Referee report on "Vacua, Symmetries, and Higgsing of Chern-Simons Matter Theories"

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The paper discusses a novel approach to characterize the maximal branches of the moduli spaces of vacua of 3d $\mathcal{N}=4$ Chern-Simons-matter theories.

Early examples where analyzed by Jafferis-Yin and Hosomichi-Lee-Lee-Lee-Park. However, due to the inherent difficulties of a pure field-theoretic analysis, most recent checks focused on the study of Hilbert series, plus (sporadic) examples of sphere partition functions for suitably chosen Chern-Simons levels.

The combination of brane setups, magnetic quiver and Hilbert series dualization employed in the paper allows for a more systematic approach. In particular, for theories that are dual to 3d N=4 quivers without CS couplings, the authors construct the Hasse diagram of the two maximal branches thanks to a detailed analysis of the duality map, combined with the more standard techniques to analyze the moduli spaces of vacua without CS terms. The authors observe from the brane setup that certain monopole operators must have equal fluxes w.r.t gauge nodes between the two NS5-branes, which is a neat and efficient way to uncover this feature, that would be hard to systematize by constructing gauge-invariant monopole operators "by hand".

For theories which do not admit a description without CS couplings, this work proposes a magnetic quiver. The proposal is motivated by a number of explicit examples. The authors further extend the magnetic quiver proposal to theories with only $\mathcal{N}=3$ supersymmetry, one magnetic quiver per maximal branch, and successfully check their proposal against earlier work of Assel. However, in this less supersymmetric case the authors only discusses Abelian examples.

The content is innovative and interesting for theoretical physicists working in the field; it may also be of interest to mathematicians working on the foundations of 3d $\mathcal{N}=4$ Coulomb branches. The paper is clearly written, the necessary steps are explained in detail but in a neat and not verbose manner. Besides the main claims about moduli spaces of vacua of CS-matter theories, the paper includes various additional interesting results, such as the explicit calculations in Appendix B and the no-go theorem in Appendix C.7.

Overall, I consider that the content and presentation of the work meet the acceptance criteria. I strongly recommend the paper be published in SciPost Physics, provided some minor clarifications are incorporated:

- In the computation of the superconformal index, it would be good to explain how the sum over magnetic fluxes is dealt with.

  For example in Eq.(2.6) the authors compute the index perturbatively in x, thus each term should include a sum over topological sectors from the value of the gauge curvature on S^2 (e.g. the sum over l in [Ref. 52, Eq.(2.1)]). What is the fugacity associated to such summation (e.g. the analogue of the fugacity w in [Ref. 52, Eq.(2.1)])? Are the results at each order in x a resummation of the contribution from (infinitely many) topological sectors? Or do only finitely many values of the sum contribute at each order?

The same applies to the other computations of the index. I would expect that, for p unitary gauge nodes, each order in x fixes a linear combination of the integers l_i ($i = 1, \dots, p$) and there are $p-1$ remaining series to resum.

- Relatedly, the authors exchange the perturbative expansion in x with the infinite sum over topological sectors. Do the authors also exchange the sum over topological sectors with the residue integral? If so, they should explain why this is justified.

These are the most important corrections or clarifications that I request.

Additionally, I list here minor comments and suggestions.

(1) Introduction (page 2), second paragraph, sentence "For 3d N=4 theories, $\mathcal{M}$ contains two distinguished components". It would be better to clarify there that the sentence refers to 3d N=4 theories without CS term, as the immediately preceding sentence refers to N=4 CSM theories.

(2) Section 2.1. The authors claim that D3-D3 strings from D3-brane segments ending on the opposite sides of a NS5-brane give rise to *twisted hypers*, whereas D3-D3 strings from D3-brane segments ending on the opposite sides of a (1,1)-brane give rise to a *hyper*.

Could the authors please explain why, or mention a reference?

I mean that, in the discussion of 3d N=4 theories without CS terms, it is customary to assign *hypers* to D3-D3 strings across and NS5-brane, so I would naively expect hypers and twisted hypers to appear exchanged than the conventions of this work.

(3) Section 2.1 (page 8) footnote 3. It is written that x is the fugacity for the R-symmetry, and half-integer powers of x appear. It would be useful to write explicitly in the text that conventions in which the $U(1)$-charges are half-integer quantized (as opposed to integer-quantized).

(4) Section 2.1 (page 8) figure 3, caption. It is written that "while the FI parameters are denoted in green," although the green Y_i in the picture are the *fugacities*, not what is customarily called FI parameter.

(5) Appendix C.3 (page 54) figure 38, caption. Same as the previos point for FI parameters/fugacities.

(6) Section 4 (page 34) figure 21, caption, sentence "intersected by n D5s". The D5 should be NS5.

(7) Section 4 (page 35) second line. What do the authors means by "firstly, $\mathrm{MQ}_{A/B}$ depend on the brane arrengement"? Is it simply that, from different configurations giving equivalent 3d theories, one may read off different magnetic quivers with same Coulomb branch?

(8) Section 5.1 (page 36) below Eq. (5.1) "an $A_{|\kappa|-1}$ singularity" should be an $A_{|\kappa|+1}$ singularity.

(9) Section 5.1 (page 36) bullet point above Eq. (5.3), sentence "replaces $(1,\kappa)$ by $(\kappa,1)$ 5-branes". To be very precise, I think the S-transformation gives $(\kappa,-1)$ 5-branes.

(10) Appendix C.1 (page 52) between Eq.s (C.4) and (C.5), sentence "In terms of the theory's coupling constant". It is the coupling constant of the 4d theory on the D3-brane, not the low-energy 3d theory.

(11) Appendix C.4 (page 57) figure 42. Step 2 of the figure, right-most piece of the first line of step 2. Should the identity wall be there? I thought it transformed into $\mathcal{S} \times \mathcal{S}^{-1}$.

(12) Appendix C, a minor comment on notation. I understand that the authors use boldface-I for the identity wall and boldface-1 for the $SL(2,\mathbb{Z})$ identity matrix. It is certainly a good choice. However, in figures 42 and 46, there are identities such as $\mathcal{S}\mathcal{S}^{-1} = \mathbb{I}$, while in the text (e.g. point 3 in C4) the notation is $\mathcal{S}\mathcal{S}^{-1} = \mathbb{1}$. Maybe it is worth mentioning that $\mathcal{S}$ in the figure is actually a shorthand for its representation.