Review of the manuscript: "Joint distribution of currents in the symmetric exclusion process"

Summary

The authors investigate the joint statistical properties of the currents for the symmetric simple exclusion process (SEP). In particular they obtain integral equations governing the correlations currents-density via a mapping to the integrable Ablowitz-Kaup-Newell-Segur (AKNS) equations.

The main results are summarized in Section 2: the authors derive the joint cumulant generating function of the integrated current Q_t and the generalized current J_t, as well as their correlation with the density of particles at initial and finite times. In the long time regime, these initial and finite time correlations are described by the single-argument functions $\overline{\Phi}$ and Φ. The crux of this work is the derivation of the equations for $\overline{\Phi}$ and Φ which consists in integral equations with boundary conditions given in Section 2.1 (the joint cumulant generating function is obtained through the integration of $\overline{\Phi}$ and Φ in (19)). This ultimately leads to the derivation of the statistical properties presented in Section 2.1. They are discussed and compared to existing results the literature in Section 2.2.
The system can be described using the so-called macroscopic fluctuation theory (MFT). In particular $\overline{\Phi}$ and Φ are "solutions" (Eqs. (47) and (48)) of nonlinear PDEs denoted MFT equations (Eqs. (42) and (43)) with specific boundary conditions. I am not an expert of this theoretical framework cannot judge the limitations of this approach, however I find the comparison between the direct numerical simulation of the MFT equations and the solution given in Section 2 displayed in Figure 1 particularly striking (considering that the solution is not smooth).
The solution of the MFT equations can be easily determined in the low density limit as shown in Section 4. I particularly appreciated the insertion of this simple case which shows that the problem can be mapped to linear diffusion equations via the Cole-Hopf transformation, demonstrating that the low density regime corresponds to a linear problem.
In fact, the authors show that the MFT equations map to the AKNS equations in the finite density regime in Section 5.2.1. Although these equations are nonlinear partial differential equations, they can be solved via the inverse scattering transform (IST). The IST problem investigated in this work is not standard since since it is constituted of both initial and final conditions. Interestingly the authors show in the same section that the initial and final conditions reduce to δ distributions which simplify drastically the IST procedure. The corresponding IST problem is solved in Sections 5.2.2 and 5.3.
The integral equations governing $\overline{\Phi}$ and Φ cannot be solved analytically and the authors use a perturbative approach to solve these equations in Section 6.1; the corresponding current/tracer correlations are given in Section 6.2.

Comments

Overall the manuscript is well-written and the derivations are clearly detailed. The authors present new results concerning the statistical properties of the SEP. The approach adopted here to solve the MFT equations and determine statistical properties via a mapping to integrable nonlinear PDEs is innovative, and I strongly believe that this link will be exploited further by both many body systems and integrable partial differential equations communities. The use of this mapping is well justified here since the boundary conditions for the MFT equations "transforms" into δ distributions in the initial and final conditions for the AKNS equations. Indeed, the IST procedure can be particularly complicated for generic initial conditions, but simplifies in this work. **In my humble opinion this work meets the criteria of novelty, significance and rigor of SciPost Physics and I recommend its publication.**

Please see below a list of minor comments/suggestions:

1. I believe that $\partial_\mu \hat{\psi}_\xi$ should read $\partial_\nu \hat{\psi}_\xi$ in (19).

2. It would be interesting to have a description of the numerical method used to solve the integral equations (8) and (9) leading to Figure 1. Besides does the approximation computed in Section 6.1 compare to the numerical solution of the integral equations?

3. In my humble opinion it would be more insightful to write (75) and (78) by part, similarly to (79), rather than introducing different proportionality coefficients.

4. I find the notations in (108) slightly confusing. Does $M(\pm\infty)$ correspond to the limit $t \to \infty$ of $M(x, t; k)$. If it is the case I would suggest to use the limit notation.

5. This last item is just a comment. I am wondering if the conditions (104) and (110) derived in Section 5.3 to determine the constant α and β can be obtained via the integration of (74) in space:

$$\int_{-\infty}^{+\infty} u(x,0)\,\mathrm{d}x = 1, \quad \int_{-\infty}^{+\infty} v(x,1)\,\mathrm{d}x = \alpha + \beta,$$

   which are well-defined (i.e. no singularity). This could be simplified using the mapping (69) and the boundary conditions (44) similarly to the obtention of the equations (75) and (78). My main point here is that the problem (42),(43),(44) should map to (70),(74) via the transformation (69), independently from the integrable nature of the AKNS equations.

   Although the condition (104) seems to be a direct consequence of the conservation law (42) of the MFT equations. Integrating (42) between $-\infty$ and $+\infty$ with $\partial_x q = \partial_x p = 0$ at $|x| \to \infty$ and using (47),(48) should yield (104).