathcal{N}=4$ theories, but they should be also taken into account for 5d $\mathcal{N}=1$ theories.

- While the authors claim that the RG flow discussed in the manuscript is activated by the real mass in 3d $\mathcal{N}=4$ SCFTs, it should break the supersymmetry as opposed to the triplet mass deformation. I am not sure whether and how it is consistent with the supergravity solution and a linear PDE preserving eight supercharges they consider from the gravity side. In that case, one of the S^2 factors, i.e. $\mathbb{S}^2_1(\theta_1, \varphi_1)$ and $\mathbb{S}^2_2(\theta_2, \varphi_2)$ must be broken because of the broken R-symmetry. This may lead to serious flaws in their analysis.

- The precise broken processes of the quiver gauge theories due to the mass parameters, which are schematically illustrated in Figure 1 and Figure 2 are rather vague since there is no detailed explanation about $\text{Gauge}_1 \oplus \text{Gauge}_2 \oplus$ decoupled. I would like the authors to clarify this claim more precisely by explaining how the $\text{Gauge}_1 \oplus \text{Gauge}_2 \oplus$ decoupled is determined once mass parameters turned on. For example, for $d=3$ a simple example is a balanced $\mathcal{N}=4$ quiver with a gauge group $U(N)$ and $2N$ flavors. When $\mathscr{F}$ of the fundamental hypers get the masses, what is $\text{Gauge}_1 \oplus \text{Gauge}_2 \oplus$ decoupled? Also what is the reasoning that the masses can split the gauge group?

  This issue will be related to the following discussions:

  - page 39; I cannot understand the figure illustrated after "This comes about because the breaking of gauge and flavour groups". It looks strange process. What does the arrow stand for? While the break down of the global symmetry due to the mass parameter is usual, that of the gauge group looks strange. It would be better to precisely explain how the process occurs in detail.

  - page 56; I cannot understand what the authors want to claim by rewriting the matrix integral (C.1) as (C.3a) and (C.3b). There is no reason to split N into N_1 and N_2. It looks just an “artificial” split. Also (C.3b) cannot be simply understood as “interaction” in 3d $\mathcal{N} = 4$ theories due to the contributions in the numerator. In other words, the expression including (C.3a) and (C.3b) should not be simply understood as the partition function of an alternative 3d $\mathcal{N} = 4$ theories, rather a formal change of variables of the matrix integral.

- In the Hanany-Witten brane configuration of 3d $\mathcal{N} = 4$ gauge theories, the mass parameters correspond to the positions of the flavor D5-branes relative to those of the color D3-branes between the leftmost and rightmost NS5-branes at $\eta = 0$ and $\eta = P$ rather than those of the color D3-branes. Since the color D3-branes do not end on the flavor D5-branes but rather intersect with them, I cannot understand why the real mass deformation that only splits “flavor branes” corresponds to splitting the “color branes” as discussed in section 3.1.2. On the other hand, the FI parameters describe the position of the NS5-branes on which the color D3-branes terminate. Why are they turned off?

  The following part will be related to this issue:

  - page 56; I cannot understand what Figure 9 explains. Turning on the masses, the corresponding flavor D5-branes which originally intersect with color D3-branes are separated from them. But I cannot see how and why N D3-branes further split into N_1 and N_2 D3-branes in the Hanany-Witten setup. Also what determines N_1 and N_2?

- In section 3.2.3, the study of matrix model, the FI parameters of 3d $\mathcal{N} = 4$ gauge theory are turned off. In general, they should also appear in the saddle point equation, which can be checked with the Poisson equation obtained in supergravity. I cannot see why the authors neglect the FI parameters. They should describe the reason that they ignore here. Otherwise, the statements may look what are convenient for them. If there is no clear reason, they should refine the analysis by introducing the FI parameters.

- I am not sure whether the supergravity solution dual to the real mass deformation of 5d (resp. 3d) quiver gauge theories can be simply addressed by assuming the AdS_6 (resp. AdS_4) factors in the holographic dual geometry because the 5d (resp. 3d) conformal symmetry is broken due to the mass parameters. For example, the holographic result (2.36) is found by starting from the supergravity solution with AdS_4 factor while (3.41) is the Wilson loop vev with the mass deformation which breaks the 3d conformal symmetry. This issue is discussed in section 2.7 but I cannot see why the authors still keep starting by AdS_6 (resp. AdS_4) factors. In fact, more recently it has been understood that the sphere partition functions of 3d $\mathcal{N} = 4$ SQFTs with mass and FI parameters are essentially given by the quantization parameters of 1d topological quantum mechanics resulting from the Ω-deformation. Holographically the Ω-deformation is induced by the non-vanishing flux in the supergravity background. Accordingly, the mass deformation of 3d $\mathcal{N} = 4$ SCFTs is proposed to be dual to the background with AdS_2 factor inside AdS_4 in the context of the twisted holography.

