Referee Report

July 22, 2024

In this manuscript, the authors find a new formula for the Virasoro fusion kernel at $c = 25$ by studying the known expression of $c = 1$ fusion kernel derived in terms of the connection coefficient of the Painlevé equation. The authors shows the shift equations satisfied by modular and fusion kernels, are invariant under the defined Virasoro-Wick rotation ($b \to ib$, $P \to iP$). By observing the parity of the Virasoro-Wick rotated kernels in momentum P, the author find that the kernels after the Virasoro-Wick rotation will not correspond to the physical kernels in the other domain of central charge c. The authors also analyze the meromorphicity properties of fusion, modular kernels for central charges c in different domains and their images under Virasoro-Wick rotation. Provided that the solution of shift equations is unique assuming the meromorphicity in c and momentum P, the authors conclude that the fusion, modular kernels for ($c < 1$, $b^2 \notin \mathbb{Q}$) are not meromorphic functions in momentum P, but distributions. The authors then study the Virasoro-Wick rotation of the crossing equations in Liouville theory as a concrete example. The crossing equations in Liouville theory can be written in terms of the structure constant and the fusion kernel. Given the known expression of the structure constant for timelike Liouville theory ($c \leq 1$), the Virasoro-Wick rotated fusion kernel from $c = 25$ is shown to satisfy the crossing equations of the timelike Liouville theory. Finally, the authors start by showing the relation between fusion kernels with specific values of external weights at $c = 1$ and at $c = 25$, and generalize the conclusion to generic values of external weights. By Virasoro-Wick rotating the two branches of $c = 1$ fusion kernel and then combining them into a meromorphic function, the authors propose a new formula for $c = 25$ fusion kernel. The numerical evidence shows that the new expression is consistent with the known integral formula.

This manuscript is well-written with plentiful interesting discussions and results accompanied by clear derivation and proof. I would definitely recommend this paper for publication in SciPost. The following is a few minor questions and comments which I want the authors to address before publication:

- I am slightly confused about the tetrahedral symmetry of Equation (2.17). From my understanding, the tetrahedral symmetry, which is isomorphic to S_4, can be viewed as permutations among 3-tuples (P_1, P_2, P_s), (P_3, P_4, P_s), (P_1, P_4, P_t), (P_2, P_3, P_t), i.e. four faces of tetrahedron. In Equation (2.27) of Reference [11], this symmetry is manifest, since the prefactor consists of four identical terms which exactly correspond to four 3-tuples listed above. In Equation (2.17) of this manuscript, the integral is identical to Equation (2.27) of Reference [11], while the prefactor doesn't seem to have tetrahedral symmetry manifest. Is Equation (2.17) different from Equation (2.27) of Reference [11] up to some normalization factor from the conformal block?

- In Section 3.1, the authors conjecture a series representation of fusion and modular kernels for $c < 1$ with $b^2 \notin \mathbb{Q}$, and also mention Reference [31] which propose an explicit expression along this direction. That's very fascinating. I'm curious whether there is a physical meaning for the infinite sum over k. Also, for the rational points, i.e. $b^2 \in \mathbb{Q}$, would the c and $26 - c$ kernels have some similar relations, mimicking the relations summarized in Table (4.1) between $c = 25$ and $c = 1$ kernels?

- $c > 25$ kernels preserve the natural inner product between conformal blocks. For $c < 1$ with $b^2 \notin \mathbb{Q}$, if we assume the kernels as distributions in P, does it have any implications on the inner product between $c < 1$ conformal blocks?