Report on *Integrable Deformations from Twistor Space*

In his 1985 paper titled 'Integrable and Solvable Systems, and Relations Among Them' Ward conjectured that many integrable differential equations arise as symmetry reductions of the anti-self-dual Yang-Mills (ASDYM) equations. Since then, an alternative organising principle for 2d integrable models has emerged: 4d Chern-Simons theory (CS4). Building on work of Costello, Bittleston and Skinner proposed that these two approaches might be related via a holomorphic Chern-Simons (HCS) theory on twistor space. The authors investigate this proposal in the context of λ-models: integrable deformations of the 2d Wess-Zumino-Witten (WZW) model.

They begin by identifying alternative boundary conditions for HCS on twistor space. Implementing the descent to space-time via the Penrose-Ward transformation, they obtain a novel 4d integrable field theory (IFT) depending on two group valued fields. Much like the 4d WZW model, this has two semi-local symmetries and its equations of motion follow from the ASDYM equations for the Lax. Reducing HCS by translations along a non-degenerate 2-plane gives CS4 with an unconventional set of boundary conditions and poles: the boundary values of the connection do not lie in an isotropic subalgebra of the defect algebra. Descending in the familiar way to a 2d IFT gives a multi-parameter family of λ-deformations for coupled WZW models. This model has been obtained from CS4 previously, although the description in the paper is novel. Reducing directly on space-time lands on the same 2d IFT. By adapting the translation group to the boundary conditions, it specialises to the λ-model. Finally, the authors consider RG flow of these 2d integrable models, verifying conjectures of Costello regarding the evolution of the meromorphic (1,0)-form ω.

This work represents an important contribution to a burgeoning field of study concerning integrable models in 4 space-time dimensions. Furthermore, it raises the possibility that a wider class of boundary conditions in CS4 are viable, doubtless worth further investigation. I'm happy to say that in my view the paper is clear and effectively written, and it certainly meets the SciPost criterion for originality and significance. I wholeheartedly recommend publication.

I have a few comments which the authors may wish to address:

- Although the equations of motion (EoM) of the two field 4d IFT follow from the ASDYM equations for a suitable Lax, it's not clear to me whether they arise as a partial gauge fixing. If so, this would seem to confirm that the (EoM of the) λ-deformation is an instance of the Ward conjecture.

- Above equation (3.6) the field $\hat{h}$ is extended to $\mathbb{PT} \times [0,1]$, presumably so that its restriction to $\mathbb{PT} \times \{0\}$ coincides with $\hat{h}$. Certainly further assumptions on this extension are needed. From context it seems natural to choose a smooth homotopy to a constant map. Alternatively, fixing an archipelago type gauge on $\hat{h}$ (if attainable) will supply independent homotopies to the constant map for $h, \tilde{h}$.

- I believe section 3.3 on reality conditions and parameters might be improved by including a twistor interpretation. For example: it's immediately clear that Euclidean reality conditions are incompatible with the twistor description, since Ω has a single double pole which cannot be paired with a double pole at an antipodal point. To obtain natural reality conditions on the fields one can work in split signature. Whilst β needs to be real, there's freedom in the reality conditions on $(\alpha, \tilde{\alpha})$ and $(\mu, \tilde{\mu})$. $\tilde{\alpha} = \bar{\alpha}$ seems like a particularly interesting case, as twisting by the anti-involution of $G^{\mathbb{C}}$ corresponding to the real form G implies that $\bar{h} = \tilde{h}$. I expect this reduces the complex two-field model to a real one-field model for a single $G^{\mathbb{C}}$ valued field.

I also have a couple of minor points:

- In footnote 7 reference is made to a 5d Chern-Simons theory, which could refer to a few different models. It might be less ambiguous if referred to as 5d Kähler Chern-Simons.

- I think footnote 8 would be clearer if it read 'More generally, a manifold whose boundary is a disjoint union of copies of $\mathbb{PT}$'.