Referee report on the article "Exact finite-size scaling spectra in the quenched and annealed Sherrington-Kirkpatrick spin glass"
by Ding Wang and Lei-Han Tang

The paper analyzes a spectral crossover of the annealed Sherrington- Kirkpatrick model (essentially, in its spherical version) to study how a gap develops below a certain critical temperature. With GOE ensemble of random matrices playing the role of spin interaction matrix in the model, the analysis of the paper is largely based on looking at the behaviour of the Stiltjes transform $g_N(z)$ associated with N GOE eigenvalues. Along this line the authors developed a clever treatment of the Stiltjes transform as $N \gg 1$, which eventually yields explicit predictions for the finite-size scaling of the spin condensation phenomenon in both quenched and annealed case. Those are favourably compared to numerical simulations. Given that the interest in behaviour of annealed spinglass models (ignored for long time) has been recently advocated by Foini and Kurchan, the present work provides some useful insights and should be published.

Still, I believe a few references to earlier literature should be added to the text, to put it in appropriate context. In particular, I can not fully agree with the statement in the end of the paper "The discrete set of eigenvalues at the leading edge of the spectrum can then be resolved exactly to order $N^{-2/3}$, *which appears to have been an open problem before the reported work*" . Not only a lot is known about the mean positions, but also fluctuations of the individual eigenvalues (in particular, the whole gap distribution between two largest/smalles eigenvalues) has been addressed for a related case of GUE matrices, see e.g.

- Anthony Perret, Gregory Schehr
  Near-Extreme Eigenvalues and the First Gap of Hermitian Random Matrices
  J Stat Phys (2014) 156:843–876

and also a lot is known about extreme GOE eigenvalues. In particular, appearance of the Airy equation (see eq.(38) for $Q(u)$) in describing properties of the eigenvalues at the edge of GOE is a classical result, see e.g. section 6.3 of the monograph

- L. Pastur, M. Shcherbina "Eigenvalue Distribution of Large Random Matrices" (AMS, 2011))

Yet I must admit I was not immediately able to recall seeing in the literature a direct treatment of the associated Stiltjes transform $g_N(z)$ for $N >> 1$ in the edge regime by using the scaling ansatz (29) in the ODE satisfied by $g(z)$. The final result via Airy functions, eqs. (34)-(38) is hardly surprising, but I may agree it was an elegant way of arriving to it.

What concerns the positions of resolvent poles/zeros of $Q(u)$ given in eqs. (37)-(40), it reminded me the study of positions of extreme zeroes of Hermite polynomials, described in the Theorem 5.2 in p.7826 of the paper

- Yang Chen and Mourad E H Ismail
  Ladder operators and differential equations for orthogonal polynomials
  J. Phys. A: Math. Gen. 30 (1997) 7817–7829

This is hardly a coincidence as analysis proceeds through not dissimilar Coulomb gas considerations.

Closer to the topic of the present paper, the scaling analysis at the spectral edge, section 3.2.1, looks to me overall not dissimilar to the analysis in the section IV of the paper

- Finite-size critical scaling in Ising spin glasses in the mean-field regime
  T. Aspelmeier, H. G. Katzgraber, D. Larson, M. A. Moore, Matthew Wittmann, and J Yeo
  Phys. Rev. E 93, 032123 (2016)

which I feel also deserves mentioning in the present context.