Referee report on
A range three elliptic deformation of the Hubbard model,
by Marius de Leeuw, Chiara Paletta, Balázs Pozsgay

The authors introduce an integrable model with three-sites interaction, that can be viewed as a deformation of the Hubbard model. The paper is interesting and may be published, but there are some points that need to be clarified. In particular, the exposure on the integrability of the model for the three site interaction model (section 5 of the paper), looks rather unclear:

1. I think more details are needed to see that the eq. (5.13) ensures the integrability of the model. First, I think that eq. (5.12) should be $\check{\mathcal{L}}_{abj} = P_{bj}P_{aj}\mathcal{L}_{abj}$ (I suppose that they mixed with $\mathcal{L}_{abj} = P_{aj}P_{bj}\check{\mathcal{L}}_{abj}$). Second the equation $\check{R}_{12,34} = P_{13}P_{24}R_{12,34}$ should be added.

   It would be nice also to add that eq (5.17) comes from the relation $[\check{R}_{12}\,,\,\check{R}_{34}] = 0$.

2. I don't understand the discussion following eq. (5.16):

   - Starting with the Yang-Baxter eq. for $\check{R}$, one gets directly the relation (5.26), so I don't understand the whole discussion to get it.
   - It is easy to check that eq. (5.16) is obeyed using eqs (5.23) and (5.24), I don't see the need to introduce $\mathcal{G}$, specially if it is to conclude that it corresponds to $\mathcal{L}$.
   - After eq (5.22), what means direct calculation? Is it by using the assumption that $\mathcal{G}$ is $\mathcal{L}$? Then, again, all this is not needed since we get directly (5.26) from the Yang-Baxter eqs. If it is a general fact, I don't see how they get that. By the way, in relation (5.19), using the form (5.24), you find that $\mathcal{G}$ does not depend on u_2, but does depend on u_3.
   - A priori, eqs (5.23) and (5.24) do not define the same R matrix, it is only because of eq. (5.26) that they match. It is true that there is a sentence about that in the paper, but the way it is written makes things look like (5.26) is deduced from this compatibility. I think this should be rephrased.

Apart from integrability, there are also minor points to be corrected

1. In section 2, when speaking of the symmetries of the Hubbard model, it is rather surprising to read that the periodic Hubbard model do not have $so(4)$ symmetry. I don't say what is written is wrong, it is just that the model they choose to be the Hubbard one is a bit unusual. To be fair, they indeed stress that, but I think it will be useful to remind it around eq. (2.34).

2. After eq (3.6), I would say that "The latter commutator vanish only in the case..." is closer to what they want to say.

3. Around eq (3.7) (special points), I think it would be clearer (and more precise) to say that Q_2^μ in eq (3.8) commutes with $\sum_j h^\mu_{j,j+1}$ and $\sum_j l^\mu_{j,j+1,j+2}$, separately ($\mu = \sigma, \tau$).

4. End of section 3: when the authors say that they tested all range 2 charges, does it include mixture of operators based on σ and operators based on τ? I suppose yes, but in view of the results, it would be clearer to state it explicitly.

5. Top of page 9, I don't understand why the odd number of down spin sector should not affect the thermodynamical properties of the model they study. Please detail the argument.

6. In section 4.2, when speaking of self-invariance, add somewhere 'up to a redefinition $j \to j - \frac{1}{2}$'. I am not sure that this redefinition makes the things clearer (one never knows of what model they speak of), but I leave the decision to the authors.

7. In eq (4.17), I think that $u = 8U \cos^4 \theta/2$.

8. Top of page 10, when speaking of actual Hubbard, do they mean any form of Hubbard model, or just their choice H_1 in eq (2.15)? Please specify it, and it would be more convincing if they tested the different forms of Hubbard.

9. After eq (4.21), you need to set $\alpha = 2Ue^{-2v}$ before the limit if you want the kinetic term to be kept.

10. In eq (5.6), I think a *log* is missing (at least for the consistency of the exposure), and the sentences before and after this eq. should be merged.

11. Section 5.2, when saying that the higher charges are local, it should be specified that the locality is not of nearest neighbors type, but is of longer range, and depends on the charge. It is local only in the sense that they remain finite, even in the limit $L \to \infty$.

12. In eq (5.11), is it not Tr_{ab} that you should have?

13. In appendix A, it would be nice to have also an expression of R (or $\check{R}$) in term of σ matrices. That would help the interested reader to compute $\mathcal{L}_{j,j+1,j+2}$.