Referee Report

SciPost Physics

Manuscript: *Role of impurity statistics and medium constraints in polaron–polaron interactions*

General Assessment

This manuscript presents a comprehensive theoretical study of polaron-polaron interactions, addressing a long-standing (and even controversial problem) in the physics of quantum impurities. The authors introduce a framework that shows how impurity statistics and constraints on the medium response (fixed density versus fixed chemical potential) control the sign and magnitude of effective quasiparticle interactions.

The work is technically solid, conceptually clear, with a level of detail much appreciated, and provides an overlooked but physically perspective that can be relevant for current and future experiments in both condensed matter and atomic physics.

I find therefore, the manuscript to be of clear interest to the SciPost Physics readership and I recommend acceptance of the manuscript for publication. Below I list of minor points whose discussion could further strengthen the manuscript.

Minor points and suggestions

1. **Landau functional form of the energy.**

   For fermionic impurities, the energy density is written in a Landau–Pomeranchuk form that includes an explicit kinetic–energy term $\mathcal{E}_{\mathrm{kin},\sigma} \propto n_\sigma^{5/3}$. I think this is OK, but can be a bit confusing for the following reason.

   If I understand correctly (perhaps I'm misunderstood something), the expression the authors show follows from the energy Landau's functional

   $$E = E_0 + \int d^dk\, E_{\mathrm{pol}}(\mathbf{k})\, n_{\mathbf{k}} + \tfrac{1}{2}\int d^dk\, d^dk'\, F(\mathbf{k},\mathbf{k}')\, n_{\mathbf{k}} n_{\mathbf{k}'}, \tag{1}$$

   the authors then approximate

   $$E_{\mathrm{pol}}(\mathbf{k}) = E_{\mathrm{pol}} + \frac{k^2}{2m*}, \tag{2}$$

   and it is the integral in 3D

   $$\int_{k<p_F} d\mathbf{k}\frac{k^2}{2m*} = \frac{p_F^5}{20\pi^2 m^*} = \frac{3}{5}E_F\, n_F = \frac{3}{5}\frac{(6\pi^2)^{2/3}}{2m^*}\, n_F^{5/3}. \tag{3}$$

   I think that here, some points should be clarified. Two minor points: a) the quadratic expansion of the polaron energy, then, b) the polaron mass is taken to be momentum independent. Finally and quite importantly, that in the Landau's functional form for the total energy, this term is incorporated in the single particle dispersion, and that in this form, its contribution is linear in $n_{\mathbf{k}}$. The non-linearity is for the total density n_F, which is not the relevant to obtain the mediated interaction for two impurities with momentum $\mathbf{p}$ and $\mathbf{p}'$.

This comes with perhaps my only main criticism: I find confusing the momentum labeling of the equations. For instance, in Eq. 8a and 8b, the authors derive the polaron energy, and the effective interaction from derivatives of n_σ, which is unclear (at least to me) what does n_σ means at this stage. Are the polaron energy and effective interaction independent of the polaron momentum $\mathbf{p}$? Later, they give to these quantities momentum dependence and they show that indeed, the momentum dependence is crucial. Did I miss something?

2. **Quasiparticle interactions and renormalization factors.**
The mediated interaction is obtained at second order in the impurity-medium coupling, scaling as g^2. In standard quasiparticle language, effective interactions are often renormalized by residue factors associated with the quasiparticle weight. Since the present calculation is already of order g^2, then consistently with perturbation theory the residue is taken to be one. It would be then useful to briefly mention whether is polaron-polaron interactions or impurity-impurity interactions.

   In this same line, in several expressions (e.g. around Eq. (42) and related formulas), bare impurity dispersions are used inside energy denominators, which is also consistent with perturbation theory, but could be mentioned.

3. **Role of the medium-medium interaction g_{bb}.**
In earlier works by some of the authors, it has been emphasized that the validity of perturbation theory for the single polaron depends not only on the impurity–medium coupling g but also crucially on the medium-medium interaction g_{bb}, with a breakdown of the quasiparticle picture as $g_{bb} \to 0$. In the present manuscript, the induced interaction scales as g^2/g_{bb}, suggesting that polaron-polaron interactions are likewise controlled by g_{bb}. Can the authors comment?

4. **Frequency dependence and retardation effects.**
The mediated interaction is evaluated on-shell, following the perturbative approach employed. It may nevertheless be useful to briefly comment on retardation effects and the role of finite propagation speed in the medium (for instance, the speed of sound in a Bose gas). A short discussion of how retardation may affect experimental observability.

5. **Order of limits in the low–energy, low–momentum regime.**
In the discussion of the polaron interaction in the limits $\omega \to 0$ and $|\mathbf{p} - \mathbf{p}'| \to 0$ (in particular around Eq. (66)), the order in which these limits are taken leads to different results. Am I right?.

6. **Landau quasiparticle interactions versus mediated interactions.**
(Perhaps it is now an open question) It may be beneficial to include a short conceptual discussion clarifying the distinction between Landau quasiparticle interactions and medium–mediated interactions in the present context. In particular, the manuscript emphasizes the role of direct and exchange terms under different medium constraints. From this perspective, one might ask which effective interaction should be used if one were to construct, for example, a ladder approximation for mediated interactions, and how this choice depends on the imposed constraint on the medium. Even a brief qualitative discussion would help guide future theoretical developments.

7. **Robustness of limit** $|\mathbf{p} - \mathbf{p}'| \to 0$
(Perhaps it is also an open question) Several results rely on a qualitative distinction between the cases $\mathbf{p} = \mathbf{p}'$ and $\mathbf{p} \neq \mathbf{p}'$, with an abrupt change of the polaron-polaron interaction when the two impurity momenta become exactly equal. While this reflects a genuine change in impurity degeneracy and the associated exchange structure, it would be interesting to comment on the robustness of this behavior beyond the strictly zero-temperature, infinite-lifetime limit. In particular, do finite temperature effects, finite quasiparticle lifetimes, or some other effects smooth the distinction between $\mathbf{p} = \mathbf{p}'$ and $\mathbf{p} \to \mathbf{p}'$?