Report on "Precision magnetometry exploiting excited state quantum phase transitions", by Q.Wang and U.Marzolino

This work discusses the use of excited-state quantum phase transition (ESQPT) in the Lipkin-Meshkov-Glick (LMG) model to implement metrological protocols. The authors first introduce the structure of the LMG excited-state structure and the presence of ESQPT. They then discuss the sensitivity of the system to small changes of the Hamiltonian, quantified by the quantum Fisher information (QFI), which shows super-extensive scaling with the system size. Finally, they proceed to introduce metrological protocols exploiting this scaling advantage.

This work provides a valuable contribution to the field of quantum sensing, by showing how the excited-state properties of the LMG could be used to achieve accuracy up to the Heisenberg scaling. As far as I can tell, the results are correct and new (I checked sections 1 to 3, as well as parts of section 4 and 5). I think the authors should also mention quantum sensing in systems showing a dynamical phase transition (for instance [1, 2]), since those are also related to the ESQPT studied here. Overall, I believe this work should be suitable for publication in Scipost, but I would like the following points to be adressed first:

My main concern is the claim that the superextensive scaling of the QFI comes from the presence of the ESQPT. Although I fully agree that the QFI here shows super-extensive scaling, I am not sure it couldn't be obtained otherwise. My reasoning is the following: in Fig.2, we see that the QFI scales like N^2 for *all* values of h. Even for $h = 0$, when we are seemingly far away from the transition, we still obtain super-extensive behavior. Hence, it seems to me that this N^2 scaling may not come from the ESQPT directly; rather, it comes from the fact that we pick up highly-excited states. To be more precise, consider the two Hamiltonians S_x^2/N or S_z constituting the LMG. Taken individually, they do not display any ESQPT; yet their highly-excited states are Dicke states with high dipole moment, which can exhibit super-extensive sensitivity. As the authors state themselves on p.2, one would talk about a critical behavior when we have two phases with a "normal" extensive behavior, and a different, super-extensive behavior occuring *only* at the boundary. This is the case, for instance when we consider the *ground-state* of the LMG, for which the QFI shows $O(N^{4/3})$ scaling at the critical point only, and $O(N)$ elsewhere. Here, it seems to me we can find this super-extensive behavior everywhere. I have included a more detailed discussion in the Appendix below.

To be clear, this doesn't affect the validity or interest of the results presented here; but I think the claims of the paper should be amended. These are threefold: the QFI allows to witness the presence of the ESQPT, we can design a protocol showing super-extensive scaling using excited-state preparation, and this super-extensive scaling comes from the ESQPT. It is this latter claim which I believe to be incorrect. In this regard, there are multiple statements that should be rewritten or supressed: for instance "F_h exhibits a sharp peak close to the critical energy E_c, and its maximum value... increases with the system size N" and "The superextensivity of the QFI... is therefore a signature of the ESQPT" on p.7, "both $\langle E_k | S_z^2 | E_k \rangle$ and $\left\langle \tilde{E}_k \middle| S_z^2 \middle| \tilde{E}_k \right\rangle$ scale as $O(N^2)$ at the critical point" on p.10, "their critical behavior is the key resource for enhanced precision" on p.19.

Other points that must be adressed:

- I found the way $\Sigma^*_{\mathcal{F}}$ was defined a bit confusing. From the caption of Fig.4, I take it it corresponds to the half-peak width of the QFI, but when expressed as a function of E/N? Could the authors write this down explicitely in the main text?
- More generally, I found the notations $\Sigma_{\mathcal{F}_h}(E_k)$ and $\Sigma^*_{\mathcal{F}_h}$ rather cumbersome, I would suggest something like $\Sigma_h(E_k)$ and $\Sigma^*_E(h)$ instead, to lighten up the notation and highlight which parameter one takes the width against.
- After Eq.(8), there is a somewhat complicated argument to conclude that $\langle E_k | S_z | E_k \rangle$ behaves like N^κ, with $\kappa \sim 1.02$. Instead, I would simply and immediately say $\kappa = 1$, since magnetization is an extensive quantity... To which one can add a quick comment stating that numerical analysis confirms this scaling (the 0.02 deviation is much more likely to come from finite-size effects or numerical errors in the implementations than from relevant physical effects, in my opinion).

Minor points/typos:

- After Eq.(12), it could be interesting to show $\rho(E)p(E)$ and $\mathcal{F}(E)$ on the same figure, to illustrate how they overlap when N changes.
- p.8: the notation $\hat{CU}_{2^j \Delta t}$ is fairly cumbersome, I would suggest something like $\hat{C}_U(j)$ instead
- p.9: "For instance, $p_{succ} = 0.9$ imples" → "implies"
- p.13: "also the metrological performances are probabilistic" → "the metrological performances are also probabilistic"
- p.15: "scale with N much slowly than the spacing" → "much more slowly"
- still on p.15, "we show the robustness magnetometric performances" → "the robustness of the magnetometric"
- p.15 still, "We then obtain superextesive QFI" → "superextensive"

1 Appendix

Let us write $|m\rangle$ the eigenstates of S_x; we have then:

$$S_x |m\rangle = m |m\rangle$$
$$S_z |m\rangle = \sqrt{N(N+1) - m(m-1)}\, |m-1\rangle + \sqrt{N(N+1) - m(m+1)}\, |m+1\rangle$$

If h is small enough (more precisely, for $h \ll N^{-2}$), then hS_z can be safely treated as a perturbation, and the eigenstates of the full LMG Hamiltonian are still given by $|m\rangle$, with energies m^2/N.

To compute the QFI, we need to consider terms of the form $\left(\frac{\langle m|S_z|n\rangle}{E_m - E_n}\right)^2$. Let us consider two neighboring highly-excited states, $|m\rangle$ and $|m+1\rangle$, with $m = kN$ and $k = O(1)$. Then we have directly $\langle m| S_z |n\rangle = O(N)$, and $E_{m+1} - E_m = \frac{(m+1)^2 - m^2}{N} = (2m+1)/N = O(1)$. As the authors state shortly before Eq.(4) in the main text, these scalings are ultimately responsible for the superextensive scaling of the QFI. However, as we just showed, this behavior can be obtained for *vanishingly* small value of h, deep in the $h < 1$ phase, where the ESQPT should not play any role. This suggest that the behavior which is obtained here comes purely from the excited-state structure of S_x^2/N, rather than from the competition between S_x^2/N and S_z.

Alternatively, we could also propose a protocol more in line with Ramsey interferometric protocol, in which we prepare the eigenstates of S_x^2/N, then let it evolve under a weak field S_z; in this case, we recover again a N^2 scaling, without relying on critical effects at any point. To conclude, it seems to me that the protocol discussed here may not really be called critical, in that it doesn't leverage the competition between two operators to give rise to a different behavior. Rather, its interest would lie in the preparation scheme, which allows to differ from usual approaches relying on one-axis twisting operations.

References

[1] Q. Guan and R. J. Lewis-Swan. Identifying and harnessing dynamical phase transitions for quantum-enhanced sensing. *Physical Review Research*, 3(3):033199, August 2021.

[2] Lu Zhou, Jia Kong, Zhihao Lan, and Weiping Zhang. Dynamical quantum phase transitions in a spinor Bose-Einstein condensate and criticality enhanced quantum sensing. *Physical Review Research*, 5(1):013087, February 2023.