Referee Report for SciPost Physics on the manuscript "Boundary operator expansion and extraordinary phase transition in the tricritical $O(N)$ model" by Xinyu Sun and Shao-Kai Jian

1 General remarks and recommendation

The preprint "Boundary operator expansion and extraordinary phase transition in the tricritical $O(N)$ model" by Xinyu Sun and Shao-Kai Jian (to be referred as SJ25 in the following) belongs to the series of contemporary papers written by young authors who try to discover and theoretically describe certain unusual phenomena that can happen in semi-infinite systems at bulk criticality, especially within their surface layer. They usually search for scenarios deviating from those described in the original classical paper [1] and [2] (not mentioned in SJ25). Efficient methods of the (Boundary) Conformal Field Theory and Conformal Bootstrap are involved, accompanied by extensive numerical computations.

One of such theoretical methods has been developed quite recently in [3] and [4]. The technique of computing the layer susceptibility (LS) and hence deriving the two-point correlation function via Radon transformation has been successfully applied for the extraordinary transition (EOT) within the framework of the critical ϕ^4 theory in semi-infinite geometries.

Shortly after that, Metlitski [5] showed the possibility of the new "extraordinary-log" boundary critical behavior in three-dimensional ($d = 3$) semi-infinite systems with continuous $O(N)$-symmetric order parameter. An essential feature of this scenario is the $1/(\ln x)^q$ behavior of the surface two-point function for large distances x at the bulk critical point.

In SJ25, the authors modify and generalize the works sketched in the last two paragraphs in order to find out a possibly new boundary critical behavior at the bulk *tricritical* point. The significance of the manuscript under consideration is in that they seem to have succeeded. The idea of studying the semi-infinite tricritical systems is not new as it is seen from references 28, 29, 32, 34 and 54 in SJ25. However, the originality of SJ25 is provided by application of quite recent methods by Dey, Hansen and Shpot, and Metlitski to the problem.

In my opinion, the article by Xinyu Sun and Shao-Kai Jian is original, up-to-date, scientifically sound and interesting, and thus deserves publication in SciPost Physics. It can be accepted with modifications implied by the following comments to be considered by the authors.

2 Comments and suggestions

1. In the Introduction, the authors refer to the tricritical $O(N)$ universality class in Refs. [25–34], a rather chaotic and incomplete list with emphasis on polymers (corresponding just to the specific $N \to 0$ limit). In the present statistical-physics context, a standard classical reference to the subject is [6], which is not mentioned in the list. This reference provides many more different physical applications beyond that mentioned in SJ25. Moreover, consulting this reference would help authors in writing their Conclusion section in this context.

It would also be appropriate to include the reference [54] into this list. It is just an extended version of Ref. [32]. The same could be done with [56].

2. There are numerous situations in statistical physics (disordered $O(N)$ models, systems with cubic anisotropy) where certain critical numbers N_c of the number of components of the order

parameter field appear, like in SJ25. Besides, using the term "critical flavor" for N_c, borrowed from quite a different field of high energy physics looks strange here. I would suggest not to use it.

3. On p. 5, specify that "we establish the extraordinary transition for any N..." *in* $d = 3$.

4. In (18), as well in (91) and (92), the BOE coefficient c_4^L appears with the coefficient $N - 1$, so that $N = 1$ is the special case when $c_4^L(N)$ vanishes. It is the only one of BOE coefficients that has this property. Do you understand the physical reason for this, and can you somehow comment on it?

A related suggestion: indicate that the sum over Δ in (91) runs starting with $\Delta = 4$.

5. If you wish to use the shorthand NLSM in Sec. 5.2 and throughout the paper, introduce it on p. 6 where the term "nonlinear sigma model" appears for the first time.

6. P. 7: saying that "...the expectation of the surface order does not vanish", specify the value of this limit.

At the end of the same paragraph, please explain why do you refer to [22] here.

7. I would encourage the authors to write down "the expansion of $G^T(\xi, z_1, z_2)$ as a series of ξ^n with known prefactors" if you have derived it.

