Referee report on “Refined cyclic RG in Russian Doll model” by V. Montamarri et al.

The Russian Doll model (RDM) suggested in Ref.[19,20] is a simple deformation of the well-known Richardson model for BCS superconductivity in systems with finite number of degrees of freedom. In contrast to the Richardson model, the RDM possesses an interaction that explicitly breaks the time-reversal invariance very much in the same way as magnetic flux does to the interaction matrix element of elections in a metallic ring. So, the new term in the RDM is intimately related with the twisted boundary conditions due to the flux.

An important property of the Hamiltonian of RDM (as well as the Richardson one) is that both of them have a form of a full (generically random) matrix with coupling between all sites. So, they are both zero-dimensional models. It is also important that both models are exactly solvable by Bethe ansatz and equivalent to certain spin-1/2 chains with and without twisted boundary conditions.

The initially suggested in Ref.[19,20] RDM was deterministic with constant off-diagonal terms and equidistant on-site energies (the diagonal entries). A natural extension of this model was therefore to allow for a random diagonal entries in it, keeping the off-diagonal ones still constant and thus fully correlated. Then a new question emerged about the localization properties of eigenstates in such a matrix model (starting from a single-pair sector). This question was addressed in a recent paper [24] and it was shown that the states that were localized in the Richardson model appear to become fractal (non-ergodic extended) in the RDM.

The next peculiar property of the deterministic RDM of Ref.[19,20] was that a major part of the spectrum repeated itself as the size of the matrix N changes. The period of such a repetition was proportional to lnN, which resulted in a cyclic renormalization group for the deterministic RDM.

Thus extension of the model to the random diagonal entries urges to address a question about the fate of the repetition in the spectrum in a so disordered RDM. This is the main goal of the present manuscript.

Although this model is also exactly solvable by the Bethe ansatz, it is not easy to obtain an answer to this question from it. In this manuscript the authors generalized for this purpose the RG method of Ref.[19,20] and used the so called matrix inversion trick suggested in Ref.[25]. The main result of such an analysis is that the cyclic character of RG persists for the random RDM, albeit the period is becoming a function of lnN, the disorder strength and the energy.

It is important that not all the spectrum is subject to repetition. There is a low-energy part of the spectrum, roughly within the range of variation of the random diagonal entries, which has a random nature and do not experience any repetitions. Thus the cyclic RG is only valid beyond this low-energy strip which becomes wide at strong disorder. This is true also for the deterministic RDM where the corresponding strip is determined by the “Debye energy”, although was not mentioned in Ref.[19,20].

So, the repetition in the spectrum is not exact and is an emergent property of relatively high-energy states. This implies that this repetition is not simply a sort of Aharonov-Bohm effect, similar to the one in the metallic rings thread by the flux. It has a more complex nature that has yet to be understood.

I find the subject interesting and the question addressed being a logical extension of the previous works of the authors. However, I had hard time to understand the goal of the manuscript and importance of the results. To do that even a prepared reader who is not exactly in the field has to read all the previous papers and has to have time for contemplation to establish the place of this study in a wider physical framework.

I think this is a defect of the introduction which is comprehensive by citation of the previous works but somehow not physically motivated.

I think that the paper deserves to be published in SciPost Physics but the Introduction should be rewritten to show the place of this research from a wider perspective in physics, its logics and its goal. Otherwise it becomes a peculiar issue about peculiar properties of some smart deformation of a well-known exactly solvable model.

Also in the body of the text there should be more discussion about the physics of the results obtained.

As a matter of fact, this model may have some connection to a very poorly studied problem of superconducting properties of strongly disordered materials driven through superconductor-to-insulator transition by strong magnetic field. There are very interesting and very recent experimental results on this matter which are hard to explain by existing theories.

Some of them are published and concern the superconducting density, or phase stiffness in such materials studied by the electromagnetic response in resonators.

arXiv:2311.15126,

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arXiv:2404.09855

Others, with magnetic-field driven transitions, are yet to be published. Maybe authors find it useful to cite these works as a part of motivation.

Another interesting connection may be with the Anderson localization on Random Regular Graph (RRG) which is locally similar to the Cayley tree. It is known that this problem shares a property of 1/lnN convergence with the Rosenzweig-Porter-like matrix models. The scale lnN is natural for RRG, since it has long circles of the length lnN. Thus in all Rosenzweig-Porter theories, Richardson and RDM including, there must be a hidden tree structure that leads to 1/lnN convergence. It is known that the time-reversal invariant Anderson model on RRG is equivalent to a certain Rosenzweig-Porter random matrix model. So, it is likely that the Anderson model on RRG with properly inserted phase factors on links (which break time-reversal invariance) is equivalent to the disordered RDM. It would be interesting to find such an equivalence.

Both connections mentioned above illustrate that the RDM could be a proxy of many interesting problems and this is another evidence that the research on RDM goes much beyond the Bethe ansatz community and justifies my impression that the paper should be published in SciPost Physics.