Review

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1 Summary

The authors investigate how the real-space range of the exchange interaction J between localized spins on flat-band lattices depends on (i) the average of quantum metric (QM) and (ii) the gap to the nearest dispersive bands. A striking outcome is that the B–B coupling does *not* diminish—even though the distance between neighbouring B orbitals increases with the dilution parameter n. This counter-intuitive behaviour is traced to the structure of the flat-band eigenstates, whose growing quantum metric enhances both the coupling amplitude and its decay length. The analysis is technically sound and novel enough, and I believe the work merits publication after the very minor points listed below are addressed.

- **Definition of $R = a_n$.** The notation $R = a_n$ appears in Fig. 3(a) but is never defined. Please add a one-line explanation.
- **Cross-references to the Supplementary Material.** Whenever a central statement relies on derivations in the SM, please cite the exact SM section or equation. For instance, the sentence "At large distances, the coupling is controlled by the contribution that originates from the two dispersive bands ..." explains the algebraic $1/R^4$ tail in the diamond chain, but the supporting calculation is only in SM. Adding an explicit pointer there (and in similar places) will greatly help readers.