Review Quasiperiodic Quadrupole Insulators
$\langle$2406.17602v2/Raul Liquito$\rangle$

The authors of the manuscript have studied the BBH model, a 2D quadrupole insulator featuring a second-order topological phase, under the influence of a quasiperiodic function that is incommensurate with the lattice periodicity. By calculating the quadrupole moment and energy gap sizes, they constructed a phase diagram in terms of a regular hopping parameter λ, and the quasiperiodic modulation strength W. They further study specific paths in the phase diagram to strengthen their claims about phase quasiperiodically induced transitions. They do so by additionally analyzing IPR, fractal dimension, and corner occupation probability.

In my opinion, while some of the results are novel, they may not meet the criteria required for publication in SciPost. Specifically, I find that they do not fully align with the requirement of "opening a new pathway in an existing or a new research direction, with clear potential for multi-pronged follow-up work". The manuscript offers some evidence for topologically nontrivial phases driven by the quasiperiodic modulation. However, it does not offer sufficient novelty. Moreover it has numerous issues of presentation and is severly lacking in its literature overview. I therefore do not recommend this paper for publication in its current format.

Below is a list of the most severe issues I have with the manuscript.

- The paper does not open a new research idea, but rather extends the availability of topological phases already existing with the very broad literature. Moreover, a very similar research paper was uploaded in arXiv before this one: 2406.13535 (now published as PRB **110**, 075422). They have the same overarching conclusion: the clean phase diagram is enlarged by the presence of the quasiperiodic modulation. This paper is not cited by the authors. Additionally, the new region found in the current paper was not found in the previous one. This might be because of the slightly different quasiperiodic function chosen in this paper, but this has to be clearly explained and a thorough comparison with the current work must be done.
- There are many other works exploring how quasiperiodicity affects a higher order quadrupole insulator. For example, 1904.09932 or 2001.07551. A quick online search reveals that there are several more published with similar research questions and results. A more thorough literature review must be conducted by the authors.
- There is an incomplete exposition of the material, with many definitions lacking and inconsistencies in the notation (see below for more detail).

The following list gives more details on issues I found with the manuscript

- In the introduction, the authors state that the effects of quasiperiodicity ermain unstudied in higher-order topological systems. This is false, as examplified by the above references and a very quick online search.
- Some acronyms have never been introduced, such as KPM and DOS (p.2).
- The Hamiltonian in Eq.1 introduces $\mathbf{\Psi}_{\mathbf{R}}$, but then writes that is composed of $c_{\mathbf{k},\alpha}$. This is probably a typo. Moreover, it contains a matrix Λ_y, which is not Hermitian.

- The authors state that the ϕ-dependence should disappear in the thermodynamic limit. They should either prove this statement or give a reference where it is proven.
- The notation for the Bloch Hamiltonian of the clean BBH model is not ideal. The Γ matrices overlap with the previous $\Gamma_{\mathbf{R}}$ matrix.
- The matrix $\Gamma_{\mathbf{R}}$, in Eq.2, contains a term proportional to δ, which was never defined.
- The operator $\hat{n}(\mathbf{R})$ below Eq.6 was never defined. Also, it should be with lowercase $\mathbf{r}$.
- Below Eq.8, there is a typo with $\hat{p}_i$, which should be $p_i(\mathbf{r}) = x_i\hat{n}(\mathbf{r})/L_{x_i}$.
- The authors should provide more details on the derivation of the boundary Green's function in Eq.7.
- The authors do not explain why they average over both phase twists $\boldsymbol{\theta}$ and phase shifts $\boldsymbol{\phi}$. They should also explain why they introduce twisted boundary conditions in the main text, and not only in the appendix, or at least refer to the appendix.
- In Fig.2, they repeat (a) and (b) twice.
- The authors have never defined what IPR_k is.
- The authors have not described how the the normalized localization length ξ/M is calculated from the transfer matrix matrix approach. They should clearly state its definition.

Finally, Some of the phases they claim are topological have very weak signatures, as their numerous results seem to indicate. This makes their relevance questionable:

- In Fig.2(b), region II is subpartitioned into a,b and c. There seems to be a very tiny gap with $\Delta E < 10^{-3}$, which they claim results in the peak that goes toward $Q_{xy} = 0.5$ and $P = 1$. This signal is extremely weak, as they mention themselves. Moreover, since the localization lengths are inversely proportional to this gap size, the zero modes will always hybridize and be pushed away from E=0 in the OBC system.
- The authors, claim that there are multiple reentrant transitions. From the phase diagram, one can only see one such transition. The previously mentioned region II does not offer sufficient evidence to consider it a proper transition. Moreover, it is typically the case that the very small gaps happen because of finite-size effects and disappear in the thermodynamic limit.
- The QPQI phase, which is the main novelty of this work, also suffers from a particular shortcoming: the zero-energy edge modes easily hybridize with edge states when the latter are close to $E = 0$, pulling away the zero modes and increasing their localization lengths. This breaks their degeneracy and one wonders how resilient the quantization of the quadrupole is against these effects.

Due to the significant overlap with prior studies (specifically, 2406.1353), I believe the authors could enhance their chances of publishing their findings by focusing on a specific class of SOTIs and demonstrating similar QPSOTI regions in the phase diagrams. This approach would highlight the novel aspects of their research and present it in a more universally applicable manner.