Second report on manuscript "Mapping a dissipative quantum spin chain onto a generalized Coulomb gas" by Oscar Bouverot-Dupuis

In my previous report, I had already recommended publication of the manuscript in SciPost. A few misprints numbered 2-4 were pointed out by the other referee and have been corrected in the revised version. The second referee also raised the issue of the limit $\nu \to 0$ (point 1) and the authors have responded that the Green's function of the bath operator $(-\nu^2\partial_x^2 - \partial_\tau^2 + \Omega^2)^{-1}$ does not deform confinuously into the one on the local bath operator $(-\partial_\tau^2 + \Omega^2)^{-1}$. They have also inserted a footnote in the manuscript making that argument.

I am not sure the argument of the authors is the correct one. If we insert Eqs. (5) into Eq. (12) we find

$$\mathcal{K}(x,\tau) = \frac{1}{\pi^2\nu}\int_0^{\tau_c^{-1}} d\Omega\alpha\tau_c^{s-1}\Omega^{s+1}K_0\left(\Omega\sqrt{\tau^2+(x/\nu)^2}\right) \tag{1}$$

If we let $\nu \to 0$ for $x \neq 0$, because of the exponential decay of the modified Bessel function for large argument, we obtain

$$\lim_{\nu\to 0}\mathcal{K}(x\neq 0,\tau) = 0, \tag{2}$$

while it is obvious that for $x = 0$,

$$\lim_{\nu\to 0}\mathcal{K}(0,\tau) = +\infty. \tag{3}$$

To verify that $\mathcal{K}(x,\tau)$ behaves as a Dirac delta distribution, we only need to calculate the weight

$$\int dx\mathcal{K}(x,\tau) = \frac{1}{\pi^2}\int_0^{\tau_c^{-1}} d\Omega\alpha\tau_c^{s-1}\Omega^{s+1}\int_{-\infty}^{+\infty} K_0\left(\Omega\sqrt{\tau^2+(x/\nu)^2}\right)\frac{dx}{\nu}, \tag{4}$$

$$= \frac{\alpha\tau_c^{s-1}}{\pi^2\tau^{s+1}}\int_0^{\tau/\tau_c} dw w^{s+1}\int_{-\infty}^{+\infty} du K_0(w\sqrt{1+u^2}), \tag{5}$$

and note that it is independent of ν. Moreover, when $\tau \gg \tau_c$, we can extend the w integration to $+\infty$ to find

$$\int dx\mathcal{K}(x,\tau\gg\tau_c) \sim \frac{\alpha\tau_c^{s-1}}{\pi^2\tau^{s+1}}\int_{-\infty}^{+\infty}\frac{du}{(1+u^2)^{1+s/2}}\int_0^{+\infty} dv v^{s+1}K_0(v) \tag{6}$$

$$\sim \frac{\alpha\tau_c^{s-1}}{\pi^2\tau^{s+1}} \tag{7}$$

$$\sim \frac{\alpha\tau_c^{s-1}\Gamma(s+1)}{\pi\tau^{s+1}}, \tag{8}$$

and recover for $s = 1$ the decay $1/\tau^2$ expected in an ohmic bath. So even though the bath operators do not deform continuously into each other, it is possible to recover the correct limit for $\nu \to 0$ directly from Eq. (12). In other words, the comment 1 of the other referee was incorrect, and the authors have been too cautious in their response and footnote.

I suggest the authors replace their footnote with a derivation of the delta distribution limit of the kernel from Eq. (12) and that that final version of the manuscript gets published in SciPost.