Report of the manuscript titled

“Diffusive hydrodynamics of hard rods from microscopics’

submitted to SciPost

The manuscript provides an explicit microscopic derivation of hydrodynamics (HD) equations for hard rods that describe large-scale motion. These HD equations include diffusive effects, which appear as first-order corrections in the hydrodynamic gradient expansion. Starting from a local equilibrium GGE state exhibiting large deviation scaling with respect to a variation length scale ℓ, the authors compute the evolution of the average of the empirical phase space density $\rho(x,p)$ of the quasiparticles.

Inspecting the evolution of a quasiparticle starting from (x_1,p_1), they find that the average position $\langle X_{\rm e}(t)|x_1,p_1)\rangle$ at order $O(\ell^0)$ is given by the Euler characteristics. This is accompanied by two important corrections at $O(1/\ell)$, originating from initial randomness and long-range correlations among the densities that develop over the Euler space-time scale. Incorporating these corrections into the phase space density $\rho(x,p)$ of the quasiparticles, they find ‘diffusive’ corrections at $O(1/\ell)$ to the Euler part of the current associated with the phase space density. Their explicit expression shows that this diffusive correction originates from the long-range correlations that evolve at the Euler space-time scale.

This result is strikingly different from previously derived Navier-Stokes (NS) hydrodynamics for hard rods. First, the NS equation is time-reversal asymmetric, while the new equation, coupled with the Euler equation of the two-point correlations, is time-reversal symmetric. Second, the NS equation applies to a local equilibrium state with no long-range correlation, whereas the equation derived in the current paper applies to general inhomogeneous states that can admit long-range correlations. I find the main result to be very interesting, the derivation novel, and it demonstrates an alternative mechanism of diffusion other than the Green-Kubo mechanism. It further makes important and significant improvement over the existing hydrodynamics of hard-rod gas beyond the Euler scale. I would like to recommend publication of the paper; however, I feel the writing of the paper can be improved if a few things are clarified further. Below I list these points along with a few queries.

- The new correction to the Euler GHD of hard rods is being called diffusive, although from equation (9) it is not obvious. However, its form in local equilibrium (eq. (3)) makes this clear. It would be great to clarify this point.

- The set of equations (9) and (55) being time-reversal symmetric, they do not show an ’arrow of time'. Has this also been tested numerically with the definition of entropy in eq. (38) which does not include two-point correlation in its definition.

- The correction to the average position $\langle X_{\rm e}(t)|x_1, p_1)\rangle$ of the quasiparticle tagged by the initial position and velocity (x_1, p_1) receives two corrections at $O(1/\ell)$: one from initial fluctuations and the other from long-range correlation. The former contribution is easy to understand.

  Is it possible to understand the second mechanism physically?

  Also, does the first contribution give rise to the Kubo diffusion (NS term) in Eq. (7)?

  Is the exact cancellation of the NS term and the $C^{\rm n}_{\rm LR,asym}$ term universal? If possible, it would be useful to make some intuitive comment on why such an exact cancellation occurs.

- Page (9) after Eq. (13): In order to obtain a macroscopic forward derivative in Eq. (9), one has to evaluate the RHS of eq. (9) at $t = 0^+$ in macroscopic scale. This calculation has also been shown in eq. (D.20). Is it feasible to get the same solution by solving the coupled Eqs. (7) and (55) starting from a local GGE state, instead of the alternative computation presented in Appendix D?

- Would one find such a 'diffusion' term originating from $C^{\rm n}_{\rm LR}$ in non-integrable systems as well that support ballistic transport?

- Typos:
  Eq. (11): It seems a t is missing from the argument $\rho_{\rm e}(..., y - v^{\rm eff}\epsilon, q)$,
  Eq. (55): ++
  Caption of Fig. (3): It seems the reference to the theoretical prediction is not correct.

- I feel moving the discussion on $X(t, x_1, p_1)$ along with Eqs. (34) and (35) immediately after Eq. (30) would be better for the readers.

- p12, after Eq. (38): I feel it would be useful for the reader to elaborate the following statement a little more: "However, because the structure of the state remains invariant under time and the hard rods dynamics do not have any memory, this immediately implies that (38) holds at all times."