For nonequilibrium systems modeled by continuous-time Markov jump processes, graph-theoretic methods have proven to be useful for establishing thermodynamic structures and response theories. This manuscript presents a generalization of response relations in Markov jump processes, extending the authors' earlier results for a single input current to the case of multiple input currents. The main contribution is a proof that the current along any edge of the network is given by a linear-affine function of the stationary currents along a selected set of edges (referred to as admissible edges). This linear-affine relation is further extended to non-stationary currents and holds in the Laplace domain, where a frequency variable conjugated to time is introduced. As consequences of the main result, the authors also show how Kirchhoff's current law–a fundamental relation of network theory–is recovered, and how the linear-affine relation connects to the linear response relation at equilibrium.

The mutual linearity of stationary currents in nonequilibrium networks was a fascinating discovery that, surprisingly, had not been observed prior to the authors' earlier work (Ref. [33]). This manuscript presents a delicate and elegant graph-theoretic and algebraic treatment that identifies the general conditions under which the mutual linearity is preserved as a linear-affine relation, even when multiple input currents are involved. All the proofs and derivations are provided in sufficient detail, and the arguments are coherent and consistent with known results. Overall, the manuscript is clearly written and well organized. Given the generality of main result, which applies to arbitrary network topologies without assuming near-equilibrium conditions, I believe this manuscript will be of significant interest to researchers in nonequilibrium statistical physics and related fields. In conclusion, I recommend the manuscript for publication in SciPost Physics, with optional revision. I provide a few minor suggestions below that may help improve the clarity and accessibility of the manuscript.

Comments (Optional)

1. The mathematical techniques and concepts used in the manuscript are not yet widely adopted among researchers in the field of nonequilibrium statistical physics. An illustration using a concrete example, such as the simple molecular motor considered in Ref. [33], would make the manuscript more accessible.
2. The meaning of the final sentence in Sec. 5 is not sufficiently clear. Could the authors elaborate on what is meant by the response relations in Eqs. (52) and (56) being "phenomenological"? Providing a hypothetical experimental setup or measurement scheme where the relation could be applied would enhance the clarity.
3. As noted in the concluding remarks, the mutual multilinearity does not apply to arbitrary macroscopic input currents. It is restricted to cases where a set of admissible edges are perturbed. However, controlling such admissible edges may not always be feasible in practical applications. Could the authors comment on the practical relevance or possible advantages of the mutual multilinearity beyond its fundamental significance?