Report on Critical spin chains and loop models with $U(n)$ symmetry, by P. Roux et al

This is an interesting paper which presents new results on $U(n)$ spin chains and loop models. I found the paper a bit difficult to read, though, and I have a number of comments and suggestions of stylistic nature that might improve the presentation and help the reader. The paper will eventually be a very valuable contribution to SciPost.

Section 1.

Subsection Phase diagrams. Are the results stated there for $U(n)$ chains obvious, well known from [11] or elsewhere, or resulting from the discussion of the present paper? This should be clearly stated.

Section 2.

Maybe, add $n > 2$ on the right of eq. (2.3);
since the authors review the theory of $U(n)$ representations, they should be precise: in eq. (2.8), what is the range of summation of indices $\nu, \bar{\rho}, \alpha, \beta, \eta, \theta$? Similar request in (2.11) and (2.12).

Subsec 2.2, line 6. "Lagrangian from fields". The authors presumably mean "Lagrangian bilinear in fields".

On page 11, why are the sites of bottom lines of diagrams called *defects*? This is not meant as the defects in, say, arxiv:2408.08241, so this may be a bit confusing.

That subsection is certainly an impressive display of mathematical erudition. But the physicist readers of SciPost may not be familiar with the walled Brauer algebra, a reference would be welcome, nor with "Specht modules": why not just recall that they are the irreducible representations of the symmetric group (labelled by Young diagrams)? And the notations in (2.26) should be explained.

Section 4.

What is the "conformal algebra $\mathfrak{C}$" on the first line of page 17? presumably $\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$, but this might be explicitly said.
Recall below (4.3) that $r \wedge r'$ is the g.c.d. of r and r: this is said only after (4.13). What is $r \wedge 0$? r, I presume, and ω_1 is nothing else than [1], but then I have difficulties with $\Omega_{(1,0)} = [\,] + U_1(\omega_1 \otimes \bar{\omega}_1 - 2) = [\,] + [1]\overline{[1]} - 2[\,] = [1]\overline{[1]} - [\,]$ which disagrees with (4.6a)?
Page 18, I found the comment in brackets about logarithmic structure particularly cryptic. Could the authors explain what they mean?
In the ingenious calculations of page 20, the factor 1/2 in front of (4.18)

seems to have disappeared in (4.20)...
Top of page 21 and of page 31, $U_{2d} = U_d(X^2 - 2)$ is confusing in view of (4.4), $U_{2d}(Y) = U_d^2(Y^2 - 2)$ (putting $Y = X + X^{-1}$) would be better.

Subsection 4.2. This is the most involved part of the paper but also the hardest to follow.
To help the reader, a graphical representation of u^2 might be added, making for example (4.27c) clear.
More importantly, the authors should elaborate a little bit about the unfamiliar "interchiral algebra", give references if introduced before, and explain why this is the relevant algebra and why they "expect" (4.33-4.34) to hold true.
On page 24, I have difficulties with the discussion that follows (4.38). Why do all e_i annihilate the last three diagrams? For me the action of e_1 on the seventh diagram produces the first. Could the authors explain?
The lines below (4.41) are also difficult to decipher: what is the "major index" of a tableau? Btw, are the T's tableaux or diagrams?
I admired the virtuosity of the authors in the final computations below (4.43) but I admit I didn't check them...

Section 5.
After the strenuous section 4, that section looks simple, clear and convincing...
In the discussion of eqs (5.13-5.14), there seems to be a swap of V_g and V_h.

Section 6.
In the Summary table appears a Rook–Brauer algebra that has not be defined before. Definition or reference ?

References: the authors adopt the strange standpoint of giving references only through their doi. This is cumbersome, and in some cases, results in ... no reference at all, for instance [24], [27], [28] or [34]. Instead of that somewhat dogmatic attitude, I suggest restoring actual references to the journals or to the arxiv. Also a link or a more explicit reference to [18] might be helpful.

Minor typos, notations, etc
one line above (2.17): defined → define
one line below (2.23): simples → simple
page 13: Walled–Brauer → walled Brauer

in eq. (4.4) and the rest of the paper, the choice of U is a *very bad* choice of notation! Up to irrelevant factors of 2, that polynomial U_m as defined in (4.4) is a Chebyshev polynomial of the *first* kind, which is universally denoted T_m, while U_m refers to Chebyshev polynomial of the second kind...
(4.23), an extra)
(4.24), $r-k \ \rightarrow \ m-k$
(4.25): is the notation g, g' really optimal?
two lines below (4.26), site $\rightarrow$ sites
three lines above (5.26), spectrums $\rightarrow$ spectra

A final question/suggestion

The authors introduce the twisted partition function $Z(g)$ on a torus, but do not discuss the effect of modular transformations on τ. There should be relations of the (generalized) Cardy type between $Z(g)$ and its modular transform. Are these relations untractable, useless, or do they tell us something on multiplicities of the representations in (4.1)?