Title: `A range three elliptic deformation of the Hubbard model`
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Referee Report

This manuscript proposes an integrable deformation of the 1D Hubbard models with the consecutive three sites interactions and violating the particle number conservation. The Hamiltonian H_3 includes the two coupling constants u and κ, where u is the coupling constant of the Hubbard model and κ is that of the three-range interaction. Sec.2 reviews the 1D Hubbard models so that the formulation fits the author's purpose. In Sec.3, the Hamiltonian H_3, including the three range interactions, are introduced. After the non-triviality and the characteristic feature of H_3 are discussed, the integrability is proved. In Sec.4, using the bond-site transformation, the authors convert the three-range interaction to the nearest-neighborhood interaction and define a new two-site model. In Sec.5, the two-site and three-site R-matrices are derived. The authors show their integrability in the sense that they satisfy the YBE. The explicit form of the two-site R-matrix is presented in App. A. They observe that the matrix entries of the two-site R-matrix are elliptic functions of the spectral parameters. Sec.6 argues the particular limits with respect to the Hubbard coupling u and U.

This article introduces the interesting three-range Hamiltonian H_3. Then, using a bond-site transformation effectively, the authors find the new elliptic solution of the YBE. These results are new and interesting. Thus, I recommend this manuscript for publication in SciPost. However, before going to the publication, I would appreciate the author providing further clarifications and expositions on the following points and improving your manuscript appropriately.

1. (2.1) seems to lack the commutation relations among the creation operators; $\{c^\dagger, c^\dagger\}$.

2. In this paper, I think that all notations $SU(2)$ and $U(1)$ stand for the Lie algebra rather than the Lie group. The Lie algebras are usually expressed by small letters such as $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$. Hence, I am afraid that the capital notations are a bit confusing to readers with pure mathematical backgrounds.

3. From (2.21) to (2.27), the authors show that two sets of the generators A's and B's satisfy the $\mathfrak{su}(2)$ algebras, respectively. For completeness, it is worth adding the commutation relations among them, i.e., $[A, B] = 0$, which concludes the whole algebra is $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$.

4. Concerning the three-range Hamiltonian H_3 in (3.2), I don't see how the authors arrived at this solution. For instance, why doesn't it have $(j, j+2)$ interaction? Why does the

three-range interaction include only σ^x, σ^z but σ^y? Some heuristic arguments would help readers.

5. The authors claim that the model given by H_3 is integrable in the sense that it has infinitely many commuting conserved charges. What do the authors mean by a transfer matrix is $t(u)$ in (5.11)? If so, I expect that the integrability here means $[t(u), t(v)] = 0$. Then, by canonical arguments, the commuting charges are obtained by expanding $t(u)$ with respect to u. Since the proof of integrability of H_3 is essential, I suggest authors to reproduce the formulation of [32] along this situation.

6. Relating to the above comment, it would be nice if authors could present the explicit expressions of the first few non-trivial conserved charges appearing in the expansion of $t(u)$.

7. The authors state that the two-site Hamiltonian H_2 in (4.8) does not include the actual Hubbard model for any choice of θ. Is there more conceptual exposition for this point? Since $H_3(\kappa)$ reduces to the Hubbard Hamiltonian $H_3(0)$ at $\kappa \to 0$, I think that it is natural to expect $H_2(\kappa)$ to become $H_2(0) = H_3(0)$ at the same limit. If not, mathematically, a bond-site transformation seems not to preserve the continuity of κ at $\kappa = 0$, i.e., $\lim_{\kappa\to 0} H_2(\kappa) \neq H_2(0)$. Why does this happen? Does this phenomenon relate to parameterization (4.7)?

8. In the second line of Sec.5, it is written as “Quantum Inverse Scattering Approach.” But, I feel that “Quantum Inverse Scattering Method (QISM)” is more common. It is not a request but my suggestion for consideration.

9. In (5.6), I think either t^{-1} or log is missing.

10. In (5.12), $\check{L}$ should also have three subscripts. In [32], $\check{L}$ is defined as $L = PP\check{L}$. Do authors use a different convention here?

11. (A.1) presents the explicit form of the R-matrix. Mathematically, the matrix expression only makes sense by specifying the ordering of the basis and the action of the liner operator on each base. These two points should be clarified.

12. Relating to 7., doesn’t the R-matrix (A.1) reduce to the R-matrix of trigonometric type in [24] in any limit of κ?

13. Throughout the manuscript, the terms two/three-site and 2/3-site are used. It would be better to integrate these into one of them.