3 Formatting and level of grammar

The overall formatting of the manuscript is good, in particular I welcome a short presentation of the main results in a separate section following the Introduction. Some further suggestions concerning the formatting are:

1. Move the text on p. 3 starting from "The methods developed in this study lay a foundation..." until the end of the paragraph to the end of Conclusions. Anyway, the final sentence in Sec. 6 is almost a direct copy of this fragment.

2. On p. 15, the sentence including (70) is a "copy-paste" of the last sentence in Appendix B2. This repetition is not necessary. It is enough to show the final result (70) in the main text, and in B2 — just refer to it.

One has to check whether there are no other similar repetitions in the text.

Apparently, the paper has been written by a "big boss" (BB) and a young "hard worker" (HW). The manuscript is a compilation of portions written by BB and HW. The parts by BB and HW are different in style and of different grammar level. There are too many grammatical issues in the part of HW. On the other hand, the style of BB is a kind of spoken-language-in-a-hurry and tends to be inaccurate.

— For example: in the second paragraph on p. 21 we learn that "the bulk action will flow to the normal boundary condition". Is it possible for the action to flow to the boundary condition?

— And: "The normal boundary condition is actually equivalent to the extraordinary transition". Can really a boundary condition be equivalent to any transition?

— Or: "the IR description of the extraordinary": extraordinary "what"?

No double check and final reading of the submitted version has been done by the authors. Apparently, HW was already too tired for this, and BB did not find time for that.

Apart from grammatical errors and inconsistencies in mathematical formulations — to be corrected by the authors, I have noticed several essential misprints:

1. P. 19: While this can be done explicitly for the critical $O(N)$ model with $N = 2$
   → While this can be done explicitly for the critical $O(N)$ model with $n = 2$

2. P. 32, the first line below (144): $b = \frac{n-2}{2\pi} \quad \to \quad b = \frac{N-2}{2\pi}$

3. P. 33, (146): $\frac{5}{2}$ in Bessel functions should be indices, as in (43)

4. P. 33, (151): the argument in the second ${}_5F_4$ has to be 1 instead of y

5. P. 33 and 34: Eq. (3.13) from [3] should be changed to (3.15) – (3.16): for some reason, the equations' enumeration in the journal article [3] and its ArXiv version are different.

6. Check the consistency of notations: For example, in equations (138) and (140), as well as in (141) and (143), the same function is denoted by H and H_d.

7. The term "Callan-Symanczyk equation" is copy-pasted from the paper by Metlitski ([5], p.10) without checking. First: the correct spelling is *Symanzik*. Second: the Callan-Symanzik equations are the ones used in the *massive* field theory. In the context of SJ25 it is appropriate to use the term "the renormalization group equation" instead.

I wish HW enough strength and BB enough time to produce the edition of their article publishable in SciPost Physics.

References

[1] H. W. Diehl, *Field–theoretical approach to critical behaviour at surfaces*, in *Phase Transitions and Critical Phenomena* (C. Domb and J. L. Lebowitz, eds.), vol. 10, pp. 75–267. Academic Press, London, 1986.

[2] H. W. Diehl, *The theory of boundary critical phenomena*, *Int. J. Mod. Phys. B* **11** (1997) 3503–3523.

[3] M. A. Shpot, *Boundary conformal field theory at the extraordinary transition: The layer susceptibility to $O(\varepsilon)$*, *J. High Energy Phys.* **2021** (2021) 055.

[4] P. Dey, T. Hansen and M. Shpot, *Operator expansions, layer susceptibility and two-point functions in BCFT*, *J. High Energy Phys.* **2020** (2020) 051.

[5] M. A. Metlitski, *Boundary criticality of the $O(N)$ model in $d = 3$ critically revisited*, *SciPost Phys.* **12** (2022) 131.

[6] I. D. Lawrie and S. Sarbach, *Theory of tricritical points*, in *Phase Transitions and Critical Phenomena* (C. Domb and J. L. Lebowitz, eds.), vol. 9, pp. 1 –161. Academic Press, London, 1984.