Minor comments:

There are further minor comments.

1. page 1; In addition to the references [10-17], there are references which give refined checks of dualities between AdS geometries and CFTs for $d = 3$. For example, there are various works using the Fermi-gas approach;

   - M. Marino and P. Putrov, " ABJM theory as a Fermi gas, " J. Stat. Mech. 1203 (2012) P03001, arXiv:1110.4066 [hep-th].
   - M. Mezei and S. S. Pufu, " Three-sphere free energy for classical gauge groups, " JHEP 02 (2014) 037, arXiv:1312.0920 [hep-th].
   - A. Grassi and M. Marino, " M-theoretic matrix models, " JHEP 02 (2015) 115, arXiv:1403.4276 [hep-th].
   - Y. Hatsuda and K. Okuyama, " Probing non-perturbative effects in M-theory, " JHEP 10 (2014) 158, arXiv:1407.3786 [hep-th].

   In particular, the second, third and fourth study 3d $\mathcal{N} = 4$ gauge theories dual to AdS_4 geometry in M-theory.

2. page 2; "when the mass is very large compared to the inverse of the sphere"

3. page 11; A space after eq.(2.19b) may not be necessary.

4. page12; Before eq.(2.24) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ is $-\log 2$ rather than $-\frac{\log 2}{2}$. Better to explain "→".

5. page12; While eq.(2.21) is argued to match with free energy of 5d SCFT, there is no description about (2.24). Is it confirmed eq.(2.24) agrees with that of 3d SCFT?

6. page 14; Why can the representation of the Wilson loop be simply controlled by the special value of σ as equation after (2.34)? When l and N are both large while l/N is fixed, the holographic dual of the Wilson loop will appear. Is it solve by setting L.H.S. to infinity?

7. page 14; As discussed in section 2.7, it may be more reasonable to investigate the limit $\frac{\sigma_0}{P} \to \infty$ of the Wilson loop vev.

8. page 15; It seems that the authors conclude that the holographic variable $\frac{\sigma_0}{P}$ is identified with the real mass m and that it is reliable at $\frac{\sigma_0}{P} \to \infty$. However, it is not so clear whether this limit is consistent with the original assumption that $-\infty < \sigma < \infty$ introduced in section 2.2.1 and the boundary conditions, e.g. $\hat{W}(\sigma \to \pm\infty)$ given in eq.(2.26).

9. page 20; In the second sentence of subsection 3.1.2 the "real scalar" is denoted by $\vec{\sigma}$ as a vector. While this is mentioned later on page 25 that it is $|\vec{N}|$-dimensional real scalar, it may be better to explain there.

10. page 23; Other than [105-109], there are more references on the sphere partition functions of linear quiver with 8 supercharges. For example, the reference

    - S. Benvenuti and S. Pasquetti, "3D-partition functions on the sphere: exact evaluation and mirror symmetry, " JHEP 05 (2012) 099, arXiv:1105.2551 [hep-th]

    discusses the residue calculation of the matrix integral for sphere partition functions of 3d $\mathcal{N} = 4$ gauge theories. and the reference

    - D. Gaiotto and T. Okazaki, "Sphere correlation functions and Verma modules, " JHEP 02 (2020) 133, arXiv:1911.11126 [hep-th]

    gives the universal IR formula of the sphere partition functions of 3d $\mathcal{N} = 4$ SQFTs.

11. page 28; What about the case with FI parameters of eq.(3.10), (3.11) and (3.12)?

12. page 38,39; It would be better to discuss the limit $\mu \to \infty$ of (3.29) and (3.30) in detail in which the background is reliable.

13. page 46; I am not sure whether the"wall-crossing" is an appropriate terminology here since the parameter b may depend on the representation of the Wilson loop as well as the mass parameter. Also $\kappa(b)$ would not be the observable in the gauge theory with fixed N, rank of gauge theory as it depends on N. It may lead to misleading.

14. page 46; While (3.41) has the prefactor $P^2/(2\pi^2)$, (2.36) has the $P^2/(2\pi^3)$. It is concluded that they perfectly coincide, but how does the difference come from? Is it a typo?

15. page 46; The explanation of the factorization is unclear since the form of the expression (3.11) may be modified by introducing the FI parameters.

16. page 46; It would be better to discuss the limit $\frac{\sigma_0}{P} \to \infty$ of (3.41) in detail in which the background is reliable.